JOHN BARNARD
STEVE LUND
USPAS
JANUARY 2004

II ENVELOPE EQUATIONS

PARAXIAL RAY EQUATION

ENVELOPE EQUATIONS FOR AXIALLY SYMMETRIC BEAMS

CARTESIAN EQUATION OF MOTION

ENVELOPE EQUATIONS FOR ELLIPTICALLY SYMMETRIC BEAMS

START WITH NEWTON'S EQUATION WITH THE LORENTZ FORCE:

$$\frac{d\underline{P}}{dt} = q(\underline{E} + \underline{v}\times\underline{B})$$

In cartesian coordinates:

$$\frac{d}{dt}(\gamma m\dot{x}) = \dot{\gamma} m\dot{x} + \gamma m\ddot{x} = q(E_x + \dot{y}B_z - \dot{z}B_y)$$

$$\frac{d}{dt}(\gamma m\dot{y}) = \dot{\gamma} m\dot{y} + \gamma m\ddot{y} = q(E_y + \dot{z}B_x - \dot{x}B_z)$$

$$\frac{d}{dt}(\gamma m\dot{z}) = \dot{\gamma} m\dot{z} + \gamma m\ddot{z} = q(E_z + \dot{x}B_y - \dot{y}B_x)$$

In cylindrical coordinates: (use $\frac{d\hat{e}_r}{dt} = \hat{e}_\theta\dot{\theta}$; $\frac{d\hat{e}_\theta}{dt} = -\hat{e}_r\dot{\theta}$)

$$\frac{d}{dt}(\gamma m\dot{r}) - \gamma m r\dot{\theta}^2 = q(E_r + r\dot{\theta}B_z - \dot{z}B_\theta) \qquad (I)$$

$$\frac{1}{r}\frac{d}{dt}(\gamma m r^2\dot{\theta}) = q(E_\theta + \dot{z}B_r - \dot{r}B_z) \qquad (II)$$

$$\frac{d}{dt}(\gamma m\dot{z}) = q(E_z + \dot{r}B_\theta - r\dot{\theta}B_r) \qquad (III)$$

When $\frac{\partial}{\partial\theta} = 0$:

$$\underline{E} = -\nabla\Phi - \frac{\partial\underline{A}}{\partial t} = \hat{e}_r\left[-\frac{\partial\Phi}{\partial r} - \frac{\partial A_r}{\partial t}\right] + \hat{e}_\theta\left[-\frac{\partial A_\theta}{\partial t}\right] + \hat{e}_z\left[-\frac{\partial\Phi}{\partial z} - \frac{\partial A_z}{\partial t}\right]$$

$$\underline{B} = \nabla\times\underline{A} = \hat{e}_r\left[-\frac{\partial}{\partial z}A_\theta\right] + \hat{e}_\theta\left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right] + \hat{e}_z\left[\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta)\right]$$

$$qr(E_\theta + \dot{z}B_r - \dot{r}B_z) = q\left(-\frac{\partial rA_\theta}{\partial t} - \dot{z}\frac{\partial rA_\theta}{\partial z} - \dot{r}\frac{\partial}{\partial r}(rA_\theta)\right)$$

$$= -q\left[\frac{\partial}{\partial t} + \underline{v}\cdot\frac{\partial}{\partial \underline{x}}\right](rA_\theta)$$

$$= -q\frac{d(rA_\theta)}{dt} \qquad (IV)$$

Eqtn II & IV $\Longrightarrow$ $\frac{d}{dt}(\gamma m r^2\dot{\theta} + qrA_\theta) = 0$

$$\underline{p} = p_r \hat{e}_r + p_\theta \hat{e}_\theta + p_z \hat{e}_z$$

where
$$p_r = \gamma m \dot{r}$$
$$p_\theta = \gamma m r \dot{\theta}$$
$$p_z = \gamma m \dot{z}$$

$$\frac{d\underline{p}}{dt} = \dot{p}_r \hat{e}_r + p_r \dot{\hat{e}}_r + \dot{p}_\theta \hat{e}_\theta + p_\theta \dot{\hat{e}}_\theta + \dot{p}_z \hat{e}_z$$

$$\Rightarrow \frac{d\underline{p}}{dt} = (\dot{p}_r - p_\theta \dot{\theta})\hat{e}_r + (p_r \dot{\theta} + \dot{p}_\theta)\hat{e}_\theta + \dot{p}_z \hat{e}_z$$

WHERE WE HAVE USED:

[redacted] $\frac{d\hat{e}_r}{dt} = \hat{e}_\theta \dot{\theta}$ & $\frac{d\hat{e}_\theta}{dt} = -\hat{e}_r \dot{\theta}$

$$\Rightarrow \frac{d\underline{p}}{dt} = \left[\frac{d}{dt}(\gamma m \dot{r}) - \frac{d}{dt}(\gamma m \dot{r} \dot{\theta}^2)\right]\hat{e}_r$$

$$+ \underbrace{\left[\gamma m \dot{r} \dot{\theta} + \frac{d}{dt}(\gamma m r \dot{\theta})\right]}_{= \frac{1}{r}\frac{d}{dt}(\gamma m r^2 \dot{\theta})} \hat{e}_\theta$$

$$+ \frac{d}{dt}(\gamma m \dot{z})$$

CONSERVATION OF CANONICAL ANGULAR MOMENTUM

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DEFINE $p_\theta = \gamma m r^2 \dot{\theta} + q r A_\theta$

$$\frac{d}{dt} p_\theta = 0$$

(CONSERVATION OF CANONICAL ANGULAR MOMENTUM)

