Lecture 2

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Mathematical Formulation of the Two Dimensional Magnetic Field

Introduction

- An understanding of magnets is not possible without understanding some of the mathematics underpinning the theory of magnetic fields. The development starts from Maxwell's equation for the three-dimensional magnetic fields in the presence of steady currents both in vacuum and in permeable material.
- For vacuum and in the absence of current sources, the magnetic fields satisfy Laplace's equation.
- In the presence of current sources (in vacuum and with permeable material) the magnetic fields satisfy Poisson's equation. Although three dimensional fields are introduced, most of the discussion is limited to two dimensional fields.
  - This restriction is not as limiting as one might imagine since it can be shown that the line integral of the three dimensional magnetic fields, when the domain of integration includes all regions where the fields are non-zero, satisfy the two dimensional differential equations.

Maxwell's Steady State Magnet Equations

$$\vec{\nabla} \times \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu\mu_0 \vec{J}$$

$\vec{B}$ in *Tesla*

$\vec{J}$ in $\frac{Amps}{m^2}$

μ is the permeability of air $= 1$

μ_0 is the permeability of vacuum $= 4\pi \times 10^{-7} \frac{T\ m}{Amp}$

Function of a Complex Variable

- The derivation of the expressions to show that a function of the complex variable, *F* where *F* is a function of the two dimensional complex space coordinate $z=x+iy$ is developed from Maxwell's equations. This function satisfies Laplace's and Poisson's equations. The development of these expressions are developed in sections 2.1 to 2.5 (pages 19 to 25 of the text).
- This function is used to describe different two dimensional magnetic fields and their error terms.

$$\vec{\nabla} \times \vec{B} = \mu\mu_0 \vec{J} \quad \Rightarrow \quad \nabla^2 \times F = \mu\mu_0 J \qquad \text{Poisson's Equation}$$

$$\vec{\nabla} \times \vec{B} = 0 \quad \Rightarrow \quad \nabla^2 \times F = 0 \qquad \text{Laplace's Equation}$$

Vector and Scalar Potentials

- The function can be expressed as $F=A+iV$ where
  - A, the *vector* potential is the real component
  - V, the *scalar* potential V is the imaginary component
- An ideal pole contour can be computed using the *scalar equipotential*.
- The field shape can be computed using the *vector equipotential*.

- The two dimensional vector components of the magnetic field can be computed from the function.
- Certain *characteristics* of the magnetic field can be determined by symmetry conditions using the function.
- The two dimensional characterization of the magnetic field is a subset of the formulation for the three dimensional magnetic field.
  - The *integrated* three dimensional field distribution can be completely characterized by the two dimensional complex function.
- The concepts covered in this lecture will be useful later when discussing;
  - Conformal mapping.
  - Field perturbations.
  - Magnetic Measurements.

Fundamental Relationships

The complex coordinate is, $z = x + iy$ where $i = \sqrt{-1}$

z can be expressed as $z = |z| e^{i\theta}$

where $|z| = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1} \frac{y}{x}$

z can also be expressed in polar coordinates.

$$z = |z| e^{i\theta} = |z| (\cos\theta + i \sin\theta)$$

LaPlace's Equation

- The function *F* describing the two dimensional magnetic field in vacuum satisfies LaPlace's equation. The equation is satisfied in the absence of current sources and permeable material.

$$\nabla^2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0$$

- *One useful* function (among many) which satisfies this equation is a function of the complex variable.
- This function is useful since it describes multipole magnets and their error terms.

$$F = Cz^n = C|z|^n e^{in\theta}$$

$$z = x + iy$$

$$i = \sqrt{-1}$$

Homework #1

- Prove that *F* satisfies LaPlace's Equation.

$$F = Cz^n$$

$$\nabla^2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0$$

- Hint

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial z}\frac{dz}{dx}$$

$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial z}\frac{dz}{dy}$$

Vector and *Scalar* Potentials

- In this section, we describe a *subset of functions* which describe a class of magnets. These magnets are important in the synchrotron business since they supply the particle optics to steer, focus and correct the particle beam orbit. This subset of functions are the functions which are components of the Taylor's series of the complex space coordinate $z=x+iy$.
- Although specific functions are used to describe ideal fields, the full Taylor series expansion is used to characterize the desired field as well as the unavoidable error fields.
- In general, $F=Cz^n$ describes a class of two dimensional magnetic fields in air and in the absence of permeable material where n is any integer and C can be a *real* or *complex* constant.

$$F = A + iV$$

- A=*Vector Potential*
- V=*Scalar Potential*.

$$A = \operatorname{Re} F$$

$$V = \operatorname{Im} F$$

- Much can be learned about the magnet characteristics from the *Function of the complex variable*.
  - The pole shape can be determined.
  - Flux Lines can be mapped.

