J.J. Barnard and S.M. Lund
US PAS, Jan. 2004

Transverse Particle Equations of Motion

Outline:

§1 Particle Equations of Motion
- Derivation of transverse equations
  - Basic form
  - Including bending and dispersive terms

§2 Transverse Particle Equations of Motion in Linear Focusing Channels
- Continuous Focusing
- Quadrupole Focusing
- Solenoidal Focusing

§3 Description of Applied Focusing Fields.
- Overview
- Multipole descriptions

§4 Linear Equations of Motion without Space Charge, Acceleration, and Momentum Spread
- Hill's equation
- Orbit stability and eigenvalue structure.

§5 Floquet's Theorem and the Phase-Amplitude form of the Particle Orbit.
- Floquet's theorem
- Phase amplitude form of the particle orbit
- Particle phase advance.

§6 The Courant-Snyder Invariant and Single-Particle Emittance
- Derivation - Courant Snyder Invariant.
- Interpretation and uses.

§7 The Betatron Formulation of the Particle Orbit
- Formulation
- Envelope of particle orbits.

§8 Momentum Spread Effects
- Overview and equations
- Dispersion function

§9 Acceleration and Normalized Emittance.
- Transformation of orbit equations to standard form
- Normalized emittance.

§10 Example Design - The High Current Transport Experiment from LBNL

References

J.J. Barnard and S.M. Lund
USPAS, Jan 2004.

§1 Particle Equations of Motion

The Lorentz force on a particle of charge q and mass m at coordinate $\vec{x}(t)$ is given by: (MKS units)

$$\boxed{\frac{d}{dt}\vec{p}_i(t) = q\left[\vec{E}(\vec{x}_i,t) + \vec{v}_i(t)\times\vec{B}(\vec{x}_i,t)\right] \qquad i = \text{particle index},\ t = \text{time}}$$

$\vec{p}_i(t) = m\,\gamma_i(t)\,\vec{v}_i(t)$ — particle momentum

$\vec{v}_i(t) = c\,\vec{\beta}_i(t)$ — particle velocity, c = speed of light.

$\gamma_i(t) = \dfrac{1}{\sqrt{1-\vec{\beta}_i^{\,2}(t)}}$

where

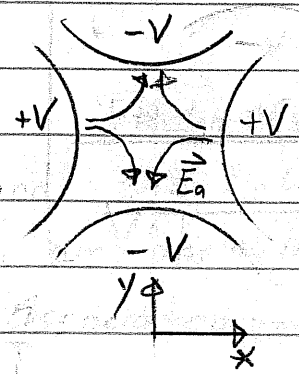
	Total-Field		Applied-Field		Self-Field
Electric Field:	$\vec{E}(\vec{x},t)$	=	$\vec{E}_a(\vec{x},t)$	+	$\vec{E}_s(\vec{x},t)$
Magnetic Field:	$\vec{B}(\vec{x},t)$	=	$\vec{B}_a(\vec{x},t)$	+	$\vec{B}_s(\vec{x},t)$

Applied Field sources:

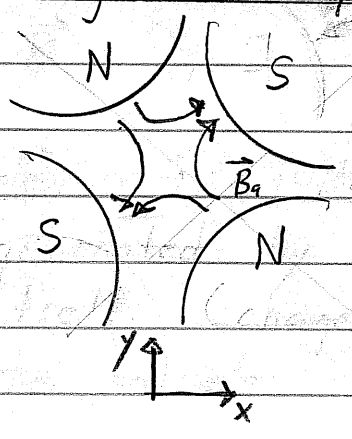
- Focusing Optics
- Dipole Bends
- Accelerating Fields

Focusing Optics (Examples)

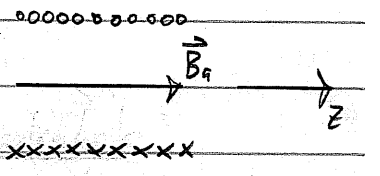
Electric Quadrupole



Magnetic Quadrupole

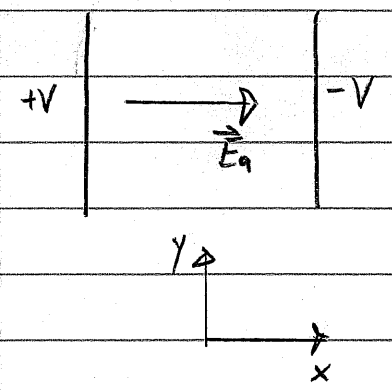


Solenoid

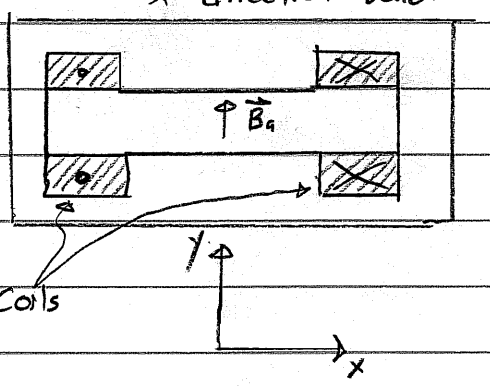


Dipole Bends (Examples)

Electric Dipole
x-direction bend

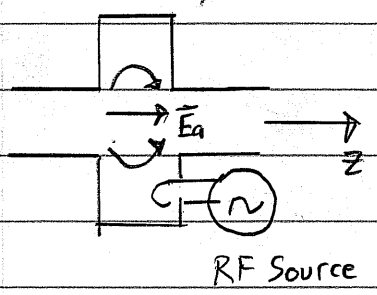


Magnetic Dipole
x-direction bend.

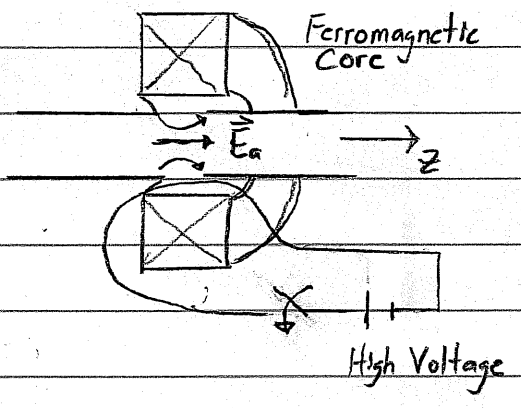


Accelerating Fields (Examples)

RF Cavity



Induction Cell

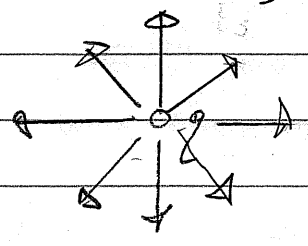


Self-Fields.

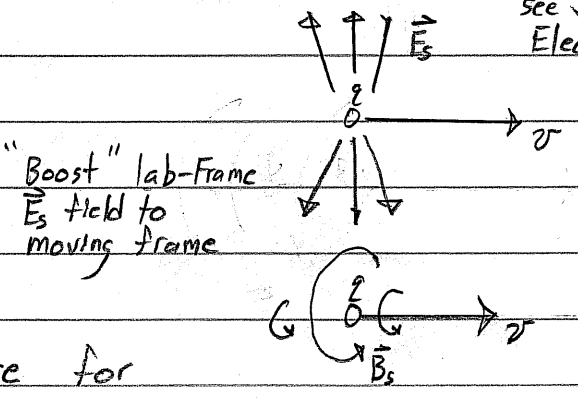
Are generated by the distribution of beam particles

- charges
- currents

Particle at Rest $\vec{E}_s$, $\vec{B}_s = \vec{0}$



Particle in motion (relativistic)
see Jackson, Classical Electrodynamics.



Superimpose for particles in beam distribution.

Accelerating particles also radiate.

$\vec{E}$ and $\vec{B}$ satisfy the Maxwell Equations, and the linear structure of the Maxwell Equations can be exploited to express:

Applied Fields (often quasi-static)

$$\nabla \cdot \vec{E}_a = \frac{\rho_a}{\varepsilon_0} \qquad \nabla \times \vec{B}_a = \mu_0 \vec{J}_a + \frac{1}{c^2}\frac{\partial \vec{E}_a}{\partial t}$$

$$\nabla \times \vec{E}_a = -\frac{\partial}{\partial t}\vec{B}_a \qquad \nabla \cdot \vec{B}_a = 0$$

$$\frac{1}{\mu_0 \varepsilon_0} = c^2$$

ρ_a = applied charge density

$\vec{J}_a$ = applied current density

+ Boundary Conditions on $\vec{E}_a, \vec{B}_a$

Self Fields

$$\nabla \cdot \vec{E}_s = \sum_i q_i \frac{\delta(\vec{x}-\vec{x}_i(t))}{\varepsilon_0} \qquad \nabla \times \vec{B}_s = \mu_0 \sum_{i=1}^{N} q_i \vec{v}_i \delta(\vec{x}-\vec{x}_i(t)) + \frac{1}{c^2}\frac{\partial \vec{E}_s}{\partial t}$$

$$\nabla \times \vec{E}_s = -\frac{\partial}{\partial t}\vec{B}_s \qquad \nabla \cdot \vec{B}_s = 0$$

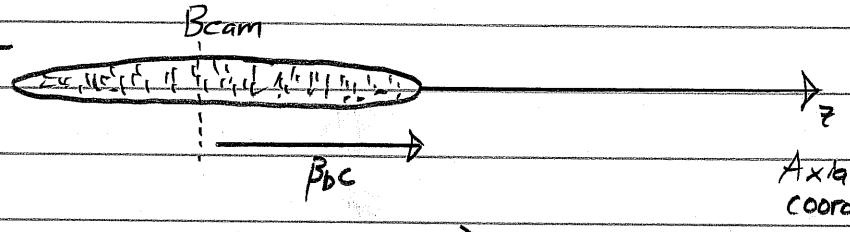
$\sum_{i=1}^{N}$ = sum over all N particles in beam

$\delta(\vec{x})$ = Dirac-delta function

+ Boundary Conditions on $\vec{E}_s, \vec{B}_s$ from material structures, radiation conditions etc.

In accelerator beams, there is ideally a single species of ion, many particles, and the motion of the "bunch" is highly directed:

<u>Directed Motion</u>



ion velocity

$$\frac{d\vec{x}_i}{dt} = \vec{v}_i = \beta_b c\,\hat{z} + \delta\vec{v}_i$$

β_b = Mean axial Relativistic factor of beam

$$\underline{\underline{|\delta\vec{v}_i| \ll \beta_b c}} \Rightarrow \underline{\text{Paraxial Approx.}}$$

<u>Many Particles</u>

$$\underline{\underline{\sum_i q_i \delta(\vec{x} - \vec{x}_i(t)) \simeq \rho(\vec{x}, t)}}$$ continuous charge density

$$\underline{\underline{\sum_i q_i \vec{v}_i \delta(\vec{x} - \vec{x}_i(t)) \simeq \beta_b c\, \rho(\vec{x}, t)\hat{z}}}$$ continuous axial current density

Moreover, the evolution is typically sufficiently slow for heavy ions where in the Maxwell equations we can neglect radiation and approximate the self-field Maxwell equations (see Intro. lecture by J.J. Barnard, Electrostatic approximation) as:

$$\boxed{\begin{aligned} \vec{E}_s &= -\nabla\phi \\ \vec{B}_s &= \nabla \times \vec{A}\,; \quad \vec{A} = \frac{\beta_b}{c}\phi\,\hat{z} \\ \nabla^2\phi &= \frac{\partial}{\partial\vec{x}}\cdot\frac{\partial}{\partial\vec{x}}\phi = \frac{-\rho}{\varepsilon_0} \quad \text{and Boundary conditions on } \phi \end{aligned}}$$

The force acting on the beam particles can be resolved into applied and self-field components:

$$\boxed{\vec{F}_i(\vec{x}_i,t) = q\vec{E}(\vec{x}_i,t) + q\vec{v}_i(t)\times\vec{B}(\vec{x}_i,t)}$$

<u>Applied:</u>

$$\vec{F}_a = q\vec{E}_a + q\vec{v}_i\times\vec{B}_a$$

<u>Self:</u>

$$\vec{F}_s = q\vec{E}_s + q\vec{v}_i\times\vec{B}_s$$

The self-field force can then be expressed as (see also J.J. Barnard, Intro. Lectures):

$$\vec{F}_s = q\left(\vec{E}_s + \vec{v}_i\times\vec{B}_s\right)$$

$$\simeq q\left[-\nabla\phi + \beta_b c\,\hat{z}\times\left(\nabla\times\frac{\beta_b}{c}\phi\right)\right]$$

$$\vec{F}_s \simeq -q(1-\beta_b^2)\nabla_\perp\phi - q\frac{\partial\phi}{\partial z}\hat{z}$$

$$\boxed{F_s \simeq \frac{-q}{\gamma_b^2}\nabla_\perp\phi - q\frac{\partial\phi}{\partial z}\hat{z}}$$

where

$$\boxed{\gamma_b \equiv \frac{1}{\sqrt{1-\beta_b^2}}}$$

is the axial relativistic factor of the beam

The particle equations of motion become:

Transverse:

$$\boxed{\frac{d}{dt}\left(m\gamma_i \vec{v}_{\perp i}\right) \simeq q\vec{E}_\perp^a + q\beta_b c\,\hat{z}\times\vec{B}_\perp^a + qB_z^a\,\vec{v}_{\perp i}\times\hat{z} - \frac{q}{\gamma_b^2}\nabla_\perp\phi}$$

applied terms

self term

Longitudinal

$$\boxed{\frac{d}{dt}\left(m\gamma_i v_{z_i}\right) \simeq qE_z^a - q\left(v_{x_i}B_y^a - v_{y_i}B_x^a\right) - q\frac{\partial\phi}{\partial z}}$$

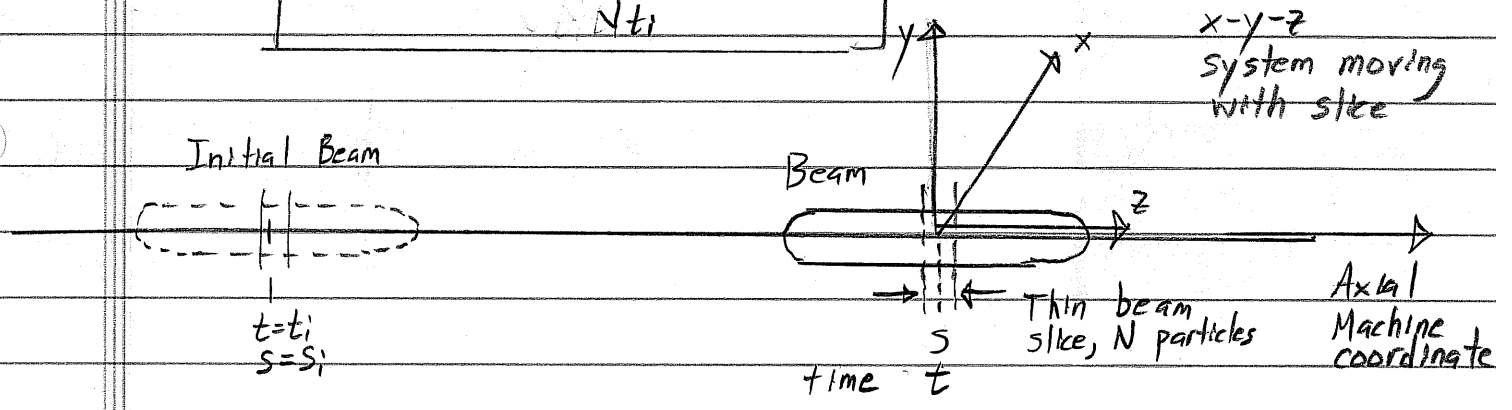
Applied terms

self term

In the remainder of this lecture, we only analyze transverse dynamics. Longitudinal dynamics will be covered in lectures by J. J. Barnard.

In transverse accelerator dynamics, it is convenient to employ the axial coordinate of a slice of beam particles in the accelerator as the independent variable:

$$\boxed{\begin{aligned} \beta_b &= \sum_{\substack{i=1 \\ \text{slice}}}^{N} \frac{v_{zi}}{c} \\ s &\equiv s_i + c\int_{t_i}^{t} d\tilde{t}\, \beta_b(\tilde{t}) \end{aligned}}$$



Transform:

$$dt = \frac{ds}{\beta_b c} \quad \Rightarrow \quad v_x = \frac{dx_i}{dt} \simeq \beta_b c \frac{dx_i}{ds} \equiv \beta_b c\, x_i'$$

Denote: etc.

$$\boxed{' \equiv \frac{d}{ds}}$$

$$\boxed{\begin{aligned} v_x &= \frac{dx}{dt} \simeq \beta_b c\, x' \\ v_y &= \frac{dy}{dt} \simeq \beta_b c\, y' \end{aligned}}$$

x', y' ~ angles trajectory makes with z-axis.

Paraxial Approx:

$x'^2, y'^2 \ll 1$

$\Rightarrow \frac{v_x^2}{c}, v_y^2 \ll v_z^2$

Transform transverse particle equation of motion:

$$\frac{d}{dt}\left(m \gamma_i \frac{d\vec{x}_{\perp i}}{dt} \right) = m v_{zi} \frac{d}{ds}\left(\gamma_i v_{zi} \frac{d}{ds} \vec{x}_{\perp i} \right)$$

$$= m \gamma_i v_{zi}^2 \frac{d^2}{ds^2} \vec{x}_{\perp i} + m \frac{d}{ds} \vec{x}_{\perp i} \, v_{zi} \frac{d}{ds}\left(\gamma_i v_{zi} \right)$$

But:

$$m \gamma_i v_{zi}^2 \frac{d^2}{ds^2} \vec{x}_{\perp i} \simeq m \gamma_b \beta_b^2 c^2 \frac{d^2}{ds^2} \vec{x}_{\perp i} = m \gamma_b \beta_b^2 c^2 \vec{x}_{\perp i}''$$

$$m \frac{d}{ds} \vec{x}_{\perp i} \, v_{zi} \frac{d}{ds}\left(\gamma_i v_{zi} \right) \simeq m \frac{d}{ds} \vec{x}_{\perp i} \, \beta_b c \frac{d}{ds}\left(\gamma_b \beta_b c \right)$$

$$\simeq m \beta_b c \left(\gamma_b \beta_b c \right)' \vec{x}_{\perp i}'$$

$$\Rightarrow \boxed{m \frac{d}{dt}\left(\gamma_i \frac{d\vec{x}_{\perp i}}{dt} \right) \simeq m \gamma_b \beta_b^2 c^2 \left[\vec{x}_{\perp i}'' + \frac{1}{\gamma_b \beta_b} \frac{d}{ds}\left(\gamma_b \beta_b \right) \vec{x}_{\perp i}' \right]}$$

Also,

$$\boxed{q B_z^a \vec{v}_{\perp i} \times \hat{z} = q B_z^a \beta_b c \, \vec{x}_{\perp i}' \times \hat{z}}$$

Using these results the transverse particle equations of motion become

Drop particle "i" subscripts:

Transverse Particle Equations of Motion

$$\vec{x}_\perp'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} \vec{x}_\perp' = \frac{q}{m\gamma_b \beta_b^2 c^2} \vec{E}_\perp^a + \frac{q}{m\gamma_b \beta_b c} \hat{z} \times \vec{B}_\perp^a$$

$$+ \frac{q B_z^a}{m\gamma_b \beta_b c} \vec{x}_\perp' \times \hat{z} - \frac{q}{\gamma_b^3 \beta_b^2 c^2} \nabla_\perp \phi$$

$\vec{E}_a$ = Applied Electric Field
$\vec{B}_a$ = Applied Magnetic Field

$$\nabla^2 \phi = \frac{\partial}{\partial \vec{x}} \cdot \frac{\partial}{\partial \vec{x}} \phi = \frac{-\rho}{\varepsilon_0}$$

and boundary conditions on ϕ

$$' \equiv \frac{d}{ds}$$

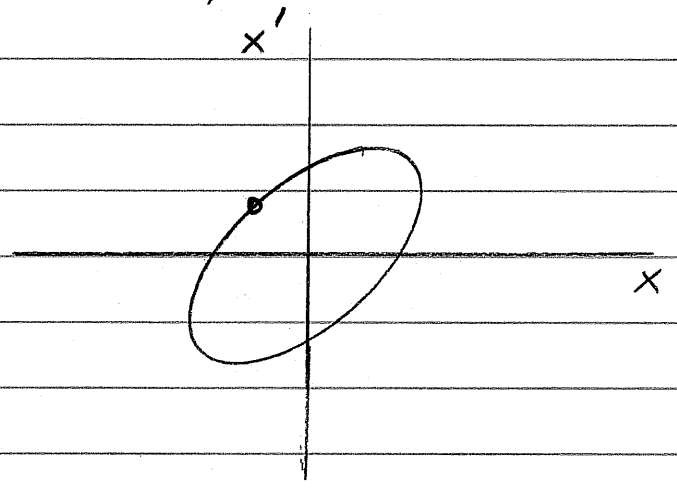
Note:

- γ_b factors different in applied and self-field terms.
- In $-\frac{q}{\gamma_b^3 \beta_b^2 c^2} \nabla_\perp \phi$ contributions to γ_b^3 come from:
  - $\gamma_b \Rightarrow$ kinematics
  - $\gamma_b^2 \Rightarrow$ leading order self-magnetic field corrections.

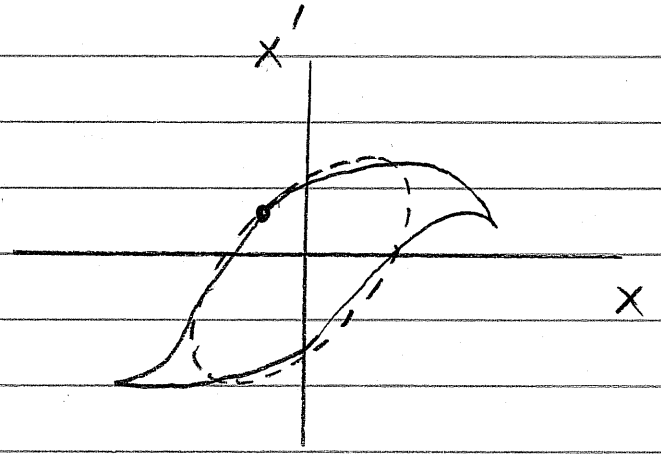
Overview:

Much transverse accelerator physics centers on understanding the evolution of beam particles in 4-dimensional $x-x'$, $y-y'$ phase-space.

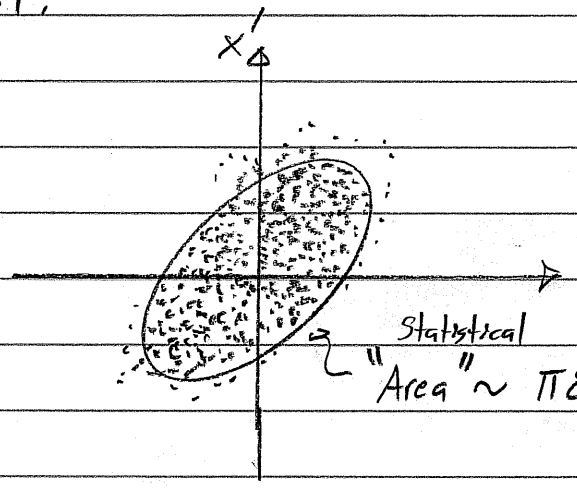
Often phase-space projections in $x-x'$ and $y-y'$ are analyzed



When forces acting on particles are <u>linear</u>, particles tend to move on ellipses of constant area (which may twist and distort) in phase-space. Nonlinear forces can distort orbits and cause undesireable effects, growing effective phase-space area.



The effective phase-space volume of a distribution of beam particles is of fundamental interest.



Effective area measure in x-x' phase-space is the x-emittance:

Statistical "Area" $\sim \pi \varepsilon_x$: $\varepsilon_x = 4\left[\langle x^2\rangle_\perp \langle x'^2\rangle_\perp - \langle x x'\rangle^2\right]^{1/2}$

We will find in statistical beam descriptions:

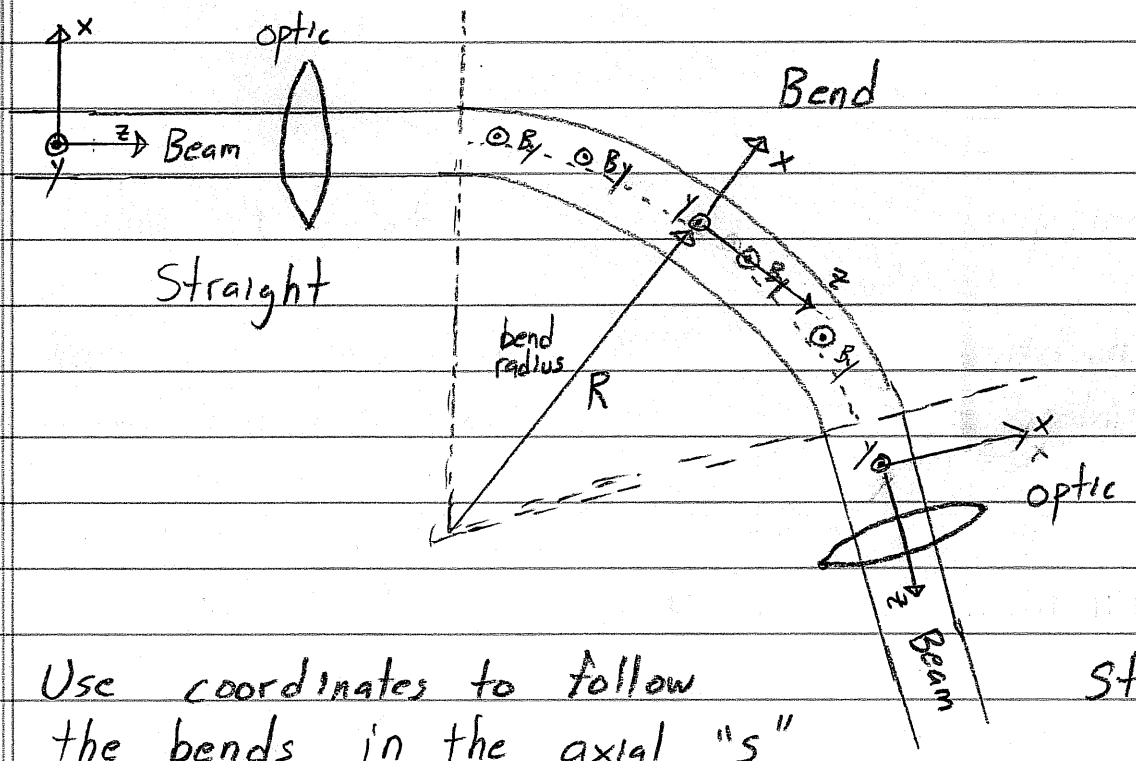
Larger phase-space areas (larger emittances) $\Rightarrow$ Harder to achieve small beam spots on target.