NOTE THAT THE FLUX ENCLOSED BY A CIRCLE OF RADIUS r

$$\psi = \int \underline{B} \cdot d\underline{A} = \int (\nabla \times \underline{A}) \cdot dA = \oint \underline{A} \cdot d\underline{l} = 2\pi r A_\theta$$

$$p_\theta = \gamma m r^2 \dot{\theta} + \frac{q}{2\pi}\psi$$

IS CONSERVED ALONG AN ORBIT IN AXISYMMETRIC GEOMETRIES

"EXTERNAL"
ELECTRIC & MAGNETIC FIELDS WITH RADIAL SYMMETRY

(REISER SECTION 3.3)

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CONSIDER THE EXTERNAL FIELD: (TIME STEADY VACUUM SOLUTION)

$\nabla \times \underline{B} = 0$ & $\nabla \times \underline{E} = 0$

$\nabla \cdot \underline{B} = 0$ & $\nabla \cdot \underline{E} = 0$ $\Rightarrow$ $\underline{E}$ & $\underline{B}$ $= -\nabla \Phi$

Let $\Phi = \sum_{\nu=0}^{\infty} f_{2\nu}(z) r^{2\nu}$ $\quad \nabla^2 \Phi = 0 \Rightarrow \Phi = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} \frac{\partial^{2\nu} f(0,z)}{\partial z^{2\nu}} \left(\frac{r}{2}\right)^{2\nu}$

$$\Rightarrow \Phi = \Phi(0,z) - \frac{1}{4}\frac{\partial^2 \Phi(0,z)}{\partial z^2} r^2 + \frac{1}{64}\frac{\partial^4 \Phi(0,z)}{\partial z^4} r^4$$

Let $B_z(0,z) = B(z)$ & Let $\Phi(0,z) = V(z)$

$$B_z(r,z) = B(z) - \frac{r^2}{4}\frac{\partial^2 B}{\partial z^2} + \frac{r^4}{64}\frac{\partial^4 B}{\partial z^4} + \ldots$$

$$B_r(r,z) = -\frac{r}{2}\frac{\partial B}{\partial z} + \frac{r^3}{16}\frac{\partial^3 B}{\partial z^3} + \ldots$$

$$\Phi(r,z) = V(z) - \frac{1}{4}V'' r^2 + \frac{r^4}{64}\frac{\partial^4 V}{\partial z^4}$$

$$\psi \cong \pi r^2 B(z)$$

$$\nabla^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{\partial^2 \phi}{\partial z^2} = 0$$

$$f(r,z) = \sum_{\nu=0}^{\infty} f_{2\nu}(z)\, r^{2\nu} \quad = f_0 + f_2 r^2 + f_4 r^4$$

$$\sum_{\nu=1}^{\infty} \left[2\nu + 2\nu(\nu-1)\right] f_{2\nu}\, r^{2\nu-2} + \sum_{\nu=0}^{\infty} f''_{2\nu}\, r^{2\nu} = 0$$

Paraxial Ray Equation

$$(\mathrm{I}) \Rightarrow \underset{\substack{\uparrow \\ \text{INERTIAL}}}{\frac{d}{dt}(\gamma m \dot{r})} - \underset{\substack{\uparrow \\ \text{CENTRIFUGAL}}}{\gamma m r \dot{\theta}^2} = q\Big(\underset{\substack{\uparrow \\ E_r \\ \text{external}}}{\frac{V''}{2} r} + \underset{\substack{\uparrow \\ v_\theta B_z \\ \text{external}}}{r\dot{\theta} B}\Big) + q\Big(\underset{\substack{\nwarrow \quad \nearrow \\ \text{SELF FIELDS}}}{E_r^{self} - v_z B_\theta^{self}}\Big)$$

Now use s as independent variable $\quad v_z\,dt = ds$

$$v_z \frac{d}{ds}(\gamma m v_z r') - \gamma m v_z^2 r \theta'^2 = q\left(\frac{V''}{2} r + r v_z \theta' B\right) + q\left(E_r^{self} - v_z B_\theta^{self}\right)$$

Expanding 1st term and $v_z \cong v$, and dividing by $\gamma m v^2$:

$$r'' - r\theta'^2 + \frac{\gamma'}{\beta^2 \gamma} r' = \frac{q}{\gamma m \beta^2 c^2}\left(\frac{V''}{2} r + r\beta c\, \theta' B + E_r^{self} - v_z B_\theta^{self}\right) \qquad (P1)$$

