Lecture 3

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Conformal Mapping

Introduction

- This section introduces conformal mapping.
  - The means of ensuring dipole field quality is reviewed.
  - Conformal mapping is used to extend the techniques of ensuring *dipole* field quality to *quadrupole* field quality.
  - Conformal mapping can be used to analyze and/or optimize the quadrupole or sextupole pole contours in by using methods applied to dipole magnets.
- Conformal mapping maps one magnet geometry into another.
- This tool can be used to extend knowledge regarding *one* magnet geometry into *another* magnet geometry.

Mapping a Quadrupole into a Dipole

- The quadrupole pole can be described by a hyperbola;

$$xy = \frac{V}{2C} = \text{A Constant}$$

Where V is the *scalar potential* and C is the coefficient of the function, F, of a complex variable.

The expression for the hyperbola can be rewritten;

$$xy = \frac{h^2}{2}$$

We introduce the complex function;

$$w = u + iv = \frac{z^2}{h} = \frac{(x + iv)^2}{h}$$

Rewriting; $$w = u + iv = \frac{x^2 - y^2}{h} + i\frac{2xy}{h}$$

$$u = \operatorname{Re} w = \frac{x^2 - y^2}{h}$$

$$v = \operatorname{Im} w = \frac{2xy}{h} = h$$ since $$xy = \frac{h^2}{2}$$

Therefore; $$w = \frac{x^2 - y^2}{h} + ih$$

the equation of a *dipole* since the imaginary (vertical) component is a constant, h.

Mapping a Dipole into a Quadrupole

- In order to map the dipole into the quadrupole, we use the polar forms of the functions; $w = |w|e^{i\varphi}$ and $z = |z|e^{i\theta}$

\ Since $w = \dfrac{z^2}{h}$ was used to convert the quadrupole into the dipole, $z^2 = hw = h|w|e^{i\varphi}$.

$z = \sqrt{h|w|}e^{i\frac{\varphi}{2}} = |z|e^{i\theta}$ therefore; $|z| = \sqrt{h|w|}$ and $\theta = \dfrac{\phi}{2}$

Finally;

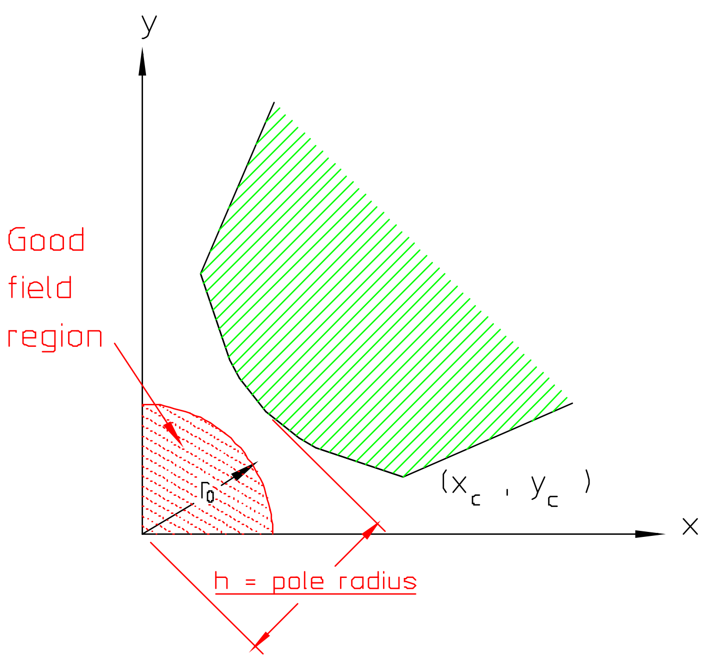
$$x = |z|\cos\theta = \sqrt{h|w|}\cos\frac{\phi}{2}$$

$$y = |z|\sin\theta = \sqrt{h|w|}\sin\frac{\phi}{2}$$

Quadrupole Field Quality

- The figure shows the pole contour of a quadrupole and its required good field region.

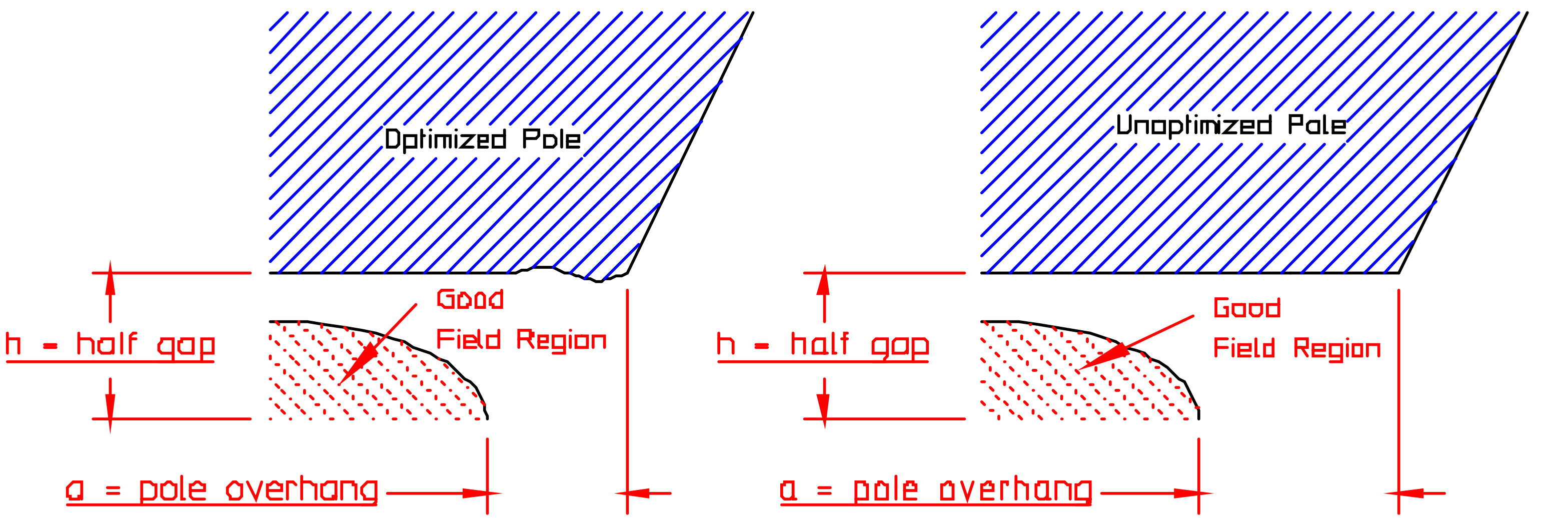
The *pole cutoff*, the point at which the *unoptimized* or *optimized* quadrupole hyperbolic pole is truncated, also determines the potential field quality for the two dimensional unsaturated quadrupole magnet.