Quadrupole Example

- C is a real constant
- n=Field index

  =2 for quadrupoles
- A=*Vector* Potential

  =Real (F)
- V=Scalar Potential

  =Imaginary (F)

$$
\begin{aligned}
F &= Cz^2 \\
&= C\left(x + iy\right)^2 \\
&= C\left(x^2 + i2xy - y^2\right)
\end{aligned}
$$

$$A = C\left(x^2 - y^2\right)$$

$$V = C2xy$$

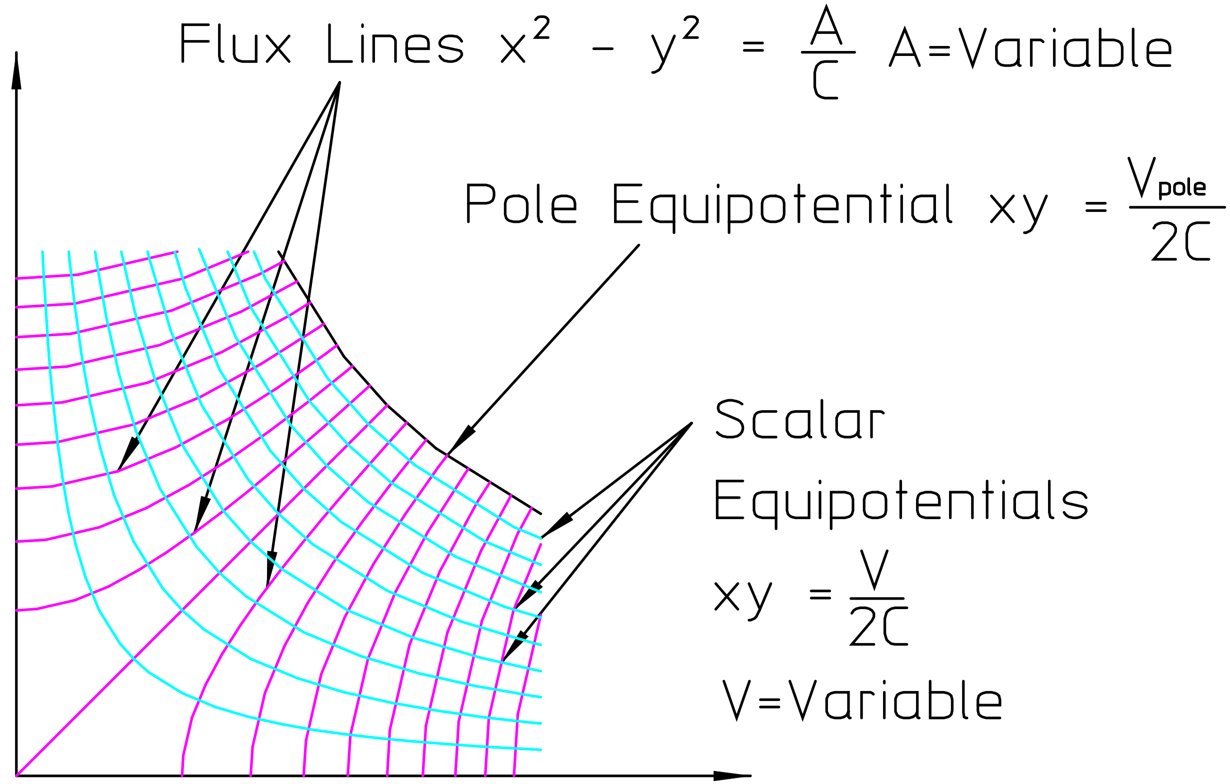
- *Vector* Equipotential Hyperbolic curve with asymptotes at +- 45 deg.

$$x^2 - y^2 = \frac{A}{C}$$

- *Scalar* Equipotential Hyperbolic curve with asymptotes along the x and y axes.

$$xy = \frac{V}{2C}$$

Quadrupole *Equipotentials*



Homework #2

- Find the expressions for the poles and the flux lines for a dipole. (Hint n=1)
- Find the expressions for the poles and the flux lines for a *skew* quadrupole.

  (Hint $C=iC$, imaginary number)

Sextupole Example

- For the sextupole case, the function of a complex variable is written in polar form.
  - This case is presented to illustrate that both polar and Cartesian coordinates can be used in the computation.

$$
\begin{aligned}
F &= A + iV \\
&= Cz^3 = C|z|^3 e^{i3\theta} \\
&= C|z|^3 (\cos 3\theta + i \sin 3\theta)
\end{aligned}
$$

$$A = C|z|^3 \cos 3\theta$$

$$V = C|z|^3 \sin 3\theta$$

Vector Potentials

$$|z|_{VectorPotential} = \left(\frac{A}{C\cos 3\theta}\right)^{\frac{1}{3}}$$

$$x_{VectorPotential} = |z|\cos\theta = \left(\frac{A}{C\cos 3\theta}\right)^{\frac{1}{3}} \cos\theta$$

$$y_{VectorPotential} = |z|\sin\theta = \left(\frac{A}{C\cos 3\theta}\right)^{\frac{1}{3}} \sin\theta$$