Smaller phase-space areas (smaller emittances) $\Rightarrow$ Easier to achieve small beam spots on target

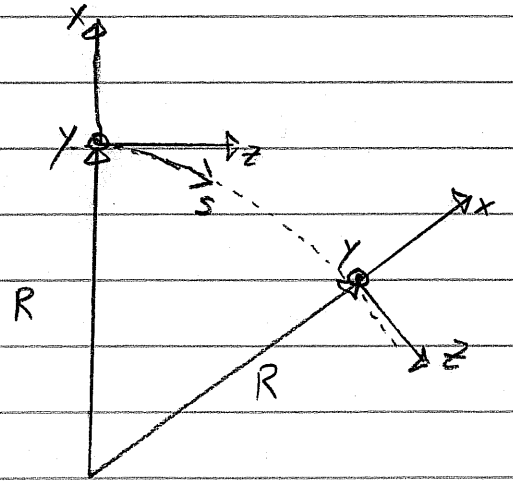
Much of advanced accelerator physics seeks to understand and control emittance growth due to nonlinear applied fields and space-charge effects. In the remainder of this lecture we will review basic accelerator physics for the transverse dynamics of a particle moving in applied fields. Later we will generalize concepts to include space-charge.

Bent Coordinate System to analyze particle equations of motion with dipole bends and spread in axial particle momentum

The previous equations of motion can be applied to dipole bends provided the x-y-z coordinate system is fixed. In practice, it can be more convienient to employ coordinates that follow the beam in a bend.



Use coordinates to follow the bends in the axial "s" coordinate.



x-y-z local cartesian system

Using this system, dipoles are adjusted given the mean beam energy / momentum to bend the required amount for a bend radius R. Usually, bends are only in one plane, say the x-plane, and are implemented by a <u>dipole</u> B_y^a or E_x^a applied field.

Denote:

$$\boxed{p_o = m\gamma_b\beta_b c \quad = \quad \text{design momentum}}$$

Then the x-bending radius R of the design momentum particle $p_o = \text{const}$ in a constant applied magnetic field $\vec{B}^a = B_y^a\hat{y} = \text{const}$ is given by

$$\boxed{\frac{1}{R} = \frac{q B_y^a}{p_o}} \qquad \begin{array}{l} B_y^a = \text{const} \\ R = \text{const.} \end{array}$$

Often this is expressed in terms of the "particle rigidity" $[B\rho]$ (read as one symbol and called "B-rho").

$$\boxed{[B\rho] \equiv \frac{p_o}{q} = \frac{m\gamma_b\beta_b c}{q}}$$

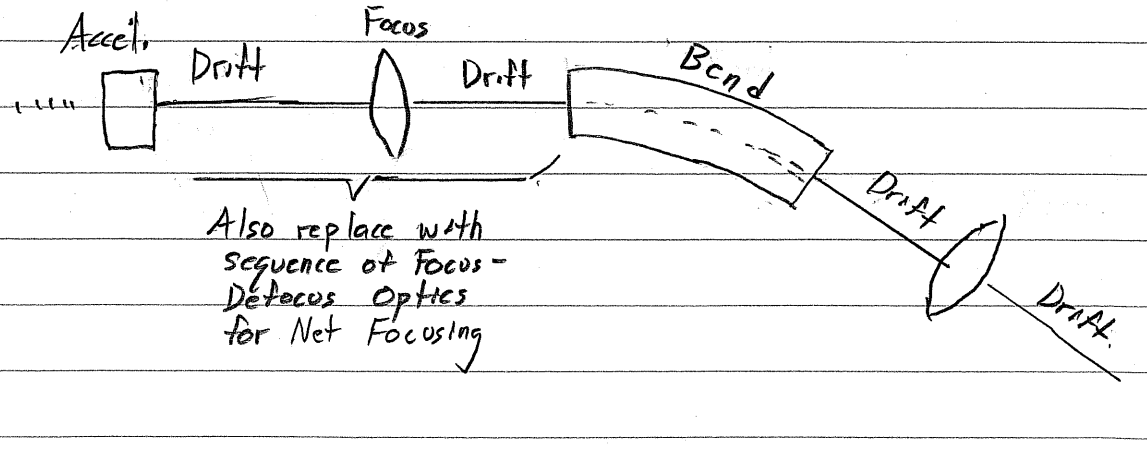
Then

$$\boxed{\frac{1}{R} = \frac{B_y^a}{[B\rho]}}$$

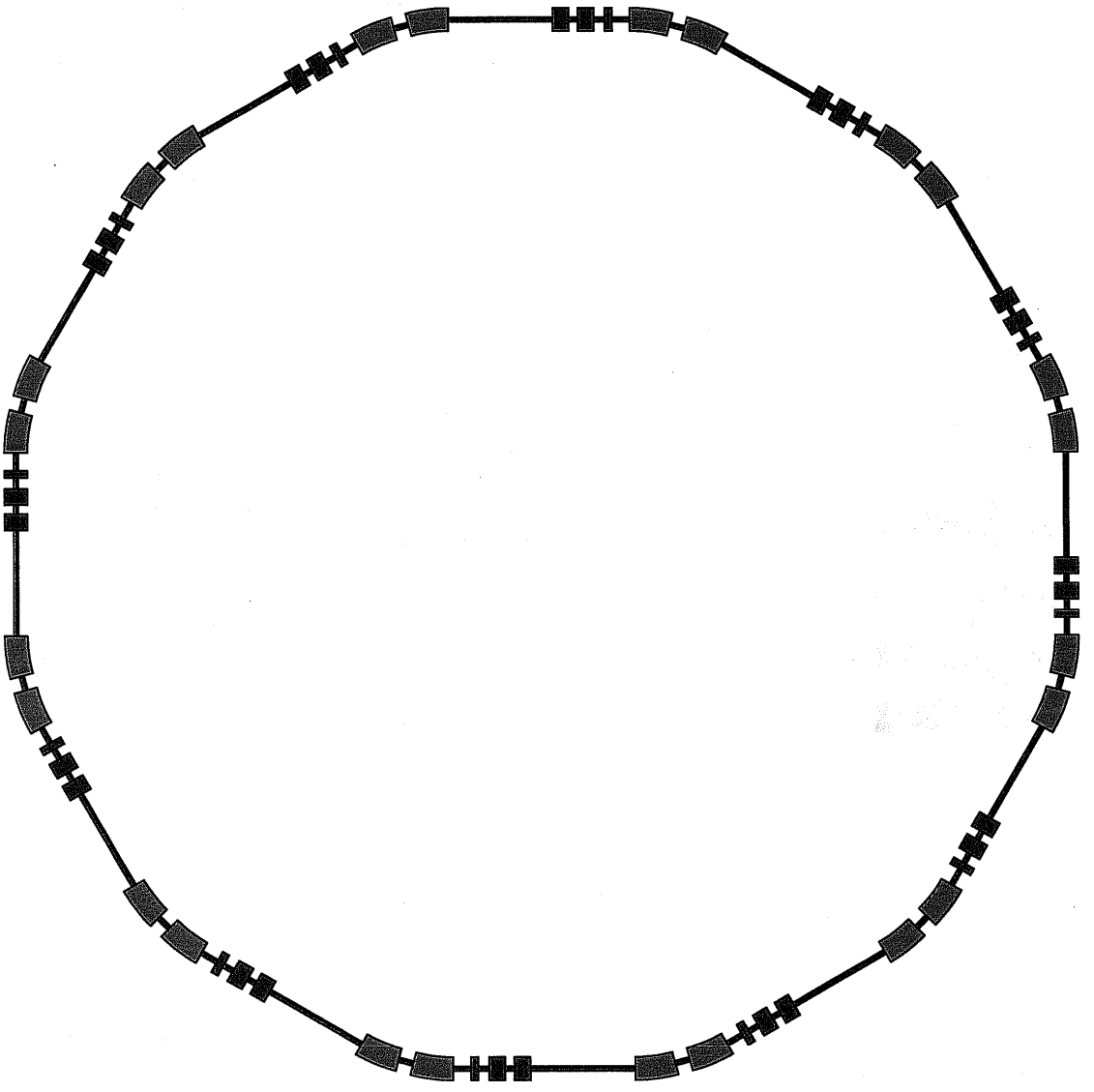
Comment: Different design formulas will result for electric bends with $E_x^a \neq 0$

In straight sections $R \to \infty$, and bends can be positive or negative radius R (direction of curve) depending on the sign of B_y^a and $[B\rho]$. Practical machines are often made from discrete element dipoles and straight sections with seperated function optics. (See the sketch of a practical ring on the next page). For a ring, such dipoles are tuned with the particle rigidity such that the orbit makes a complete lap through the machine. Dipoles must be adjusted as the particles gain energy to maintain a closed orbit in the machine aperture. The need to synchronously adjust bending dipoles and focusing elements with the gains in particle energy is the origin of the name "Synchrotron" for such machines. The total bending strength of a synchrotron in tesla-meters limits ultimate achievable particle energy/momentum in the machine.

Typical Seperated Function Lattice Period



SIS – 18 Synchrotron
GSI, Germany
18 Tesla–Meter Bending Strength



Acceleration Cells
Not Shown

Diplole Bending Magnets

Quadrupole Focusing Magnets (Triplets)

Off momentum effects:

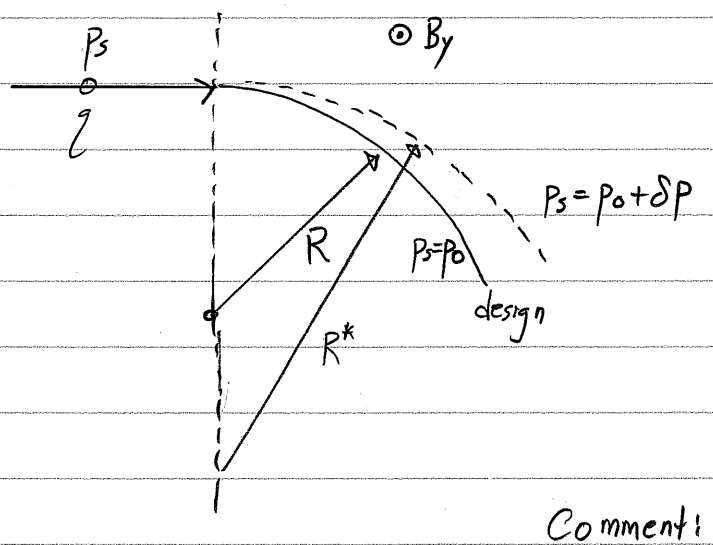
$$\boxed{p_s = m\gamma_b\beta_b c = \text{design momentum.}}$$

For momentum errors:

$$p_s = p_0 + \delta p$$

$$p_0 = m\gamma_b\beta_b c = \text{"design-momentum"}$$
$$\delta p = \text{"off-momentum"}$$

This will modify the particle equations of motion, particularly in cases where there are bends since particles with different momenta will be bent at different radii.



Comment: It is not usual to have acceleration in bends. However, dipoles bends and quadrupole focusing magnets are sometimes combined.

Without presenting algebraic detail (see standard accelerator texts such as Edwards and Syphers) the particle equations of motion for a coasting beam with design momentum p_o become (the full derivation is beyond the scope of this class).

$$x''(s) + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x'(s) + \left[\frac{1}{R^2(s)}\frac{1-\delta}{1+\delta}\right]x(s) = \frac{\delta}{1+\delta}\frac{1}{R(s)} + \frac{q}{m\gamma_b\beta_b^2c^2}\frac{E_x^q}{(1+\delta)^2}$$

$$-\frac{q}{m\gamma_b\beta_b c}\frac{B_y^q}{1+\delta} + \frac{q B_s^q}{m\gamma_b\beta_b c}\frac{y'(s)}{1+\delta} - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{1}{1+\delta}\frac{\partial\phi}{\partial x}$$

$$y''(s) + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y'(s) = \frac{q}{m\gamma_b\beta_b^2c^2}\frac{E_y^q}{(1+\delta)^2} + \frac{q}{m\gamma_b\beta_b c}\frac{B_x^q}{1+\delta}$$

$$\frac{-q B_s^q}{m\gamma_b\beta_b c}\frac{x'(s)}{1+\delta} - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{1}{1+\delta}\frac{\partial\phi}{\partial y}$$

$$\delta \equiv \frac{\delta p}{p_o} = \text{Fractional Momentum Error.}$$

$p_o = m\gamma_b\beta_b c$ = design momentum, $\frac{1}{R} = \frac{B_y^q\big|_{dipole}}{[B\rho]}$, $[B\rho] = p_o/q$

Comments:

- Design bends only in x and E_x^q, B_y^q contain <u>no</u> dipole field terms. These components are set consistent with the design bend radius $R(s)$
- x and y equations differ significantly due to the bends modifying the x-equation when $R(s) \not\to \infty$.
- Equations contain only low-order terms in δ.

J.J. Barnard and S.M. Lund
USPAS, Jan 2004

§2 Transverse Particle Equations of Motion in Linear Focusing Channels

The transverse particle equations of motion previously derived can be expressed as:

$$\boxed{\begin{aligned} x'' + \left[\frac{1}{\gamma_b\beta_b}\frac{d}{ds}\left(\gamma_b\beta_b\right)\right]x' &= \frac{q}{m\gamma_b\beta_b^2c^2}E_x^a - \frac{q}{m\gamma_b\beta_b c}B_y^a + \frac{q}{m\gamma_b\beta_b c}B_z^a y' \\ &\quad - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x} \\ y'' + \left[\frac{1}{\gamma_b\beta_b}\frac{d}{ds}\left(\gamma_b\beta_b\right)\right]y' &= \frac{q}{m\gamma_b\beta_b^2c^2}E_y^a + \frac{q}{m\gamma_b\beta_b c}B_x^a - \frac{q}{m\gamma_b\beta_b c}B_z^a x' \\ &\quad - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y} \end{aligned}}$$

Assumptions:

- Paraxial equations: $x'^2, y'^2 \ll 1$
- Dispersive effects neglected; $v_z = \beta_b c$ for all particles, but acceleration (β_b may vary in s) is allowed.
- Electrostatic and leading-order (in β_b) self-magnetic interactions included.
- Fixed x-y-z coordinate system (no local bends).

To complete the description of the $\perp$ particle equations of motion, the applied focusing fields

$$\boxed{\begin{array}{ll} \text{electric:} & E_x^a,\ E_y^a,\ E_z^a \\ \text{magnetic:} & B_x^a,\ B_y^a,\ B_z^a \end{array}}$$

must be specified as a function of s and the $\perp$ particle coordinates x, y. Also the consistent change in axial velocity with $\beta_b c$ must be evaluated due to E_z^a and possibly time-varying electromagnetic fields (as in RF cavities). Here we restrict ourselves to $\vec{E}_a, \vec{B}_a$ <u>resulting from applied focusing structures</u>. In intense beam accelerators and transport lattices, focusing structures are designed to optimize linear focusing with:

Linear Focusing

$$\boxed{\begin{array}{l} E_x^a \simeq (\text{function of } s) \cdot (x \text{ or } y) \\ E_y^a \simeq (\text{function of } s) \cdot (x \text{ or } y) \\ \\ B_x^a \simeq (\text{function of } s) \cdot (x \text{ or } y) \\ B_y^a \simeq (\text{function of } s) \cdot (x \text{ or } y) \\ B_z^a \simeq (\text{function of } s) \end{array}}$$

We now review some common situations that realize these forms:

1/ Continuous Focusing
2/ Quadrupole Focusing
3/ Solenoidal Focusing

Continuous Focusing

$$\boxed{\begin{cases} \vec{B}^a = 0 \\ E_x^a = -\dfrac{m\gamma_b\beta_b^2c^2}{q}k_{\beta 0}^2 x \qquad E_y^a = -\dfrac{m\gamma_b\beta_b^2c^2}{q}k_{\beta 0}^2 y \end{cases} \qquad k_{\beta 0}^2 = \text{const.}}$$

Then the $\perp$ equations of motion reduce to:

$$\boxed{\vec{x}_\perp'' + \left[\frac{1}{\gamma_b\beta_b}\frac{d}{ds}(\gamma_b\beta_b)\right]\vec{x}_\perp' + k_{\beta 0}^2\vec{x}_\perp = \frac{-q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial\vec{x}_\perp}}$$

This situation is realized by a stationary (mass $\to\infty$), partially neutralizing uniform background of charges. To see this we apply Poisson's equation to the applied field to calculate the neutralizing charge density:

$$\boxed{\rho_n = \varepsilon_0\nabla\cdot\vec{E}^a = -\frac{2m\varepsilon_0\gamma_b\beta_b^2c^2k_{\beta 0}^2}{q} = \text{const.}}$$

- Unphysical model, but very commonly employed since it simply represents the <u>average</u> action of more physical focusing fields.

- We will find that continuous focusing can provide reasonably good estimates for more realistic focusing models if $k_{\beta 0}^2$ is appropriately identified in terms of relevant factors.

In more physical situations one requires that quasistatic focusing fields in the aperture satisfy the vacuum Maxwell Equations in the beam region:

$$\boxed{\begin{array}{ll} \nabla \cdot \vec{E}_a^a = 0 & \nabla \cdot \vec{B}_a^a = 0 \\ \nabla \times \vec{E}_a^a = 0 & \nabla \times \vec{B}_a^a = 0 \end{array}}$$

These equations constrain the 3D form of the vacuum fields resulting from spatially localized lenses. But the following situations can be exploited to optimize linear focusing in practical lattices. We will cover two common cases:

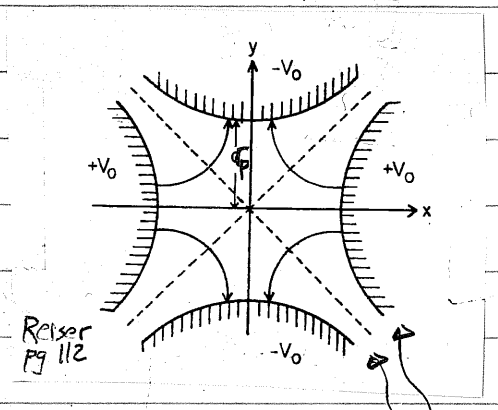
1) Alternating Gradient Quadrupoles
   - Electric Quadrupoles
   - Magnetic Quadrupoles
2) Magnetic Solenoids.

Alternating Gradient Quadrupole Focusing

A) Electric Quadrupoles: $(\vec{B}^a = 0)$

In the axial center of an electric quadrupole:

2D Fields



$$E_x^a = -\frac{2V_0}{r_p}\frac{x}{r_p} = -E_q' x$$

$$E_y^a = \frac{2V_0}{r_p}\frac{y}{r_p} = E_q' y$$

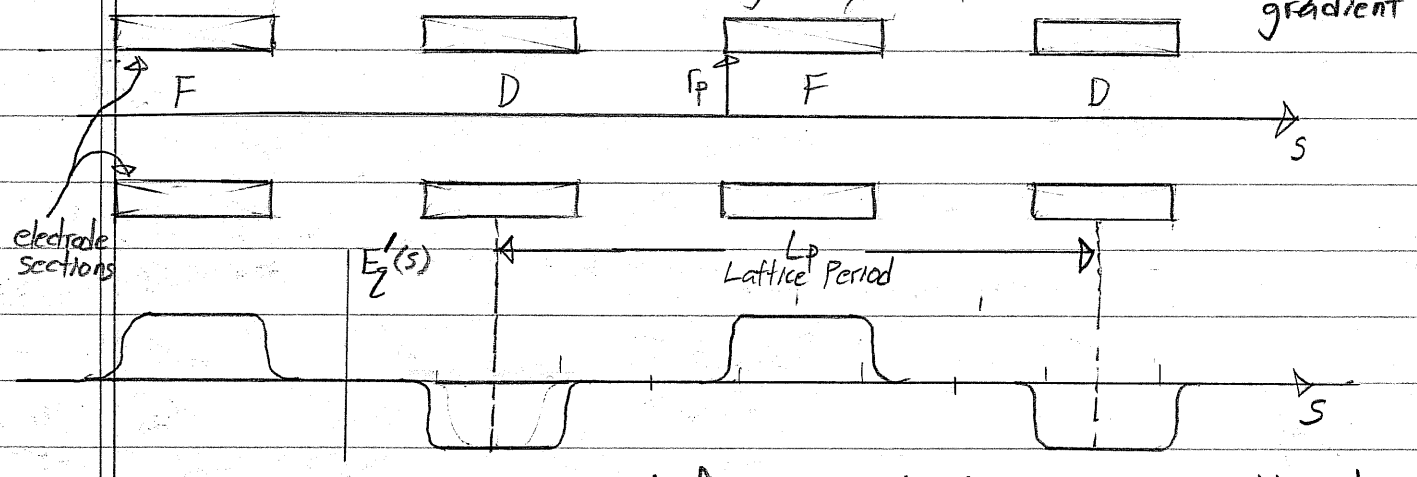
$$G = \frac{2V_0}{r_p^2} = -\frac{\partial E_x^a}{\partial y} = \frac{\partial E_y^a}{\partial x} = \text{Electric Gradient}$$

Hyperbolic Electrodes

V_0 = pole voltage

r_p = "pipe" aperture radius

For a lattice of finite length quadrupoles with alternating gradient



Net focusing in both planes by alternating "Focus" and "Defocus" quadrupoles.

Equations of motion:

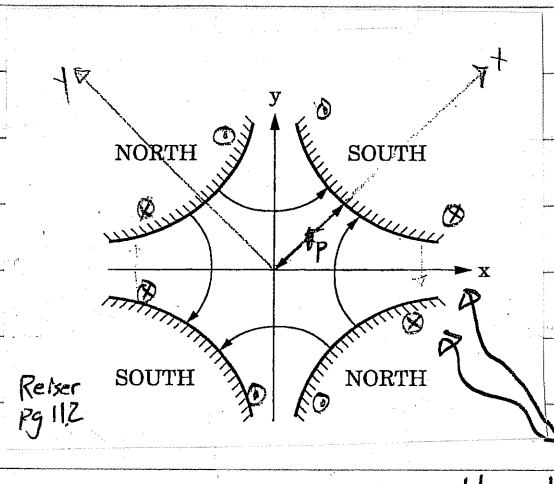
$$x'' + \left[\frac{1}{\gamma_b\beta_b}\frac{d}{ds}(\gamma_b\beta_b)\right]x' + \kappa_q(s)x = \frac{-q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$

$$y'' + \left[\frac{1}{\gamma_b\beta_b}\frac{d}{ds}(\gamma_b\beta_b)\right]y' - \kappa_q(s)y = \frac{-q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}$$

$$\kappa_q(s) \equiv \frac{qG}{m\gamma_b\beta_b^2c^2} = \frac{G}{\beta_b c\,[B\rho]}$$

B) Magnetic Quadrupoles ($\vec{E}^a = 0$)

In the center of a magnetic quadrupole:



Reiser pg 112

Hyperbolic Iron Poles.

2D Fields

$$B_x^a = B_0 \frac{y}{r_p}$$

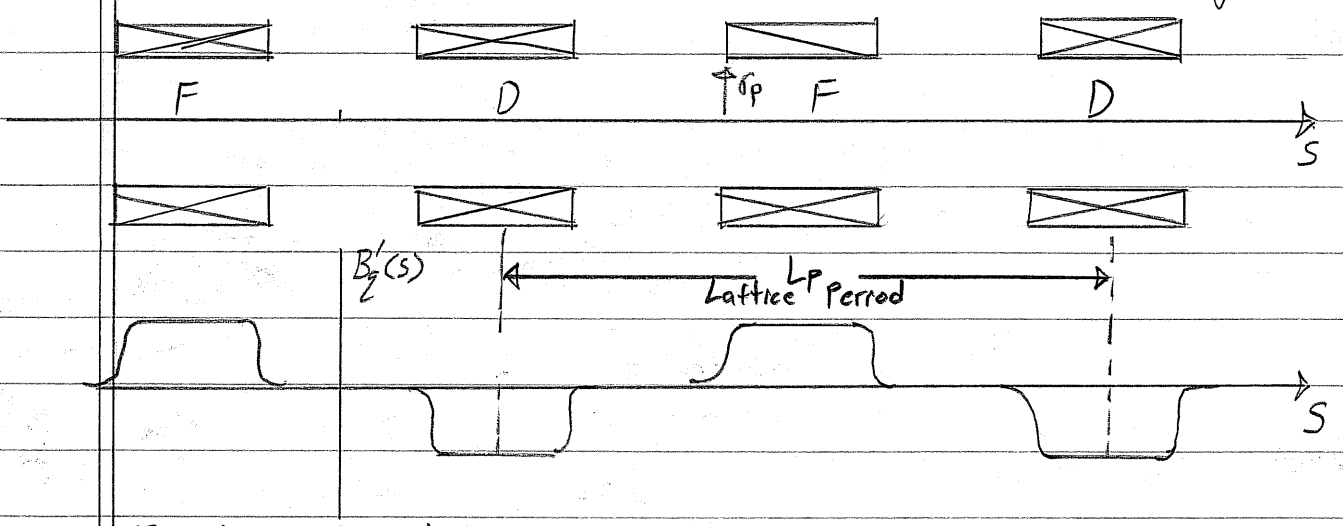
$$B_y^a = B_0 \frac{x}{r_p}$$

$$G \equiv \frac{B_0}{r_p} = \frac{\partial B_x^a}{\partial y} = \frac{\partial B_y^a}{\partial x} = \text{Magnetic Gradient}$$

$B_0 = |\vec{B}_a|_{r=r_p}$ = pole Field

r_p = "pipe" aperture radius

For a lattice of finite length quadrupoles with alternating gradient.