- The location of this pole cutoff has design implications. It affects the saturation characteristics of the magnet since the iron at the edge of the quadrupole pole is the first part of the pole area to exhibit saturation effects as magnet excitation is increased. Also, it determines the width of the gap between adjacent poles and thus the width of the coil that can be installed (for a two piece quadrupole). The field quality advantages of a two piece quadrupole over a four piece quadrupole will be discussed in a later section.

H Magnet Field Quality *Review*

- The relation between the field quality and "pole overhang" are summarized by simple equations for a *window frame* dipole magnet with fields below saturation.



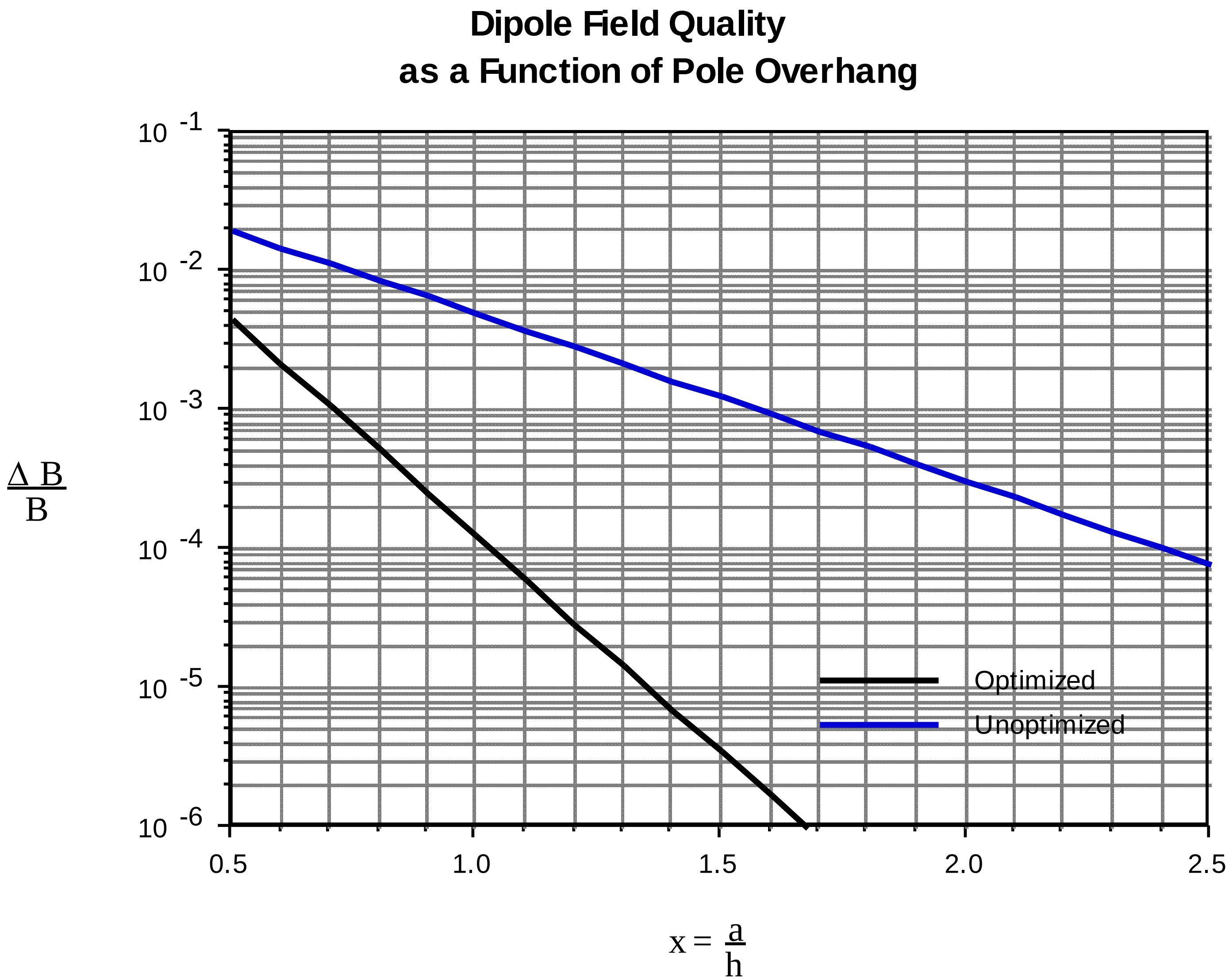
The required *pole overhang* beyond the good field region are given by the following equations.