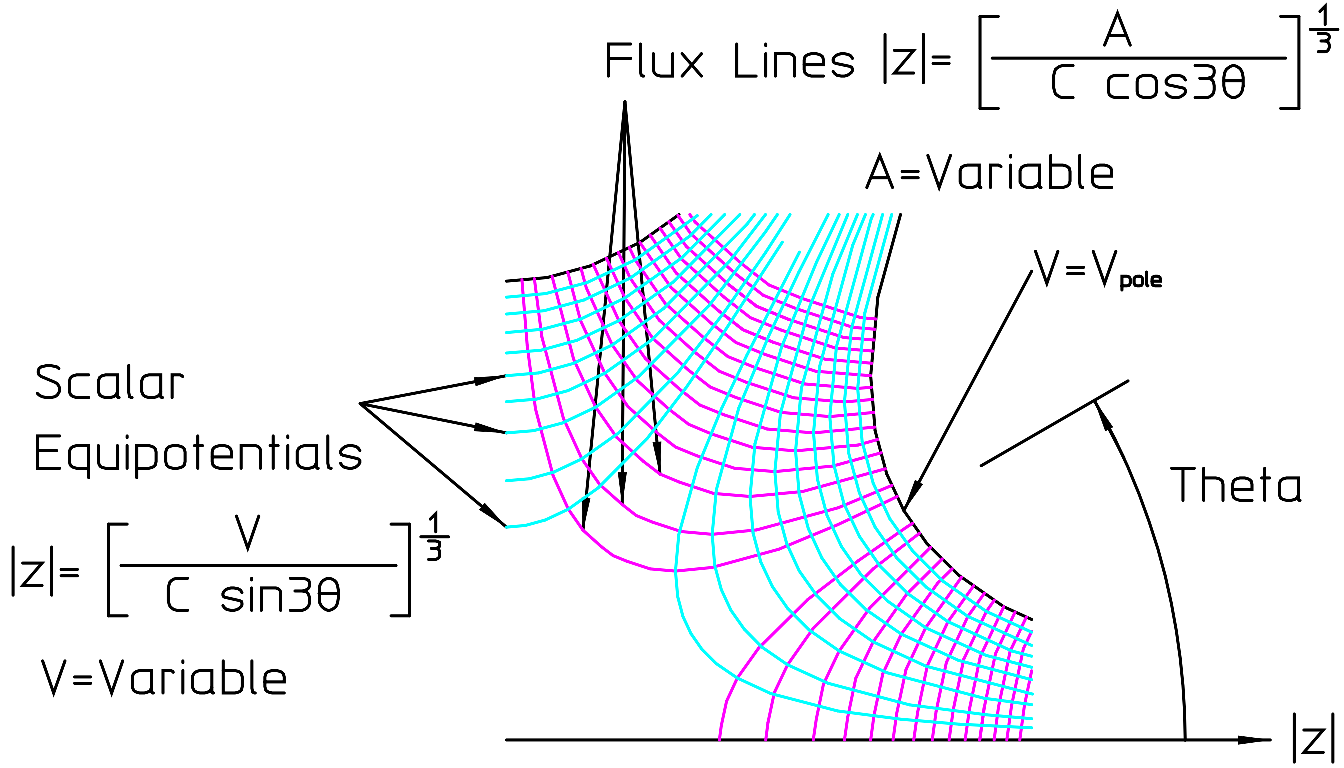
Scalar Potentials

$$|z|_{SclarPotential} = \left(\frac{V}{C\sin 3\theta}\right)^{\frac{1}{3}}$$

$$x_{ScalarPotential} = |z|\cos\theta = \left(\frac{V}{C\sin 3\theta}\right)^{\frac{1}{3}} \cos\theta$$

$$y_{ScalarPotential} = |z|\sin\theta = \left(\frac{V}{C\sin 3\theta}\right)^{\frac{1}{3}} \sin\theta$$

Sextupole Equipotentials



Multipole Magnet Nomenclature

- The dipole has two poles and field index $n=1$.
- The quadrupole has four poles and field index $n=2$.
- The sextupole has six poles and field index $n=3$.
- In general, the N-pole magnet has N poles and field index $n=N/2$.

Even Number of Poles

- Rotational periodicity does not allow odd number of poles. Suppose we consider a magnet with an odd number of poles.
- One example is a magnet with *three* poles spaced at 120 degrees. The first pole is positive, the second is negative, the third is positive and we return to the first pole which *would need to be negative to maintain the periodicity but is positive*.

Characterization of *Error Fields*

- Since $F = Cz^n$ Satisfies LaPlace’s equation,

$$F = \sum C_n z^n$$

  must also satisfy LaPlace’s equation.

- Fields of specific magnet types are characterized by the function

$$F = C_N z^N + \sum_{n \neq N} C_n z^n$$

  where the first term is the “fundamental” and the remainder of the terms represent the “error” fields.