Equations of motion:

$$x'' + \left[\frac{1}{\gamma_b\beta_b}\frac{d}{ds}(\gamma_b\beta_b)\right]x' + \kappa_q(s)\,x = \frac{-q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$

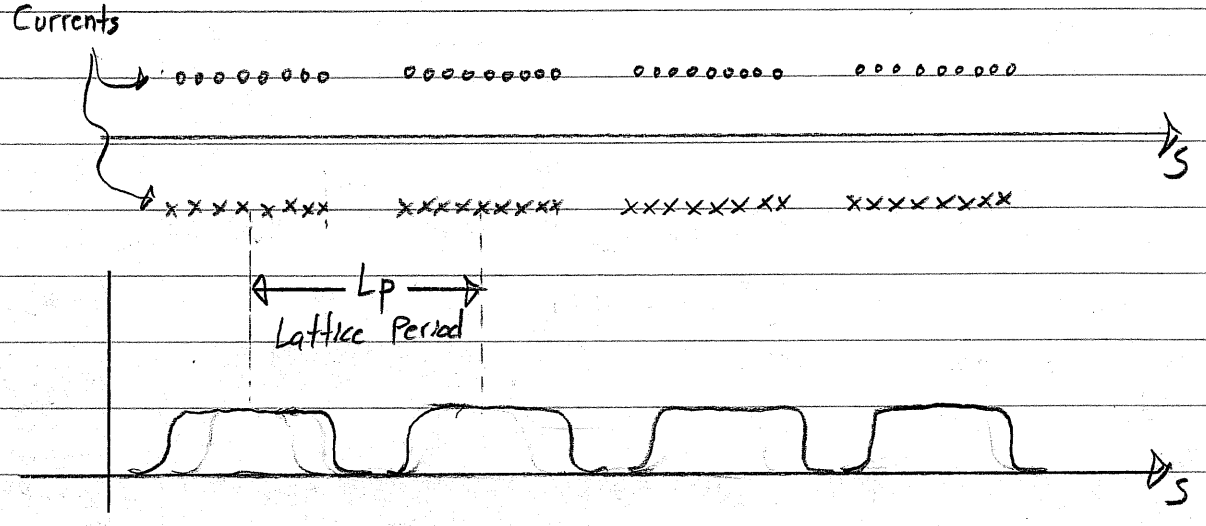
$$y'' + \left[\frac{1}{\gamma_b\beta_b}\frac{d}{ds}(\gamma_b\beta_b)\right]y' - \kappa_q(s)\,x = \frac{-q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}$$

$$\kappa_q(s) \equiv \frac{qG}{m\gamma_b\beta_b c} = \frac{G}{[B\rho]}$$

* Equations are the same as for the electric case with diff. $\kappa_q(s)$.

Solenoidal Focusing

$$\boxed{\begin{aligned} \vec{E}^a &= 0 \\ \vec{B}^a &= -\frac{1}{2} B_z^{a\prime}(s)\, \vec{x}_\perp + \hat{z}\, B_z^a(s) + \text{Higher order Terms.} \end{aligned}}$$



Equations of motion:

$$\boxed{\begin{aligned} x'' + \left[\frac{1}{\gamma_b \beta_b}\frac{d}{ds}(\gamma_b \beta_b)\right] x' &= \frac{\omega_c'(s)}{2\gamma_b \beta_b c}\, y + \frac{\omega_c(s)}{\gamma_b \beta_b c}\, y' - \frac{q}{m\gamma_b^3 \beta_b^2 c^2}\frac{\partial \phi}{\partial x} \\ y'' + \left[\frac{1}{\gamma_b \beta_b}\frac{d}{ds}(\gamma_b \beta_b)\right] y' &= \frac{-\omega_c'(s)}{2\gamma_b \beta_b c}\, x - \frac{\omega_c(s)}{\gamma_b \beta_b c}\, x' - \frac{q}{m\gamma_b^3 \beta_b^2 c^2}\frac{\partial \phi}{\partial y} \end{aligned}}$$

Here:

$$\omega_c = \frac{q B_z^a(s)}{m} = \text{cyclotron frequency of applied magnetic field}$$

These equations are linearly cross-coupled in the applied field terms

- x equation depends on $B_z' y$, $B_z y'$
- y equation depends on $B_z' x$, $B_z x'$

The linear cross-coupling in the applied field can be transformed away in the special case of no acceleration ($\gamma_b \beta_b = \text{const.}$) by a local (s-varying) transformation to a rotating "Larmor" frame (see Appendix A). If the beam is axisymmetric

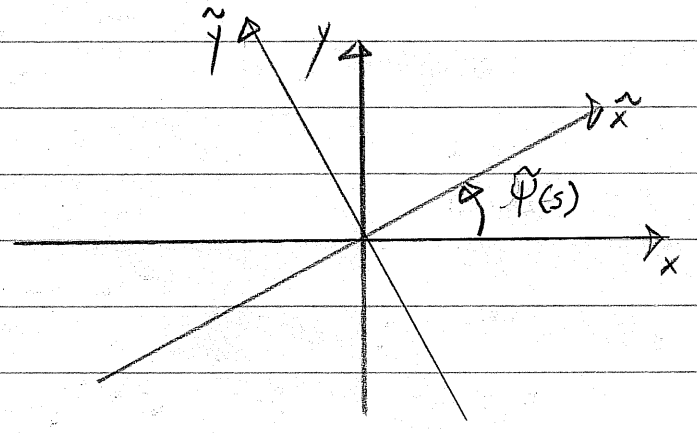
$$\frac{\partial \phi}{\partial \vec{x}_\perp} = \frac{\partial \phi}{\partial r}\frac{\partial r}{\partial \vec{x}_\perp} = \frac{\partial \phi}{\partial r}\frac{\vec{x}_\perp}{r}$$

then the equations of motion can be expressed in uncoupled form as (see Appendix A):

$$\boxed{\begin{aligned} \tilde{x}'' + \kappa(s)\,\tilde{x} &= \frac{-q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi}{\partial r}\frac{\tilde{x}}{r} \\ \tilde{y}'' + \kappa(s)\,\tilde{y} &= \frac{-q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi}{\partial r}\frac{\tilde{y}}{r} \end{aligned}}$$

* Transform valid only for coasting beam.

Here:



Rotating Frame variables with $\sim$

$$\kappa(s) = \frac{\omega_c^2(s)}{4\gamma_b^2\beta_b^2c^2}$$ Solenoid focusing strength.

$$\tilde{x}(s) = x\cos\tilde{\psi}(s) + y\sin\tilde{\psi}(s)$$
$$\tilde{y}(s) = -x\sin\tilde{\psi}(s) + y\cos\tilde{\psi}(s)$$

$$\tilde{\psi}(s) = -\frac{1}{2\gamma_b\beta_b c}\int_{s_i}^{s} d\tilde{s}\,\omega_c(\tilde{s})$$

s_i outside chain of magnets.

It has been shown that in certain reasonable approximations in a <u>linear</u> focusing channels including:

1) Continuous Focusing
2) Alternating Gradient Quadrupole Focusing
   - Electric
   - Magnetic
3) Solenoidal Focusing

that the transverse particle equations of motion can be cast in the form:

$$\boxed{\begin{aligned} x'' + \left[\frac{1}{\gamma_b\beta_b}\frac{d}{ds}(\gamma_b\beta_b)\right]x' + \kappa_x(s)\,x &= \frac{-q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x} \\ y'' + \left[\frac{1}{\gamma_b\beta_b}\frac{d}{ds}(\gamma_b\beta_b)\right]y' + \kappa_y(s)\,y &= \frac{-q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y} \\ \kappa_x(s) &= x\text{-lattice focusing function} \\ \kappa_y(s) &= y\text{-lattice focusing function} \end{aligned}}$$

- Continuous Focusing: $\kappa_x = \kappa_y = k_{\beta 0}^2 = \text{const.}$
- Quadrupole focusing: $\kappa_x(s) = -\kappa_y(s) \equiv \kappa_q(s)$
- Solinoidal focusing: $\kappa_x(s) = \kappa_y(s) \equiv \kappa(s)$
  with $\frac{d}{ds}(\gamma_b\beta_b) = 0$

Before studying space-charge effects, it is instructive to analyze these equations neglecting space-charge:

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial y} \longrightarrow 0$$

and acceleration:

$$\frac{1}{\gamma_b\beta_b}\frac{d}{ds}(\gamma_b\beta_b) \longrightarrow 0$$

In this simple case, the x- and y- equations have identical form:

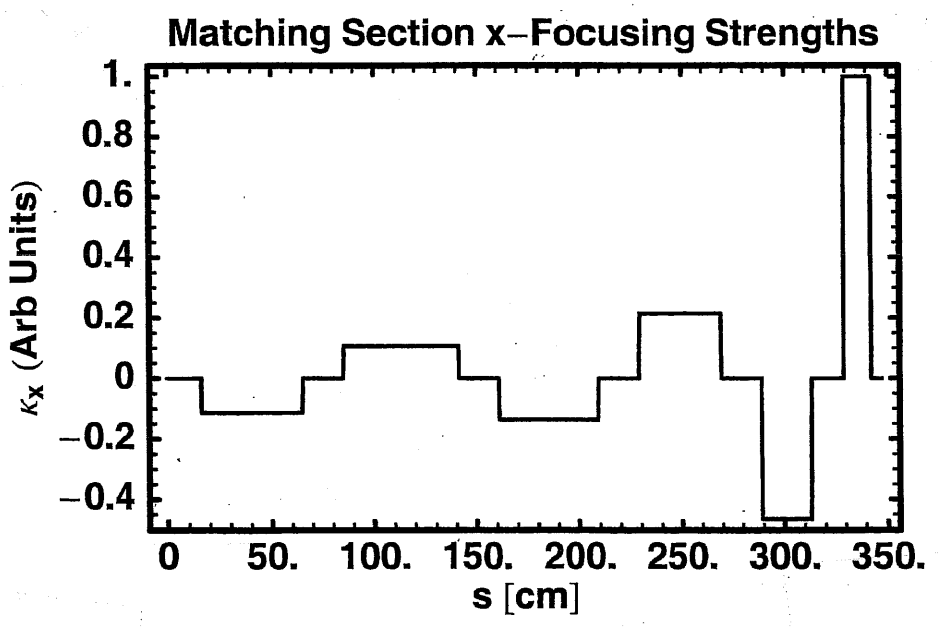
$$\boxed{\begin{aligned} x'' + \kappa_x(s)\,x &= 0 \\ y'' + \kappa_y(s)\,y &= 0 \end{aligned}}$$

Later we will show (KV Model) that linear space charge forces may also be included in κ_x and κ_y.

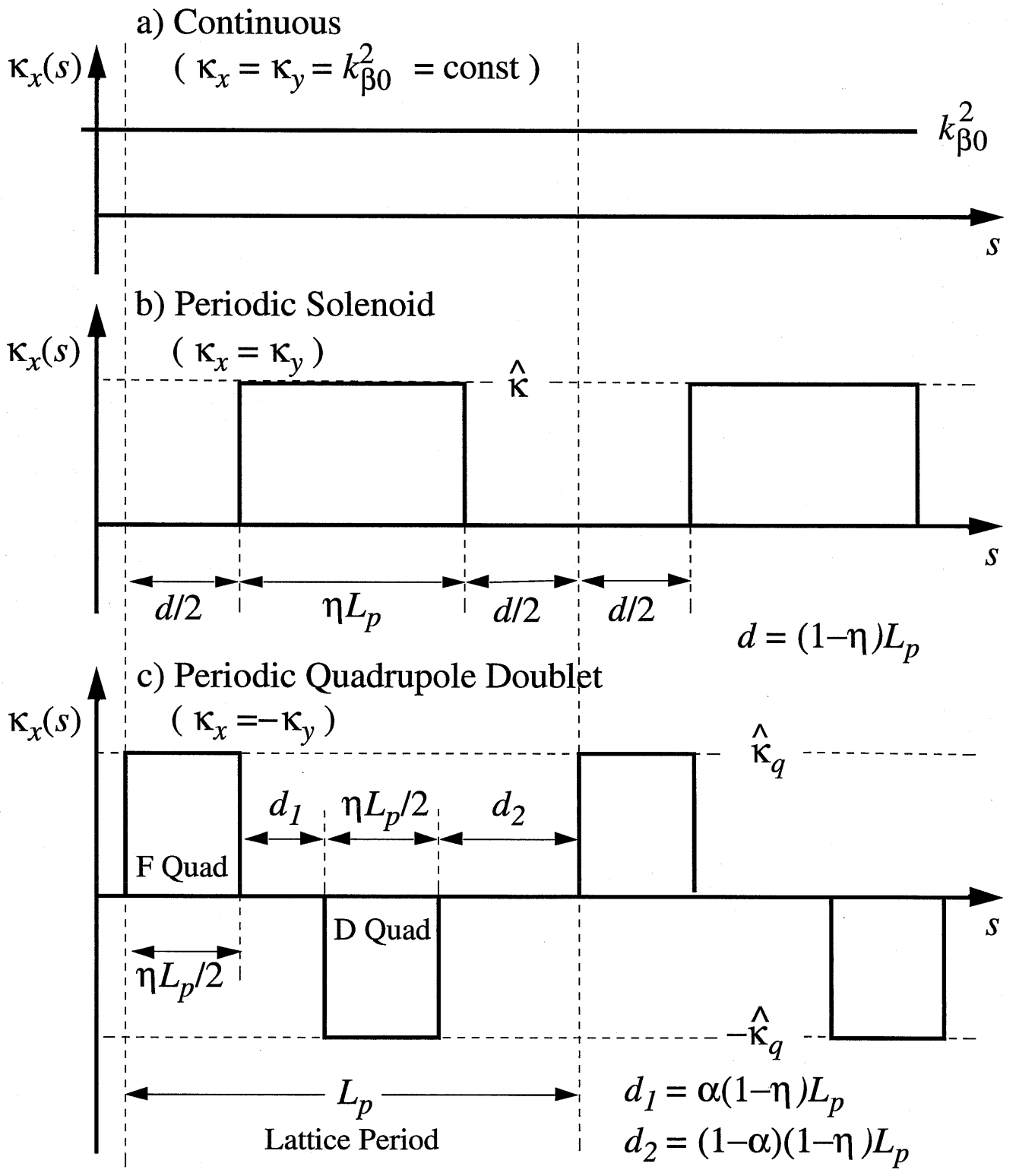
Usually, κ_x and κ_y will be periodic functions of s.

$$\boxed{\begin{aligned} \kappa_x(s+L_p) &= \kappa_x(s) \\ \kappa_y(s+L_p) &= \kappa_y(s) \end{aligned} \qquad L_p = \text{lattice period}}$$

But this <u>need not</u> be the case. Most of the analysis that follows is presented for the periodic case for simplicity in interpretation, but holds in general for matching sections and other non-periodic lattices where $\kappa_x(s)$ and $\kappa_y(s)$ may change form strongly



Piecewise Constant Linear Focusing Functions for Periodic Lattices



The equations presented in this section apply to a single particle moving in a beam. In the remaining sections we will carry out analysis of the equations neglecting space-charge effects ($\phi \to 0$) as is conventional in the standard theory of low-intensity accelerators. What we learn from this treatment will later aid analysis of high intensity beams where space-charge cannot be neglected. For example, we will later see that the x and y particle equations of motion can be applied to analyze the evolution of the beam in the absence of image charges:

$$x \longrightarrow x_c = \langle x \rangle_\perp \qquad \text{x-Centroid}$$
$$y \longrightarrow y_c = \langle y \rangle_\perp \qquad \text{y-Centroid}$$

By making appropriate variable substitutions, much of the analysis that follows can be applied to intense beams.

J.J. Barnard and S.M. Lund, USPAS Jan. 2004

Appendix A: Transformation of Solenoidal Focused Particle Equations of Motion to Uncoupled Form

A1

Solenoid Equations of motion:

$$\boxed{\begin{aligned} x'' + \left[\frac{1}{\gamma_b\beta_b}\frac{d}{ds}(\gamma_b\beta_b)\right]x' &= \frac{\omega_c'(s)}{2\gamma_b\beta_b c}y + \frac{\omega_c(s)}{\gamma_b\beta_b c}y' - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x} \\ y'' + \left[\frac{1}{\gamma_b\beta_b}\frac{d}{ds}(\gamma_b\beta_b)\right]y' &= -\frac{\omega_c'(s)}{2\gamma_b\beta_b c}x - \frac{\omega_c(s)}{\gamma_b\beta_b c}x' - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y} \end{aligned}}$$

To simplify algebra introduce the complex coordinate

$$\boxed{z = x + iy}$$

Then the two equations can be expressed as a single complex equation:

$$z'' + \left[\frac{1}{\gamma_b\beta_b}\frac{d}{ds}(\gamma_b\beta_b)\right]z' + \frac{i\omega_c'}{2\gamma_b\beta_b c}z + \frac{i\omega_c}{\gamma_b\beta_b c}z' = \frac{-q}{m\gamma_b^3\beta_b^2c^2}\left(\frac{\partial\phi}{\partial x} + i\frac{\partial\phi}{\partial y}\right)$$

For axisymmetric potential ϕ:

$$\frac{\partial\phi}{\partial x} + i\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial r}\frac{z}{r} \qquad r = \sqrt{x^2+y^2}$$

and the equation of motion becomes:

$$\boxed{z'' + \left[\frac{1}{\gamma_b\beta_b}\frac{d}{ds}(\gamma_b\beta_b)\right]z' + \frac{i\omega_c'}{2\gamma_b\beta_b c}z + \frac{i\omega_c}{\gamma_b\beta_b c}z' = \frac{-q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{z}{r}}$$

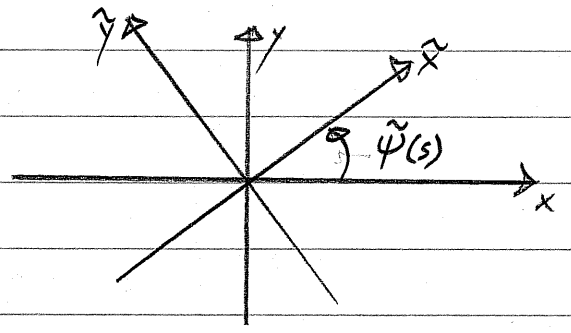
For the case of no acceleration, the equations reduce to:

$$\boxed{z'' + \frac{i\omega_c'}{2\gamma_b\beta_b c}z + \frac{i\omega_c}{\gamma_b\beta_b c}z' = \frac{-q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{z}{r}}$$

Following Wiedemann, Vol. II pg 82, S.M. Lund A2/

Introduce the complex variable $\tilde{z}$ and a (real) phase function $\tilde{\psi}(s)$ by:

$$\boxed{\tilde{z} = z\, e^{-i\tilde{\psi}(s)} \equiv \tilde{x} + i\tilde{y}}$$



Note:
$|\tilde{z}| = r$

Then

$$z = \tilde{z}\, e^{i\tilde{\psi}}$$
$$z' = (\tilde{z}' + i\tilde{\psi}'\tilde{z})\, e^{i\tilde{\psi}}$$
$$z'' = (\tilde{z}'' + 2i\tilde{\psi}'\tilde{z}' + i\tilde{\psi}''\tilde{z} - \tilde{\psi}'^2\tilde{z})\, e^{i\tilde{\psi}}$$

and the complex equations of motion become:

$$\boxed{\tilde{z}'' + \left[2i\tilde{\psi}' + i\frac{\omega_c}{\gamma_b\beta_b c}\right]\tilde{z}' + \left[i\tilde{\psi}'' - \tilde{\psi}'^2 - \frac{\omega_c}{\gamma_b\beta_b c}\tilde{\psi}' + i\frac{\omega_c'}{2\gamma_b\beta_b c}\right]\tilde{z} = \frac{-q}{m\gamma_b^3\beta_b^2 c^2}\frac{\partial\phi}{\partial r}\frac{\tilde{z}}{r}}$$

Choose $\tilde{\psi}(s)$ to elliminate the $\tilde{z}'$ coefficient:

$$\boxed{\tilde{\psi}' = -\frac{\omega_c}{2\gamma_b\beta_b c}}$$

Then $\tilde{\psi}'' = \dfrac{-\omega_c'}{2\gamma_b\beta_b c}$ and the equation reduces to:

$$\boxed{\tilde{z}'' + \frac{\omega_c^2}{4\gamma_b^2\beta_b^2 c^2}\tilde{z} = \frac{-q}{m\gamma_b^3\beta_b^2 c^2}\frac{\partial\phi}{\partial r}\frac{\tilde{z}}{r}}$$

Or using $\tilde{z} = \tilde{x} + i\tilde{y}$, the equations are now decoupled in the $\tilde{x}$, $\tilde{y}$ variables!

Uncoupled form same as for continuous or quadrupole focusing.

$$\boxed{\begin{aligned} \tilde{x}'' + \kappa_s(s)\tilde{x} &= \frac{-q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{\tilde{x}}{r} \\ \tilde{y}'' + \kappa_s(s)\tilde{y} &= \frac{-q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{\tilde{y}}{r} \\ \text{Solenoidal focusing force} \quad \kappa_s(s) &\equiv \frac{\omega_c^2(s)}{4\gamma_b^2\beta_b^2c^2} \qquad \omega_c(s) = \frac{qB_z(s)}{m} \end{aligned}}$$

The transformation can be effected by integrating the equation for $\tilde{\psi}'$

$$\boxed{\tilde{\psi}(s) = -\frac{1}{2\gamma_b\beta_b c}\int_{s_i}^{s} d\tilde{s}\ \omega_c(\tilde{s})}$$

Where s_i some value of s where the solenoidal field $B_z(s)$ is zero. (i.e., outside the lattice of magnets). Then the transformation is given by:

$$\boxed{\begin{aligned} \tilde{x} &= \mathrm{Re}\left[(x+iy)e^{-i\tilde{\psi}(s)}\right] = x\cos\tilde{\psi}(s) + y\sin\tilde{\psi}(s) \\ \tilde{y} &= \mathrm{Im}\left[(x+iy)e^{-i\tilde{\psi}(s)}\right] = -x\sin\tilde{\psi}(s) + y\cos\tilde{\psi}(s) \end{aligned}}$$

Since $\tilde{\psi}' = -\frac{1}{2}\left(\frac{\omega_c}{\gamma_b\beta_b c}\right)$, note that the $\tilde{x}, \tilde{y}$ frame is rotating at 1/2 the local (s-varying) cyclotron frequency. This rotating frame is refered to as the "Larmor" frame. For a

uniform solenoid with $B_z(s) = \text{const}$, $\tilde{\psi}' = -\frac{1}{2}\left(\frac{\omega_c}{\gamma_b \beta_b c}\right) = \text{const}$, and the Larmor frame is uniformly rotating as is well-known.