$$x = \frac{a}{h} = \frac{\text{"pole overhang"}}{\textit{half}\ \text{gap}}$$

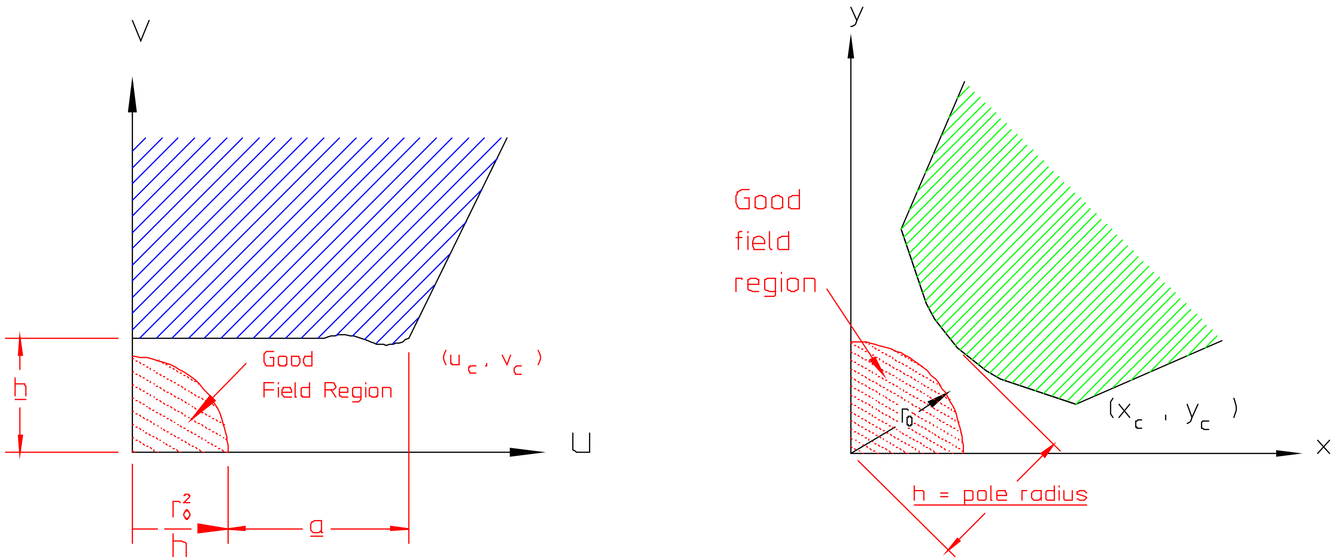
$$x_{unoptimized} = \left(\frac{a}{h}\right)_{unoptimized} = -0.36 \ln \frac{\Delta B}{B} - 0.90$$

$$x_{optimized} = \left(\frac{a}{h}\right)_{optimized} = -0.14 \ln \frac{\Delta B}{B} - 0.25$$

- The relations can be presented graphically.



- Given; (u_c, v_c) satisfying *dipole* uniformity requirements.
- Find; (x_c, y_c) satisfying the same requirements for *quadrupoles*.



$$w = \frac{|z|^2}{h} \quad \Rightarrow \quad r_{good\ field\ region} = \frac{r_0^2}{h}$$

For the Dipole;

$$a_{unoptimized} = -h\left[0.36\ln\frac{\Delta B}{B} + 0.90\right] = -h[\textit{unoptimized}\ \text{factor}]$$

$$a_{optimized} = -h\left[0.14\ln\frac{\Delta B}{B} + 0.25\right] = -h[\textit{optimized}\ \text{factor}]$$

Therefore;

$$u_c = \frac{r_0^2}{h} + a = \frac{r_0^2}{h} - h[factor] \quad \text{and} \quad v_c = \text{h}$$

Substituting a unitless (normalized) good field region, $\rho_0 = \dfrac{r_0}{h}$

and using the conformal mapping expressions,

$$x = |z|\cos\theta = \sqrt{h|w|}\cos\frac{\phi}{2}$$

$$y = |z|\sin\theta = \sqrt{h|w|}\sin\frac{\phi}{2}$$

and the half angle formulae,

$$\cos\frac{\phi}{2} = \sqrt{\frac{1+\cos\phi}{2}}$$

$$\sin\frac{\phi}{2} = \sqrt{\frac{1-\cos\phi}{2}}$$

and substituting,

$$\frac{|w_c|}{2h} = \frac{\sqrt{u_c^2 + v_c^2}}{2h} = \sqrt{\left(\frac{u_c}{2h}\right)^2 + \left(\frac{v_c}{2h}\right)^2}$$

$$= \sqrt{\frac{1}{4}\left(\frac{r_0^2}{h^2} - [factor]\right)^2 + \left(\frac{h}{2h}\right)^2} = \frac{1}{2}\sqrt{\left(\frac{r_0^2}{h^2} - [factor]\right)^2 + 1}$$

we get,

$$\frac{x_c}{h} = \sqrt{\frac{1}{2}\sqrt{\left(\rho_0^2 - [factor]\right)^2 + 1} + \frac{1}{2}\left(\rho_0^2 - [factor]\right)}$$

$$\frac{y_c}{h} = \sqrt{\frac{1}{2}\sqrt{\left(\rho_0^2 - [factor]\right)^2 + 1} - \frac{1}{2}\left(\rho_0^2 - [factor]\right)}$$

Substituting the appropriate factors for the *unoptimized* and *optimized* dipole cases, we get finally for the quadrupoles;

$$\frac{x_{c\,\mathrm{unoptimized}}}{h} = \sqrt{\frac{1}{2}\sqrt{\left(\rho_0^2 - \left[0.36\ln\frac{\Delta B}{B} + 0.90\right]\right)^2 + 1} + \frac{1}{2}\left(\rho_0^2 - \left[0.36\ln\frac{\Delta B}{B} + 0.90\right]\right)}$$