Using canonical momentum, eliminate θ' via

$$\theta' = \frac{p_\theta - \frac{q\psi}{2\pi}}{\gamma m r^2 \beta c} = \frac{p_\theta}{\gamma m r^2 \beta c} - \frac{qB}{2\gamma\beta m c} \equiv \frac{p_\theta}{\gamma m r^2 \beta c} - \frac{\omega_c}{2\gamma\beta c}$$

where we define $\omega_c = \dfrac{qB}{m}$

Adding the two θ' terms in the equation (P1)

$$-r\theta'^2 - \frac{r\omega_c \theta'}{\gamma\beta c} = \frac{-p_\theta^2}{\gamma^2 m^2 r^3 \beta^2 c^2} + \frac{p_\theta \omega_c}{\gamma^2 m \beta^2 c^2 r} - \frac{r\omega_c^2}{4\gamma^2\beta^2 c^2}$$

$$\qquad - \frac{p_\theta \omega_c}{\gamma^2 m \beta^2 c^2 r} + \frac{r\omega_c^2}{2\gamma^2 \beta^2 c^2}$$

$$= \frac{-p_\theta^2}{\gamma^2 m^2 r^3 \beta^2 c^2} + \frac{r\omega_c^2}{2\gamma^2\beta^2 c^2}$$

So equation (P1) becomes:

$$r'' + \frac{\gamma'}{\beta^2\gamma} r' = \frac{q}{\gamma m \beta^2 c^2}\left(\frac{V''}{2} r\right) + \frac{r\omega_c^2}{2\gamma^2\beta^2c^2} + \frac{P_\theta^2}{\gamma^2 m r^3 \beta^2 c^2} + \frac{q}{\gamma m \beta^2 c^2}\left[E_r^{self} - v_z B_\theta^{self}\right] \quad (P2)$$

Now $\gamma' mc^2 = q\dfrac{\underline{E}\cdot\underline{v}}{v_z}$ so $\gamma'' \approx -\left(V'' + \dfrac{\partial^2\phi^{self}}{\partial z^2}\right)\dfrac{q}{mc^2}$

CALCULATING $\dfrac{q}{\gamma m\beta^2c^2}\left[\dfrac{V''}{2} r + E_r^{self} - v_z B_\theta^{self}\right]$:

$$\nabla^2\phi^{self} = -\frac{\rho}{\varepsilon_0} \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = -\frac{\rho}{\varepsilon_0} - \frac{\partial^2\phi^{self}}{\partial z^2}$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = -\frac{r\rho(r)}{\varepsilon_0} - \frac{r\partial^2\phi^{self}}{\partial z^2}$$

$$r\frac{\partial\phi}{\partial r} = -\frac{1}{2\pi\varepsilon_0}\int_0^r 2\pi r\,\rho(r)\,dr - \frac{r^2}{2}\frac{\partial^2\phi}{\partial z^2}$$

$$= -\frac{1}{2\pi\varepsilon_0}\lambda(r) - \frac{r^2}{2}\frac{\partial^2\phi^{self}}{\partial z^2}$$

$$\Rightarrow E_r^{self} = \frac{\lambda(r)}{2\pi\varepsilon_0 r} + \frac{r}{2}\frac{\partial^2\phi^{self}}{\partial z^2}$$

$$\nabla\times\underline{B} = \mu_0\underline{J} \Rightarrow 2\pi r B_\theta = \mu_0\int_0^r 2\pi r\, J_z(r)\,dr = \mu_0 v_z\lambda(r)$$

$$B_\theta^{self} = \frac{\mu_0 v_z\lambda(r)}{2\pi r} = \frac{v_z}{c^2}\frac{\lambda(r)}{2\pi\varepsilon_0 r}$$

$$\left[\frac{V''}{2}r + E_r^{self} - v_z B_\theta^{self}\right] = \left[\frac{r}{2}\underbrace{\left(V'' + \frac{\partial^2\phi^{self}}{\partial z^2}\right)}_{-\frac{mc^2}{q}\gamma''} + \underbrace{\left(1-\frac{v_z^2}{c^2}\right)}_{1/\gamma^2}\frac{\lambda(r)}{2\pi\varepsilon_0 r}\right]$$

So equation (Pz) becomes: "THE PARAXIAL RAY EQUATION:"

$$\underbrace{r'' + \frac{\gamma'}{\beta^2\gamma} r'}_{\text{INERTIAL}} + \underbrace{\frac{\gamma''}{2\beta^2\gamma} r}_{\substack{E_r \\ \text{(CONVERGENCE OF FIELD LINES)}}} + \underbrace{\left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r}_{\substack{V_\theta B_z \\ -\text{CENTRIFUGAL}}} - \underbrace{\left(\frac{p_\theta}{\gamma\beta mc}\right)^2 \frac{1}{r^3}}_{\text{CENTRIFUGAL}} - \underbrace{\frac{q}{\gamma^3 m v_z^2} \frac{\lambda(r)}{2\pi\epsilon_0 r}}_{\text{SELF FIELD}} = 0$$

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Moment Equations

Vlasov eqtn: $\frac{\partial f}{\partial s} + x' \frac{\partial f}{\partial x} + x'' \frac{\partial f}{\partial x'} + y' \frac{\partial f}{\partial y} + y'' \frac{\partial f}{\partial y'} = 0$