Note that at the initial condition with $B_z = 0$,

$$\tilde{x}(s=s_i) = x(s=s_i) = x_i$$
$$\tilde{x}'(s=s_i) = x'(s=s_i) = x_i'$$

$$\tilde{y}(s=s_i) = y(s=s_i) = y_i$$
$$\tilde{y}'(s=s_i) = y'(s=s_i) = y_i'$$

The rotation matrix of the transformation can be explicitly worked out:

$$\tilde{x} = x\cos\tilde{\psi} + y\sin\tilde{\psi}$$
$$\tilde{x}' = -x\tilde{\psi}'\sin\tilde{\psi} + x'\cos\tilde{\psi} + y\tilde{\psi}'\cos\tilde{\psi} + y'\sin\tilde{\psi}$$

$$\tilde{y} = -x\sin\tilde{\psi} + y\cos\tilde{\psi}$$
$$\tilde{y}' = -x\tilde{\psi}'\cos\tilde{\psi} - x'\sin\tilde{\psi} - y\tilde{\psi}'\sin\tilde{\psi} + y'\cos\tilde{\psi}$$

collecting terms and using $\tilde{\psi}' = -\frac{1}{2}\frac{\omega_c}{\gamma_b\beta_b c}$;

$$\boxed{\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} = \begin{pmatrix} \cos\tilde{\psi} & 0 & \sin\tilde{\psi} & 0 \\ \frac{\omega_c}{2\gamma_b\beta_b c}\sin\tilde{\psi} & \cos\tilde{\psi} & -\frac{\omega_c}{2\gamma_b\beta_b c}\cos\tilde{\psi} & \sin\tilde{\psi} \\ -\sin\tilde{\psi} & 0 & \cos\tilde{\psi} & 0 \\ \frac{\omega_c}{2\gamma_b\beta_b c}\cos\tilde{\psi} & -\sin\tilde{\psi} & \frac{\omega_c}{2\gamma_b\beta_b c}\sin\tilde{\psi} & \cos\tilde{\psi} \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{x}' \\ \tilde{y} \\ \tilde{y}' \end{pmatrix}}$$

J.J. Barnard and S.M. Lund
USPAS, Jan. 2004

§3 Description of Applied Focusing Fields

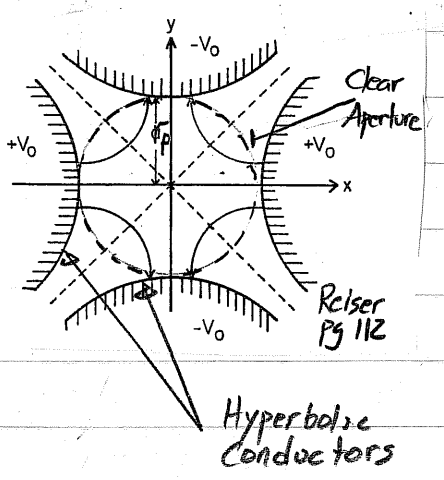
Applied fields for focusing, bending, and acceleration enter the equations of motion via

$\vec{E}^a$: Applied Electric Field
$\vec{B}^a$: Applied Magnetic Field

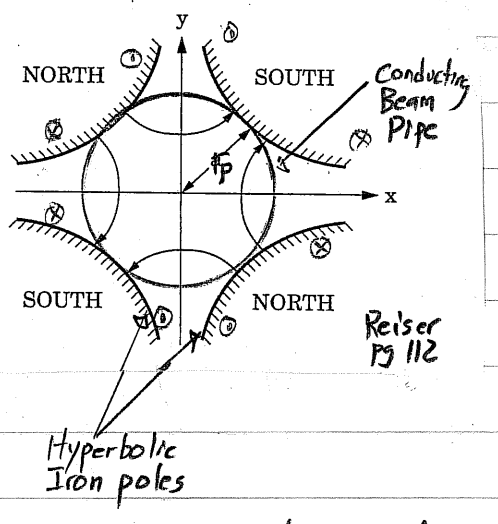
Generally, these fields are produced by sources outside an aperture of radius r_p. For example, for electric and magnetic quadrupoles:

Transverse Cross-Sections

Electric Quadrupole



Magnetic Quadrupole



The fields within the aperture of these and similar elements satisfy the vacuum Maxwell Equations:

$$\boxed{\begin{aligned} \nabla \cdot \vec{E}^a &= 0 & \nabla \cdot \vec{B}^a &= 0 \\ \nabla \times \vec{E}^a &= -\frac{\partial \vec{B}^a}{\partial t} & \nabla \times \vec{B}^a &= \frac{1}{c^2}\frac{\partial \vec{E}^a}{\partial t} \end{aligned}}$$

In many cases of interest, the fields are static or a quasistatic approximation can be employed and the $\partial/\partial t$ terms neglected. Then the equations for $\vec{E}^a$ and $\vec{B}^a$ within the aperture are of the same form with:

$$\boxed{\begin{array}{ll} \nabla \cdot \vec{E}^a = 0 & \nabla \cdot \vec{B}^a = 0 \\ \nabla \times \vec{E}^a = 0 & \nabla \times \vec{B}^a = 0 \end{array}}$$

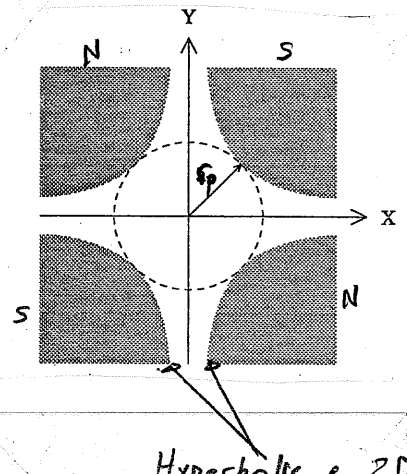
Static Vacuum Maxwell Equations

In general, elements are tuned to limit the strength of nonlinear field terms, so the beam experiences primarily linear applied fields. We will find later that this allows better preservation of beam quality. However, this removal of nonlinear field terms <u>cannot</u> be done absolutely in 3D geometries. There will always be some nonlinear fields in finite 3D structures as a result of the properties of the vacuum Maxwell equations. Also, even in lower-dimensional idealizations of real structures, deviations from ideal symmetry will result from practical concerns and such deviations will produce further nonlinear field terms. As an example of this, 2D iron magnet with infinite pole extent must be truncated leading to nonlinear focusing terms even in 2D:

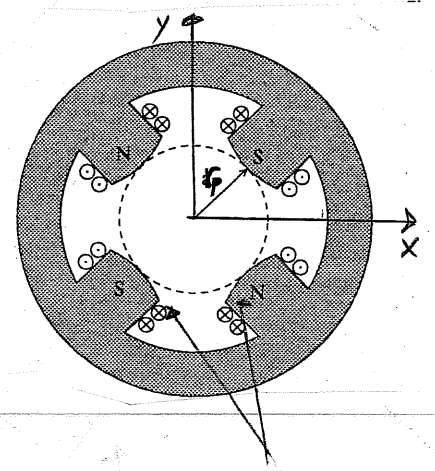
Cross-Sections of Quadrupole Magnets

(Conte and Mackay, Intro. to the Physics of Particle Accelerators, World Scientific, 1991 pg 59.)

Ideal | **Practical**



Hyperbolic iron poles : 2D Field Ideal Quadrupole



Truncated iron poles : 2D Field Ideal Quadrupole + Error Terms

Magnetic optic and accelerating structures are both extensive topics and it is not possible to cover them in this intro. (survey) class. In the following we will illustrate some simple methods of field error characterization of magnetic optics with transverse field components (dipoles, quadrupoles, sextupoles, ...) This treatment is not comprehensive.

Consider magnetic optics with:

$$\vec{E}^a = 0 \quad \text{and} \quad \vec{B}^a \simeq \vec{B}^a_\perp$$

Electric optics can be treated analogously since the static vacuum Maxwell Equations are the same form for $\vec{E}^a$ and $\vec{B}^a$.

Magnetic Forces

$$\vec{F}_\perp^q \simeq q\beta_b c \cdot \hat{z} \times \vec{B}_\perp^q$$

$$\vec{B}_\perp^q = \hat{x} B_x^q + \hat{y} B_y^q \qquad \perp \text{ Field}$$

$$\boxed{\begin{aligned} F_x^q &\simeq -q\beta_b c \cdot B_y^q \\ F_y^q &\simeq q\beta_b c \, B_x^q \end{aligned}}$$

Field components can be Taylor-series expanded about the element center $\vec{x}_\perp = 0$

$$B_x^q = \underbrace{B_x^q(0)}_{\text{Bend}} + \underbrace{\frac{\partial B_x^q}{\partial y}(0)\, y}_{\substack{\text{Normal}\\ \text{Linear Focus}}} + \underbrace{\frac{\partial B_x^q}{\partial x}(0)\, x}_{\substack{\text{Skew}\\ \text{Linear Focus}}}$$

$$\text{Nonlinear Terms} \left\{ \begin{aligned} &+ \frac{1}{2}\frac{\partial^2 B_x^q}{\partial x^2}(0)\, x^2 + \frac{\partial^2 B_x^q}{\partial x \partial y}(0)\, xy + \frac{1}{2}\frac{\partial^2 B_x^q}{\partial y^2}(0)\, y^2 \\ &+ \cdots\cdots \end{aligned} \right.$$

$$B_y^q = \underbrace{B_y^q(0)}_{\text{Bend}} + \underbrace{\frac{\partial B_y^q}{\partial x}(0)\, x}_{\substack{\text{Normal}\\ \text{Linear Focus}}} + \underbrace{\frac{\partial B_y^q}{\partial y}(0)\, y}_{\substack{\text{Skew}\\ \text{Linear Focus}}}$$

$$\text{Nonlinear Terms} \left\{ \begin{aligned} &+ \frac{1}{2}\frac{\partial^2 B_y^q}{\partial x^2}(0)\, x^2 + \frac{\partial^2 B_y^q}{\partial x \partial y}(0)\, xy + \frac{1}{2}\frac{\partial^2 B_y^q}{\partial y^2}(0)\, y^2 \\ &+ \cdots\cdots \end{aligned} \right.$$

Sources of Undesired Applied Field Components:

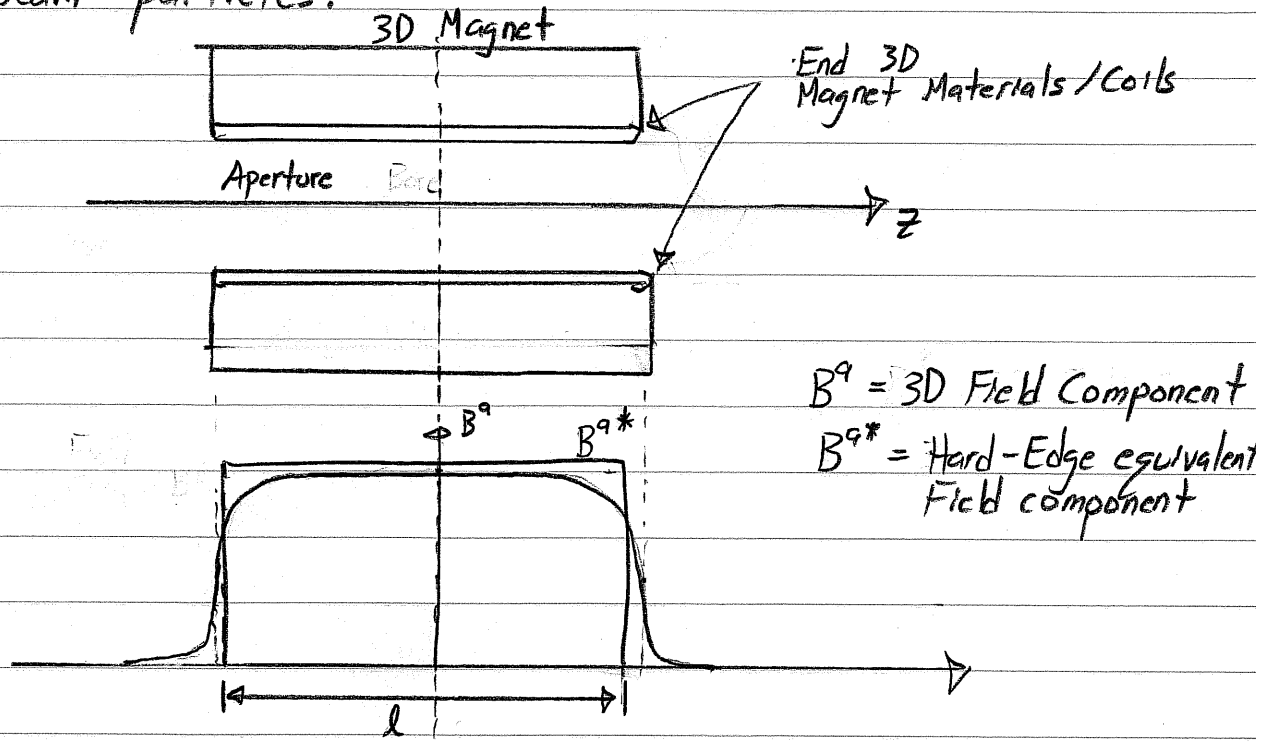
1/ Systematic errors or nonideal geometry associated with the intrinsic structure of the magnet.

2/ Random construction errors in individual magnets.

3/ Alignment of magnets giving field projections in unwanted directions

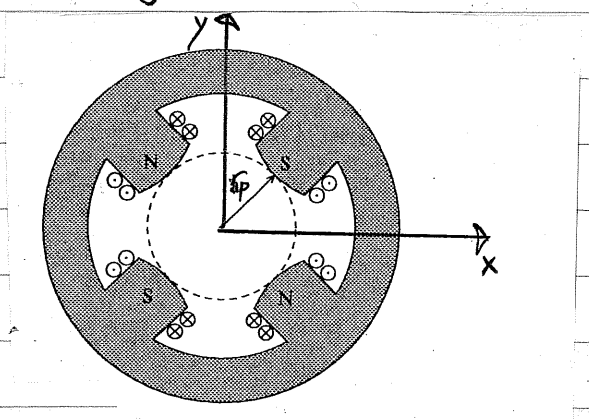
4/ Excitation errors (currents in coils not correct values and/or balanced).

Real 3D magnets can often be modeled for sufficient accuracy by 2D hard-edge equivalent magnets that give same impulse to the beam particles.



Mid-Plane 2D Structure that generates B^{q*}

ℓ = Hard-Edge length



Many perscriptions exist for calculating ℓ and optimal hard-edge models for various purposes.

A reasonable perscription to calculate ℓ for a physical quadrupole magnet with a linear field gradient G is:

$$\boxed{\begin{aligned} G(z) &\equiv \left.\frac{\partial B_x^a}{\partial y}\right|_{x=y=0} \\ \ell\, G(z=0) &= \int_{-\infty}^{\infty} dz\; G(z) \end{aligned}}$$

ℓ = "effective" magnet length

In many cases it is sufficient to characterize the field errors in 2D hard-edge equivalent as:

$$\boxed{\begin{aligned} B_x(x,y) &= \frac{1}{\ell}\int_{-\infty}^{\infty} dz\; B_x^a(x,y,z) \\ B_y(x,y) &= \frac{1}{\ell}\int_{-\infty}^{\infty} dz\; B_y^a(x,y,z) \end{aligned}}$$

2D Effective Fields 3D Fields

Operating on the vacuum Maxwell equations with $\int_{-\infty}^{\infty}\frac{dz}{\ell}\cdots$ yields the (exact) 2D field equations:

$$\boxed{\begin{aligned} \frac{\partial B_x(x,y)}{\partial x} &= -\frac{\partial B_y(x,y)}{\partial y} \\ \frac{\partial B_x(x,y)}{\partial y} &= \frac{\partial B_y(x,y)}{\partial x} \end{aligned}}$$

These equations can be recognized as the Cauchy-Riemann conditions for a complex field variable

$$\boxed{\underset{\sim}{B} = B_y + i B_x}$$

Here, $i = \sqrt{-1}$

to be an analytical function of the complex variable

$$\boxed{\underset{\sim}{z} = x + iy}$$

$i = \sqrt{-1}$

It follows that $\underset{\sim}{B}(z)$ can be analyzed using the full power of the highly developed theory of analytical functions of a complex variable.

Expand $\underset{\sim}{B}(z)$ as a Laurent series within the vacuum aperture as:

$$\underset{\sim}{B}(z) = B_y + i B_x = \sum_{n=1}^{\infty} \underset{\sim}{B_n} \left(z / r_p \right)^{n-1}$$

$\underset{\sim}{B_n}$ = const — complex constants (multipole coefficients) giving the structure of the field.

n = multipole index

r_p = aperture "pipe" radius

The multipole coefficients can be resolved as:

$$\underset{\sim}{B_n} = b_n + i a_n \qquad a_n, b_n \text{ real constants}$$

$b_n \Rightarrow$ "normal" multipoles

$a_n \Rightarrow$ "skew" multipoles (rotated)

Index	Name	Normal Field	
$n=1$	Dipole	$B_x = 0$,	$B_y = b_1$
$n=2$	Quadrupole	$B_x = b_2\, y / r_p$	$B_y = b_2\, x / r_p$
$n=3$	Sextupole	$B_x = b_3\, 2xy / r_p^2$,	$B_y = b_3\, (x^2 - y^2) / r_p^2$
$n=4$	Octupole		
⋮	⋮	⋮	⋮

Higher order multipole field components leading to nonlinear focusing forces decrease rapidly within the aperture. To see this use a polar representation of $\underset{\sim}{z}$ and $\underset{\sim}{B}_n$:

$$\underset{\sim}{z} = x + iy = r e^{i\theta} \qquad r = \sqrt{x^2+y^2}$$
$$\theta = \mathrm{ArcTan}[y, x]$$

similarly:

$$\underset{\sim}{B}_n = |\underset{\sim}{B}_n| \, e^{i\psi}$$

Thus the n-th order multipole term scales as

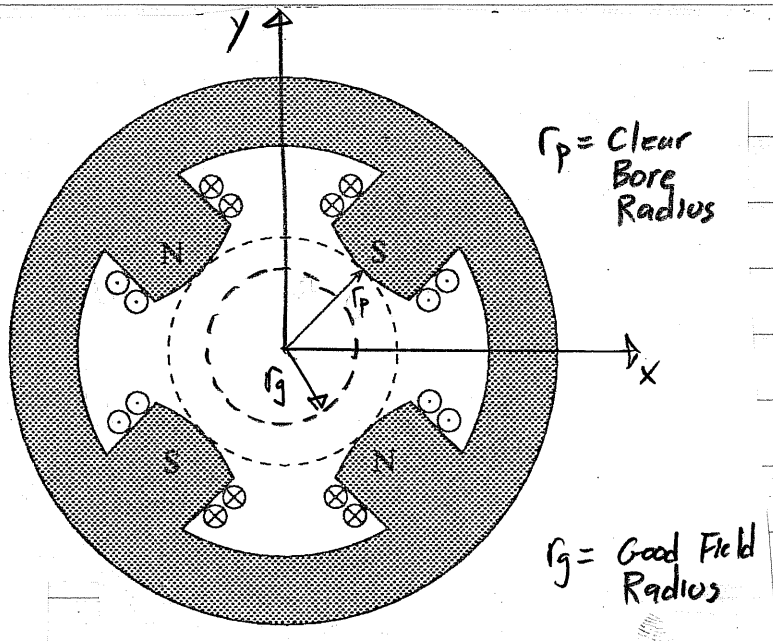
$$\boxed{\underset{\sim}{B}_n \left(\underset{\sim}{z}/r_p\right)^{n-1} = |\underset{\sim}{B}_n| \left(\frac{r}{r_p}\right)^{n-1} e^{i[(n-1)\theta + \psi]}}$$

- Unless the coefficient $|\underset{\sim}{B}_n|$ is very large, high-order terms will become small rapidly as (r/r_p) decreases

Better linear field quality magnets can be made simply by making the clear bore r_p larger or alternatively smaller beam cross-sections can be employed.

- Larger bore magnets cost more. So design becomes trade-off between cost and performance.

Generally, a practical magnet design will have a "good-field" radius that maximum field errors are specified on. In a good magnet design this can be 70% or more of the clear bore to the begining of magnet material structures.



Beam particle orbits are designed to remain within radius r_g.

Field error statements are readily generalized to 3D since

$$\nabla \cdot \vec{B}^a = 0 \qquad \Rightarrow \qquad \nabla^2 \vec{B}^a = 0$$
$$\nabla \times \vec{B}^a = 0$$

each component of $\vec{B}^a$ satisfies a Laplace equation within the vacuum aperture. Therefore, field errors can only decrease when moving deeper within a source free region of 3D space.

S. M. Lund 11/

Example 2D Nonideal Magnets from Permanent Magnet blocks

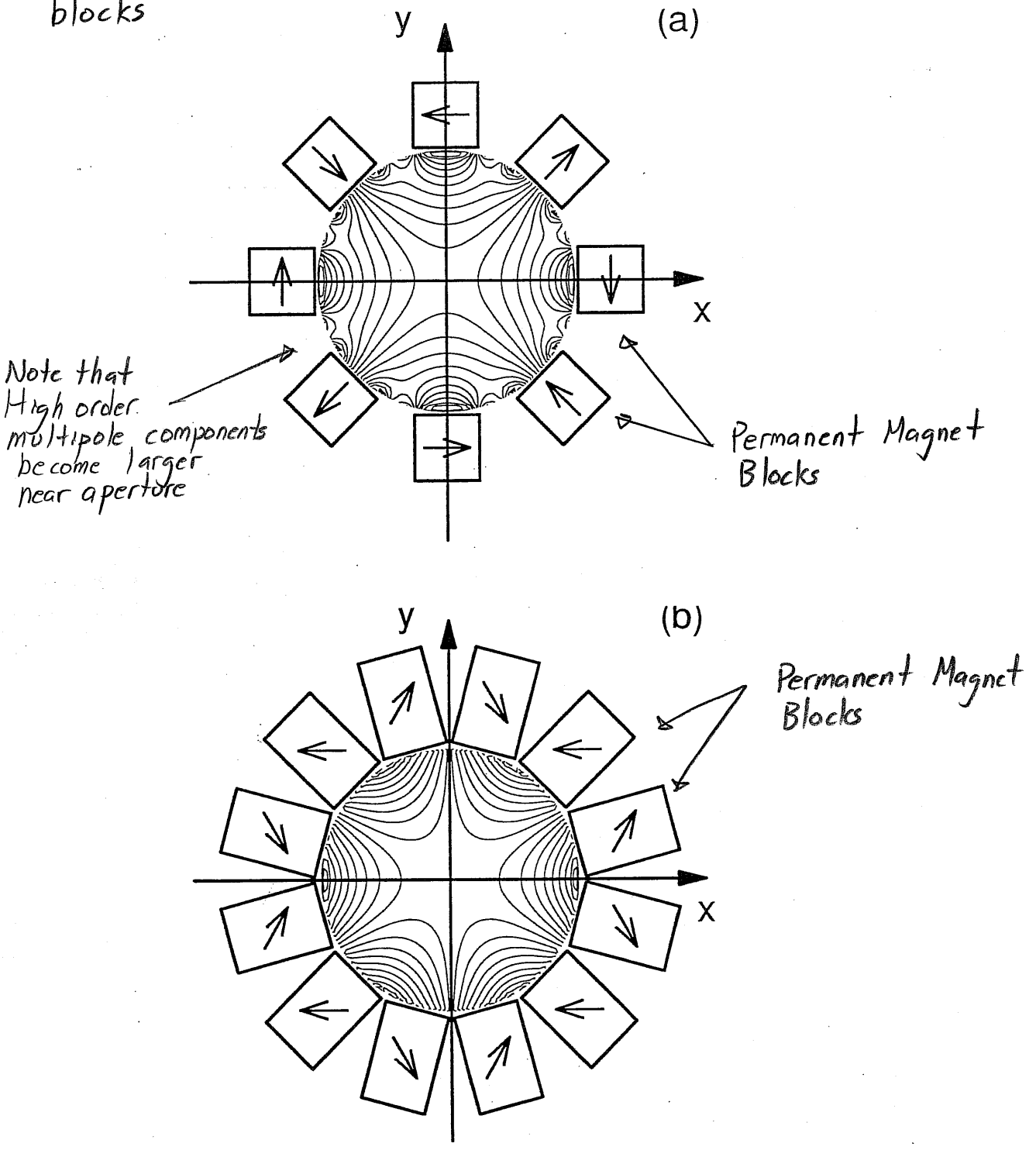


Fig. E1. Cross-sections of a Halbach array quadrupole magnet formed from $M = 8$ square-block magnets (a), and a sextupole magnet formed from $M = 12$ rectangular-block magnets (b).

J.J. Barnard and S.M. Lund
USPAS Jan. 2004

See Reiser § 3.8.2
Pgs 145–153

§4 Linear Equations of Motion without Space-Charge, Acceleration, and Momentum Spread

Neglect: – Space charge effects: $\phi \to 0$
– Nonlinear Applied Focusing: $\vec{B}^a, \vec{E}^a$ contain only linear focusing terms
– Acceleration: $\beta_b \gamma_b = \text{const.}$
– Momentum Spread Effects

Then the transverse particle equations of motion can be expressed in the form of Hill's equation:

$$\boxed{x''(s) + \kappa(s)\, x(s) = 0}$$

x = $\perp$ coordinate, possibly formed from various coordinates (even longitudinal coordinates can be cast in this form at some levels of approx.)

s = axial coordinate of ref. particle.

$$x' = \frac{dx}{ds}$$

$\kappa(s)$ = lattice focusing function.

For a periodic lattice:

$$\boxed{\begin{aligned} \kappa(s+L_p) &= \kappa(s) \\ L_p &= \text{Lattice Period.} \end{aligned}}$$

For a ring, one also has the superperiodicity condition:

$$\boxed{\begin{aligned} \kappa(s+C) &= \kappa(s) \quad * \\ C &= N L_p \qquad N = \text{superperiodicity.} \end{aligned}}$$

* This distinction matters when there are random error fields in the ring.