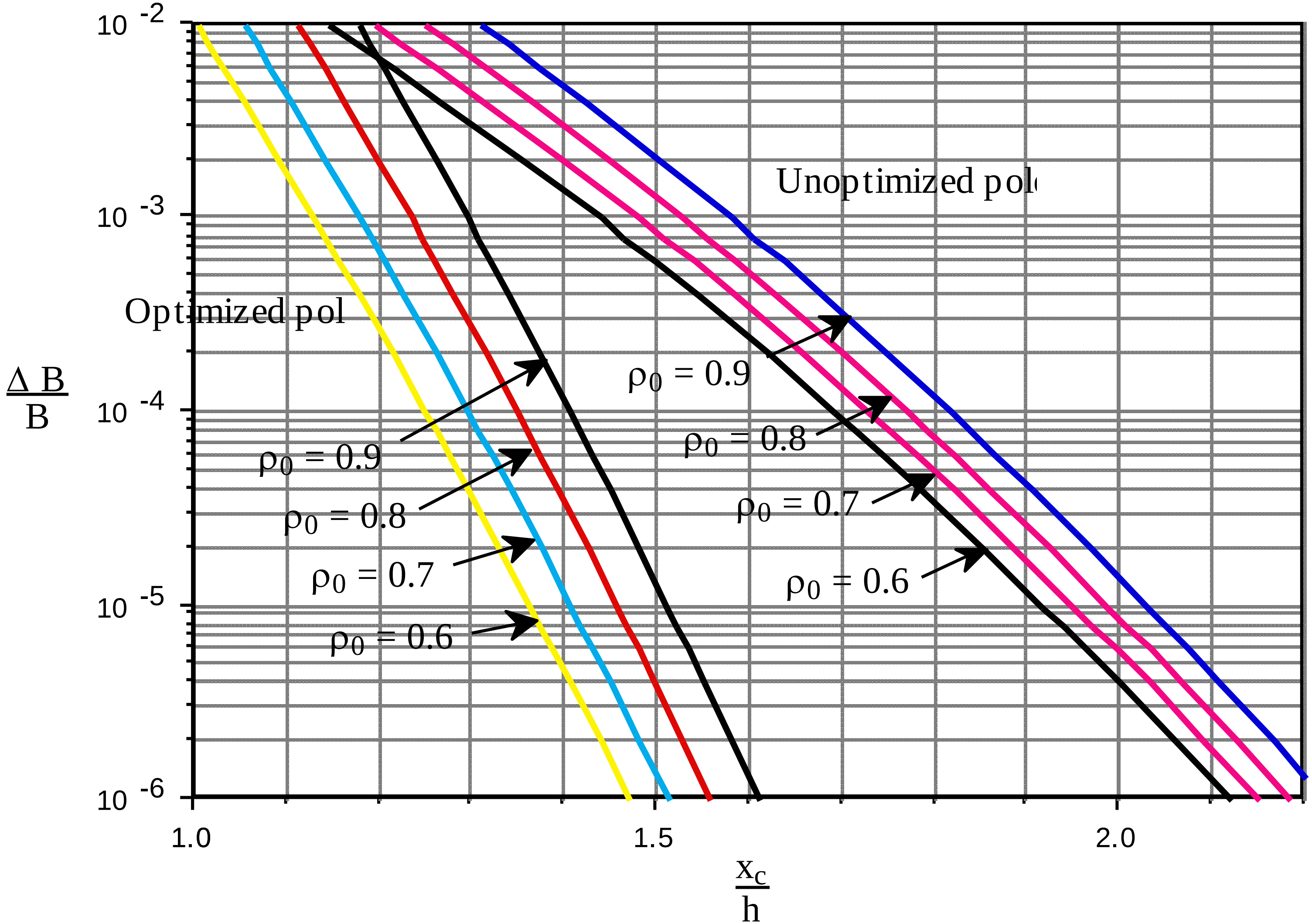
$$\frac{y_{c\,\mathrm{unoptimized}}}{h} = \sqrt{\frac{1}{2}\sqrt{\left(\rho_0^2 - \left[0.36\ln\frac{\Delta B}{B} + 0.90\right]\right)^2 + 1} - \frac{1}{2}\left(\rho_0^2 - \left[0.36\ln\frac{\Delta B}{B} + 0.90\right]\right)}$$

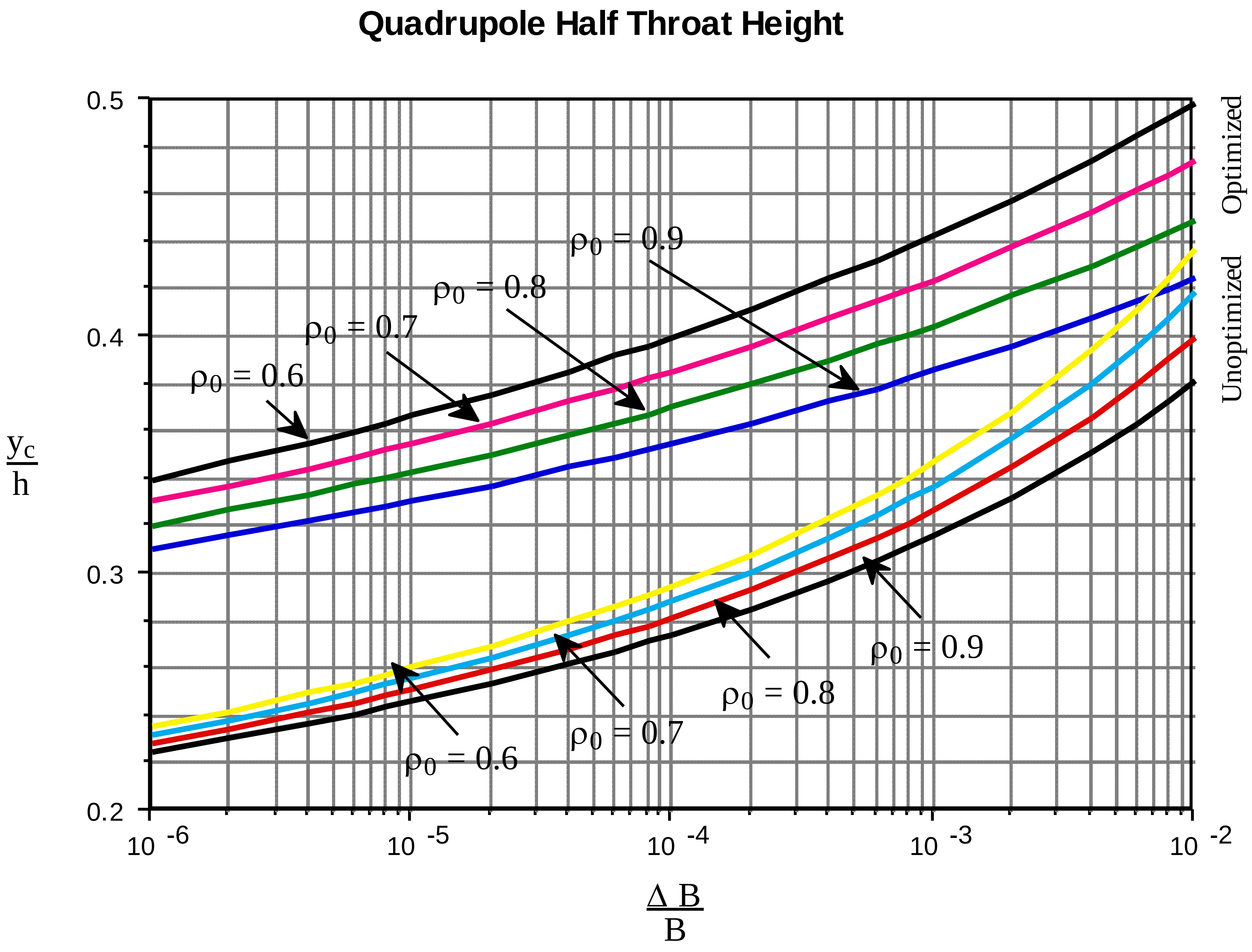
$$\frac{x_{c\,\mathrm{optimized}}}{h} = \sqrt{\frac{1}{2}\sqrt{\left(\rho_0^2 - \left[0.14\ln\frac{\Delta B}{B} + 0.25\right]\right)^2 + 1} + \frac{1}{2}\left(\rho_0^2 - \left[0.14\ln\frac{\Delta B}{B} + 0.25\right]\right)}$$

$$\frac{y_{c\,\mathrm{optimized}}}{h} = \sqrt{\frac{1}{2}\sqrt{\left(\rho_0^2 - \left[0.14\ln\frac{\Delta B}{B} + 0.25\right]\right)^2 + 1} - \frac{1}{2}\left(\rho_0^2 - \left[0.14\ln\frac{\Delta B}{B} + 0.25\right]\right)}$$