Let $g = g(x, x', y, y')$; $N = \iiiint f \, dx\, dx'\, dy\, dy'$

MULTIPLY VLASOV equation by g & $\frac{1}{N}\iiiint dx\, dx'\, dy\, dy'$

$$\int dx\, dx'\, dy\, dy' \left[g \frac{\partial f}{\partial s} + \overbrace{g x' \frac{\partial f}{\partial x} + g x'' \frac{\partial f}{\partial x'}}^{\text{INTEGRATE BY PARTS}} + g y' \frac{\partial f}{\partial y} + g y'' \frac{\partial f}{\partial y'} \right] = 0$$

$$\Rightarrow \frac{d}{ds}\langle g \rangle + \underbrace{\frac{1}{N}\iiint g f \Big|_{x=-\infty}^{\infty}}_{\to 0} - \underbrace{\frac{1}{N}\iiiint \frac{\partial g}{\partial x} f x'}_{= \langle x' \frac{\partial g}{\partial x} \rangle} + \ldots = 0$$

$$\Rightarrow \frac{d}{ds}\langle g \rangle = \langle x' \frac{\partial g}{\partial x} \rangle + \langle x'' \frac{\partial g}{\partial x'} \rangle + \langle y' \frac{\partial g}{\partial y} \rangle + \langle y'' \frac{\partial g}{\partial y'} \rangle$$

But $\frac{dg}{ds} = \frac{\partial g}{\partial x} x' + \frac{\partial g}{\partial x'} x'' + \frac{\partial g}{\partial y} y' + \frac{\partial g}{\partial y'} y''$

$$\Rightarrow \frac{d}{ds}\langle g \rangle = \langle g' \rangle$$

So $\frac{d}{ds}\langle x^2 \rangle = 2\langle x x' \rangle$

$\frac{d}{ds}\langle x'^2 \rangle = 2\langle x' x'' \rangle$ etc...

$\frac{d}{ds}\langle x x' \rangle = \langle x x'' \rangle + \langle x'^2 \rangle$

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ENVELOPE EQUATION - AXISYMMETRIC BEAMS

Let $r_b^2 = 2\langle r^2\rangle = 2(\langle x^2\rangle + \langle y^2\rangle) = 4\langle x^2\rangle$ for an axisymmetric beam

$$2 r_b r_b' = 4\langle r r'\rangle \quad \Rightarrow \quad r_b' = \frac{2\langle r r'\rangle}{r_b}$$

$$r_b'' = \frac{2\langle r r''\rangle + 2\langle r'^2\rangle}{r_b} - \frac{2\langle r r'\rangle}{r_b^2}\left(\frac{2\langle r r'\rangle}{r_b}\right)$$

$$= \frac{2\langle r r''\rangle}{r_b} + \frac{4\langle r^2\rangle\langle r'^2\rangle - 4\langle r r'\rangle^2}{r_b^3}$$

WHAT IS $\langle r r''\rangle$?

Using eq. P1 & $\langle p_\theta\rangle^2 = \gamma^2 m^2 \beta^2 c^2 \langle r^2\theta'\rangle^2 + \frac{q^2 B^2}{4}\langle r^2\rangle^2 + (\gamma m \beta c)(qB)\langle r^2\rangle\langle r^2\theta'\rangle$

$$\langle r r''\rangle = \langle r^2\theta'^2\rangle - \frac{\langle r^2\theta'\rangle^2}{\langle r^2\rangle} + \frac{\gamma''\langle r^2\rangle}{2\beta^2\gamma} - \frac{q}{\gamma^3 m \beta^2 c^2}\left\langle \frac{\partial\varphi^{self}}{\partial r} r\right\rangle$$
$$- \frac{\omega_c^2}{4\gamma^2\beta^2 c^2}\langle r^2\rangle + \frac{\langle p_\theta\rangle^2}{(\gamma m \beta c)^2\langle r^2\rangle} - \frac{\gamma'}{\beta^2\gamma}\langle r r'\rangle$$

$$\frac{\lambda(r)}{2\pi\epsilon_0 r}$$

what is $\left\langle \frac{\partial\varphi^{self}}{\partial r} r\right\rangle$? **(actually what is $\left\langle \frac{\lambda(r)}{2\pi\epsilon_0 r} r\right\rangle$?**

$$\nabla\cdot E = \frac{\rho}{\epsilon_0} \quad \Rightarrow \quad 2\pi r \frac{\partial\varphi^{self}}{\partial r} = -\frac{1}{\epsilon_0}\int_0^r 2\pi r\, \rho(r)\, dr$$

IT CAN BE SHOWN $\left\langle \frac{\partial\varphi^{self}}{\partial r} r\right\rangle = \frac{-\lambda}{4\pi\epsilon_0}$ $\underline{\underline{\text{if}}}$ $\rho(r,\theta) = \rho(r)$

$$\frac{\lambda(r)}{2\pi\epsilon_0 r}$$

Envelope Equation — continued

Substituting expression for $\langle r r'' \rangle$ into r_b'' equation:

$$\boxed{r_b'' + \frac{\gamma'}{\beta^2 \gamma} r_b' + \frac{\gamma''}{2\beta^2\gamma} r_b + \left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r_b - \frac{4\langle p_\theta\rangle^2}{(\gamma m \beta c)^2 r_b^3} - \frac{\varepsilon_r^2}{r_b^3} + - \frac{Q}{r_b} = 0}$$

Where $\varepsilon_r^2 = 4\left(\langle r^2\rangle\langle r'^2\rangle - \langle r r'\rangle^2 + \langle r^2\rangle\langle r^2\theta'^2\rangle - \langle r^2\theta'\rangle^2\right)$

Note that $\varepsilon_r^2 = \varepsilon_x^2 - 4\langle r^2\theta'\rangle^2$ (if $\rho = \rho(r)$ only)