The equation of motion is linear and the (unique) solution with

$$\boxed{\begin{aligned} x(s=s_i) &= x(s_i) \\ x'(s=s_i) &= x'(s_i) \end{aligned}}$$

can be expressed in matrix form as:

$$\boxed{\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} \equiv \overline{M}(s|s_i) \begin{pmatrix} x(s_i) \\ x'(s_i) \end{pmatrix} \equiv \begin{pmatrix} C(s|s_i) & S(s|s_i) \\ C'(s|s_i) & S'(s|s_i) \end{pmatrix} \begin{pmatrix} x(s_i) \\ x'(s_i) \end{pmatrix}}$$

where $C(s|s_i)$ and $S(s|s_i)$ are "cosine"-like and "sine"-like principal trajectories satisfying:

$$C''(s|s_i) + \kappa(s)\, C(s|s_i) = 0 \; ; \quad C(s_i|s_i) = 1 \; , \quad C'(s_i|s_i) = 0$$

$$S''(s|s_i) + \kappa(s)\, S(s|s_i) = 0 \; ; \quad S(s_i|s_i) = 0 \; , \quad S'(s_i|s_i) = 1$$

An important property of this linear motion is that the Wronskian is conserved:

$$\boxed{\begin{aligned} W(s|s_i) \equiv \det \overline{M}(s|s_i) &= \det \begin{vmatrix} C(s|s_i) & S(s|s_i) \\ C'(s|s_i) & S'(s|s_i) \end{vmatrix} \\ &= C(s|s_i)\, S'(s|s_i) - C'(s|s_i)\, S(s|s_i) = 1 \end{aligned}}$$

// <u>Proof:</u> (abbreviate notation)

$$\begin{aligned} & -S(C'' + \kappa C) = 0 \\ + \quad & C(S'' + \kappa S) = 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} & CS'' - SC'' + \kappa(\overbrace{CS - SC}^{0}) = 0 \\ & \frac{dW}{ds} = 0 \end{aligned}$$

$$\Rightarrow \quad W(s) = W(s_i) = C_i S_i' - C_i' S_i = 1 \cdot 1 - 0 \cdot 0 = 1 \quad \checkmark \quad //$$

// Example

For a continuously focused system:

$$\kappa(s) = k_{\beta 0}^2 = \text{const.}$$

$$x'' + k_{\beta 0}^2 x = 0$$

$$C(s|s_i) = \cos[k_{\beta 0}(s-s_i)]$$
$$S(s|s_i) = \frac{\sin}{k_{\beta 0}}[k_{\beta 0}(s-s_i)]$$

$$\begin{pmatrix} x(s) \\ x(s') \end{pmatrix} = \begin{pmatrix} \cos[k_{\beta 0}(s-s_i)] & \frac{\sin}{k_{\beta 0}}[k_{\beta 0}(s-s_i)] \\ -k_{\beta 0}\sin[k_{\beta 0}(s-s_i)] & \cos[k_{\beta 0}(s-s_i)] \end{pmatrix} \begin{pmatrix} x(s_i) \\ x'(s_i) \end{pmatrix}$$

$$\bar{W} = \cos^2[k_{\beta 0}(s-s_i)] + \sin^2[k_{\beta 0}(s-s_i)] = \underline{1} \qquad //$$

We now examine particle stability. First, $\bar{M}$ must be the same in any period of a periodic lattice:

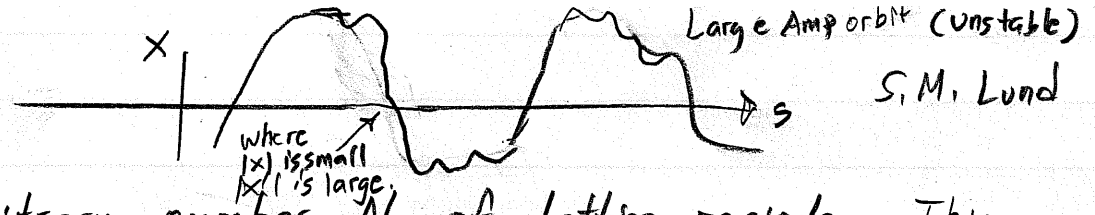
$$\bar{M}(s+L_p|s_i+L_p) = \bar{M}(s|s_i)$$

For a propogation distance s satisfying

$$NL_p \le s-s_i \le (N+1)L_p$$

$$\bar{M}(s|s_i) = \bar{M}(s-NL_p|s_i)\bar{M}(NL_p|s_i) = \bar{M}(s-NL_p|s_i)\bar{M}(s_i+L_p|s_i)^N$$

For a particle to have a stable orbit $x(s)$ and $x'(s)$ should remain bounded on propogation through an



arbitrary number N of lattice periods. This is equivalent to requiring that the elements of $\overline{M}$ remain bounded on propagation through any number of lattice periods, i.e.;

$$\overline{M} \equiv [M_{ij}]$$

$$\boxed{\lim_{N\to\infty} \left| M_{ij}(s_i+L_p|s_i)^N \right| < \infty \quad \Rightarrow \text{stable motion.}}$$

To analyze this condition we examine the eigenvectors of $\overline{M}$ for transport through one lattice period, as defined by

$$\boxed{\begin{gathered} \overline{M}(s_i+L_p|s_i)\cdot\vec{E}(s_i) = \lambda\,\vec{E}(s_i) \\ \lambda = \text{eigenvalue} \\ \vec{E}(s_i) = \text{eigenvector} \end{gathered}}$$

This equation will have two independent eigenvalues and eigenvectors. Abbreviate notation as

$$\overline{M}(s_i+L_p|s_i) \equiv \begin{pmatrix} C(s_i+L_p|s) & S(s_i+L_p|s_i) \\ C'(s_i+L_p|s) & S'(s_i+L_p|s_i) \end{pmatrix} \equiv \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \text{ etc.}$$

Non zero solutions exist when

$$\det\begin{vmatrix} C-\lambda & S \\ C' & S'-\lambda \end{vmatrix} = \lambda^2 - (C+S')\lambda + (CS'-C'S) = 0$$

Note: $CS'-C'S = W = 1$ (Wronskian Condition)

Denote: $C+S' = \text{Trace } M \equiv 2\cos\delta_0$

$$\Rightarrow \quad \lambda^2 - 2\lambda\cos\delta_0 + 1 = 0$$

2 solutions $\Rightarrow$

$$\boxed{\begin{gathered} \lambda_\pm = \cos\delta_0 \pm i\sin\delta_0 \qquad \text{eigenvalues.} \\ i \equiv \sqrt{-1} \\ \vec{E}_\pm \quad \text{corresponding eigenvectors} \end{gathered}}$$

$$\boxed{\overline{M}\vec{E}_\pm = \lambda_\pm \vec{E}_\pm}$$

Note $\lambda_+\lambda_- = 1$
$\Rightarrow \lambda_- = 1/\lambda_+$

Denote an initial condition by

$$\vec{X}_i \equiv \begin{pmatrix} X(s_i) \\ X'(s_i) \end{pmatrix}$$

This initial condition may be expanded in the eigenvectors:

$$\vec{X}_i = \alpha_+ \vec{E}_+ + \alpha_- \vec{E}_-$$

$\alpha_\pm$ constants

Then

$$\overline{M}^N \vec{X}_i = \alpha_+ \overline{M}^N \vec{E}_+ + \alpha_- \overline{M}^N \vec{E}_-$$
$$= \alpha_+ \lambda_+^N \vec{E}_+ + \alpha_- \lambda_-^N \vec{E}_-$$

Thus, if $\lambda_\pm^N$ remains bounded the motion is stable. This will always be the case if $|\lambda_\pm| \leq 1$, corresponding to δ_0 real with $|\cos\delta_0| \leq 1$, or equivalently

$$\boxed{\frac{1}{2}\left|\text{Trace } \overline{M}(s_i+L_p|s_i)\right| = \frac{1}{2}\left|C(s_i+L_p|s_i) + S(s_i+L_p|s_i)\right| = |\cos\delta_0| \leq 1}$$

For stable motion.

// Example

For continuous focusing:

$$\frac{1}{2}\left|\text{Trace}\left[M(s_i+L_p|s_i)\right]\right| = \left|\cos\left[k_{p0}L_p\right]\right| \leq 1$$

Motion is stable for all k_{p0} real. //

In a periodic focusing system this condition will place restrictions on the lattice structure that are generally thought of in terms of phase advance limits. Accelerator lattices are almost always tuned for single particle stability. Because even for intense beam machines, when image charges are neglected, the beam centroid obeys single particle equations of motion and instability would result in undesireable, large beam centroid excursions.

- Space charge effects can introduce additional stability constraints on the lattice
  - Envelope modes may be unstable (lowest order collective mode)
  - Higher order space charge modes may be unstable.
- Nonlinear focusing terms in the applied field and machine construction errors etc. may also induce instabilities.

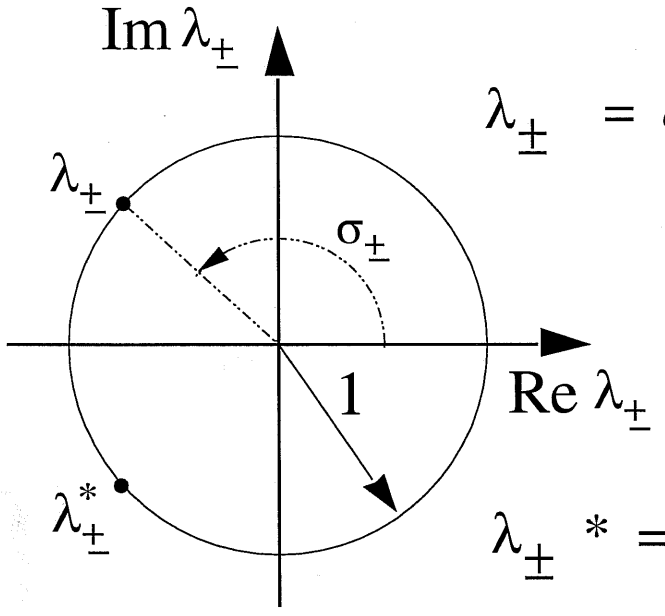
In the Homework problems, a simple example of a stability limit for a thin lens "FODO" periodic focusing channel is analyzed.

Later, we will show that for stable orbits that σ_0 can be interpreted as the phase-advance of single particle oscillations moving in the linear applied field.

Possible Eigenvalue Symmetries for 2x2 Transfer Matrices

a) Stable

Occurs for $0 \leq \delta_0 \leq 180^\circ$ / period

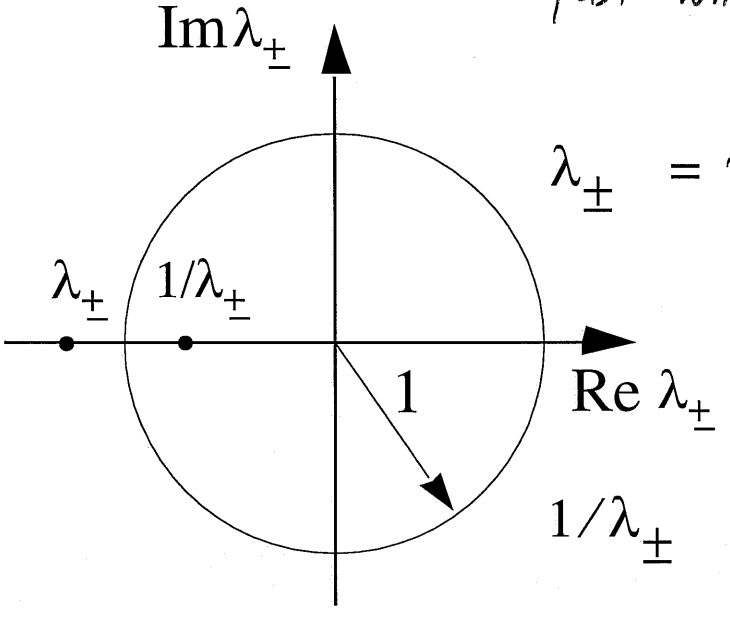


$$\lambda_\pm = e^{i\sigma_\pm}$$

$$\lambda_\pm{}^* = 1/\lambda_\pm = e^{-i\sigma_\pm}$$

b) Unstable, Lattice Resonance

Occurs in bands after focusing strengths are increased past where $\delta_0 = 180^\circ$/period.



$$\lambda_\pm = \gamma_\pm e^{-i\pi}$$

$$1/\lambda_\pm = (1/\gamma_\pm)e^{-i\pi}$$

For more info see A. Dragt, "Lectures on Nonlinear Orbit Dynamics"
in "Physics of High Energy Accelerators" AIP Conf. Proc. No. 87 (AIP, New York, 1982) p. 147

J.J. Barnard and S.M. Lund
USPAS Jan. 2004

See: Reiser §3.8.2 pgs 145-153
Reiser

§5 Floquet's Theorem and the Phase-Amplitude Form of the Particle Orbit

Floquet's Theorem:

$$x''(s) + \kappa(s)x(s) = 0$$

$$\kappa(s+L_p) = \kappa(s)$$

Similiar statement valid for non-periodic $\kappa(s)$.

Has two independent solutions that can be expressed (except where $\overline{M}(s_i|s_i+L)$ has degenerate eigenvalues, but a similar statement applies in this case) as:

$$x_1(s) = w(s)e^{i\mu s} \qquad \mu = \text{const} \equiv \cos\sigma_0$$

$$x_2(s) = w(s)e^{-i\mu s} \qquad = \frac{1}{2}\text{Trace } \overline{M}(s_i|s_i+L_p)$$

where

$$w(s+L_p) = w(s) \quad \text{is periodic.}$$

As a consequence of Floquet's theorem, any stable or nondegenerate unstable solution can be expressed in the phase-amplitude form:

$$x(s) = A(s)\cdot\cos\psi(s) \qquad A(s) = \text{Amplitude}$$

$$A(s+L_p) = A(s) \qquad \psi(s) = \text{Phase.}$$

$$\Rightarrow x' = A'\cos\psi - A\psi'\sin\psi$$

initial conditions
$$\begin{cases} x(s_i) = A(s_i)\cos\psi(s_i) \\ x'(s_i) = A'(s_i)\cos\psi(s_i) - A(s_i)\psi'(s_i)\sin\psi(s_i) \end{cases}$$

$$\begin{cases} x = A\cos\psi \\ x' = A'\cos\psi - A\psi'\sin\psi \\ x'' = A''\cos\psi - 2A'\psi'\sin\psi - A\psi''\sin\psi - A\psi'^2\cos\psi \end{cases}$$

$$\boxed{x'' + \kappa x = (A'' - A\psi'^2 + \kappa A)\cos\psi - (2A'\psi' + A\psi'')\sin\psi = 0}$$

We are free to make a restriction between A and ψ and we take:

$$A\psi'' + 2A'\psi' = 0 \quad \Rightarrow \quad \boxed{(A^2\psi')' = 0}$$

Then it follows from the transformed equation of motion that we must have ($\cos\psi \neq 0$ for all ψ)

$$\boxed{A'' - A^2\psi'^2 + \kappa A = 0}$$

The equation $(A^2\psi')' = 0$ can be integrated:

$$A^2\psi' = \text{const.}$$

One commonly rescales $A(s)$ so that this constant can be taken as unity:

$$\boxed{\begin{aligned} &A(s) \equiv A_i \cdot w(s) \quad ; \quad A_i = \text{const.} \\ &\Rightarrow \quad w^2\psi' = 1 \quad \Rightarrow \quad \psi(s) = \psi_i + \int_{s_i}^{s} \frac{d\bar{s}}{w^2(\bar{s})} \end{aligned}}$$

Then:

$$\boxed{\begin{aligned} &x(s) = A_i\, w(s) \cdot \cos\psi(s) \\ &\begin{cases} w(s+L_p) = w(s) \\ w''(s) + \kappa(s)\, w(s) - \dfrac{1}{w^3(s)} = 0 \end{cases} \\ &\psi'(s) = \frac{1}{w^2(s)} \qquad \psi(s) = \psi_i + \int_{s_i}^{s} \frac{d\bar{s}}{w^2(\bar{s})} \equiv \psi_i + \Delta\psi(s) \end{aligned}}$$

Here, A_i and ψ_i are constants set by the initial conditions at $s = s_i$. Note that $w(s)$ will be a <u>unique</u> periodic function satisfying the nonlinear amplitude equation $w'' + \kappa w - 1/w^3 = 0$. $w(s)$ may be regarded as a special, periodic function characterizing the lattice. The sign of $w(s)$ may be taken as positive definite to remove any phase ambiguity in relating the initial conditions A_i and ψ_i to $x(s_i)$ and $x'(s_i)$.

Aside

Let $w(s)$ be positive at some point. Then the equation

$$w'' + \kappa w - \frac{1}{w^3} = 0$$

insures that w can never vanish or change sign since whenever w becomes small, $w'' \approx 1/w^3 > 0$ can become arbitrarily large to turn w before it reaches zero. Thus to fix phases we conveniently require that $w > 0$. //

In this phase-amplitude formulation, the particle trajectory is:

$$\boxed{\begin{aligned} x(s) &= A_i\, w(s) \cos \psi(s) \\ x'(s) &= A_i\, w'(s) \cos \psi(s) - \frac{A_i}{w(s)} \sin \psi(s) \end{aligned}}$$

Here, we have used $\psi' = 1/w^2$. Then the initial particle coordinates $x(s_i)$ and $x'(s_i)$ are related to the amplitude and phase parameters by:

$$\boxed{\begin{aligned} x(s_i) &= A_i\, w_i \cos \psi_i \\ x'(s_i) &= A_i\, w_i' \cos \psi_i - \frac{A_i}{w_i} \sin \psi_i \\ & \\ w_i &\equiv w(s_i) \quad ; \quad w_i' \equiv w'(s_i) \\ & \\ A_i \cos \psi_i &= x(s_i)/w_i \\ A_i \sin \psi_i &= x(s_i)\, w_i' - x'(s_i)\, w_i \end{aligned}}$$

TOPS FORM 7527
Made in U.S.A.

Note in this formulation that the amplitude parameters

$$W_i = W(s_i)$$
$$W_i' = W'(s_i)$$

depend <u>only</u> on the (periodic) lattice properties, and are independent of the particle coordinates. This is likewise true for the phase advance from the initial particle coordinate:

$$\Delta\psi \equiv \int_{s_i}^{s} \frac{d\bar{s}}{W^2(s)}$$

Note in addition from

$$W'' + \kappa W - \frac{1}{W^3} = 0$$

that W has dimension [meters$^{1/2}$].

TOPS
FORM 7527
Made in U.S.A.

These relations may then be used to identify the principal functions in the transfer matrix analysis of the particle orbit:

$$\boxed{\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \overline{M}(s|s_i)\begin{pmatrix} x(s_i) \\ x'(s_i) \end{pmatrix} = \begin{pmatrix} C(s|s_i) & S(s|s_i) \\ C'(s|s_i) & S'(s|s_i) \end{pmatrix}\begin{pmatrix} x(s_i) \\ x'(s_i) \end{pmatrix}}$$

Some algebra identities (in the problem sets):

$$C(s|s_i) = \frac{W(s)}{W_i}\cos\Delta\Psi(s) - W_i' W(s)\sin\Delta\Psi(s)$$

$$S(s|s_i) = W_i W(s)\sin\Delta\Psi(s)$$

$$C'(s|s_i) = \left(\frac{W'(s)}{W_i} - \frac{W_i'}{W(s)}\right)\cos\Delta\Psi(s) - \left(\frac{1}{W_i W(s)} + W_i W'(s)\right)\sin\Delta\Psi(s)$$

$$S'(s|s_i) = \frac{W_i}{W(s)}\cos\Delta\Psi(s) + W_i W'(s)\sin\Delta\Psi(s)$$

where $\Delta\Psi(s) \equiv \int_{s_i}^{s} \frac{d\bar{s}}{W^2(\bar{s})}$ $\qquad W_i = W(s=s_i)$, $\quad W_i' = W'(s=s_i)$

TOPS FORM 7527

Made in U.S.A.

Now we can now connect σ_0 for a stable orbit to the advance in phase Ψ of the orbit through one lattice period:

$$\cos\sigma_0 \equiv \frac{1}{2}\,\mathrm{Trace}\,\overline{M}(s_i+L_p|s_i)$$

$$= \frac{1}{2}\left[C(s_i+L_p|s_i) + S'(s_i+L_p|s_i)\right]$$

$$= \frac{1}{2}\left(\frac{W(s_i+L_p)}{W_i} + \frac{W_i}{W(s_i+L_p)}\right)\cos\Delta\Psi(s_i+L_p)$$

$$+ \frac{1}{2}\left(W_i\,W'(s_i+L_p) - W_i'\,W(s_i+L_p)\right)\sin\Delta\Psi(s_i+L_p)$$

But $W(s_i+L_p) = W_i$, $W'(s_i+L_p) = W_i'$ $\Rightarrow$ coefficient of $\cos\Delta\Psi(s_i+L_p)$ is 1, coefficient of $\sin\Delta\Psi(s_i+L_p)$ is 0

$$\boxed{\cos\sigma_0 = \cos\Delta\Psi(s_i+L_p) = \frac{1}{2}\,\mathrm{Trace}\,\overline{M}(s_i+L_p|s_i)}$$

Recall $\frac{1}{2}\left|\mathrm{Trace}\,\overline{M}(s_i+L_p|s_i)\right| \le 1 \Rightarrow$ Stability

Thus σ_0 is identified as the phase advance of a stable particle orbit through one lattice period.

$$\boxed{\sigma_0 = \Delta\Psi(s_i+L_p) = \int_{s_i}^{s_i+L_p}\frac{d\bar{s}}{W^2(\bar{s})}}$$

Note that σ_0 is independent of s_i since $W(s)$ is periodic. The stability criteria $\frac{1}{2}\left|\mathrm{Trace}\,\overline{M}(s_i+L_p|s_i)\right| = |\cos\sigma_0| \le 1$ is now connected to the phase advance of the particle through one period of the focusing lattice.

TOPS FORM 7527
Made in U.S.A.

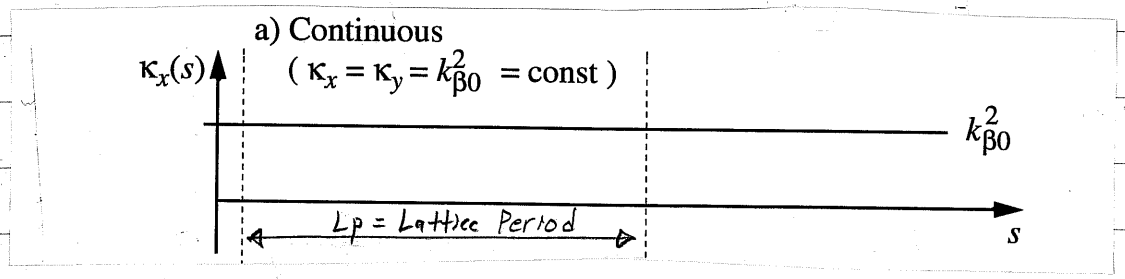
The phase advance is an extremely useful dimensionless measure to characterize the focusing strength of lattices. Much of conventional accelerator physics centers on focusing strength and suppression of resonance effects. The phase advance is a natural parameter to employ in these situations to present results in a readily interpretable and generalizable manner.

We now present phase advance formulas for σ_0 for 3 classes of periodic lattices with piecewise constant $\kappa_x(s)$ and $\kappa_y(s)$

a) Continuous Focusing
b) Solenoidal Focusing
c) Quadrupole Doublet focusing

These formulas are derived from transfer matrix analysis and direct integration of the equations of motion:

1/ Continuous Focusing

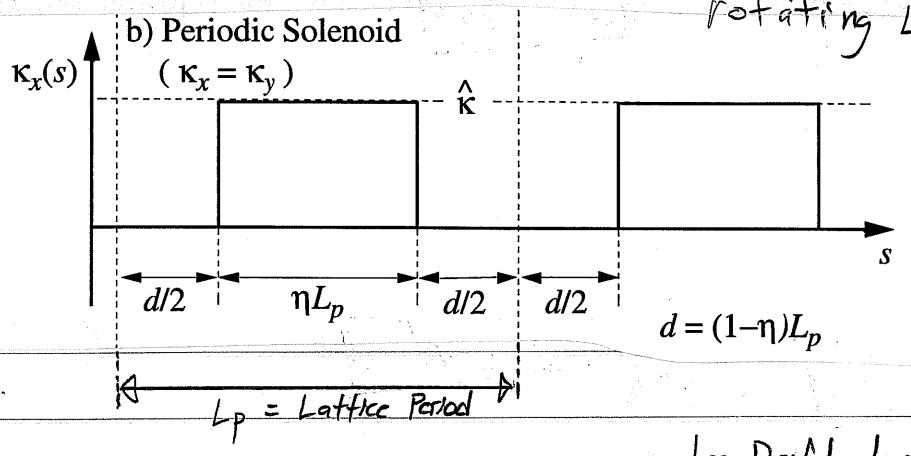


$$\boxed{\sigma_0 = k_{\beta 0} L_p}$$

The "lattice period" here is an arb. length for phase accumulation

b) Periodic Solenoidal Focusing

Phase advance must be interpreted in the rotating Larmor frame. See earlier lectures.