- The equations are graphed in a variety of formats to summarize the information available in the expressions. The expressions are graphed for both the *optimized* and *unoptimized* pole to illustrate the advantages of pole edge shaping in order to enhance the field. The quality at various good field radii are computed since the beam typically occupies only a fraction of the aperture due to restrictions of the beam pipe.

Quadrupole Field Quality as a Function of Pole Cutoff

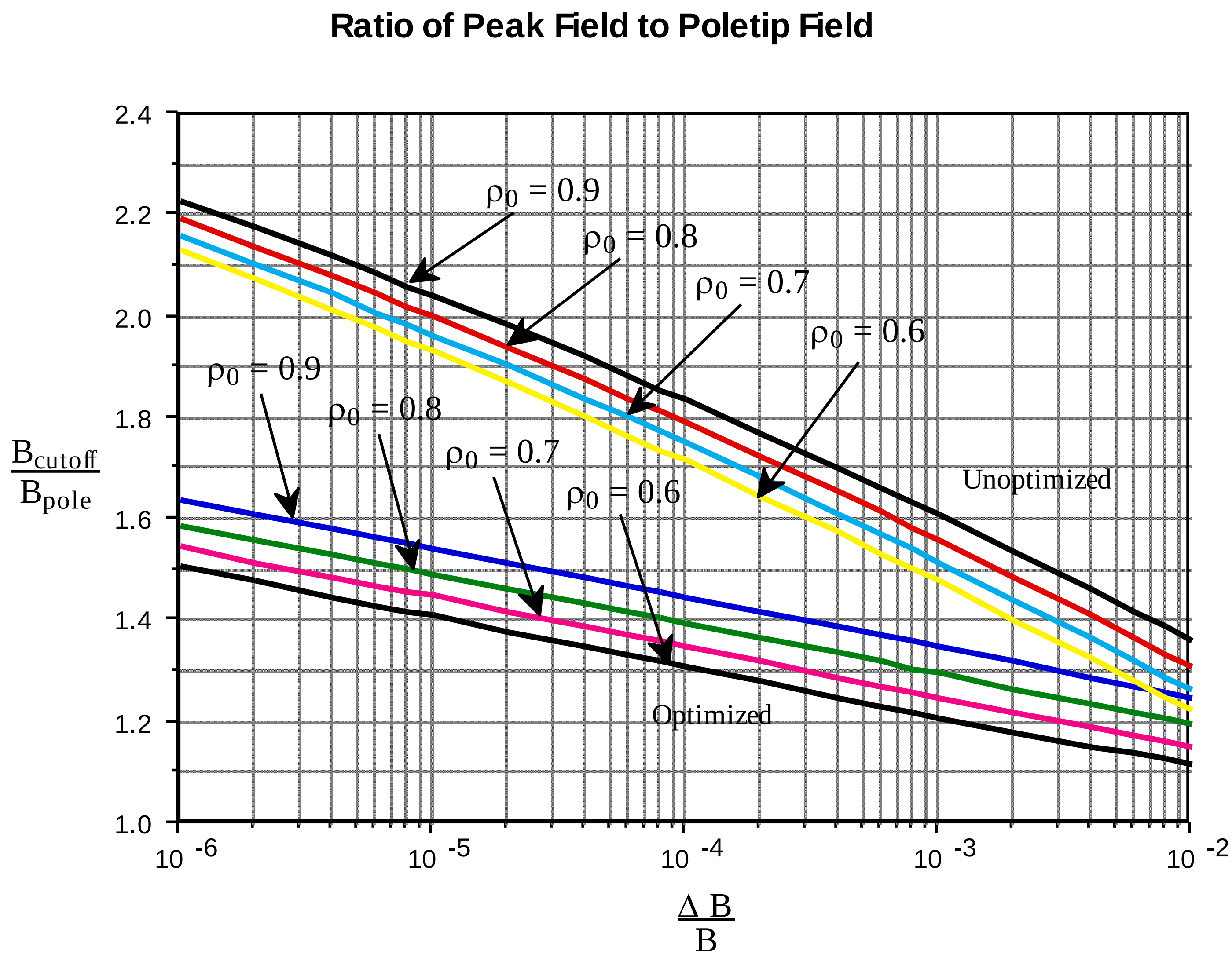




Quadrupole Half Throat Height
0.5
0.4
0.3
0.2
$\frac{y_c}{h}$
10 -6
10 -5
10 -4
10 -3
10 -2
$\frac{\Delta B}{B}$
ρ_0 = 0.9
ρ_0 = 0.8
ρ_0 = 0.7
ρ_0 = 0.6
ρ_0 = 0.9
ρ_0 = 0.8
ρ_0 = 0.7
ρ_0 = 0.6
Optimized
Unoptimized

Since the field for the quadrupole varies with the radius;