$$F_{dipole} = C_1 z + \sum_{n \neq 1} C_n z^n$$

$$F_{quadrupole} = C_2 z^2 + \sum_{n \neq 2} C_n z^n$$

$$F_{sextupole} = C_3 z^3 + \sum_{n \neq 3} C_n z^n$$

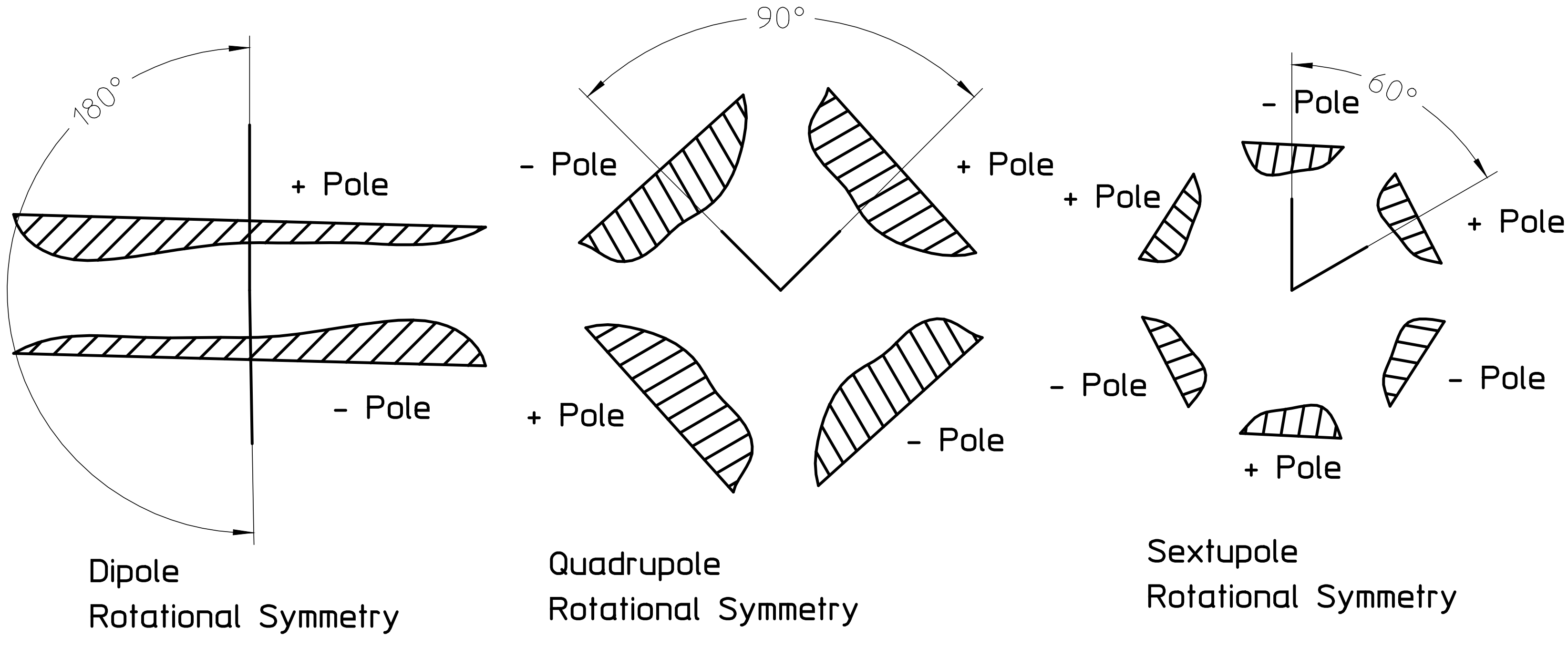
$$F_{Npole} = C_{2N} z^{2N} + \sum_{n \neq N} C_n z^n$$

$$ErrorFields = \sum_{n \neq N} C_n z^n$$

Allowed Multipole Errors

- The error multipoles can be divided among *allowed* or *systematic* and *random* errors.
- The *systematic* errors are those inherent in the design and subject to *symmetry* and *polarity* constraints.
- *Symmetry* constraints require the errors to *repeat* and *change polarities* at angles spaced at π/N, where N is the index of the fundamental field.

- In the figure, the poles are *not* symmetrical about their respective centerlines. This is to illustrate *rotational* symmetry of the N poles.



- Requiring the function to repeat and change signs according to the symmetry requirements:

$$\Delta F_{Npole}\left(\theta+\frac{\pi}{N}\right)=-\Delta F_{Npole}(\theta)$$

- Using the "polar" form of the function of the complex variable:

$$\Delta F(\theta)=\sum C_n z^n=\sum C_n |z|^n e^{in\theta}=\sum C_n |z|^n (\cos n\theta+i\sin n\theta)$$

$$\Delta F\left(\theta+\frac{\pi}{N}\right)=\sum C_n |z|^n e^{in\left(\theta+\frac{\pi}{N}\right)}$$

$$=\sum C_n |z|^n \left[\cos n\left(\theta+\frac{\pi}{N}\right)+i\sin n\left(\theta+\frac{\pi}{N}\right)\right]$$

- In order to have alternating signs for the poles, the following two conditions must be satisfied.

$$\cos n\left(\theta + \frac{\pi}{N}\right) \equiv -\cos n(\theta) \qquad \sin n\left(\theta + \frac{\pi}{N}\right) \equiv -\sin n(\theta)$$

- Rewriting;

$$\cos n\left(\theta + \frac{\pi}{N}\right) = \cos n(\theta)\cos\frac{n\pi}{N} - \sin n(\theta)\sin\frac{n\pi}{N} \equiv -\cos n(\theta)$$

$$\sin n\left(\theta + \frac{\pi}{N}\right) = \sin n(\theta)\cos\frac{n\pi}{N} + \cos n(\theta)\sin\frac{n\pi}{N} \equiv -\sin n(\theta)$$

- Therefore;

$$\sin\frac{n\pi}{N}=0 \qquad \Rightarrow \qquad \frac{n}{N}=1,2,3,4,\cdots,\text{all integers.}$$

$$\cos\frac{n\pi}{N}=-1 \qquad \Rightarrow \qquad \frac{n}{N}=1,3,5,7,\cdots,\text{all odd integers.}$$

- The more restrictive condition is;