EXAMPLES OF

SYSTEMS WITH AXIAL SYMMETRY

- PERIODIC SOLENOIDS

- EINZEL LENSES

- CONTINUOUS FOCUSING

EXAMPLES OF

SYSTEMS WITHOUT AXIAL SYMMETRY

- ELECTRIC OR MAGNETIC QUADRUPOLE

$\Rightarrow$ USE CARTESIAN COORDINATES WITH ELLIPTICAL SPACE CHARGE SYMMETRY

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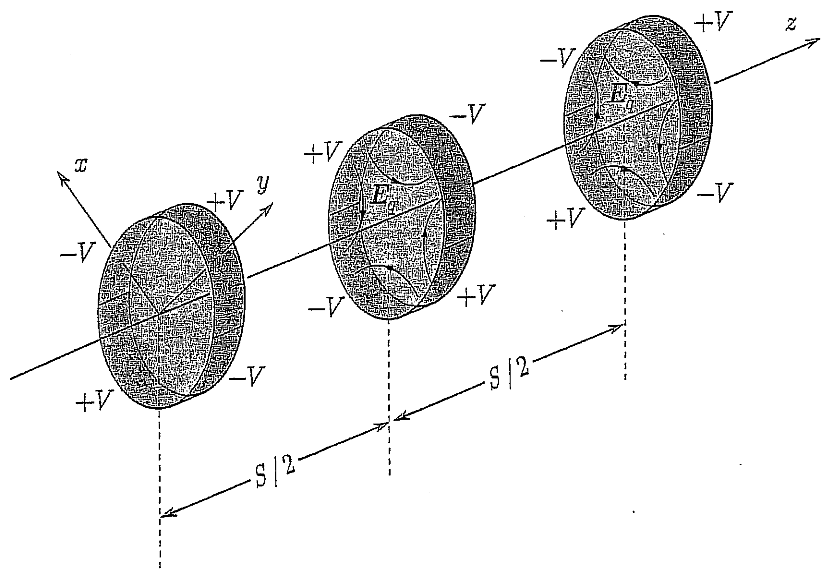

Figure 3.3. Schematic of conductor configuration with applied voltages producing an alternating-gradient quadrupole electric field with axial periodicity length S.

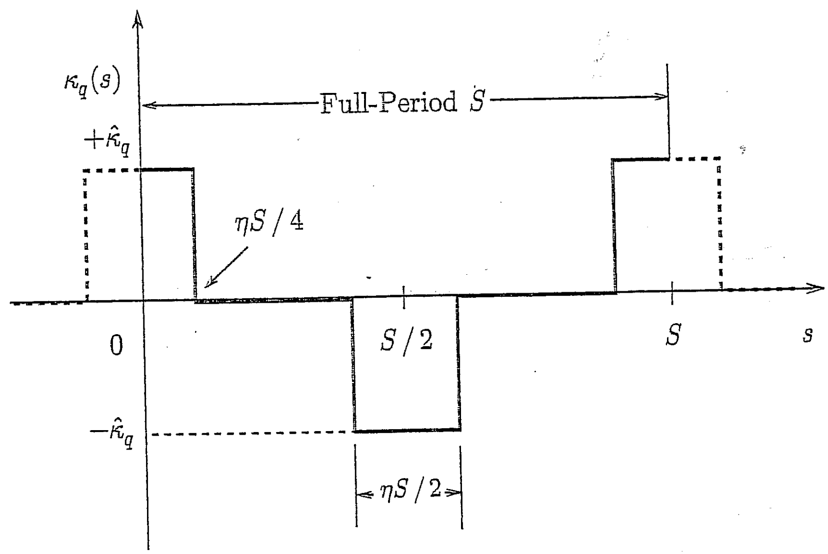

Figure 3.7. Alternating step-function model of a periodic quadrupole lattice with filling factor η for the lens elements. The figure shows a plot of the quadrupole coupling coefficient $\kappa_q(s)$ versus s for one full period (S) of the lattice. Such a configuration is often called a FODO transport lattice (acronym for focusing-off-defocusing-off).

FIGURES FROM DAVIDSON & QIN 2003

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figure from
Davidson & Qin, 2003.

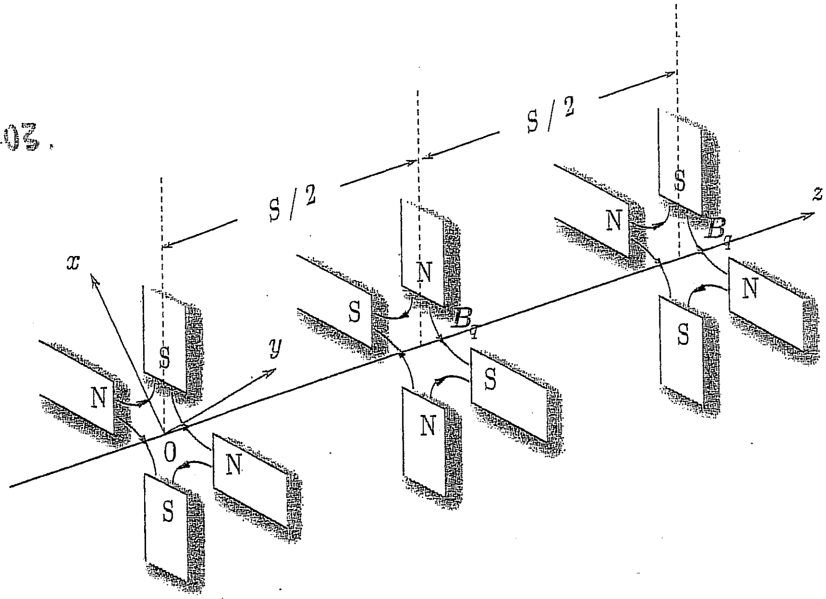

Figure 3.1. Schematic of magnet sets producing an alternating-gradient quadrupole field with axial periodicity length S.