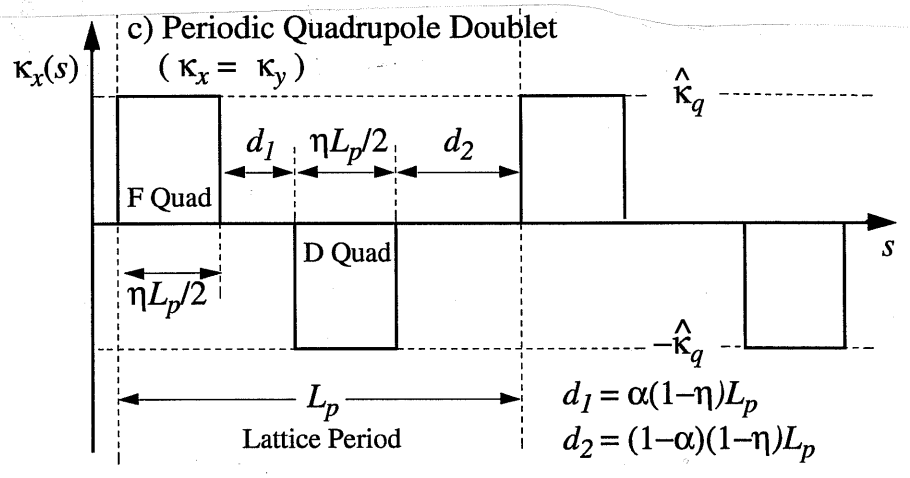
d = Drift length

$\eta \in [0,1]$ = occupancy of solenoid in period.

$$\cos\sigma_0 = \cos(2\Theta) - \frac{1-\eta}{\eta}\Theta\sin(2\Theta)$$

$$\Theta \equiv \sqrt{\hat{\kappa}}\eta L_p/2$$

c) Periodic Quadrupole Doublet Lattice



$$\cos\sigma_0 = \cos\Theta\cosh\Theta + \frac{1-\eta}{\eta}\Theta(\cos\Theta\sinh\Theta - \sin\Theta\cosh\Theta) - 2\alpha(1-\alpha)\frac{(1-\eta)^2}{\eta^2}\Theta^2\sin\Theta\sinh\Theta$$

$$\Theta \equiv \sqrt{|\hat{\kappa}_q|}\eta L_p/2$$

d_1, d_2 drift lengths

$\eta \in (0,1]$ = occupancy of quads in period

$\alpha \in [0,1]$ = syncopatic parameter

α measures how close focusing (F) and defocusing (D) quadrupoles are to eachother in the lattice.

$$\alpha = 0: \quad d_1 = 0, \quad d_2 = (1-\eta) L_p$$
$$\alpha = 1: \quad d_1 = (1-\eta) L_p, \quad d_2 = 0$$

$$\alpha = \frac{1}{2} \Rightarrow d_1 = d_2 = (1-\eta)\frac{L_p}{2} \equiv \text{"FODO" lattice.}$$

with equal drifts.

The range $\alpha \in [\frac{1}{2}, 1]$ can be mapped to $\alpha \in [0, \frac{1}{2}]$ by relabeling lattice quantities. Therefore, take:

$$\alpha \in [0, \tfrac{1}{2}]$$

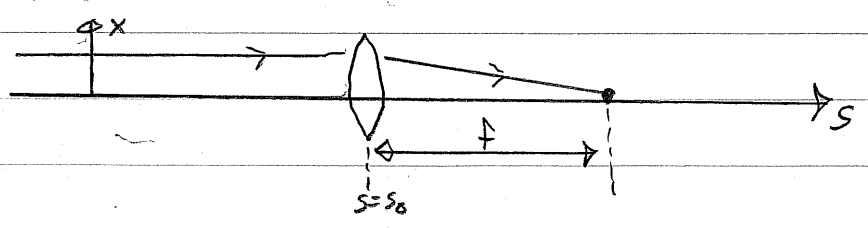
Thin Lens limit

Convenient to simply understand scaling

$$\kappa_x(s) = \frac{1}{f}\delta(s - s_0)$$

s_0 = optic location
f = const, focal length.

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s=s_0^+} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s=s_0^-}$$



The thin-lens limit of "thick" solenoids and quadrupoles can be obtained by taking

Solenoids: $\hat{\kappa} \equiv \dfrac{1}{\eta f L_p}$; then $\lim_{\eta \to 0}$

Quadrupoles: $\hat{\kappa}_q \equiv \dfrac{2}{\eta f L_p}$; then $\lim_{\eta \to 0}$

This obtains:

$$\cos\sigma_0 = \begin{cases} 1 - \frac{1}{2}\frac{L_p}{f}, & \text{thin-lens solenoids,} \\ 1 - \frac{\alpha}{2}(1-\alpha)(\frac{L_p}{f})^2, & \text{thin-lens quadrupoles.} \end{cases}$$

This formula can also be derived directly from thin lens transfer matrices:

$$\cos\sigma_0 = \frac{1}{2}\mathrm{Tr}\begin{bmatrix} 1 & L_p \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} = 1 - \frac{1}{2}\frac{L_p}{f},$$

Solenoid lattice

$$\cos\sigma_0 = \frac{1}{2}\mathrm{Tr}\begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}\begin{bmatrix} 1 & \alpha L_p \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix} \times \begin{bmatrix} 1 & (1-\alpha)L_p \\ 0 & 1 \end{bmatrix} = 1 - \frac{\alpha}{2}(1-\alpha)\left(\frac{L_p}{f}\right)^2,$$

Quadrupole doublet lattice.

In many cases it is also desireable to have simple formulas for design that relate magnet parameters to σ_0. For example, if the formula for $\cos\sigma_0$ is expanded to leading order in $\theta = \sqrt{|\hat{\kappa}_q|}\,\eta L_p/2$ for quadrupole doublet focusing, we obtain:

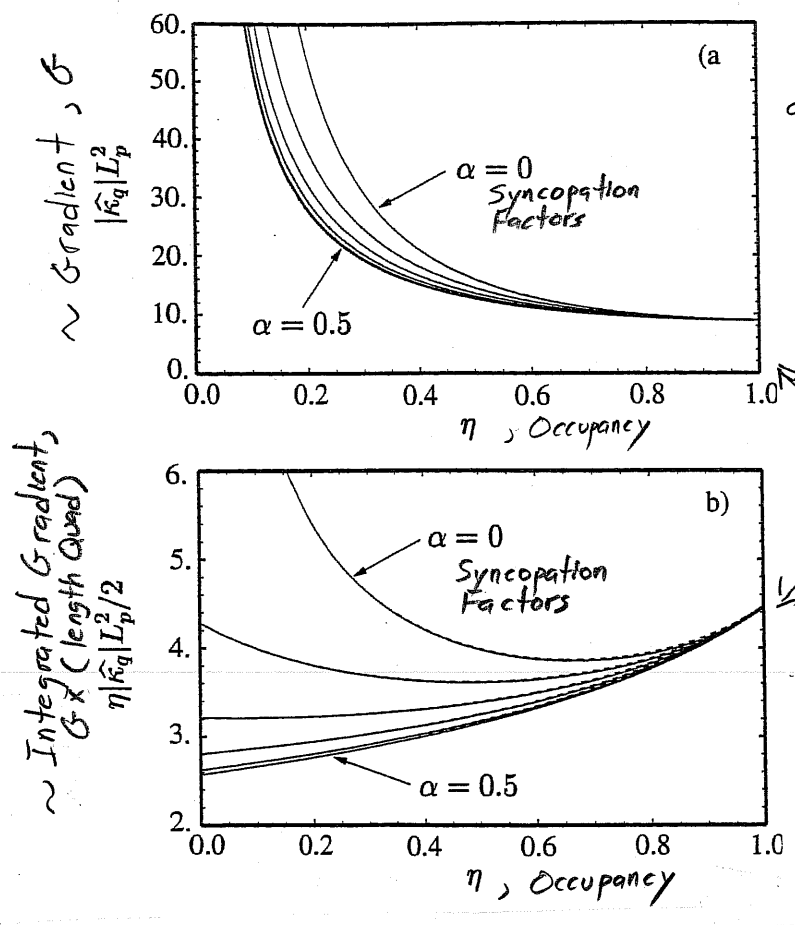
$$\cos\sigma_0 = 1 - \frac{(\eta\widehat{\kappa_q}L_p^2)^2}{32}\left[\left(1-\frac{2}{3}\eta\right) - 4\left(\alpha-\frac{1}{2}\right)^2(1-\eta)^2\right].$$

where

$$\hat{\kappa}_q = \begin{cases} \dfrac{\hat{G}}{[B\rho]} & ; \text{ Magnetic Quadrupoles} \\ \dfrac{\hat{G}}{\beta_b c\,[B\rho]} & ; \text{ Electric Quadrupoles} \end{cases} \qquad \hat{G} = \text{Field Gradient.}$$

Plot eqn:

For $\sigma_0 = 80^\circ$/period



$\alpha = 0, 0.1, 0.2, 0.3, 0.4, 0.5$

Dashed Curves Exact results (almost overlay)

J.J. Barnard and S.M. Lund
USPAS Jan 2004

See: Reiser §3.8.2 pgs 145-153

§6 The Courant-Snyder Invariant and Single-Particle Emittance

Constants of the motion simplify interpretation of dynamics

// Example Continuous Focusing System $\leftrightarrow$ simple harmonic oscillator.

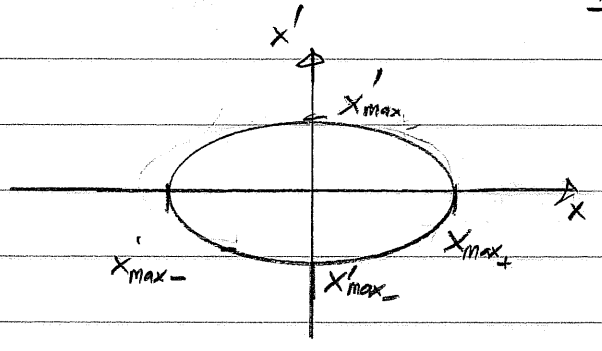
$$x'' + k_{\beta 0}^2 x = 0 \quad ; \quad k_{\beta 0}^2 = \text{const.}$$

$$H = \frac{x'^2}{2} + \frac{k_{\beta 0}^2}{2} x^2 = \text{const.}$$

Particle motion is constrained to move on an ellipse in $x-x'$ phase space with max extents

$$x_{max} \Rightarrow x' = 0 \quad : \quad x_{max_\pm} = \pm\sqrt{2H/k_{\beta 0}^2}$$

$$x'_{max} \Rightarrow x = 0 \quad : \quad x'_{max_\pm} = \pm\sqrt{2H}$$



//

For the forced oscillator described by Hill's equation

$$x''(s) + K(s)\, x(s) = 0$$

there is also an invariant called the Courant-Snyder invariant after it's discoverers. [Courant and Snyder co invented strong (AG) focusing in their seminal paper that stimulated many of the advances in modern accel. physics. Christofolos

TOPS
FORM 7527
Made in U.S.A.

also independently realized the concept of strong focusing via a more heuristic analysis]. This invariant is heavily used in modern accelerator physics and in the analysis of forced oscillators.

Phase amplitude form of particle trajectory:

$$\begin{cases} x(s) = A_i\, w(s) \cos\psi(s) \\ x'(s) = A_i\, w'(s) \cos\psi(s) + \dfrac{A_i}{w(s)} \sin\psi(s) \end{cases}$$

re-arrange:

$$\begin{cases} x/w = A_i \cos\psi \\ wx' - w'x = A_i \sin\psi \end{cases}$$

Note:
$$w^2\left(\frac{x}{w}\right)' = wx' - xw'$$

Add squares of eqns to obtain the Courant-Snyder invariant

$$\boxed{\left(\frac{x}{w}\right)^2 + (wx' - w'x)^2 = A_i^2(\cos^2\psi + \sin^2\psi) = A_i^2 = \text{const.}}$$

or $\left(\frac{x}{w}\right)^2 + w^4\left(\frac{x}{w}\right)'^2 = A_i^2 = \text{const.}$

To better interpret this invariant $A_i^2 = \text{const}$, we expand the expression isolating terms in x^2, xx', and x'^2.

$$\left(\frac{1}{w^2} + w'^2\right)x^2 + 2(-ww')\,xx' + w^2 x'^2 = A_i^2 = \text{const}$$

This is the equation of an ellipse in x-x' phase space. The coefficients of x^2, xx', and x'^2 are functions of the lattice only (not particle initial conditions). It is convienient to use "Twiss" parameter notation

TOPS FORM 7527 Made in U.S.A.

and label these coefficients as:

$$\boxed{\begin{aligned} \beta(s) &\equiv w^2(s) \\ \alpha(s) &\equiv -w(s)\,w'(s) \\ \gamma(s) &\equiv \frac{1}{w^2(s)} + w'(s)^2 = \frac{1+\alpha^2(s)}{\beta(s)} \end{aligned}}$$

Then the Courant-Snyder Invariant is then expressed as:

$$\boxed{\gamma x^2 + 2\alpha x x' + \beta x'^2 = A_i^2 = \text{const.}}$$

The "area" of this ellipse in xx' phase space is:

$$\boxed{\text{Area} = \int_{\text{ellipse}} dx\,dx' = \frac{\pi A_i^2}{\sqrt{\gamma\beta - \alpha^2}} = \pi A_i^2 \equiv \pi \varepsilon}$$

Analytic Geometry Formula.

where ε is the single particle emittance which corresponds to the phase space area enclosed by the particle orbit <u>divided by π.</u>

// <u>Aside</u> Units

The definition of emittance is not unique. Some prefer to call $\pi\varepsilon$ the emittance. Many also quote emittance values in units of mm-mrad with a "π" in the units as a reminder, i.e., $\varepsilon = 10\ \pi$-mm-mrad, etc.

//

TOPS FORM 7527 Made in U.S.A.

From these identifications:

$$\boxed{\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon = \text{const.}}$$

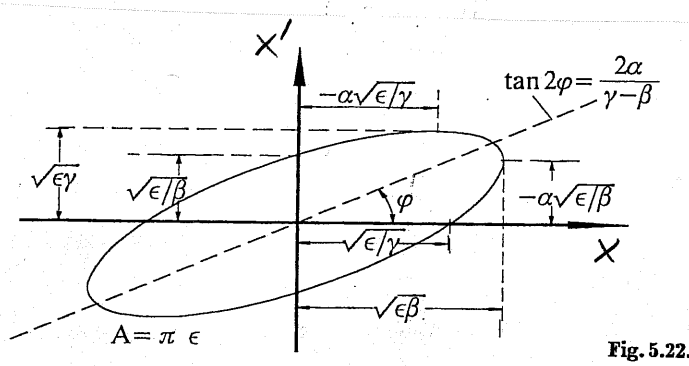


Fig. 5.22. Phase space ellipse

152

From Wiedemann

Note:

- The ellipse will <u>rotate</u> and <u>change shape</u> as the particle advances through the focusing lattice. But the <u>area</u> of the ellipse will not change. ($\pi\varepsilon = \text{const}$).

- The location of the particle on the ellipse and the size of the ellipse depends on the initial conditions of the particle.

- The <u>orientation</u> of the ellipse is <u>independent</u> of the particle initial conditions. <u>All</u> particles move on <u>nested ellipses</u>.

TOPS
FORM 7527
Made in U.S.A.

Aside: Lattice Maps

The Courant-Snyder invariant helps us understand the phase space evolution of the particles. Knowing how the ellipse transforms (twists and rotates without changing area) is equivalent to knowing the dynamics.

General s:

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon$$

Initial $s = s_i$!

$$\gamma_i x_i^2 + 2\alpha_i x_i x_i' + \beta_i x_i'^2 = \varepsilon$$

$$\beta_i \equiv \beta(s_i)$$
$$\alpha_i \equiv \alpha(s_i)$$
$$\gamma_i = \gamma(s_i)$$

Identify coefficients of the transport matrix:

$$\begin{bmatrix} x(s) \\ x'(s) \end{bmatrix} = \begin{bmatrix} C(s|s_i) & S(s|s_i) \\ C'(s|s_i) & S'(s|s_i) \end{bmatrix} \begin{bmatrix} x_i \\ x_i' \end{bmatrix}$$

$$(S'^2\gamma_i - 2S'C'\alpha_i + C'^2\beta_i)x^2 +$$
$$2(-SS'\gamma_i + S'C\alpha_i + SC'\alpha_i - CC'\beta_i)xx' +$$
$$(S^2\gamma_i - 2SC\alpha_i + C^2\beta_i)x'^2 = \varepsilon$$

-or-

$$\begin{bmatrix} \beta \\ \alpha \\ \gamma \end{bmatrix} = \begin{bmatrix} C^2 & -2SC & S^2 \\ -CC' & (S'C+SC') & -SS' \\ C'^2 & -2S'C' & S'^2 \end{bmatrix} \begin{bmatrix} \beta_i \\ \alpha_i \\ \gamma_i \end{bmatrix}$$

TOPS FORM 7527 Made in U.S.A.

// <u>Example</u> Particle Hamiltonian and energy of a kicked oscillator

$$x'' + \kappa(s)\, x = 0 \quad ; \quad \kappa(s) \neq \text{const.}$$

$$H = \frac{x'^2}{2} + \frac{\kappa x^2}{2} \neq \text{const}$$

to see this:

$$\frac{dx}{ds} = \frac{\partial H}{\partial x'} = x'$$

$$\frac{dx'}{ds} = -\frac{\partial H}{\partial x} = -\kappa x$$

$$\frac{dH}{ds} = \frac{\partial H}{\partial s} + \frac{\partial H}{\partial x}\frac{dx}{ds} + \frac{\partial H}{\partial x'}\frac{dx'}{ds} = \frac{\partial H}{\partial s} = \frac{\kappa' x^2}{2} \neq \text{const.}$$

The energy of a kicked oscillator is not conserved. Do not confuse the energy of an oscillator with the Courant-Snyder invariant. <u>Only</u> in a continuous focusing system (simple harmonic oscillator) will they be simply related:

$$\kappa(x) = k_{\beta 0}^2 = \text{const} \qquad \text{Continuous focusing}$$

$$\Rightarrow \quad H = \frac{x'^2}{2} + \frac{k_{\beta 0}^2}{2} x^2 = \text{const.}$$

$$w'' + k_{\beta 0}^2 w - \frac{1}{w^3} = 0$$

$$\Rightarrow \quad w = \sqrt{\frac{1}{k_{\beta 0}}} = \text{const} \qquad \text{solution.}$$

$$\beta(s) = w^2 = 1/k_{\beta 0} = \text{const}$$

$$\alpha(s) = -ww' = 0$$

$$\gamma(s) = \frac{1}{w^2} + w'^2 = k_{\beta 0} = \text{const.}$$

$$\varepsilon = k_{\beta 0} x^2 + \frac{x'^2}{k_{\beta 0}} = \frac{2}{k_{\beta 0}}\left(\frac{1}{2}x'^2 + \frac{k_{\beta 0}^2}{2}x^2\right) = \text{const.}$$

$$= 2H/k_{\beta 0} = \text{const.}$$

//

FORM 7527
Made in U.S.A.

Example: Ellipse Maps:

Drift: length s

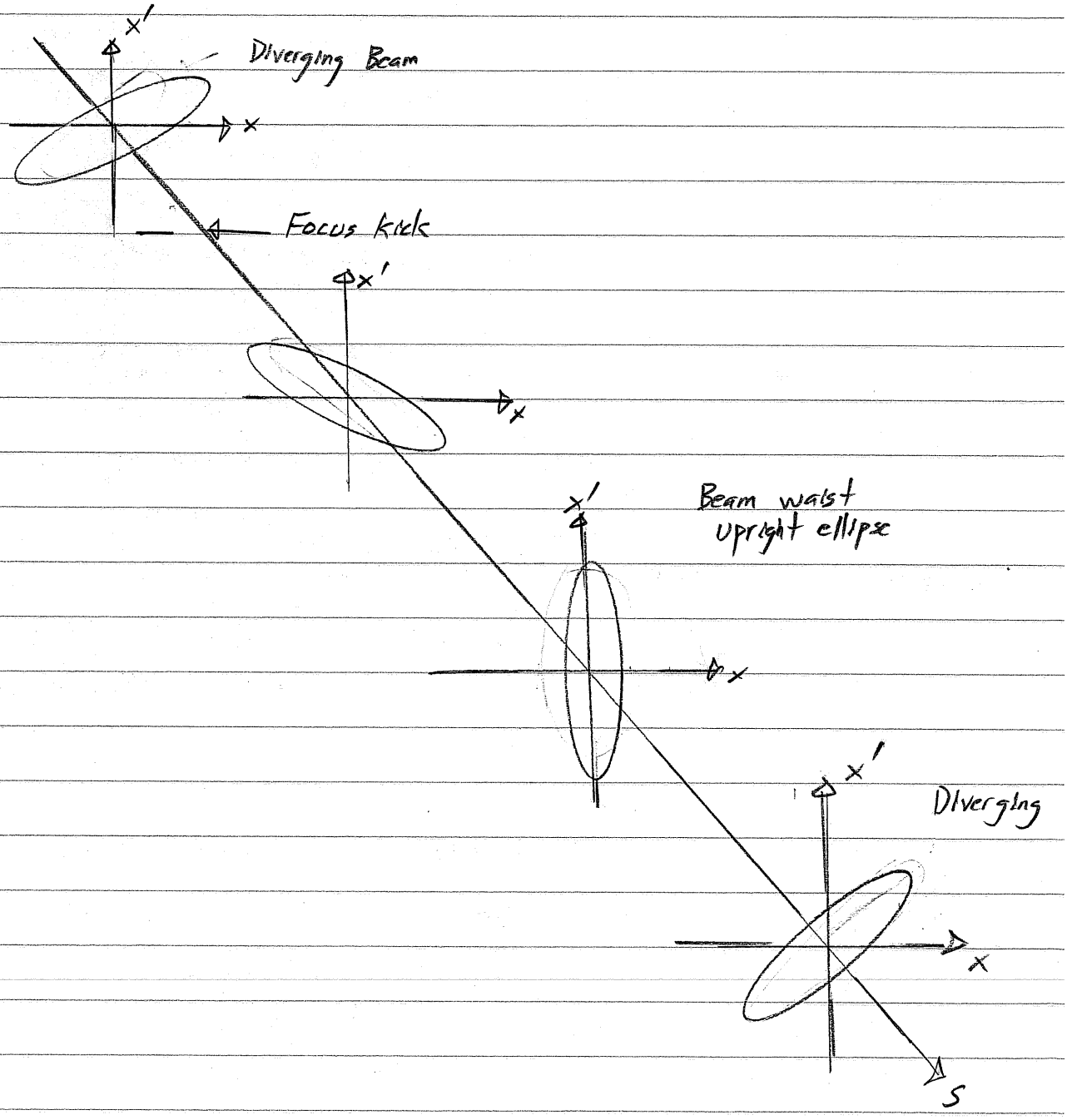
$$\begin{bmatrix} c_1 & s_1 \\ c_1' & s_1' \end{bmatrix} = \begin{bmatrix} 1 & s-s_i \\ 0 & 1 \end{bmatrix}$$

$$\beta = \beta_0 - 2\alpha_i(s-s_i) + \gamma_i(s-s_i)^2$$
$$\alpha = \alpha_i - \gamma_i(s-s_i)$$
$$\gamma = \gamma_i$$

Focusing lense (discreet kick)

$$\begin{bmatrix} c_1 & s_1 \\ c_1' & s_1' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$$

$f > 0$ focusing strength.



TOPS
FORM 7527
Made in U.S.A.

J.J. Barnard and S.M. Lund
USPAS Jan. 2004

Sec: Reiser §3.8.2 pgs. 144-153

§7 Betatron Formulation of Particle Orbit

The phase amplitude form we originally chose

$$\boxed{x(s) = A_i\, w(s) \cos \psi(s)}$$

is not a unique choice and w has unusual dimensions of meters$^{1/2}$. Due to this and the utility of the Twiss parameters in describing the Courant-Snyder invariant and the orientation of the phase space ellipse on which the particle moves, it is convenient to take an alternative phase-amplitude representation with:

$$\boxed{\begin{aligned} x(s) &= \sqrt{\varepsilon}\sqrt{\beta(s)} \cos \psi(s) \\ \beta\text{-tron function:} \quad & \beta(s) \equiv w^2(s) \\ \text{emittance:} \quad & \varepsilon = A_i^2 = \text{const.} \\ \text{phase:} \quad & \psi(s) = \psi_i + \int_{s_i}^{s} \frac{d\bar{s}}{\beta(\bar{s})} = \psi_i + \Delta\psi(s) \end{aligned}}$$

Here, $\beta(s) = w^2$ has dimensions of meters.
Note that the initial phase ψ_i will differ from the initial phase in the previous phase-amplitude form $x = A_i\, w \cos(\psi_0 + \Delta\psi)$ since $\sqrt{\varepsilon} > 0$ (we choose not to distinguish the ψ_i for notational simplicity), but the phase advance:

$$\Delta\psi(s) = \int_{s_i}^{s} \frac{d\bar{s}}{w^2(\bar{s})} = \int_{s_i}^{s} \frac{d\bar{s}}{\beta(\bar{s})}$$

is identical in both formulations.

FORM 7527
Made in U.S.A.

In this case the β-tron function satisfies:

$$\boxed{\begin{gathered}\frac{1}{2}\beta(s)\beta''(s) - \frac{1}{4}\beta'^2(s) + \beta^2(s)\kappa(s) = 1 \\ \beta(s+L_p) = \beta(s) \\ \beta(s) > 0.\end{gathered}}$$

From the particle orbit equation

$$x = \sqrt{\varepsilon\beta}\cos\psi,$$

the maximum possible excursion of the particle, $x_m = \mathrm{Max}[|x|]$ occurs where $\cos\psi = \pm 1$ and

$$\boxed{x_m(s) = \sqrt{\varepsilon_m\beta(s)} = \sqrt{\varepsilon_m}\, W(s)}$$

ε_m = max single-particle emittance.

Since all particles move on nested ellipses, the maximum possible excursion of all particles is given by this expression with the maximum single-particle emittance. This excursion is the minimum possible aperture of the accelerator structures to prevent single particle loss. Resonance effects due to nonlinear focusing fields, space-charge, image charges, imperfect vacuum, and finite particle lifetime all influence the practical aperture choice.

TOPS
FORM 7527
Made in U.S.A.

From the equation for w:

$$w''(s) + \kappa(s)\, w(s) - \frac{1}{w^3(s)} = 0$$

$$w(s+L_p) = w(s) \quad ; \quad w(s) > 0$$

we obtain the equation for the maximum envelope of particle orbits $x_m = \sqrt{\varepsilon_m}\, w$

$$\boxed{\begin{gathered} x_m''(s) + \kappa(s)\, x_m(s) - \frac{\varepsilon_m^2}{x_m^3(s)} = 0 \\ x_m(s+L_p) = x_m(s) \\ x_m > 0 \end{gathered}}$$

Note the similarity of this equation to the statistical envelope equations derived in the intro lectures by J.J. Barnard

- When an extra term is added to include linear space-charge forces this equation is the commonly used KV envelope equation.