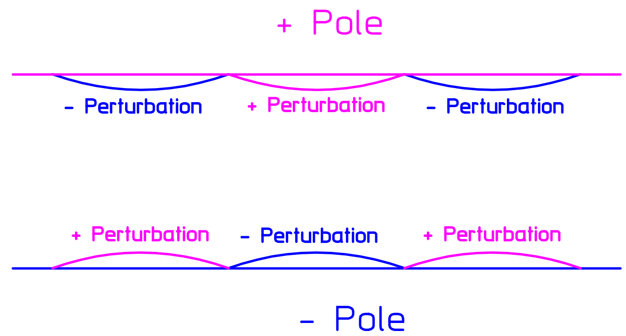
$$\frac{n}{N}=\text{all odd integers }=2m+1$$

$$\text{where } m=0,1,2,3,4,\cdots,\text{all integers.}$$

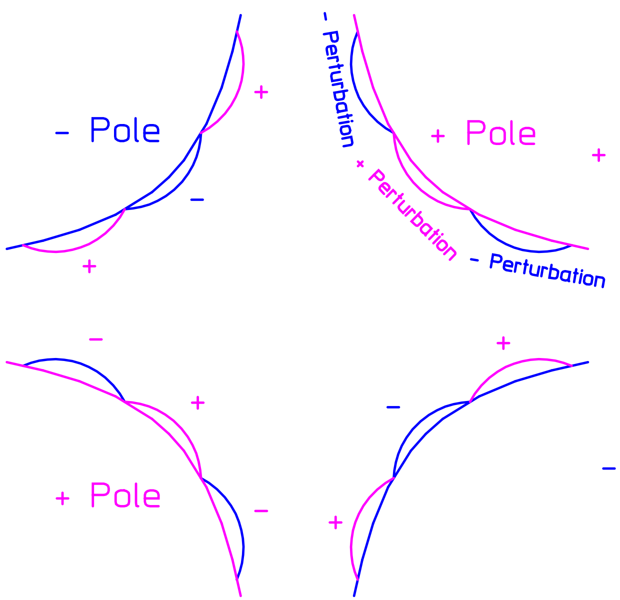
- Rewriting;

$$n_{allowed}=N(2m+1) \quad \text{where} \quad m=0,1,2,3,4,\cdots,\text{all integers.}$$

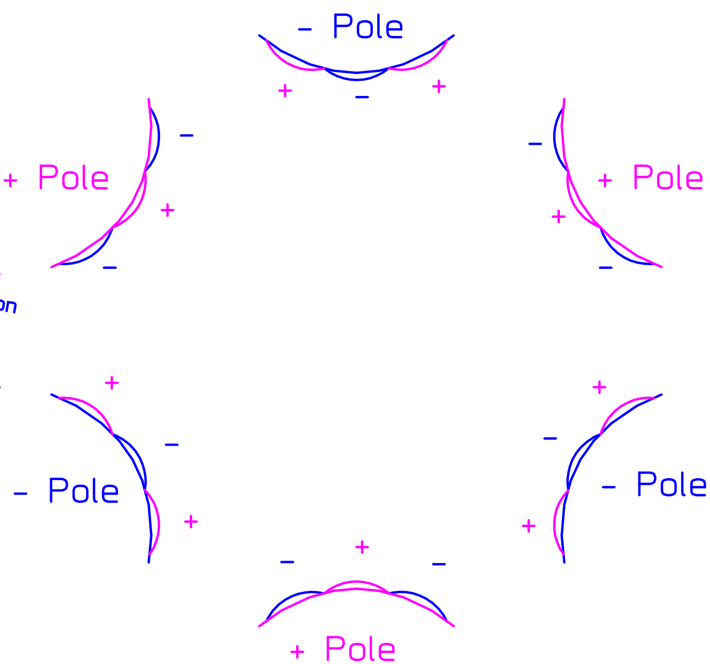
- Thus, the error multipoles allowed by rotational symmetry are;
  - For the dipole, $N=1$, the allowed error multipoles are $n=3, 5, 7, 9, 11, 13, 15, \ldots$
  - For the quadrupole, $N=2$, the allowed error multipoles are $n=6, 10, 14, 18, 22, \ldots$
  - For the sextupole, $N=3$, the allowed error multipoles are $n=9, 15, 21, 27, 33, 39, \ldots$



Dipole



Quadrupole



Sextupole

Magnetic Field from the Function of the Complex Variable

- The field is a *vector* with both *magnitude* and *direction*. The vector can be described in *complex notation* since the x and y components can be described as the real and imaginary components of the complex function. Therefore, the field can be described as a function of *F(z)*.

Dipole Example

- The complex conjugate of the field is given by;