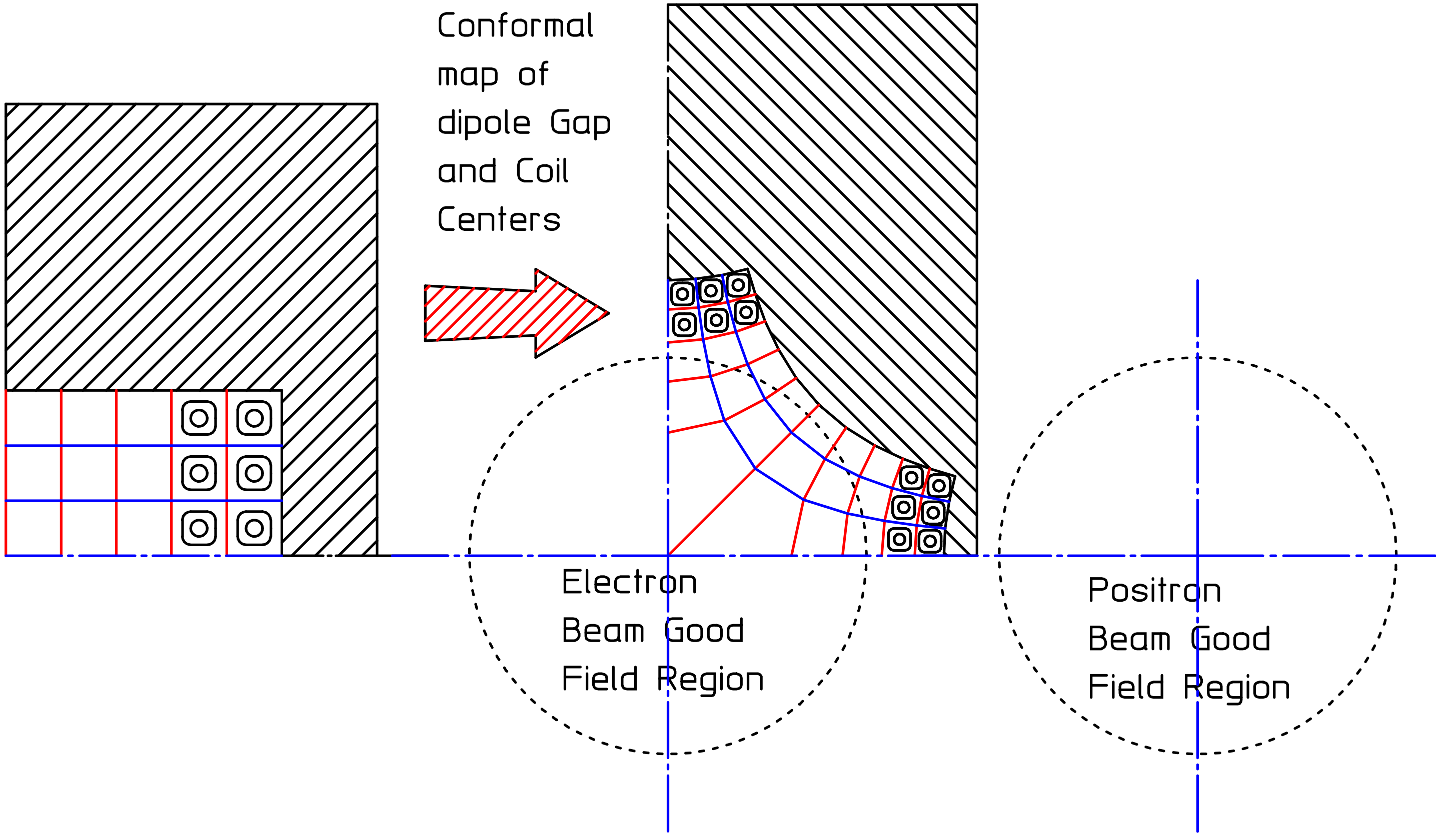
$$\frac{B_{cutoff}}{B_{pole}} = \frac{\sqrt{x_c^2 + y_c^2}}{h}$$



The Septum Quadrupole

- PEPII is a positron electron collider. In order to maximize the the number of collisions and interactions, the two beams must be tightly focused as close to the interaction region as possible. At these close locations where the final focus quadrupoles are located, the two crossing beams are very close to each other. Therefore, for the septum quadrupoles, it is not possible to take advantage of the potential field quality improvements provided by a generous pole overhang. It is necessary to design a quadrupole by using knowledge acquired about the performance of a good field quality dipole. This dipole is the window frame magnet.

- The conformal map of the *window frame dipole* aperture and the centers of the separate conductors is illustrated.
- The conductor shape does not have to be mapped since the current acts as a point source at the conductor center.



Other Uses for Conformal Maps

- Programs such as *POISSON* compute the two dimensional distribution of the vector potential. The vector potential function is computed using a relaxation method (for *POISSON*) or a modified matrix inversion (for *PANDIRA*) among neighboring mesh points defined by the magnet geometry.

- The magnetic field distribution is then computed from the derivative of the vector potential.

$$H^* = H_x - iH_y = iF'(z) \Rightarrow \quad \mathrm{H_x} = \frac{\partial \mathrm{A}}{\partial y} \quad \mathrm{H_y} = -\frac{\partial \mathrm{A}}{\partial x}$$

•For an ideal dipole field (H_y=constant and/or H_x=*constant*) the *vector potential* function is a linear function of z.

• For a *quadrupole* field, the *vector potential* function is a *quadratic* function of z.

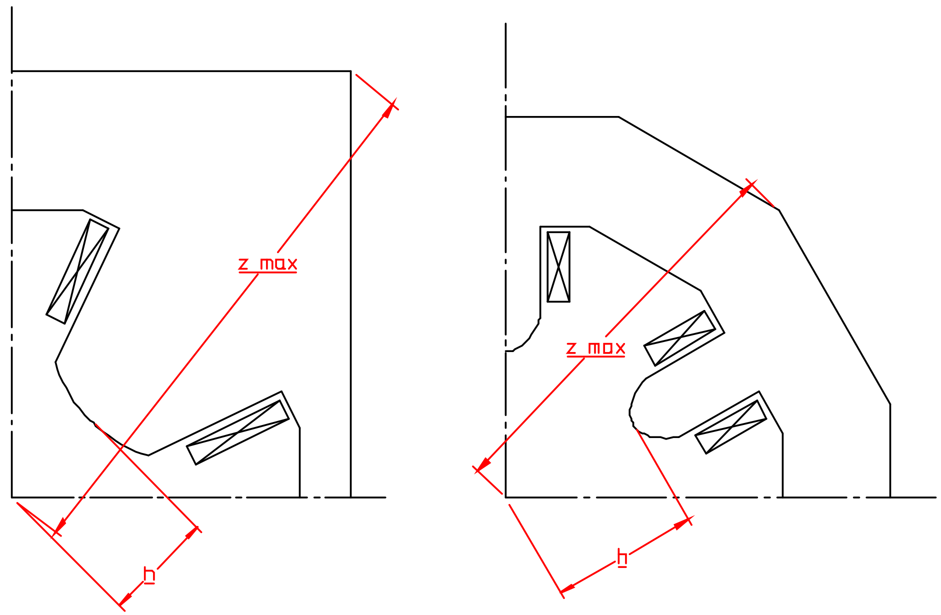
•The vector potential for a sextupole is a cubic function of z.