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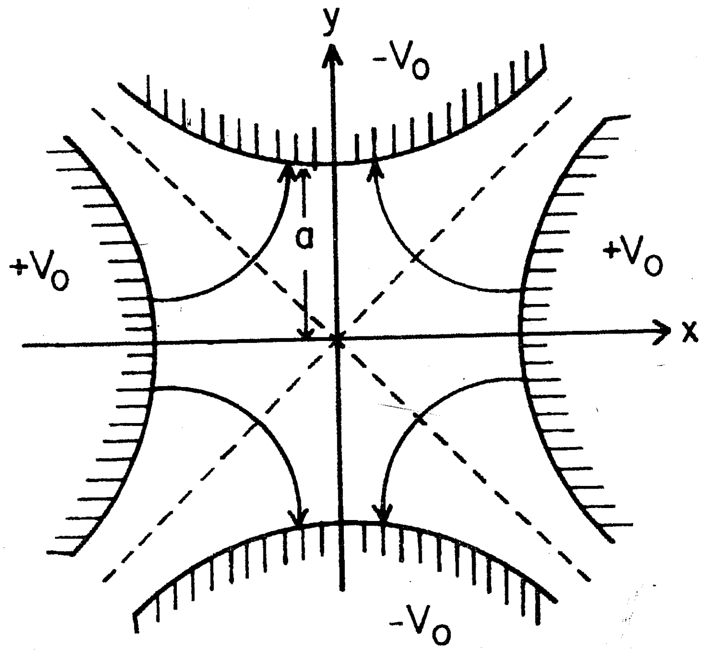
FROM REISER, p.112

$E_x = -E'x$

$E_y = E'y$



$F_x = -qE'x$

$F_y = qE'y$

ELECTROSTATIC QUADS

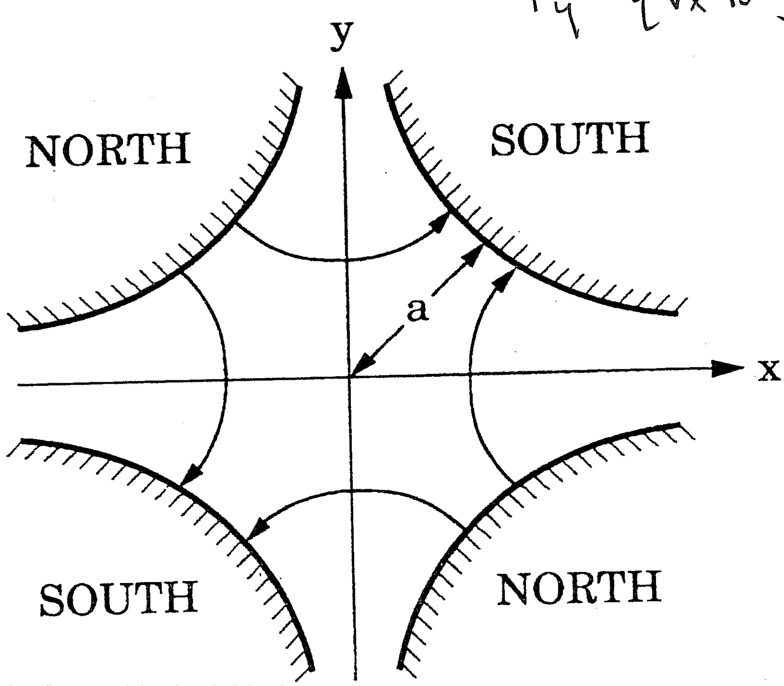
Figure 3.15. Electrodes and force lines in an electrostatic quadrupole.

$B_x = B'y$

$B_y = B'x$

$F_x = -qV_z B'x$

$F_y = qV_x B'y$



MAGNETIC QUADS

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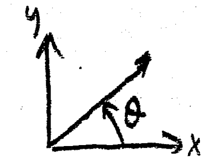
QUADRUPOLE FOCUSING

Now, relax radial symmetry:

For $\nabla \cdot E = 0$ or $\nabla \times B = 0$

EXPAND FIELD IN CYLINDRICAL "MULTIPOLES":

$$E_r, B_r = \sum_{n=1}^{\infty} f_n r^{n-1} \cos(n\theta)$$



$E_x = E_r \cos\theta - E_\theta \sin\theta$
$E_y = E_r \sin\theta + E_\theta \cos\theta$

$$E_\theta, B_\theta = \sum_{n=1}^{\infty} - f_n r^{n-1} \sin(n\theta)$$

$n=1 \Rightarrow$ dipole $\begin{cases} E_r = f_1 \cos\theta \\ E_\theta = -f_1 \sin\theta \end{cases} \Rightarrow \begin{matrix} E_x = f_1 \\ E_y = 0 \end{matrix}$

$n=2 \Rightarrow$ quadrupole $\begin{cases} E_r = f_2 r \cos 2\theta \\ E_\theta = -f_2 r \sin 2\theta \end{cases} \Rightarrow \begin{matrix} E_x = f_2 x \\ E_y = -f_2 y \end{matrix}$

NOTE: ABOVE EXPANSION IS VALID WHEN E & $B \neq$ function(z).
FOR MAGNETS OF FINITE AXIAL EXTENT, FOR EACH FUNDAMENTAL n-pole, A SET OF HIGHER ORDER MULTIPOLES WITH SAME AZIMUTHAL SYMMETRY ARE REQUIRED TO SATISFY $\nabla^2\phi = 0$.