$$B^* = B_x - iB_y = iF'(z)$$

$$B^* = iF'(z) = i\frac{d}{dz}\left(C_n z^n\right) = inC_n z^{n-1}$$

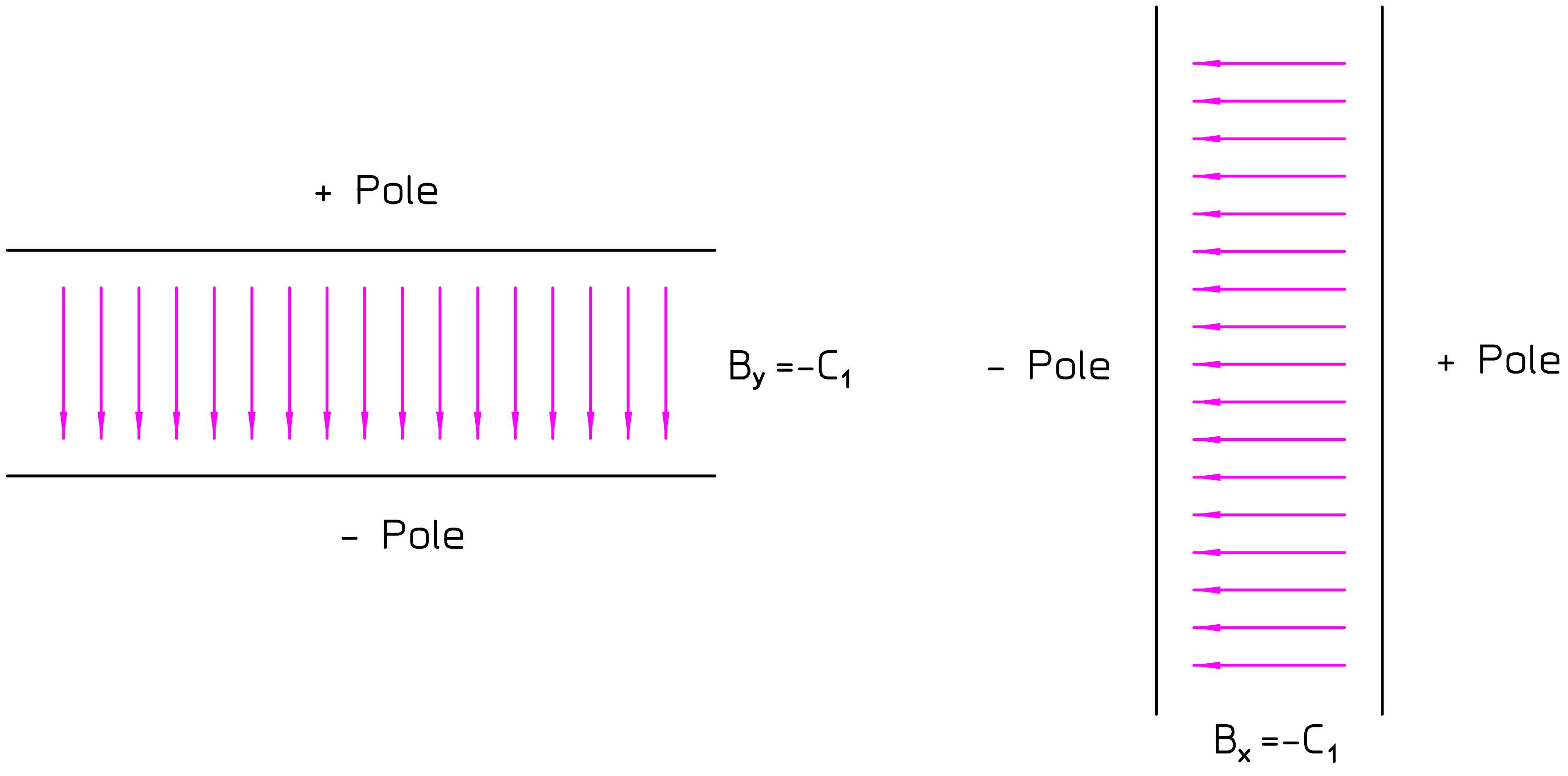
$$B_1^* = i\frac{d}{dz}C_1 z = iC_1$$

Suppose; $C_1 = \mathrm{Re}\, Number$

$$B_1^* = B_x - iB_y = iC_1 \Rightarrow \begin{array}{l} B_x = 0 \\ B_y = -C_1 \end{array}$$

Suppose; $C_1 = \mathrm{Im}\, Number$

$$B_1^* = B_x - iB_y = i \times iC_1 \Rightarrow \begin{array}{l} B_x = -C_1 \\ B_y = 0 \end{array}$$