Comment:

The use of β for this function along with α, β, γ for Twiss parameters does not often result in confusion with relativistic factors (often denoted β, γ rather than β_b and γ_b as used here) because the context usually makes the intent clear. Often relativistic factors are absorbed into definitions of focusing strengths κ too, etc.

J.J. Barnard and S.M. Lund
USPAS, Jan 2004

See Reiser §5.4.10 pgs 437–447

§8 Momentum Spread Effects

Up to this point, we have analyzed particle dynamics with momentum

$$\boxed{p_s = m\gamma_b\beta_b c = \text{const.}}$$

for all particles in a beam slice. In reality, there will be a finite spread of particle momenta in the slice:

$$\boxed{\begin{aligned} p_s &= p_0 + \delta p \\ p_0 &= m\gamma_b\beta_b c = \text{design momentum} \\ \delta p &= \text{"off" momentum} \end{aligned}}$$

Typical values for momentum spread are:

$$\frac{|\delta p|}{p_0} \sim 10^{-2} \text{ to } 10^{-6}$$

in a beam with a single species of particles and a machine with conventional sources and accelerating structures. This spread of momentum can modify particle orbits, particularly in machines with dipole bends where the bending radius of a particle varies strongly with the particle momentum.

To better understand this effect, we analyze the particle equations of motion with leading order dispersive effects:

$$x''(s) + \left[\frac{1}{R(s)^2} \frac{1-\delta}{1+\delta} + \frac{\kappa_x(s)}{(1+\delta)^n} \right] x(s) = \frac{\delta}{1+\delta} \cdot \frac{1}{R(s)}$$

$$y''(s) + \frac{\kappa_y(s)}{(1+\delta)^n} y(s) = 0$$

where:

$R(s)$ = Local bend radius for design momentum p_0 ($R \to \infty$ in straight sections)

$$\frac{1}{R} = \frac{B_y^a|_{dipole}}{[B\rho]}$$

$$[B\rho] = p_0/q$$

$$\delta \equiv \frac{\delta p}{p_0} = \text{Fractional momentum error.}$$

$$n = \begin{cases} 1 \; ; & \text{Magnetic Quadrupoles} \\ 2 \; ; & \text{Solenoids, Electric Quadrupoles} \end{cases}$$

Here we have neglected:

- Space charge effects: $\phi \to 0$
- Nonlinear applied focusing: $\vec{E}^a, \vec{B}^a$ contain only linear focusing terms.
- Acceleration: $p_0 = m\gamma_b\beta_b = \text{const.}$

Also, the bends are set from the design particle orbit (see previous discussions).

Terms resulting from the momentum spread can be lumped into two classes, chromatic and dispersive:

<u>Chromatic terms</u>: present in both x- and y-equations

$$k_x \cdot x \longrightarrow \frac{k_x\, x}{(1+\delta)^n}$$

$$k_y \cdot y \longrightarrow \frac{k_y\, y}{(1+\delta)^n}$$

where $k_{x,y}$ remains defined by the design momentum.

Particle momentum changes vary the focal strength of optics and will thereby modify the bundle of particles.

- Generally not important (corrections not large) except for very small focal spots when the beam is focused onto a target or when momentum spreads are large.

- Lectures by J.J. Barnard will overview consequences of this effect in final focus optics. We will neglect these terms here.

<u>Dispersive terms</u>: only in the x-equation. results from dipole bends in x

Result from particles being bent at different radii when the momentum deviates from the design value. Dispersive terms contain the bend radius R:

$$\frac{1}{R^2(s)}\,\frac{1-\delta}{1+\delta}\,x(s) \quad ; \quad \frac{\delta}{1+\delta}\,\frac{1}{R(s)}$$

Generally, bend radii R are large and δ is small, and we can take to leading order:

$$\left[\frac{1}{R^2}\,\frac{1-\delta}{1+\delta} + k_x\right] x \simeq k_x x$$

$$\frac{\delta}{1+\delta}\,\frac{1}{R} \simeq \frac{\delta}{R}$$

and the equations become:

$$\boxed{\begin{aligned} x''(s) + k_x(s)\,x(s) &= \frac{\delta}{R(s)} \\ y''(s) + k_y(s)\,y(s) &= 0 \end{aligned}}$$

At this level of approx., no change in y-equation

Generally, the x-equation is solved for periodic lattices by exploiting the linear structure of the equation and resolving:

$$\boxed{x = \underset{\text{Particular Sol.}}{x_p} + \underset{\text{Homogeneous sol.}}{x_h}}$$

where

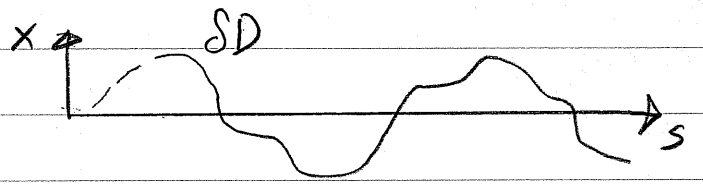
x_h is the general solution to:

$$\boxed{x_h''(s) + K_x(s)\, x_h(s) = 0}$$

and x_p is the periodic solution to:

$$\boxed{\begin{aligned} x_p &= \delta \cdot D \\ D''(s) + K_x(s)\, D(s) &= \frac{1}{R(s)} \\ D(s+L_p) &= D(s) \end{aligned}}$$

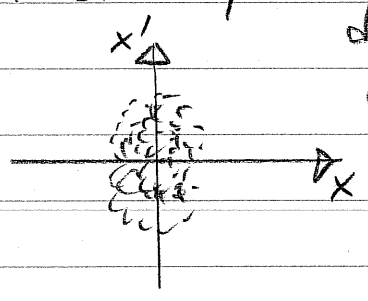
This convenient resolution can always be made. D is called the dispersion function and $\delta \cdot D$ measures the offset of the particle orbit with respect to the design orbit.

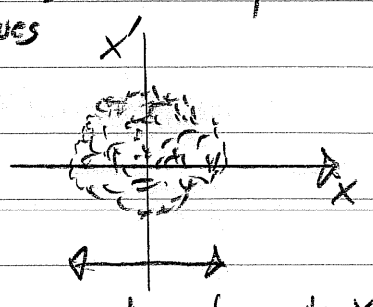


$x = 0 \Rightarrow$ Design orbit.

Bundle of particles $D=0$ — Bundle of particles $D \neq 0$

distribution of δ values





broadens in x

Note: $[D]$ = meters

$\Rightarrow$ $\delta \cdot D$ gives orbit offset in meters.

// Example – Continuous Focusing in a continuous bend.

$$K_x(s) = k_{\beta 0}^2 = \text{const.}$$
$$R(s) = R = \text{const.}$$

$$\Rightarrow \quad D'' + k_{\beta 0}^2 D = \frac{1}{R} \quad \Rightarrow \quad D = \frac{1}{k_{\beta 0}^2 R}$$

In periodic lattices D will oscillate with (approx.) average value:

$$D \sim \frac{L_p^2}{\sigma_0^2 R}$$

//

Many rings are designed to focus the dispersion function to small values in straight sections even though the lattice has strong bends. Quadrupole triplet focusing lattices often are employed for rings since the optics allow enough freedom to do this. This is desireable since it allows smaller beam size at locations where $D \simeq 0$ and these locations can be used to more easily insert and extract (kick) the beam into and from the ring.

J.J. Barnard and S.M. Lund
USPAS, Jan 2004

§9 Acceleration and Normalized Emittance

If the beam is accelerated, Hill's equation for the single-particle orbit becomes:

$$\boxed{x''(s) + \underbrace{\frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}}_{\text{Acceleration term}} x'(s) + \kappa(s)\, x(s) = 0}$$

Neglects:

- Space charge effects: $\phi \to 0$
- Nonlinear applied focusing: $\vec{E}^a, \vec{B}^a$ contain only linear focusing terms
- Momentum spread effects.

Also, implicit in this equation, is that any energy variations in $\kappa(s)$ are compensated for in focusing optics if the lattice is periodic or otherwise incorporated in the s-dependence.

In practical applications the changes induced by the acceleration term are slow and the fractional energy change induced by individual acceleration cells is small (except possibly near the injector).

The acceleration term

$$\frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} > 0 \quad \Rightarrow \quad \text{damped harmonic oscillator equation in } x.$$

and oscillations in the particle orbit will damp. $\gamma_b\beta_b$ will be a function of s perscribed by the acceleration schedule of the machine – derived

consistently from machine gap structures in the lattice, practical accelerating field breakdown limits. Even with acceleration, we will find that the particle orbit has a Courant-Snyder invariant (normalized emittance) if coordinates are appropriately chosen to compensate for the acceleration induced damping of the particle phase-space area.

Transform:

$$\boxed{\tilde{X} \equiv \sqrt{\gamma_b \beta_b}\, X}$$

"Guess" can be motivated qualitatively based on conjugate variable arguments - see Intro. lecture J.J. Barnard

Then

$$X = \frac{\tilde{X}}{\sqrt{\gamma_b \beta_b}} \quad \Longrightarrow \quad \tilde{X} = \sqrt{\gamma_b \beta_b}\, X$$

$$X' = \frac{\tilde{X}'}{\sqrt{\gamma_b \beta_b}} - \frac{1}{2}\frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)^{3/2}}\tilde{X} \qquad \tilde{X}' = \sqrt{\gamma_b \beta_b}\, X' + \frac{1}{2}\frac{(\gamma_b \beta_b)'}{\sqrt{\gamma_b \beta_b}} X$$

$$X'' = \frac{\tilde{X}''}{\sqrt{\gamma_b \beta_b}} - \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)^{3/2}}\tilde{X}' + \frac{3}{4}\frac{(\gamma_b \beta_b)'^2}{(\gamma_b \beta_b)^{5/2}}\tilde{X} - \frac{1}{2}\frac{(\gamma_b \beta_b)''}{(\gamma_b \beta_b)^{3/2}}\tilde{X}$$

substituting these factors, the particle equation of motion with acceleration becomes:

$$\boxed{\tilde{X}'' + \left[\kappa + \frac{1}{4}\frac{(\gamma_b \beta_b)'^2}{(\gamma_b \beta_b)^2} - \frac{1}{2}\frac{(\gamma_b \beta_b)''}{(\gamma_b \beta_b)} \right] \tilde{X} = 0}$$

The transformed equation can be expressed as

$$\boxed{\begin{aligned} &\tilde{x}'' + \tilde{\kappa}\tilde{x} = 0 \\ &\tilde{\kappa} \equiv \kappa + \frac{1}{4}\frac{(\gamma_b\beta_b)'^2}{(\gamma_b\beta_b)^2} - \frac{1}{2}\frac{(\gamma_b\beta_b)''}{(\gamma_b\beta_b)} \end{aligned}}$$

where $\tilde{\kappa}$ is a corrected focusing constant including additional acceleration terms beyond those already contained in κ:

$$\kappa = \begin{cases} \pm\kappa_q = \dfrac{qG}{m\gamma_b\beta_b^2c^2} \; ; & G = \text{Electric Quadrupole Gradient} \\ \pm\kappa_q = \dfrac{qG}{m\gamma_b\beta_b c} \; ; & G = \text{Magnetic Quadrupole Gradient} \\ \dfrac{\omega_c^2}{4\gamma_b^2\beta_b^2c^2} = \dfrac{qB_z^a}{4m\gamma_b^2\beta_b^2c^2} \; ; & B_z^a = \text{Solenoidal Magnetic Field.} \end{cases}$$

Note that the transformed equation has the same form as usual Hill's equation with no acceleration

$$x'' + \kappa x = 0$$

Therefore, all previous analysis on Courant-Snyder invariants in $x-x'$ phase space can be immediately applied to the $\tilde{x}-\tilde{x}'$ phase-space for an accelerating beam. We then have:

$$\boxed{\tilde{\varepsilon} = \pi\cdot(\text{area traced by orbit in } \tilde{x}-\tilde{x}' \text{ phase-space}) = \text{const.}}$$

Phase space areas are related as

$$d\tilde{x} \otimes d\tilde{x}' = |J| \, dx \otimes dx'$$

where J is the Jacobian

$$J = \det \begin{vmatrix} \frac{\partial \tilde{x}}{\partial x} & \frac{\partial \tilde{x}}{\partial x'} \\ \frac{\partial \tilde{x}'}{\partial x} & \frac{\partial \tilde{x}'}{\partial x'} \end{vmatrix} = \det \begin{vmatrix} \sqrt{\gamma_b \beta_b} & 0 \\ \frac{1}{2} \frac{(\gamma_b \beta_b)'}{\sqrt{\gamma_b \beta_b}} & \sqrt{\gamma_b \beta_b} \end{vmatrix}$$

$$= \gamma_b \beta_b$$

Thus:

$$\boxed{d\tilde{x} \otimes d\tilde{x}' = \gamma_b \beta_b \, dx \otimes dx'}$$

Based on this area transform, if we define an (instantaneous) area of the orbit traced in x-x' to be $\pi \cdot \varepsilon$ (regular emittance):

$$\boxed{\begin{aligned} \tilde{\varepsilon} &= \gamma_b \beta_b \, \varepsilon = \text{const.} \\ &\equiv \text{"Normalized" Emittance.} \equiv \varepsilon_n \end{aligned}}$$

The $\gamma_b \beta_b$ factor compensates for the acceleration induced damping of particle oscillations.

// Aside:

The equation of motion with acceleration was derived by approximating:

$$\frac{d}{dt}\left(\gamma \frac{dx}{dt}\right) = \frac{q}{m} E_x^a - \frac{q}{m} v_z B_y^a + \underbrace{\frac{q}{m} v_y B_z^a}_{\to 0}$$

No Solenoidal Field.

using:

$$\frac{d}{dt}\left(\gamma \frac{dx}{dt}\right) \simeq \gamma_b \beta_b^2 c^2 \left[x'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} x' \right]$$

Changes in $\gamma_b \beta_b$ are calculated from the longitudinal particle equation

$$\frac{d}{dt}\left(\gamma \frac{dz}{dt}\right) = \frac{q}{m} E_z^a + \underbrace{\frac{q}{m} v_x B_y^a - \frac{q}{m} v_y B_x^a}_{\text{Weak change in particle longitudinal energy, neglect.}}$$

and

$$\frac{d}{dt}\left(\gamma \frac{dz}{dt}\right) \simeq c^2 \beta_b (\gamma_b \beta_b)'$$

$$\frac{q}{m} E_z^a \simeq -\frac{q}{m} \frac{\partial \phi^a}{\partial z}\bigg|_{r=0}$$

where ϕ^a is the accelerating potential in a quasi-static approximation resulting from an axisymmetric gap.

Also:

$$E_x^a \simeq E_x^a\bigg|_{\text{Focusing Optics}} - \frac{\partial \phi^a}{\partial x}$$

$\perp$ Focusing from Accel.

The longitudinal equation then becomes:

$$\beta_b(\gamma_b\beta_b)' \simeq \frac{-q}{mc^2}\frac{\partial\phi^q}{\partial z}\bigg|_{r=0}$$

But:

$$\gamma_b' = \left(\frac{1}{\sqrt{1-\beta_b^2}}\right)' = \gamma_b^3\beta_b\beta_b'$$

$$\Rightarrow \beta_b(\gamma_b\beta_b)' = (1-\gamma_b^2\beta_b^2)\gamma_b\beta_b\beta_b' = \gamma_b^3\beta_b\beta_b' = \gamma_b'$$

and the longitudinal equation can be integrated!

$$\gamma_b' = \frac{-q}{mc^2}\frac{\partial\phi^q}{\partial z}\bigg|_{r=0}$$

$$\boxed{\gamma_b + \frac{q}{mc^2}\phi^q(r=0,z) = \text{const.}}$$

This result can be used to cast terms occuring in the transverse equation of motion as

$$\boxed{x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' \simeq \frac{q}{m\gamma_b\beta_b^2c^2}E_x^q - \frac{q}{m\gamma_b\beta_bc}B_y^q - \frac{q}{m\gamma_b\beta_b^2c^2}\frac{\partial\phi^q}{\partial x}}$$

Applied Focus Electric Quads etc

Magnetic Focus

Transverse Foc From Accel. gap.

In the quasi-static approximation, the axisymmetric field of an accelerating gap can be expanded as: (see J.J. Barnard, Intro lectures and Reiser Sec. 3.3):

$$\boxed{\begin{aligned} E_z^a &= -\frac{\partial \phi^a}{\partial z} \\ \phi^a &= V(z) - \frac{1}{4}\frac{\partial^2 V(z)}{\partial z^2}(x^2+y^2) \\ &\quad + \frac{1}{64}\frac{\partial^4 V(z)}{\partial z^4}(x^2+y^2)^2 + \cdots\cdots \end{aligned}}$$

where $V(z)$ is the on-axis potential due to gap accelerating structures.

Using these results it is possible to cast certain acceleration terms in more convenient forms – both in the equation for x and the transformed equation for $\tilde{x}$. An example of this can be found in the intro. lectures by J.J. Barnard.

//

§10

High Current Transport Experiment (HCX) at LBNL Phase I Lattice, Electric Quadrupole Doublet Focus

8-16-2001

■ Particle Parameters

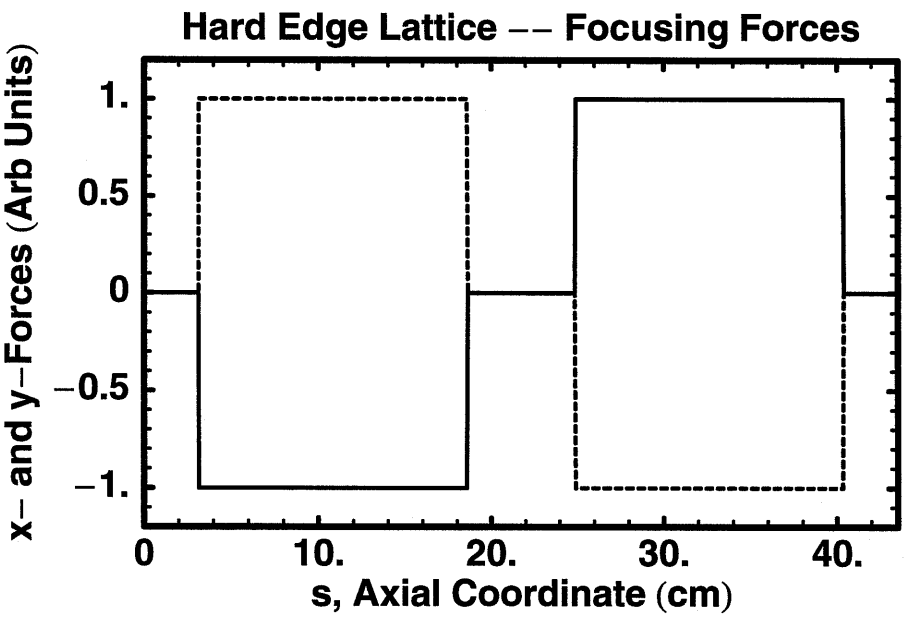
Ion Mass [amu]	38.9637
Charge State Z_{ion}	1
Particle Kinetic Energy, $\mathcal{E}_b$ [keV]	1800.
Mean Axial Velocity, V_b [m/sec]	2.986×10^{6}
Relativistic Factors:	
Beta, $\beta_b = V_b / c$	9.952×10^{-3}
Gamma, $\gamma_b = 1 / \sqrt{1 - \beta_b^2}$, $\gamma_b - 1$	4.952×10^{-5}
Gamma-Beta, $\gamma_b \beta_b$	9.952×10^{-3}
Rigidity, [$B\rho$] [Tesla-m]	1.206

■ Beam Parameters

Current, I_b [mA]	590.
Line Charge Density, λ_b [μC/m]	0.1976
Dimensionless Perveance, K	9.866×10^{-4}
Flat-Top Pulse:	
Duration, τ_b [μs]	4.
Bunch Length, $V_b \tau_b$ [cm]	1194.
, $V_b \tau_b$ [lattice periods]	27.44
Particles, N	1.473×10^{13}
Total Kinetic Energy, $\mathcal{E}$ [Joules]	4.248

■ Transport Lattice --Hard Edge Focusing

Lattice Period, 2L [cm]	43.52
Focusing Quadrupoles:	
Occupancy, Hard Edge Field, η_q	0.71
Syncopation Factor, α_q ($\alpha_q = 1/2 \Rightarrow$ FODO)	0.5
Hard Edge Field Length (Per Quad), l_q [cm]	15.45
Magnetic Quadrupoles (equivalent if electric):	
Gradient, $\lvert dB_{qx}/dy \rvert$ [Tesla/cm]	0.6379
Integrated Gradient, $l_q \lvert dB_{qx}/dy \rvert$ [Tesla]	9.856
Vacuum Pipe Radius, r_p [cm]	2.3
Electric Quadrupoles (equivalent if magnetic):	
Gradient, $\lvert dE_{qx}/dx \rvert$ [Volts/cm^2]	19.05×10^3
Clear Aperture Radius, r_p [cm]	2.3
Aperture Clearance, $r_p = \alpha_p \, \mathrm{Max}[r_x, r_y] + \Delta_p$	
α_p	1.25
Δ_p [cm]	0.5591
Axial Length Free (diag., pump, etc.) [cm]	2.
Axial Distance to Start Hard-Edge Q1 [cm]	3.155
Number Lattice Periods, N_{lat}	20
Transport Length, N_{lat} (2L) [m]	8.704
Beam Transit Time, $\tau_{res} = N_{lat}\,(2\,L)/v_b$ [μs]	2.915

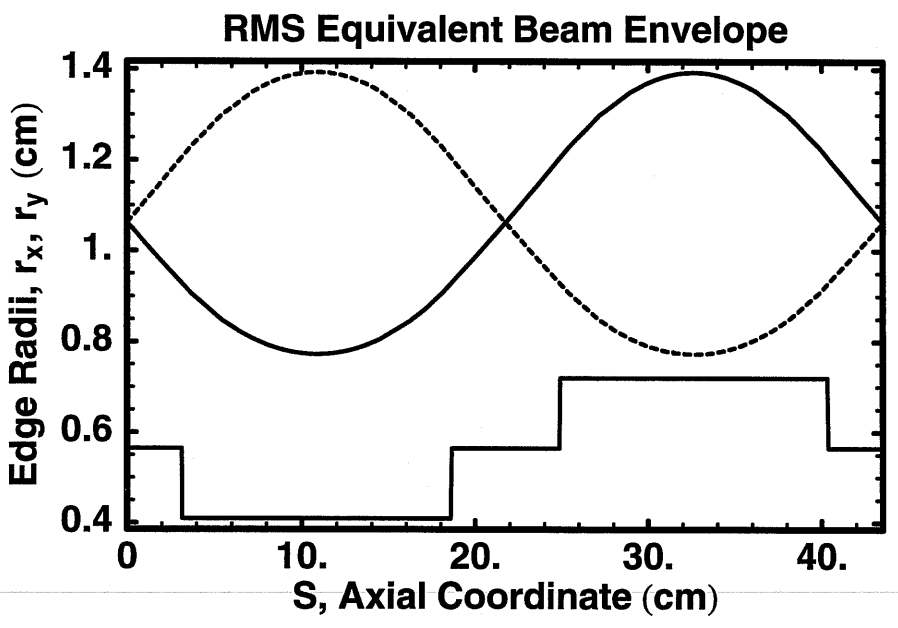


■ Transport Lattice --Transverse Beam Envelope, Mid-Pulse

	x-Horizontal	y-Vertical
Emittances, RMS Edge [mm-mr] :		
Normalized, ϵ_{nx}, ϵ_{nx}	0.5233	0.5233
Unnormalized, ϵ_x, ϵ_y	52.58	52.58
Radii, $r_x = 2\langle x^2\rangle^{1/2}$, $r_y = 2\langle y^2\rangle^{1/2}$:		
Avg (Lattice Period), $\overline{r_x}$, $\overline{r_y}$ [cm]	1.073	1.073
Max, Max[r_x], Max[r_y] [cm]	1.393	1.393
s-locations of Maxs [cm]	32.64	10.88
Min, Min[r_x], Min[r_y] [cm]	0.7728	0.7729
s-locations of Mins [cm]	10.88	32.64
Angles, r_x', r_y' :		
Max, Max[r_x'], Max[r_y'] [mrad]	47.09	47.1
s-locations of Maxs [cm]	24.91	3.155
Min, Min[r_x'], Min[r_y'] [mrad]	-47.09	-47.09
s-locations of Mins [cm]	40.37	18.61
Matching Conditions:		
Radii, $r_x[0]$, $r_y[0]$ [cm]	1.062	1.062
Angles, $r_x'[0]$, $r_y'[0]$ [mrad]	-44.1	44.11

Round Beam Symmetry Points:

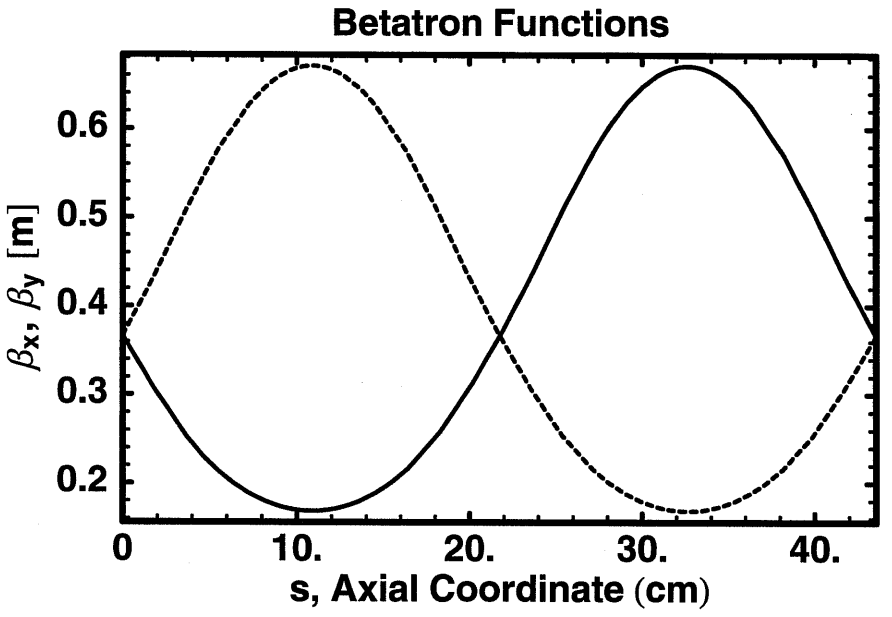
1st Crossing Point	
Beam Radius, $r_x = r_y$ [cm]	1.062
s-location [cm]	21.76
2nd Crossing Point	
Beam Radius, $r_x = r_y$ [cm]	1.062
s-location [cm]	43.52

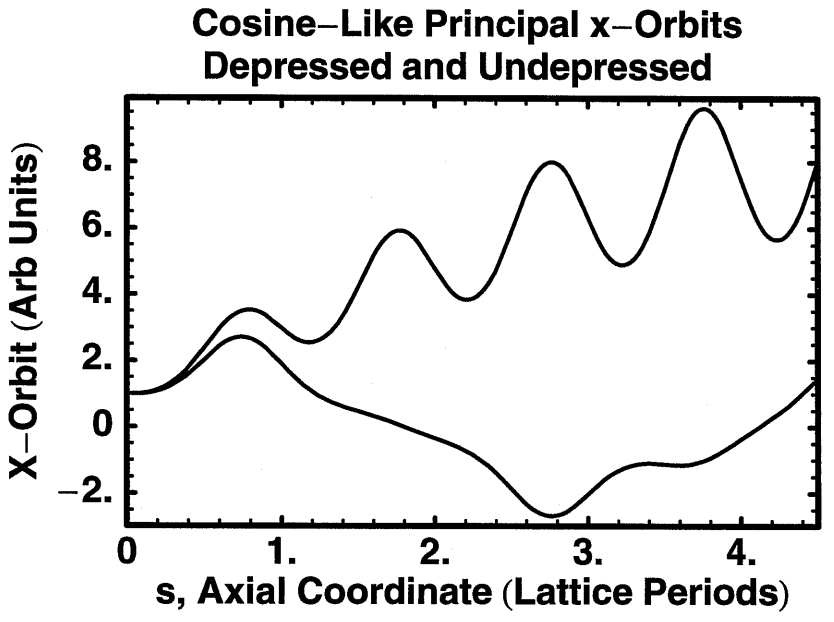


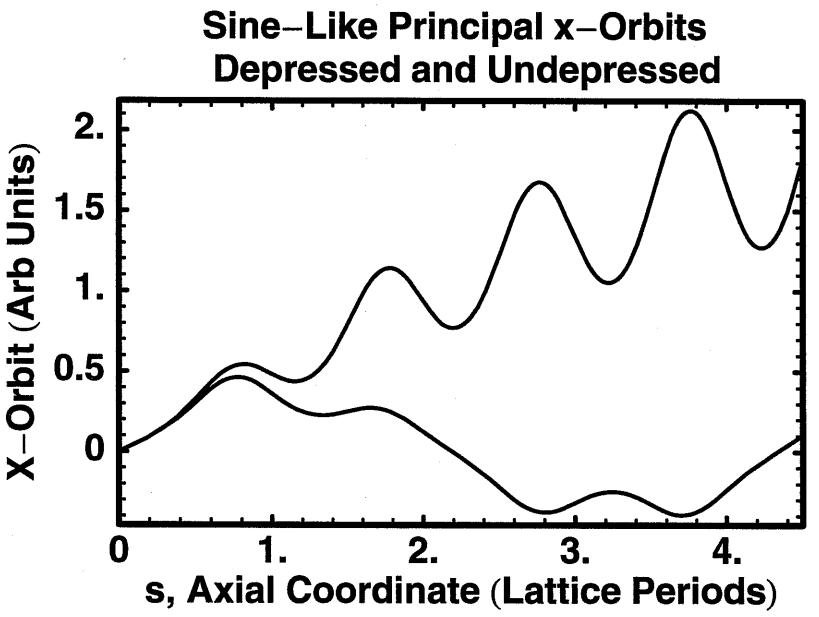
Average:	
Beam Radius, $r_b = \sqrt{\overline{r_x}\,\overline{r_y}}$ [cm]	1.05
$(\overline{r_x} + \overline{r_y})/2$ [cm]	1.073
Aperture Current Density, $I/(\pi r_p^2)$ [mA/cm^2]	35.5
Beam Current Density, $I/(\pi r_b^2)$ [mA/cm^2]	170.4
Beam Intensity $\mathcal{E}/(\pi r_b^2 \tau_b)$ [kW/cm^2]	306.7

■ Transport Lattice --Transverse Beam Dynamics, Mid-Pulse,KV Equiv

	x-Horizontal	y-Vertical
Particle Phase Advances [deg/period]:		
Undepressed, σ_{0x}, σ_{0y}	80	80
Depressed, σ_x, σ_y	13.7	13.73
Tune Depression, σ_x/σ_{0x}, σ_y/σ_{0y}	0.1712	0.1716
Number of Betatron Oscillations:		
Undepressed, $\omega_{\beta 0x}\,\tau_{res}/(2\pi)$, $\omega_{\beta 0y}\,\tau_{res}/(2\pi)$	4.44	4.44
Depressed, $\omega_{\beta x}\,\tau_{res}/(2\pi)$, $\omega_{\beta y}\,\tau_{res}/(2\pi)$	0.761	0.763
Plasma Frequencies, σ_{px}, σ_{py} [deg/period]	111.	111.
Number of Plasma Oscillations:		
$\omega_{px}\,\tau_{res}/(2\pi)$, $\omega_{py}\,\tau_{res}/(2\pi)$	6.19	6.19
Velocity Spreads, RMS, $v_{tx,ty} = \langle (v_{x,y} - V_{x,y})^2 \rangle^{1/2}$:		
Max, Max$[v_{tx}/V_b]$, Max$[v_{ty}/V_b]$	0.0034	0.0034
Min, Min$[v_{tx}/V_b]$, Min$[v_{ty}/V_b]$	0.00189	0.00189
Avg (Lattice Period), $\overline{v_{tx}}/V_b$, $\overline{v_{ty}}/V_b$	0.00256	0.00256
Temps, Spatial Avg, $T_{x,y} = m\langle (v_{x,y} - V_{x,y})^2 \rangle$ [eV]:		
Max, Max$[T_x]$, Max$[T_y]$	41.7	41.6
Min, Min$[T_x]$, Max$[T_y]$	12.8	12.8
Avg (Lattice Period), $\overline{T_x}$, $\overline{T_y}$	23.6	23.6
Betatron Functions, β_x, β_y :		
Avg (Lattice Period), $\overline{\beta_x}$, $\overline{\beta_y}$ [m]	0.3939	0.3939
Max, Max$[\beta_x]$, Max$[\beta_y]$ [m]	0.6707	0.6707
s-locations of Maxs [cm]	32.64	10.88
Min, Min$[\beta_x]$, Min$[\beta_y]$ [m]	0.1684	0.1684
s-locations of Mins [cm]	10.88	32.64


Betatron Functions
0.6
0.5
0.4
0.3
0.2
β_x, β_y [m]
0
10.
20.
30.
40.
s, Axial Coordinate (cm)


Cosine–Like Principal x–Orbits
Depressed and Undepressed
8.
6.
4.
2.
0
–2.
X–Orbit (Arb Units)
0
1.
2.
3.
4.
s, Axial Coordinate (Lattice Periods)

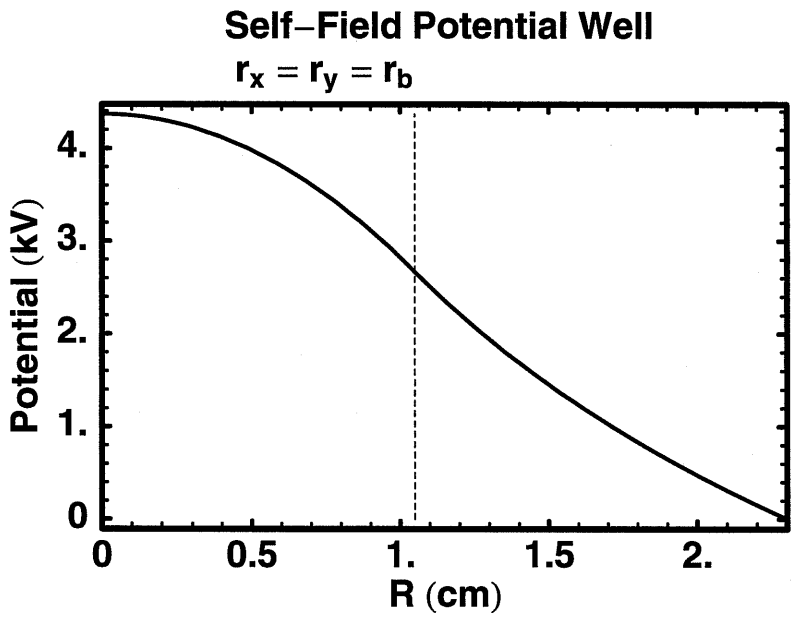


Envelope Modes, Continuous Focusing Model:

Frequencies [deg/period]:	
Breathing Mode	115.
Quadrupole Mode	83.5
Envelope Oscillations:	
Breathing Mode	6.38
Quadrupole Mode	4.64
Max Envelope Excursion [cm]:	
± 1% Mismatch	1.41
± 10% Mismatch	1.53

Average Radius Beam Properties, $r_b = \sqrt{\overline{r_x\, r_y}}$:

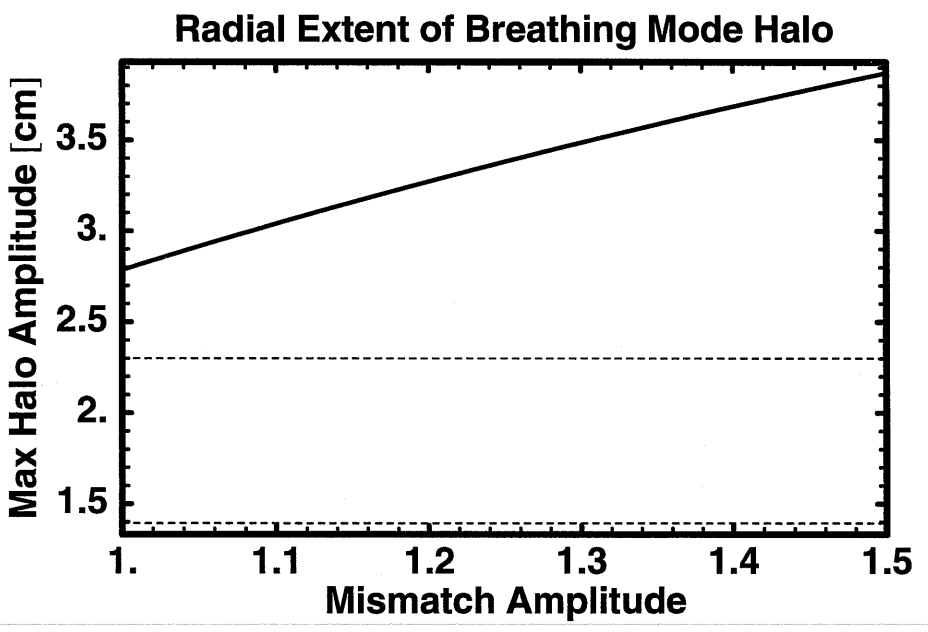
Space-Charge Potential, $\bar{\phi}(r)$ [kV] :	
Edge Beam, $\bar{\phi}(r = r_b)$	2.7
Center Beam, $\bar{\phi}(r = 0)$	4.4
Number density, $\overline{n_b}$ [# / cm^3]	3.41×10^9
Plasma Frequency, $\overline{\omega_{pb}}$ [rad/sec]	1.24×10^7
Plasma Wavenumber, $\overline{k_{pb}} = \overline{\omega_{pb}} / V_b$ [rad/cm]	0.0414
Temperature, T [eV]	22.6
Debye Length, $\overline{\lambda_d}$ [cm]	0.0605
Thermal Beam Width, $r_b / \overline{\lambda_d}$	17.4



Beam Halo -- Approx Core-Particle Model Scaling

x_{max} = Max resonant particle amplitude

Mismatch Amplitude = $\mathrm{Max}[r_x]_{mismatched}/\mathrm{Max}[r_x]_{matched}$

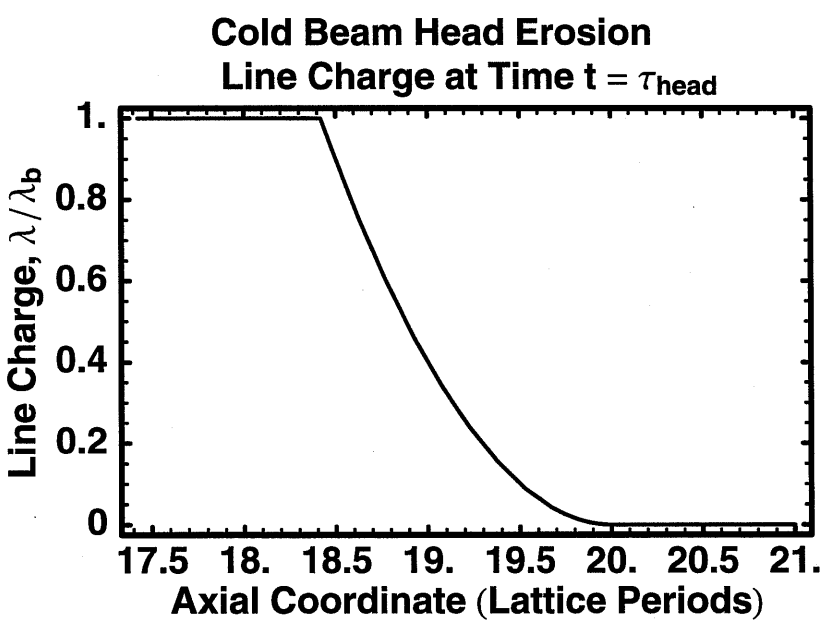


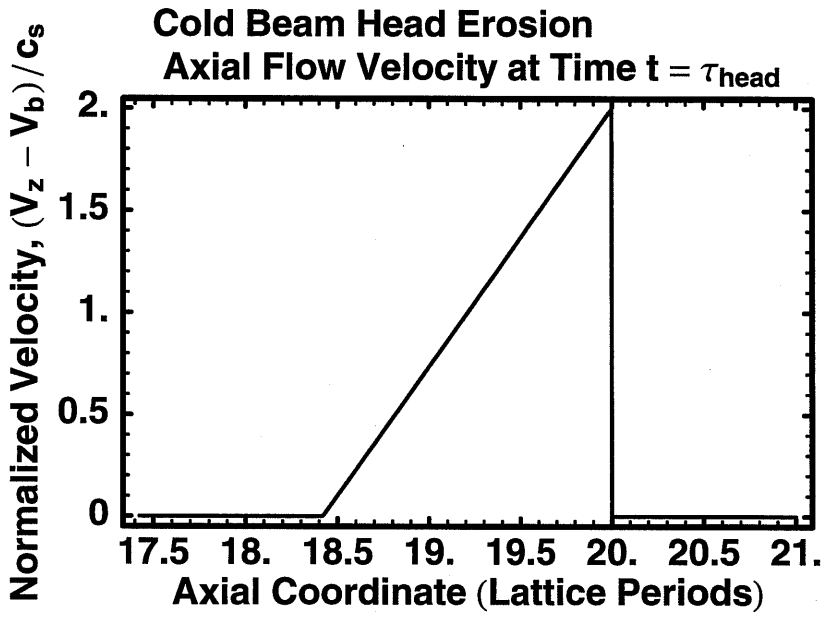
■ Transport Lattice --Longitudinal Beam Dynamics, Mid-Pulse

Vel Spread, RMS, $v_{tz} / V_b = \langle (v_z - V_b)^2 \rangle^{1/2} / V_b$	0.00181
v_{tz} [cm/μsec]	0.54
Temp, $T_z = m \langle (v_z - V_b)^2 \rangle$ [eV]	11.8
Thermal Anisotropy, $T_z / [(\overline{T_x} + \overline{T_y}) / 2]$	0.5
Average g-Factor, $g = 2 \ln[r_p / r_b]$	1.57
Normalized Cold Space-Charge Wave:	
Characteristic Wave Speed, c_s / V_b	0.0278
, c_s [cm/μsec]	8.3
Fast Wave, Phase Velocity, V^+ / V_b	1.03
Beam-Frame Propagation, $(V^+ - V_b)\, \tau_{res}$ [cm]	24.2
Slow Wave, Phase Velocity, V^- / V_b	0.972
Beam-Frame Propagation, $(V^- - V_b)\, \tau_{res}$ [cm]	-24.2

■ Transport Lattice --Longitudinal Beam Dynamics, Bunch Head

Cold Beam Head Erosion Without Confinement:	
Max Head Velocity, $2\, c_s$ [cm/μsec]	16.6
Lattice Transit Time [μsec]:	
Bunch Head, τ_{head}	2.76
Start of Mid-Pulse, τ_{fill}	3.
Length of Head Erosion, $l_{erode} = 3\, c_s\, \tau_{fill}$ [cm]	74.7
, l_{erode} [lattice periods]	1.72
Duration of Erosion Zone, $\tau_{erode} = \tau_{fill} - \tau_{head}$ [μsec]	0.237





■ Transport Lattice --Quadrupoles, Electric

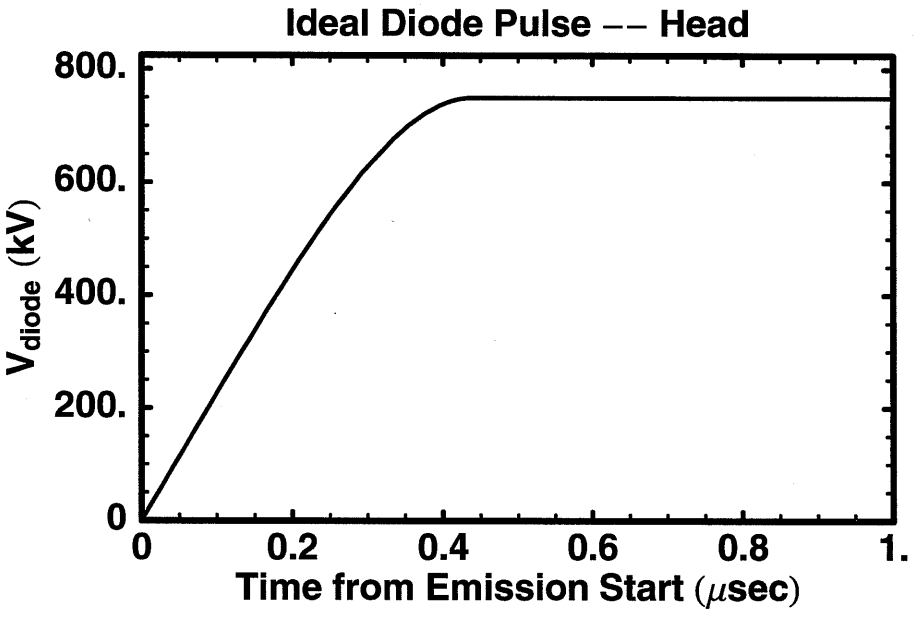
Clear Aperture Radius, r_p [cm]	2.3
Axial Length [cm]:	
Hard-Edge Field, l_q	15.45
Endplates,	1.27
Quadrupole Assembly,	19.76
Free in Period (for Diag., Accel., Steer, Pump)	2.
Quadrupole Rods:	
Lengths [cm],	15.45
Radii [cm],	2.63
Endplate-Rod Axial Gaps [cm],	1.77
Biases (Bipolar, Rod to Ground) [kV],	50.4
Maximum Working Biases (for spec. gaps) [kV]:	
Endplate-Rod Gap	66.5
Rod-Rod Gap	77.

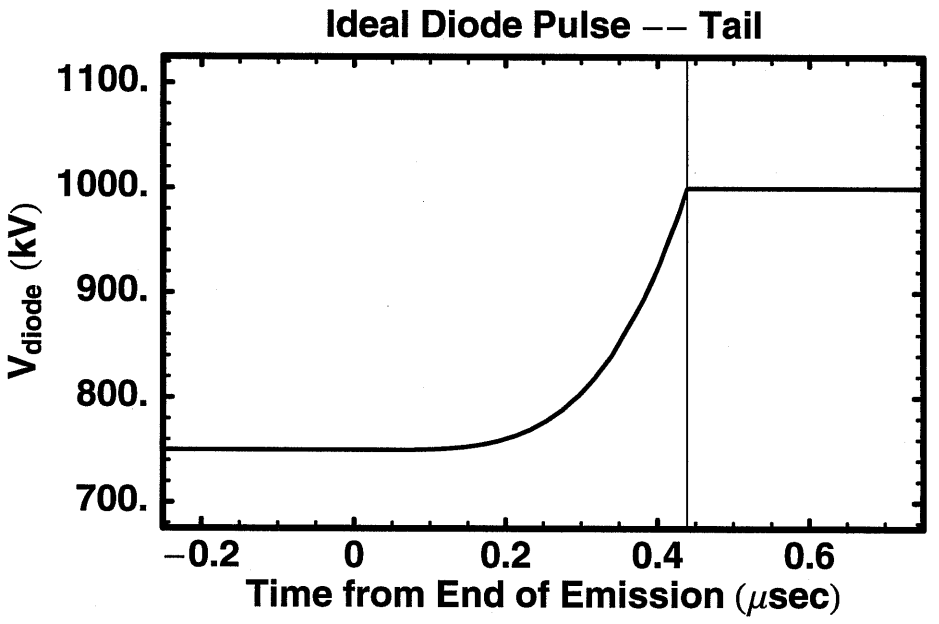
■ Injector

Beam current and envelope data are taken from mid–pulse,3D PIC simulations of the modified ESQ injector carried out with the WARP code.

System:		
Particle Kinetic Energy, $\mathcal{E}_b$ [keV]	1800.	
Current, I_b [mA]	590.	
	x-Horizontal	y-Vertical
Expected Envelope Parameters:		
Radii, r_x, r_y [cm]	3.836	3.653
Angles, r_x', r_y' [mrad]	-37.92	45.38

```
Source:
  Radius [cm]                        5.08
  Radius of Curvature [cm]           20.3
  1-D Equivalent Radius [cm]         5.12
  Area [cm²]                         82.4
  Current Density [mA/cm²]           7.16
  Current [mA]                       590.
  Temperature [°C]                   950.
              [eV]                   0.105
Effective Diode Geometry:
  1-D Equivalent Length [cm]         28.2
  Voltage [kV]                       750.
  Particle Transit Time [μs]         0.439
  Emittance Dilution Factor          3.
```




Ideal Diode Pulse -- Tail
1100.
1000.
900.
800.
700.
V_{diode} (kV)
−0.2
0
0.2
0.4
0.6
Time from End of Emission (μsec)

References

The course textbook Reiser provides a solid introduction to many of the topics covered for further reading including more advanced treatments, we suggest the following books:

Floquet's Theorem, Courant-Snyder Invariants, Hill's Equation, Dispersion Function....

* H. Wiedemann, "Particle Accelerator Physics, Basic Principles and Linear Beam Dynamics." Springer Verlag, Berlin, 1993.

* Highly Recommended

Resonances and Solenoidal Focusing

H. Wiedemann, "Particle Accelerator Physics II, Nonlinear and Higher-Order Beam Dynamics." Springer Verlag, Berlin, 1995.

D.A. Edwards and M.J. Syphers, "An Introduction to the Physics of High Energy Accelerators," John Wiley & Sons, Inc., New York, 1993.

Acknowledgement:

The authors wish to acknowledge the role of Guliano Franchetti (CERN) in educating us in many of the methods presented here.

some additional references:

Particle equations of motion with bends and momentum spread:

D.A. Edwards and M.J. Syphers, "An Introduction to the Physics of High Energy Accelerators," John Wiley & Sons, New York, 1993.

Original classic paper on strong alternating gradient focusing

E.D. Courant and H.S. Snyder, "Theory of the Alternating Gradient Synchrotron" Annals of Physics 3, 1 (1958).

More mathematical treatment of transfer maps and stability. A standard, high quality reference

A. Dragt, "Lectures on Nonlinear Orbit Dynamics," in "Physics of High Energy Accelerators," edited by R.A. Carrigan, F.R. Hudson, and M. Month (AIP Conf. Proc. No. 87 (AIP, New York, 1982) p. 147.

Recent review containing related information on phase advances, orbit stability, etc.

S.M. Lund and B. Bukh, "Stability Properties of the Transverse Envelope Equations Describing Intense Ion Beam Transport," to appear Phys. Rev. Special Topics - Accelerators and Beams, 2004.