- When computing the field distribution, it is necessary to compute the *derivative* by interpolating the distribution of the vector potential function among several mesh points. The precision of the field calculations depends on the mesh density and the continuity of the interpolated values of the vector potential.
- Since the dipole function is simple (a linear distribution), the potential precision of field calculations is much higher than for quadrupole (quadratic) or sextupole (cubic) fields. (An accurate estimate of the derivatives for a linear distribution of a potential function can be obtained from fewer values from "neighboring" mesh points than for a quadratic or cubic distribution.)

Therefore, when high precision computations for magnetic field distribution have been required, a conformal transformation is often employed to convert the quadrupole and/or sextupole geometry to a dipole configuration.

$$w = \frac{z^2}{h} \text{ for a quadrupole}, \quad w = \frac{z^3}{h^2} \text{ for a sextupole.}$$

However, there is a *problem* in the mapping of the quadrupole and sextupole to the dipole space.

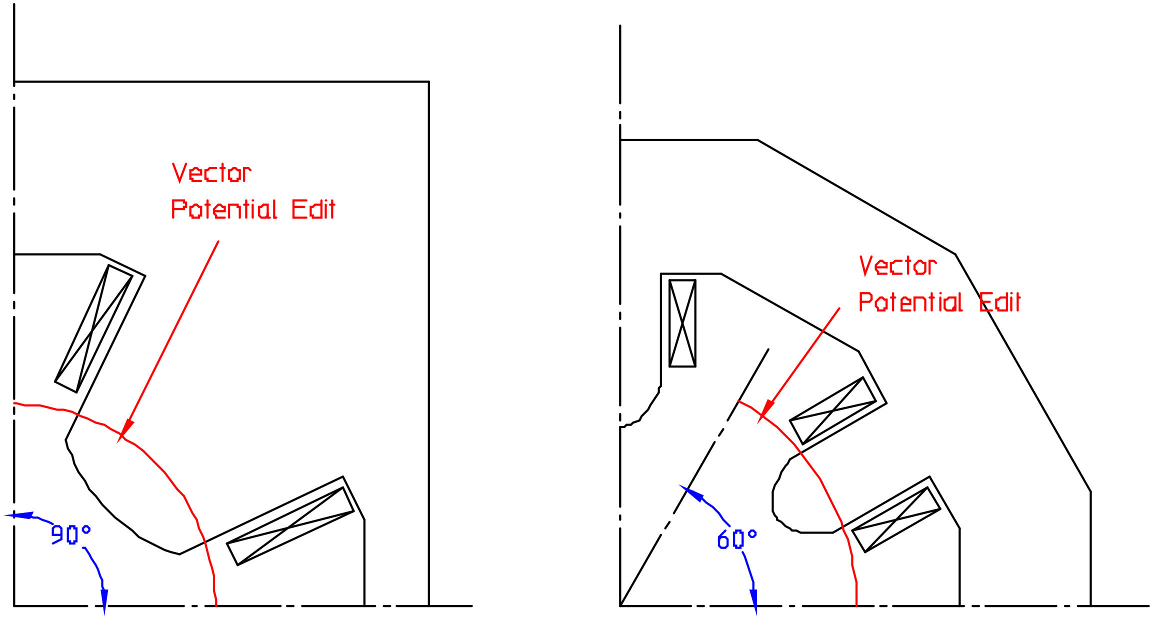


Typically, $\frac{z_{\max}}{h} > 1$

Therefore, $$\left(\frac{w}{h}\right)_{dipole} = \left(\frac{z}{h}\right)^2_{\text{quadrupole}} >> 1$$ and

$$\left(\frac{w}{h}\right)_{dipole} = \left(\frac{z}{h}\right)^3_{\text{sextupole}} >> 1$$ in the mapped space.

- When mapping from the quadrupole or sextupole geometries to the dipole space, the POISSON computation is initially made in the original geometry and a vector potential map is obtained at some reference radius which includes the pole contour.



The vector potential values are then mapped into the dipole (w) space and used as boundary values for the problem.

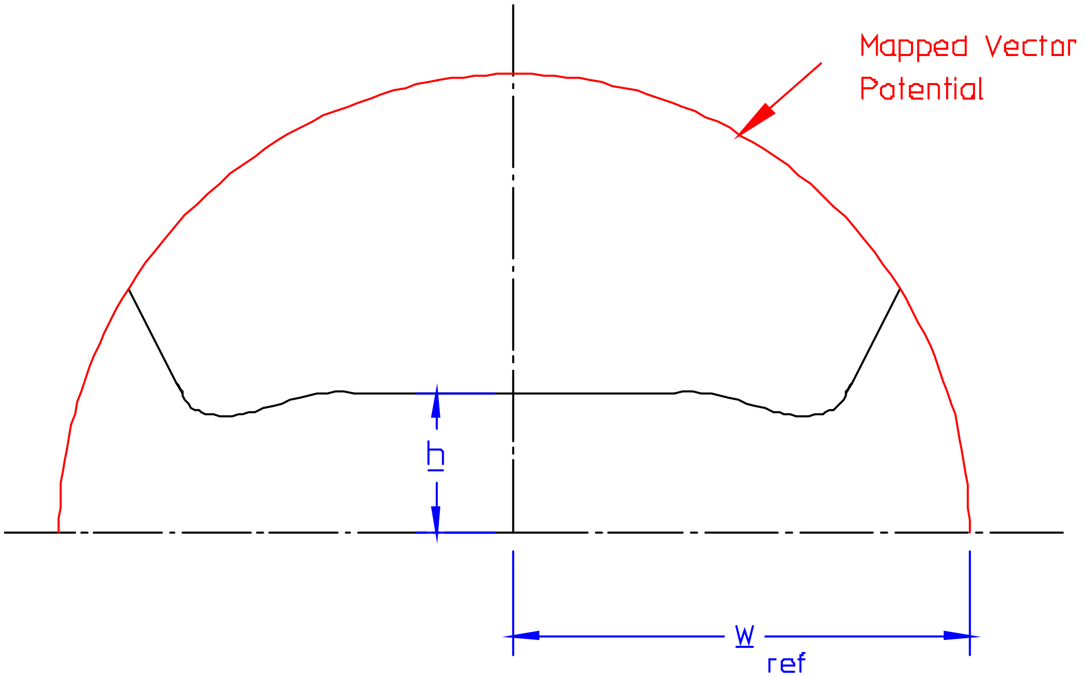
$$A(w_{ref}, \phi) = a(r_{ref}, \theta)$$