+ Pole
$B_y = -C_1$
- Pole
- Pole
+ Pole
$B_x = -C_1$

Homework #3

From the function; $F = C_2 z^2$

Find B_x and B_y using $B^* = iF'(z)$

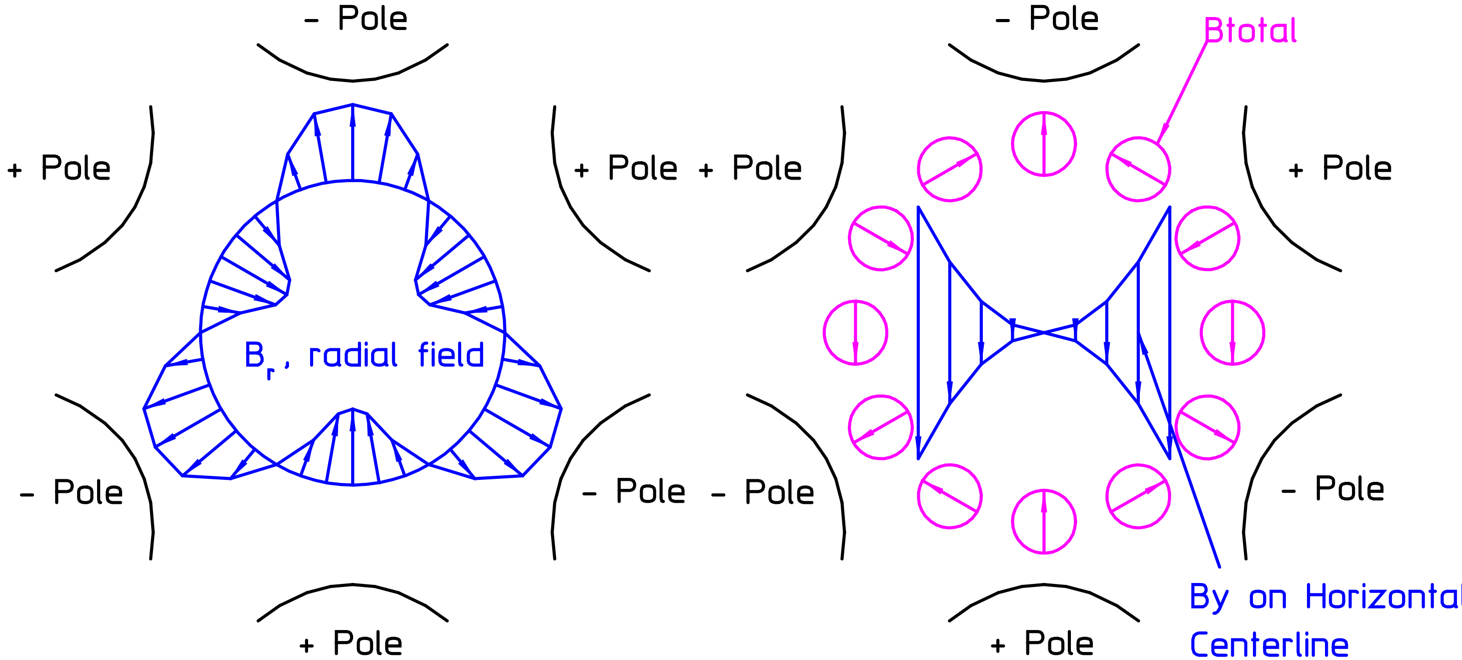
Sextupole Example

$B_3^* = i3C_3 z^2$ a quadratically varying field.

$$B_3^* = B_x - iB_y = i3C_3 z^2 = i3C_3 |z|^2 e^{i2\theta} = i3C_3 |z|^2 \left(\cos 2\theta + i \sin 2\theta\right)$$

$$\Rightarrow \quad \begin{aligned} B_x &= -3C_3 |z|^2 \sin 2\theta \\ B_x &= -3C_3 |z|^2 \cos 2\theta \end{aligned}$$

$$\begin{aligned} B_r &= B_x \cos\theta + B_y \sin\theta \\ &= -3C_3 |z|^2 \left(\sin 2\theta \cos\theta + \cos 2\theta \sin\theta\right) \\ &= -3C_3 |z|^2 \sin 3\theta \end{aligned}$$



- Pole
+ Pole
+ Pole
- Pole
- Pole
+ Pole
B_r, radial field
- Pole
+ Pole
+ Pole
- Pole
- Pole
+ Pole
Btotal
By on Horizontal Centerline

The Curl Equation

- We postulate that the *B* field can be completely determined by the vector potential *A*.

$$B^* = B_x - iB_y = iF'(z) = i\frac{d}{dz}(A + iV)$$

$$B_x - iB_y = i\frac{dA}{dz} = i\left(\frac{\partial A}{\partial x}\frac{dx}{dz} + \frac{\partial A}{\partial y}\frac{dy}{dz}\right) = i\left(\frac{\partial A}{\partial x} - \frac{\partial A}{\partial y}i\right) = \frac{\partial A}{\partial y} + i\frac{\partial A}{\partial x}$$

$$B_x = \frac{\partial A}{\partial y}$$

$$B_y = -\frac{\partial A}{\partial x}$$

- This is consistent with the vector equation;

$$\vec{i}B_x + \vec{j}B_y + \vec{k}B_z = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = Curl\,\vec{A}$$

$$= \vec{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \vec{j}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \vec{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Where in two dimensions;

$A_x = A_y = 0$ and $A = A_z$

- In general, the three dimensional field *vector* can be written as the vector equation;

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{B} = curl\left(\vec{A}\right)$$

- The Vector Potential *A* is a *vector quantity*.

The Divergence Equation

- We postulate that the *B* field can be completely determined by the *Scalar* potential *V*.

$$B_x - iB_y = -\frac{dV}{dz} = -\frac{\partial V}{\partial x}\frac{dx}{dz} - \frac{\partial V}{\partial y}\frac{dy}{dz} = -\frac{\partial V}{\partial x} + i\frac{\partial V}{\partial y}$$

$$B_x = -\frac{\partial V}{\partial x}$$

$$B_y = -\frac{\partial V}{\partial y}$$

- This is consistent with;

$$\vec{i}B_x + \vec{j}B_y + \vec{k}B_z = -\vec{i}\frac{\partial V}{\partial x} - \vec{j}\frac{\partial V}{\partial y} - \vec{k}\frac{\partial V}{\partial z}$$

- In general, the 3D field *vector* can be written as the vector equation,

$$\vec{B} = -\vec{\nabla} V$$

$$\vec{B} = -div V$$

The Scalar Potential *V* is a *scalar quantity*.

