Eigenvalues, Eigenvectors and the Similarity Transformation

Eigenvalues and the associated eigenvectors are 'special' properties of *square* matrices. While the eigenvalues parameterize the dynamical properties of the system (timescales, resonance properties, amplification factors, etc) the eigenvectors define the vector coordinates of the normal modes of the system. Each eigenvector is associated with a particular eigenvalue. The general state of the system can be expressed as a linear combination of eigenvectors. The beauty of eigenvectors is that (for square symmetric matrices) they can be made orthogonal (decoupled from one another). The normal modes can be handled independently and an orthogonal expansion of the system is possible.

The decoupling is also apparent in the ability of the eigenvectors to diagonalize the original matrix, A, with the eigenvalues lying on the diagonal of the new matrix, Λ. In analogy to the inertia tensor in mechanics, the eigenvectors form the principle axes of the solid object and a *similarity transformation* rotates the coordinate system into alignment with the principle axes. Motion along the principle axes is decoupled.

The matrix mechanics is closely related to the more general singular value decomposition. *We will use the basis sets of orthogonal eigenvectors generated by SVD for orbit control problems*. Here we develop eigenvector theory since it is more familiar to most readers.

Square matrices have an eigenvalue/eigenvector equation with solutions that are the eigenvectors x_λ and the associated eigenvalues λ:

$$Ax_\lambda = \lambda x_\lambda$$

The special property of an eigenvector is that it transforms into a scaled version of itself under the operation of A. Note that the eigenvector equation is *non-linear* in both the eigenvalue (λ) and the eigenvector (x_λ). The usual procedure is to first identify the eigenvalues and then find the associated eigenvectors. To solve for the eigenvalues, we write the equation in the form

$$(A-\lambda I)x_\lambda=0$$

The equation shows that the eigenvectors x_λ lie in the nullspace of $A-\lambda I$. From the theory of linear algebra, non-trivial solutions require that we choose λ so that matrix $A-\lambda I$ *has* a nullspace, that is, it must have determinant zero:

$$\det(A-\lambda I)=0.$$

The expression for the determinant should be familiar from linear algebra or from the theory of linear differential equations where an n^{th} order equation has been Laplace-transformed, broken up into series of first order equations and arranged in matrix form.

The *spectrum* of eigenvalues is found by solving for the roots of the characteristic polynomial or secular equation $\det(A-\lambda I)=0$. In general there will be as many eigenvalues as the rank of matrix A. Repeated eigenvalues indicate linear dependence within the rows and columns of A.

Once the eigenvalues are known, the associated eigenvectors are found by solving for x_λ in the eigenvector equation:

$$Ax_\lambda = \lambda x_\lambda$$

or

$$(A-\lambda I)x_\lambda=0$$

where λ is now a known quantity. Numerically, the eigenvectors are often found using elimination. MATLAB makes it easy

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A=randn(4,4);
A=A'*A;
[X,D]=eig(A)
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In terms of linear algebra, the eigenvectors span the nullspace of $A-\lambda I$. The dimension of the basis set is equal to the number of eigenvalues, which is equal to the rank of the original matrix A. More importantly the eigenvectors form an orthogonal set of vectors that can be used to expand the motion of the system. Each eigenvector is a normal mode of the system and acts independently. Solving for the eigenvalue/eigenvector pairs allows us to represent the system in terms of a linear superposition of normal modes.

LINEAR SUPERPOSITION

Analogous to Fourier series analysis which can decompose a waveform onto a set of sinusoidal basis functions, in this case the ‘basis functions’ are eigenvectors of the matrix A. To expand a vector in the vector subspace we form inner products. Let y be a vector in the subspace spanned by eigenvectors x_i.

$$y = \Sigma a_i x_i$$

The coefficients a_i are found by taking the inner product of both sides of the equation with each eigenvector one at a time. Assuming ortho-*normal* eigenvectors ($x_i^* x_j=\delta_{ij}$) yields

$$a_i = y^* x_i$$

The a_i coefficients are *projections* of the vector onto the coordinate axes of the eigenspace.

By linear superposition,

$$y = \Sigma a_i x_i = \Sigma(y^* x_i)x_i$$

The analogy to quantum mechanics is the expansion of a wavefunction ψ on the set of orthogonal basis functions u:

$$\psi(r) = \Sigma a_i u_i(r)$$
$$a_i = \int \psi(r) u_i^*(r) d^3 r$$
$$\psi(r) = \Sigma_i (\int \psi(r) u_i^*(r) d^3 r) u_i(r)$$

DYNAMICAL SYSTEMS

In dynamical problems (linear, time-invariant differential equations) we have a system of equations

$$\dot{x} = Ax$$

We can skirt the issue of Fourier/Laplace transformations by assuming exponential solutions of the form

$$x = e^{\lambda t} v$$
$$\dot{x} = \lambda e^{\lambda t} v = A e^{\lambda t} v$$

or

$$\lambda v = Av$$

which is the eigenvalue/eigenvector problem by definition. The eigenvalue/eigenvector pairs are orthogonal and the system evolves as

$$x = c_1 e^{\lambda_1 t} v^1 + ... + c_n e^{\lambda_n t} v^n$$

The coefficients $c_1 \ldots c_n$ are determined by taking inner products of both sides of the equation with eigenvectors v^i at time t=0. For driven systems, convolution integrals are required to develop the particular solution.

SIMILARITY TRANSFORMATIONS

Before leaving eigenvectors, lets examine how the matrix of eigenvectors leads to the diagonalization of matrix A leaving the eigenvalues of A on the diagonal. Assembling the eigenvectors column-wise into a matrix X, the eigenvector equations can be written

$$AX = X\Lambda$$

where Λ is a diagonal matrix with eigenvalues on the diagonal. Note that the matrix X is invertible because the columns are linearly independent. Pre-multiplying both sides by X^{-1} demonstrates the diagonalization of A with eigenvalues on the diagonal:

$$X^{-1}AX = \Lambda.$$

This is the similarity transformation that rotates the original coordinate system onto the eigenvector coordinate system leaving the eigenvalues on the diagonal of the new matrix, Λ. Decoupling of the eigenstates shows up as zero elements in the off-diagonal elements of Λ. As Cleve puts it, 'the eigenvalue decomposition is an attempt to find a similarity transformation to diagonal form'.

Similarly,

$$A=X\Lambda X^{-1}$$

which provides a first look at the more general form produced by singular value decomposition,

$$A=UWV^T.$$

We will see how SVD operates on the orbit response matrix R to produce two sets of *orthonormal* orbit- and corrector eigenvectors that will be used as expansion bases for the orbit and corrector column vectors. The main conceptual difference is that SVD works with rectangular and rank-deficient matrices and generates a separate eigenspace for the orbit and correctors. SVD also produces real, positive singular values (eigenvalues) that can be truncated to control properties of the solution. The key is still orthogonality of eigenvectors, decomposition into eigenvectors, and eigenvalue scaling.

MATLAB Example: Eigenvalues, Eigenvectors and Similarity Transformation

>>edit eig_1