$$w_{ref} = \left(\frac{r_{ref}^2}{h} \right)_{quadrupole}$$

$$\phi = 2\theta_{quadrupole}$$

$$w_{ref} = \left(\frac{r_{ref}^3}{h^2} \right)_{sextupole}$$

$$\phi = 3\theta_{sextupole}$$



Quadrupole/Sextupole Pole *Optimization*

- It is *far* easier to visualize the required shape of pole edge bumps on a dipole rather than the bumps on a quadrupole or sextupole pole.
- It is also easier to evaluate the uniformity of a constant field for a dipole rather than the uniformity of the linear or quadratic field distribution for a quadrupole or sextupole.
- Therefore, the pole contour is *optimized* in the dipole space and *mapped* back into the quadrupole or sextupole space.

- The process of pole optimization is similar to that of analysis in the dipole space.
  - Choose a quadrupole pole width which will provide the required field uniformity at the required pole radius.
    - The pole cutoff (x_c, y_c) for the quadrupole can be obtained from the graphs developed earlier using the dipole pole arguments.
    - The sextupole cutoff can be computed by conformal mapping the pole overhang from the dipole space using

$$z = \sqrt[3]{h^2 w}$$

- Select the *theoretical ideal* pole contour.

$$xy = \frac{h^2}{2}$$ for the *quadrupole*.

$$3x^2y - y^3 = h^3$$ for the *sextupole*.

■Select a practical coil geometry.

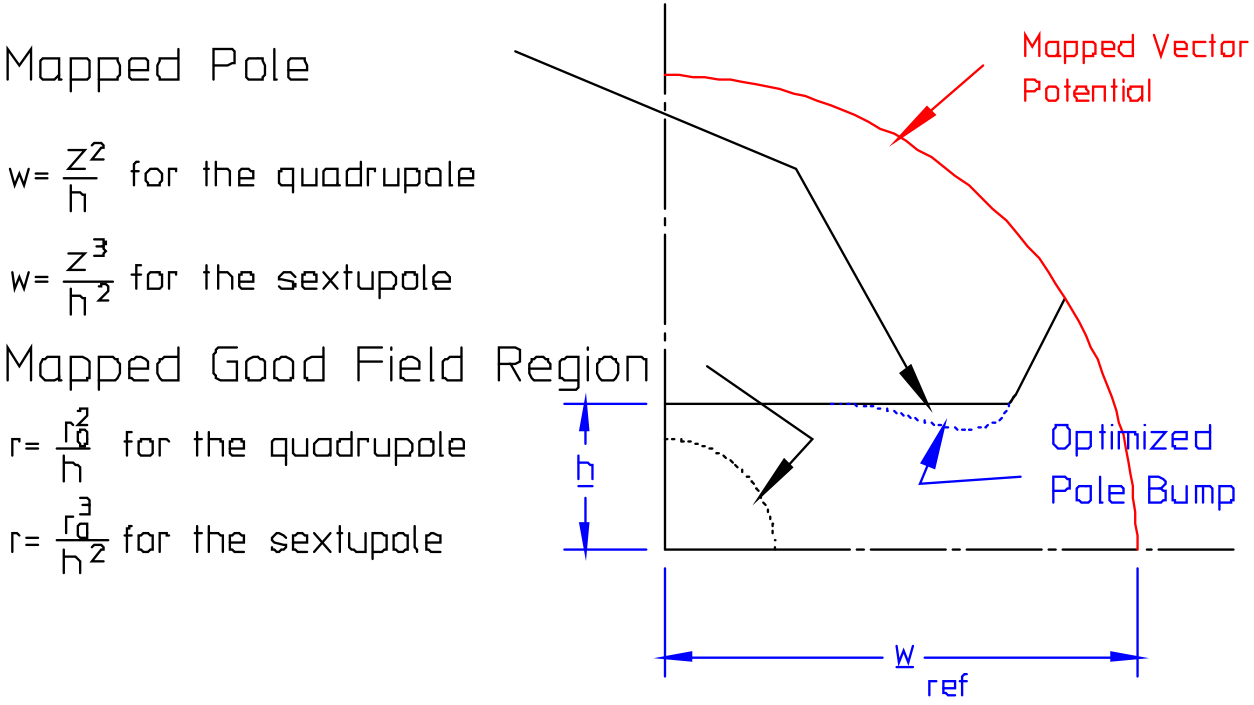
◆Expressions for the required excitation and practical current densities will be developed in a later lecture.

■Select a yoke geometry that will not saturate.

■Run POISSON (or other 2D code) in the quadrupole or sextupole space.

■From the solution, edit the vector potential values at a fixed reference radius.

- Map the vector potentials, the good field region and the pole contour.
- Design the pole bump such that the field in the mapped good field region satisfies the required uniformity.



- Map the optimized dipole pole contour back into the quadrupole (or sextupole) space.
- Reanalyze using POISSON (or other 2D code).

Closure

- The function, z^n, is important since it represents different field shapes. Moreover, by simple mathematics, this function can be manipulated by taking a root or by taking it to a higher power. The mathematics of manipulation allows for the mapping of one magnet type to another --- extending the knowledge of one magnet type to another magnet type.
- One can make a significant design effort optimizing one simple magnet type (the dipole) to the optimization of a much more difficult magnet type (the quadrupole and sextupole).
- The tools available in POISSON can be exploited to verify that the performance of the simple dipole can be reproduced in a higher order field.

Lecture 4

- Lecture 4 will cover the POISSON computer code. This session will be followed by a computer laboratory session where the lessons learned in the lecture can be applied.
- Chapter 6 should be read prior to the lecture.