Cauchy-Riemann

- Using one or the other potentials;

$$B_x = \frac{\partial A}{\partial y} = -\frac{\partial V}{\partial x} \qquad B_y = -\frac{\partial A}{\partial x} = -\frac{\partial V}{\partial y}$$

which requires $\frac{\partial A}{\partial y} = -\frac{\partial V}{\partial x} \qquad \frac{\partial A}{\partial x} = \frac{\partial V}{\partial y}$

which are the Cauchy-Riemann conditions and can only

be satisfied for $B^* = iF'(z)$

and not for $B = iF'(z)$

Complex Extrapolation

- Using the concept of the magnetic potentials, the ideal pole contour can be determined for a desired field.
- Gradient Magnet Example
  - The desired gradient magnet field requires a field at a point and a *linear* gradient.
  - Given:
    - A central field and gradient.
    - The magnet half gap, h, at the magnet axis.
  - What is the ideal pole contour?

The desired field is; $B_y = B_0 + B' x$

The *scalar* potential satisfies the relation; $B_y = -\frac{\partial V}{\partial y}$

Therefore; $V = -\int \left(B_0 + B' x \right) dy = -B_0 y - B' xy$

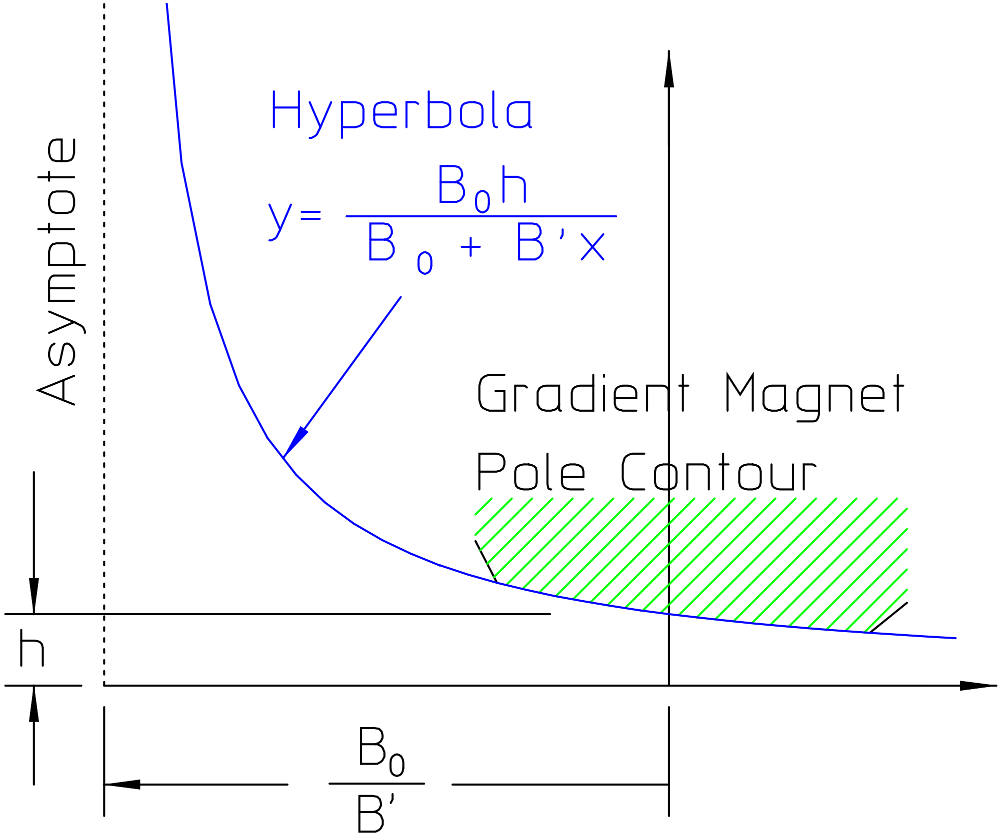
For $\left(x, y \right) = \left(0, h \right)$ on the pole surface,

$$V_{pole} = -B_0 h - B' \left(0 \times h \right) = -B_0 h$$

Therefore, the equation for the pole is, $-B_0 y - B' xy = V_{pole} = -B_0 h$

or solving for *y*, $y = \frac{B_0 h}{B_0 + B' x}$

Hyperbola $y = \dfrac{B_0 h}{B_0 + B' x}$ with asymptote at, $x = -\dfrac{B_0}{B'}$



Other Functions

- In this section, we discussed two dimensional "multipole" magnets, those with rotational symmetric fields. This is a small subset of all of the possible magnetic field distributions.
- Wigglers and undulators are magnets which are finding increasing use in light source synchrotrons. The two dimensional characterization of the magnetic fields from these magnets is *longitudinally* (rather than rotationally) periodic and represents another subset of possible magnetic fields.
- The characterization of the fields from these magnetic structures is well documented in the published literature and can be characterized by an analytic function. Although the full characterization of these fields has *not* been included in the text and should be covered in a separate course on magnetic structures, it should be emphasized that the analytic expression describing these fields can also be characterized by a father simple analytic function.

$$F = \sum A_n \sin \frac{n\pi z}{T} + B_n \sin \frac{n\pi z}{T}$$

The constants for this function are evaluated by computing the Fourier constants which satisfy the boundary conditions.

Lecture 3

- Lecture 3 will cover conformal mapping and application of the tools to extend knowledge about the simple dipole magnet to the more complex quadrupole magnet.
- Section 2.1 should be reviewed and Chapter 3 should be read.
- It would also be helpful to read a part of Chapter 6 (section 6.6) on applications of conformal mapping to POISSON calculations.