FOR EXAMPLE FOR A FUNDAMENTAL QUADRUPOLE THE FIELD MAY BE EXPANDED:

$$E_r = \sum_{\nu=0}^{\infty} f_{2,\nu}(z)\,[1+\nu]\, r^{1+2\nu} \cos[2\theta]$$

$$E_\theta = \sum_{\nu=0}^{\infty} - f_{2,\nu}(z)\, r^{1+2\nu} \sin[2\theta]$$

$$E_z = \sum_{\nu=0}^{\infty} \frac{1}{2} \frac{df_{2,\nu}}{dz} r^{2+2\nu} \cos 2\theta$$

with $f_{2,\nu+1}(z) = \dfrac{-1}{4(\nu+1)(\nu+3)} \dfrac{d^2 f_{2,\nu}(z)}{dz^2}$

SEE LUND, S. M. (1996) FOR EXAMPLE. HIF note 96-LLNL.

SPACE CHARGE TERM WITH ELLIPTICAL SYMMETRY

NOW DEFOCUSING IN ONE DIRECTION AND FOCUSING IN THE OTHER $\Rightarrow$ RADIAL SYMMETRY SHOULD BE REPLACED BY ELLIPTICAL SYMMETRY: $\rho = \rho\left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2}\right)$

CAN BE SHOWN THAT $\left\langle x \frac{\partial f}{\partial x} \right\rangle = \frac{-\lambda}{4\pi\varepsilon_0} \frac{r_x}{r_x + r_y}$

$$\left\langle y \frac{\partial f}{\partial y} \right\rangle = \frac{-\lambda}{4\pi\varepsilon_0} \frac{r_y}{r_x + r_y}$$

(PROOF TO BE SHOWN NEXT JB LECTURE).

DEFINING $Q = \frac{2\lambda}{4\pi\varepsilon_0 \gamma^3 m v_z^2}$

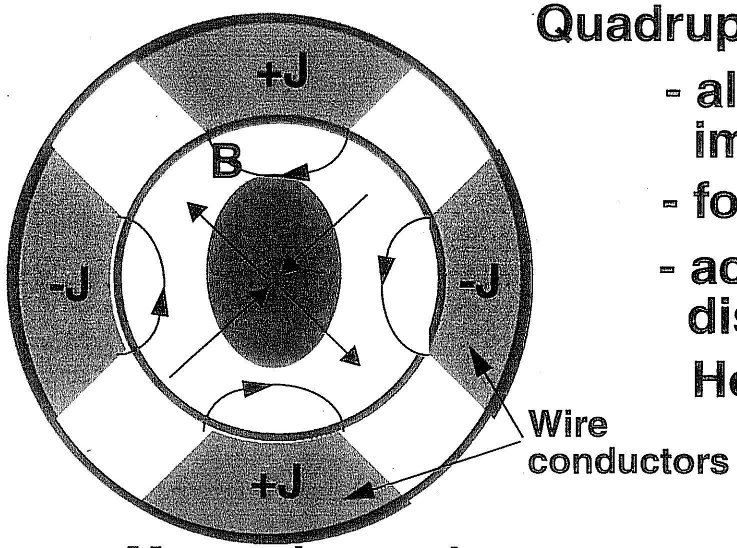
$$\Rightarrow \boxed{\begin{aligned} & r_x'' + \frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z) r_x' - \frac{2Q}{r_x + r_y} \mp \frac{B'}{[B\rho]} r_x - \frac{\varepsilon_x^2}{r_x^3} = 0 \\ & r_y'' + \frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z) r_y' - \frac{2Q}{r_x + r_y} \pm \frac{B'}{[B\rho]} r_y - \frac{\varepsilon_y^2}{r_y^3} \end{aligned}}$$

(for Electric Focusing $\frac{B'}{[B\rho]} \rightarrow \frac{qE'}{\gamma m v_z^2}$).

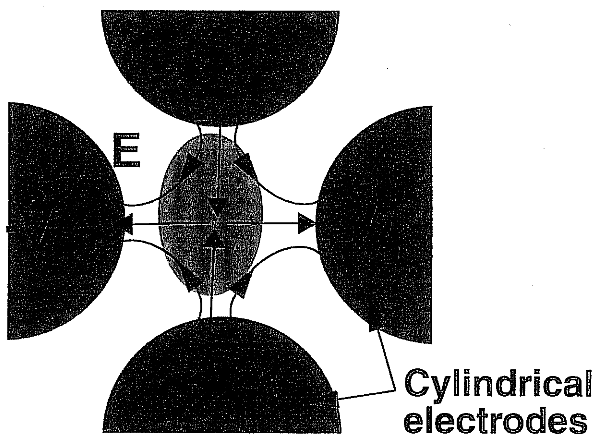
Heavy ion accelerators use alternating gradient quadrupoles to focus (confine) the beams (non-neutral plasmas)

BERKELEY LAB

Space-charge forces and thermal forces act to expand beam



Magnetic quad

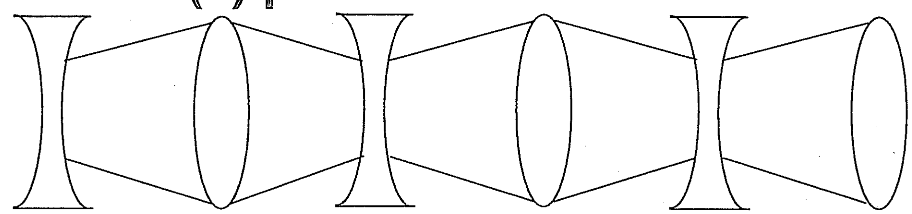


Electric quad

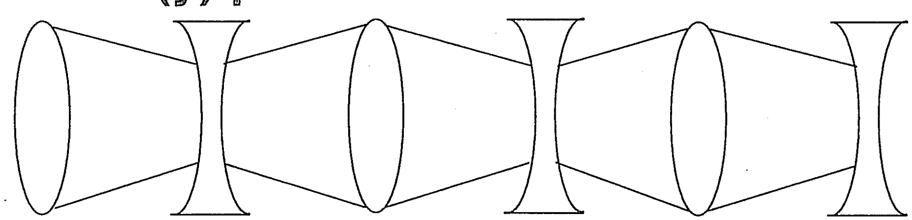
Quadrupoles (magnetic or electric):

- alternately provide inward then outward impulse
- focus in one plane and defocus in other
- act as linear lenses. (Force proportional to distance from axis).

Horizontal (x) plane:



Vertical (y) plane:



Average displacement is larger in focusing lenses so the net effect is focusing.

J. Barnard

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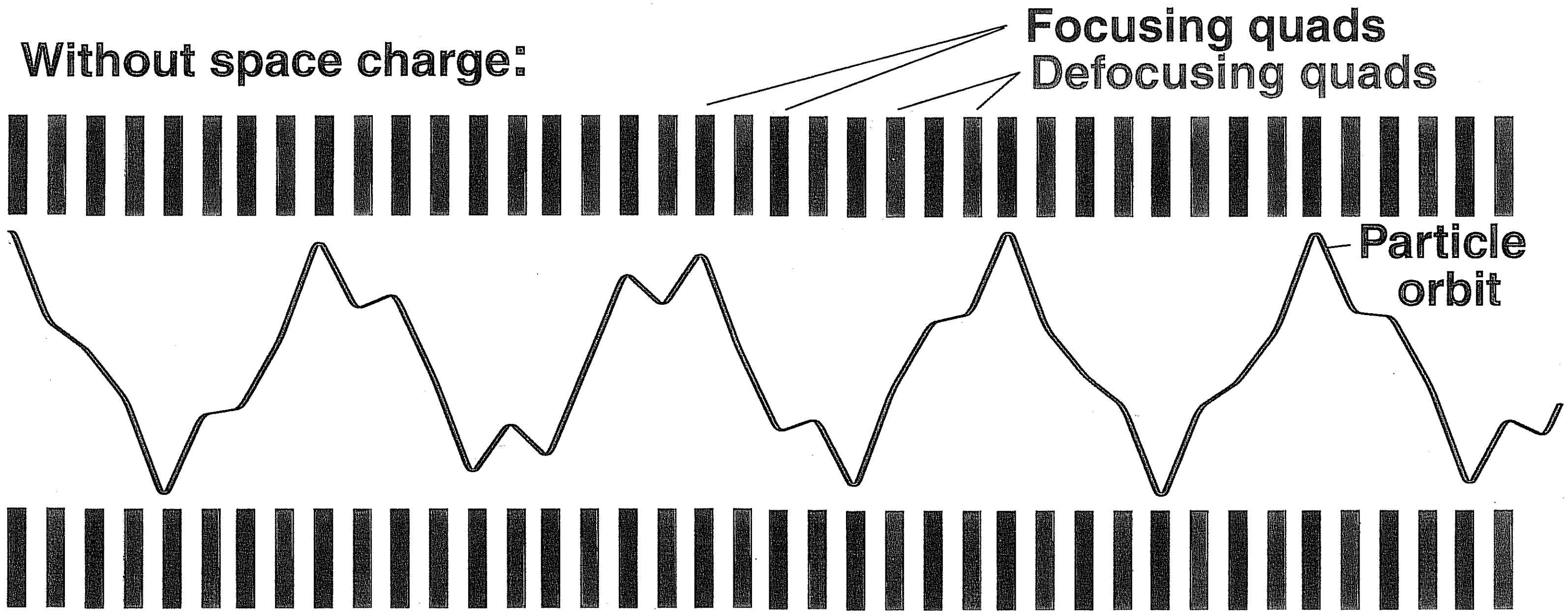
Space charge reduces betatron phase advance

BERKELEY LAB

J. BARNARD (18)

Without space charge:



With space charge: