Transverse Equilibrium Distribution Functions

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References

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§1 Vlasov Equilibria

Background Reading:
Reiser §5.3
pg 335–341

Write the Vlasov equation as:

$$\boxed{\frac{df}{ds} = \frac{\partial f}{\partial s} + \frac{d\vec{x}}{ds}\cdot\frac{\partial f}{\partial \vec{x}} + \frac{d\vec{x}'}{ds}\cdot\frac{\partial f}{\partial \vec{x}'} = 0}$$

$f = f(\vec{x}, \vec{x}', s)$ single-particle dist. func.

Single particle eqns of motion can be expressed in Hamiltonian form as:

$$\boxed{\begin{aligned} \frac{d\vec{x}}{ds} &= \frac{\partial H}{\partial \vec{x}'} \\ \frac{d\vec{x}'}{ds} &= \frac{-\partial H}{\partial \vec{x}} \end{aligned}}$$

// Example
Typical H for continuous Focusing:

$$H = \frac{1}{2}\vec{x}_\perp'^2 + k_{\beta 0}^2 \frac{\vec{x}_\perp^2}{2} + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2} \quad ; \quad k_{\beta 0}^2 = \text{const.}$$

– Other more complicated Hamiltonians for more detailed models. //

Using the Hamiltonian, express the Vlasov eqn as:

$$\boxed{\begin{aligned} \frac{df}{ds} &= \frac{\partial f}{\partial s} + \frac{\partial H}{\partial \vec{x}'}\cdot\frac{\partial f}{\partial \vec{x}} - \frac{\partial H}{\partial \vec{x}}\cdot\frac{\partial f}{\partial \vec{x}'} = 0 \\ &= \frac{\partial f}{\partial s} + \{H, f\} = 0 \end{aligned}}$$

$\{H, f\} \equiv$ Poisson Bracket.

Summary:

The system of equations describing the Vlasov evolution of the Ion beam is:

q, m = ion charge and mass.

β_b, γ_b = axial relativistic factors

$f(\vec{x}, \vec{x}', s)$ = single-particle distribution function.

$H(\vec{x}, \vec{x}', s)$ = single particle Hamiltonian

$$\frac{df}{ds} = \frac{\partial f}{\partial s} + \frac{\partial H}{\partial \vec{x}'} \cdot \frac{\partial f}{\partial \vec{x}} - \frac{\partial H}{\partial \vec{x}} \cdot \frac{\partial f}{\partial \vec{x}'} = 0$$

$$= \frac{\partial f}{\partial s} + \frac{d\vec{x}}{ds} \cdot \frac{\partial f}{\partial \vec{x}} + \frac{d\vec{x}'}{ds} \cdot \frac{\partial f}{\partial \vec{x}'} = 0$$

$$\frac{d\vec{x}}{ds} = \frac{\partial H}{\partial \vec{x}'}$$

$$\frac{d\vec{x}'}{ds} = -\frac{\partial H}{\partial \vec{x}}$$

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon_0} \quad ; \quad \rho = q \int d^3x' \, f(\vec{x}, \vec{x}', s)$$

+ Boundary conditions on ϕ.

This system is commonly used to describe the transverse evolution of a beam using the Hamiltonian! See previous lectures:

$$\vec{x}_\perp = \hat{x}x + \hat{y}y$$

$$H = H_\perp(\vec{x}_\perp, \vec{x}_\perp'; s) = \frac{1}{2}\vec{x}_\perp'^2 + \frac{\kappa_x(s)x^2}{2} + \frac{\kappa_y(s)y^2}{2} + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi$$

$\kappa_x(s)$ — x-focusing function of lattice

$\kappa_y(s)$ — y-focusing function of lattice

- Continuous Fousing:

$$\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \text{const.}$$

- Alternating Gradient (AG) quadurupole focusing

$$\kappa_x(s) = -\kappa_y(s) = \kappa_q(s)$$

- Solenoidal Focusing:

$$\kappa_x(s) = \kappa_y(s) = \kappa(s)$$

In this $\perp$ case distribution and Poisson equation become:

$$f = f_\perp(\vec{x}_\perp, \vec{x}_\perp'; s)$$

$$\nabla_\perp^2\phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -\frac{\rho}{\epsilon_0}$$

$$\rho = q\int d\vec{x}_\perp' \, f_\perp(\vec{x}_\perp, \vec{x}_\perp'; s)$$

+ boundary conditions on ϕ.

Equilibrium Conditions

Let C_i be a constant of the single particle motion. Then any function of the C_i will be a steady, equilibrium solution of Vlasov's equation with $d/ds = 0$.

$f(\{C_i\})$ equilibrium, C_i single-particle constants of motion.

$$\frac{d}{ds} f(\{C_i\}) = \sum_i \frac{\partial f}{\partial C_i} \cancelto{0}{\frac{dC_i}{ds}} = 0$$

// **Example:** Transverse distribution in a continuous focusing channel.

$$f = f(H_\perp) \quad ; \quad H_\perp = \frac{1}{2}\vec{x}_\perp'^{\,2} + \frac{k_{\beta 0}^2}{2}\vec{x}_\perp^{\,2} + \frac{q}{m\gamma_b^3\beta_b^2 c^2}\phi$$

$$k_{\beta 0} = \text{const.}$$

$$\Rightarrow \frac{\partial H_\perp}{\partial s} = 0 \qquad \text{No explicit } s\text{-dependance}$$

$$\frac{df}{ds} = \frac{\partial f}{\partial s} + \frac{\partial H}{\partial \vec{x}'}\cdot\frac{\partial f}{\partial \vec{x}} - \frac{\partial H}{\partial \vec{x}}\cdot\frac{\partial f}{\partial \vec{x}'}$$

$$= \frac{\partial f}{\partial H_\perp}\cancelto{0}{\frac{\partial H_\perp}{\partial s}} + \frac{\partial f}{\partial H_\perp}\cancelto{0}{\left(\frac{\partial H_\perp}{\partial \vec{x}'}\cdot\frac{\partial H_\perp}{\partial \vec{x}} - \frac{\partial H_\perp}{\partial \vec{x}}\cdot\frac{\partial H_\perp}{\partial \vec{x}'}\right)}$$

$$= 0$$

Any $f(H_\perp)$ taken in a continuous focusing channel will be a Vlasov equilibrium. //

Typical single particle constants:

* Transverse single particle Hamiltonian:

$$H_\perp = \frac{1}{2}\vec{x}_\perp'^2 + \frac{k_{\beta 0}^2}{2}\vec{x}_\perp^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi \quad = \text{const}$$

- Continuous focus systems with specific equilibrium distribution $f(H_\perp)$ used to calculate ϕ. Specific choices in $f(H_\perp)$ will yield different structure equilibria

* Axial kinetic energy:

$$\mathcal{E} = (\gamma - 1)mc^2 \quad = \text{const.}$$

- Systems with no axial forces, i.e., no acceleration, with an unbunched, coasting beam.

* Canonical angular momentum

$$P_\theta = xy' - yx' \quad = \text{const.}$$

- System with rotational symmetry about the longitudinal axis, e.g. continuous focusing with an azimuthally rotating beam and solenoidal focusing

// **Example:**
For a long, unbunched beam with a spread in axial velocities in a continuous focusing channel, valid equilibrium distributions can be of the form:

$$f = f_\perp(H_\perp)\, f_\parallel(\mathcal{E})$$ //

Discussion:

In plasma physics equilibria are central since system stability is often analyzed in terms of perturbations about an equilibrium motivated on physical grounds (i.e., local thermal equilibrium etc.). In the physics of intense beams this concept has proven to be a useful guide but is not as general in scope. In an accelerator, the beams are born (injected) off a source and propagate from an initial condition to the target. This initial condition will not, in general, be an equilibrium. Nevertheless, if the system can be tuned to more closely resemble a stable equilibrium during the cycle of operation, one expects less possible deliterious effects. Thus the equilibrium concept can prove useful.

Further caveats:

* For realistic focusing lattices with discrete lenses $\partial H/\partial s \neq 0$ and equilibrium structure may not exist outside of a transverse "KV" distribution that has pathological (unstable) structure.

* Higher levels of model detail render the equilibrium concept more intractable. In practice, it proves most useful in simple models for use as a general guide.

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§2 The KV Equilibrium Distribution

All we know how to solve analytically are linear equations of motion.
Previously we analyzed in detail Hill's Equations describing the ⊥ particle orbits in an applied linear focusing field:

$$\boxed{\begin{aligned} x'' + \kappa_x(s)\,x &= 0 \\ y'' + \kappa_y(s)\,y &= 0 \end{aligned}}$$

Neglecting space charge, the focusing forces κ_x, κ_y are due to linear applied fields:
Specific examples discussed in previous lectures:

- Continuous focusing

  $$\kappa_x = \kappa_y = k_{\beta 0}^2 = \text{const.}$$

- Alternating gradient quadrupole

  $$\kappa_x(s) = -\kappa_y(s) = \kappa_q(s)$$

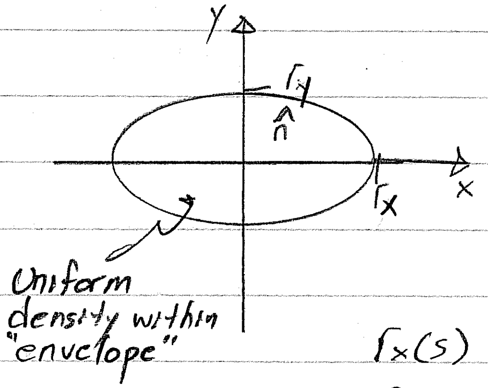
- Solenoidal focusing (after transformation)

  $$\kappa_x(s) = \kappa_y(s) = \kappa(s)$$

We now correct the equations of motion for the case where there are linear self-field corrections to the equations of motion due to beam self-fields.

- The self-consistent ⊥ distribution function that generates this situation is called the KV distribution after it's discoverers Kapchinskij and Vladimirskij.

Consider a uniform density beam with elliptical cross-section



Charge density:

$$\rho = q\int d^2x' f_\perp = \begin{cases} q\hat{n} & \frac{x^2}{r_x^2}+\frac{y^2}{r_y^2} < 1 \\ 0 & \frac{x^2}{r_x^2}+\frac{y^2}{r_y^2} > 1 \end{cases}$$

$n = \hat{n} =$ const. within beam slice (can evolve in s though)

$r_x(s)$ = x-radius of elliptical beam

$r_y(s)$ = y-radius of elliptical beam.

Charge conservation requires that:

$$\boxed{\lambda = q\,\hat{n}(s)\pi r_x(s) r_y(s) = \text{const.}}$$

Note: $\hat{n} \neq$ const if r_x, r_y evolve such that Area $= \pi r_x r_y \neq$ const

The 2-D Poisson equation is:

$$\boxed{\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = -\frac{\rho}{\varepsilon_0} = \begin{cases} \frac{-\lambda}{\pi\varepsilon_0 r_x r_y} & \frac{x^2}{r_x^2}+\frac{y^2}{r_y^2} < 1 \\ 0 & \frac{x^2}{r_x^2}+\frac{y^2}{r_y^2} > 1 \end{cases}}$$

In free-space, the solution interior to the beam ($x^2/r_x^2 + y^2/r_y^2 < 1$) that connects to an exterior solution with $\partial/\partial\theta = 0$ as $r \to \infty$ is given by (see Appendix A for proof — this is a nontrivial result contrary to comments in Reiser)

$$\boxed{\phi = \frac{-\lambda}{2\pi\varepsilon_0}\left\{\frac{x^2}{r_x(r_x+r_y)} + \frac{y^2}{r_y(r_x+r_y)}\right\} + \text{const.}}$$

Thus:

$$\boxed{\begin{aligned} -\frac{\partial\phi}{\partial x} &= \frac{\lambda}{\pi\varepsilon_0}\frac{x/r_x}{(r_x+r_y)} \propto x \quad \Rightarrow \text{linear in } x \\ -\frac{\partial\phi}{\partial y} &= \frac{\lambda}{\pi\varepsilon_0}\frac{y/r_y}{(r_x+r_y)} \propto y \quad \Rightarrow \text{linear in } y \end{aligned}}$$

continue ...

For the Hamiltonian:

$$\boxed{H_\perp = \frac{1}{2} x_\perp'^2 + \frac{\kappa_x(s)}{2} x^2 + \frac{\kappa_y(s)}{2} y^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi}$$

the particle equations of ~~of~~ motion for a particle moving within the beam are: (see construction on pg 4/)

$$x'' + \kappa_x(s)x - \frac{2Q\,x}{r_x(s)[r_x(s)+r_y(s)]} = 0$$

$$y'' + \kappa_y(s)y - \frac{2Q\,y}{r_y(s)[r_x(s)+r_y(s)]} = 0$$

$$Q \equiv \frac{q\lambda}{2\pi\varepsilon_0 m\gamma_b^3\beta_b^2c^2} = \text{const} \equiv \text{Perveance}$$

These are linear equations of motion (Hills' equations). If $r_x(s)$ and $r_y(s)$ are regarded as specified functions, then Floquet's Theorem can be applied, Courant-Snyder invariants can be identified, and phase-amplitude solutions can be constructed. However, a distribution constructed from the Courant-Snyder invariants must be found that generates the needed uniform density beam with an elliptical envelope to construct a fully self-consistent model. There is no gaurantee that such a model exists apri-ori.

We can now apply methods developed in previous lectures to analyze the equations of motion in the x- and y- planes!

Derivation of linear particle equations of motion with space charge from $H_\perp$:

$$H_\perp = \frac{1}{2}\vec{x}'^2 + \frac{\kappa_x(s)}{2}x^2 + \frac{\kappa_y(s)}{2}y^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2}$$

with:

$$\phi = \frac{-\lambda}{2\pi\epsilon_0}\left\{\frac{x^2}{r_x(s)[r_x(s)+r_y(s)]} + \frac{y^2}{r_y(s)[r_x(s)+r_y(s)]}\right\} + \text{const.}$$

dropping an overall constant we have

$$H_\perp(x,y,x',y';s) = \frac{1}{2}x'^2 + \left[\kappa_x(s) - \frac{2Q}{r_x(s)[r_x(s)+r_y(s)]}\right]\frac{x^2}{2} + \frac{1}{2}y'^2 + \left[\kappa_y(s) - \frac{2Q}{r_y(s)[r_x(s)+r_y(s)]}\right]\frac{y^2}{2}$$

where

$$Q = \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2c^2} = \text{const} \quad \text{is the perveance}$$

and we have:

$$\frac{d}{ds}x = \frac{\partial H}{\partial x'} = x'$$

$$\frac{d}{ds}y = \frac{\partial H}{\partial y'} = y'$$

$$\frac{d}{ds}x' = -\frac{\partial H}{\partial x} = -\left[\kappa_x(s) - \frac{2Q}{r_x(s)[r_x(s)+r_y(s)]}\right]x$$

$$\frac{d}{ds}y' = -\frac{\partial H}{\partial y} = -\left[\kappa_y(s) - \frac{2Q}{r_x(s)[r_x(s)+r_y(s)]}\right]y$$

$$\Rightarrow \quad x'' + \kappa_x(s)x - \frac{2Q\,x}{r_x(s)[r_x(s)+r_y(s)]} = 0$$

similiar for y

Phase amplitude solutions:

$$x(s) = A_x \cdot w_x(s) \cos\psi_x(s)$$
$$x'(s) = A_x w_x'(s) \cos\psi_x(s) - \frac{A_x}{w_x(s)} \sin\psi_x(s)$$

$$y(s) = A_y w_y(s) \cos\psi_y(s)$$
$$y'(s) = A_y w_y'(s) \cos\psi_y(s) - \frac{A_y}{w_y(s)} \sin\psi_y(s)$$

where:

$$w_x'' + \kappa_x(s) w_x - \frac{2Q}{r_x(r_x+r_y)} w_x - \frac{1}{w_x^3} = 0$$
$$\psi_x = \psi_{xi} + \int_{s_i}^{s} \frac{d\tilde{s}}{w_x^2(\tilde{s})}, \quad w_x > 0$$
$$w_y'' + \kappa_y(s) w_y - \frac{2Q}{r_y(r_x+r_y)} w_y - \frac{1}{w_y^3} = 0$$
$$\psi_y = \psi_{yi} + \int_{s_i}^{s} \frac{d\tilde{s}}{w_y^2(\tilde{s})}; \quad w_y > 0$$

and the Courant - Snyder invariants are:

$$\left(\frac{x}{w_x}\right)^2 + (w_x x' - w_x' x)^2 = A_x^2 = \text{const.}$$
$$\left(\frac{y}{w_y}\right)^2 + (w_y y' - w_y' y)^2 = A_y^2 = \text{const.}$$

As analyzed before, these equations define ellipses in phase space.

- Must construct distribution of invariants such as these giving uniform density for self-consistency of assumed self-field forms.
- The maximum particle amplitude $A_{x,max}$ and $A_{y,max}$ will define the phase-space extent of the beam.

Define:

$$\boxed{\begin{aligned} \varepsilon_x &= A_{x,max}^2 \\ \varepsilon_y &= A_{y,max}^2 \end{aligned}}$$

, emittances. – From max particle orbit in the distribution.

Then the beam-edge is given by: {see discussion of β-tron function in previous lectures}

$$\boxed{\begin{aligned} r_x(s) &= \sqrt{\varepsilon_x}\, w_x(s) \\ r_y(s) &= \sqrt{\varepsilon_y}\, w_y(s) \end{aligned}}$$

and the equations for the envelope radii are obtained from the equations for $w_x(s)$ and $w_y(s)$ as:

KV Envelope Equations

$$\boxed{\begin{aligned} r_x''(s) + \kappa_x(s) r_x(s) - \frac{2Q}{r_x(s)+r_y(s)} - \frac{\varepsilon_x^2}{r_x^3(s)} &= 0 \\ r_y''(s) + \kappa_y(s) r_y(s) - \frac{2Q}{r_x(s)+r_y(s)} - \frac{\varepsilon_y^2}{r_y^3(s)} &= 0 \end{aligned}}$$

These are the so-called RMS envelope equations. They may be thought of as consistency conditions on the beam envelope evolution of $r_x(s)$, $r_y(s)$ such that this linear space-charge model remains valid for all s.

- For the uniform density beam analyzed $\varepsilon_x = \text{const}$ and $\varepsilon_y = \text{const.}$ and the formulation is <u>exact</u>.

To complete the derivation, we must show that a distribution formed from the constants of motion yields the correct, consistent density projections.

Courant - Snyder Invariants

$$\left(\frac{x}{w_x}\right)^2 + (w_x x' - w_x' x)^2 = \text{const.} \quad ; \quad \text{Max value} = \varepsilon_x$$

$$\left(\frac{y}{w_y}\right)^2 + (w_y y' - w_y' y)^2 = \text{const.} \quad ; \quad \text{Max value} = \varepsilon_y$$

In Appendix B a proof is outlined to show that the inverted δ-function distribution (KV distribution)

$$\boxed{f_\perp(x,y,x',y',s) = \frac{\lambda}{q\pi^2 \varepsilon_x \varepsilon_y}\,\delta\left[\frac{(x/w_x)^2 + (w_x x' - w_x' x)^2}{\varepsilon_x} + \frac{(y/w_y)^2 + (w_y y' - w_y' y)^2}{\varepsilon_y} - 1\right]}$$

where:

$$\int_{-\infty}^{\infty} dx\, \delta(x) = 1 \quad ; \quad \delta(x) = 0 \quad x \neq 0.$$

is the Dirac-delta function, satisfies the required density projection:

$$\boxed{n(x,y) = \int dx'dy'\, f_\perp(x,y,x',y',s) = \begin{cases} \dfrac{\lambda}{q\pi r_x r_y} \, ; & (x/r_x)^2 + (y/r_y)^2 < 1 \\ 0 \, ; & (x/r_x)^2 + (y/r_y)^2 > 1 \end{cases}}$$

This result shows that the KV distribution is self-consistent. Note that the KV distribution is a singular, 4-dimensional hyper-surface in x, x', y, y' phase-space.

// Aside

For space charge physics, W_x and W_y are rarely used. Rather, we take

$$r_x = \sqrt{\varepsilon_x}\, W_x$$
$$r_y = \sqrt{\varepsilon_y}\, W_y$$

r_x and r_y are also related to averages over the distribution as:

$$r_x = 2\langle x^2 \rangle_\perp^{1/2}$$
$$r_y = 2\langle y^2 \rangle_\perp^{1/2}$$

where
$$\langle \cdots \rangle_\perp = \frac{\int d^2x\, d^2x' \cdots f_\perp}{\int d^2x\, d^2x'\, f_\perp}$$

$$\langle x^2 \rangle_\perp = \frac{\int d^2x\; x^2 n}{\int d^2x\; n} \; ; \quad n = \int d^2x'\, f_\perp$$

Using r_x and r_y in place of W_x and W_y, the KV distribution $f_\perp$ can be expressed as:

$$\boxed{f_\perp(x,y,x',y';s) = \frac{\lambda}{q\pi^2\varepsilon_x\varepsilon_y}\,\delta\left[\left(\frac{x}{r_x}\right)^2 + \left(\frac{r_x x' - r_x' x}{\varepsilon_x}\right)^2 + \left(\frac{y}{r_y}\right)^2 + \left(\frac{r_y y' - r_y' y}{\varepsilon_y}\right)^2 - 1\right]}$$

To save writing we will sometimes denote restricted phase-space averages over angle variables x' and y' as:

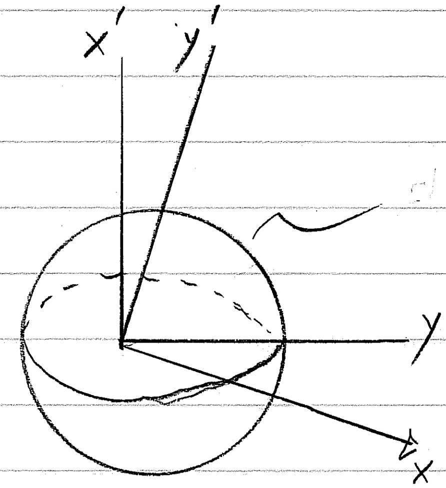
$$\boxed{\langle \cdots\cdots \rangle_{x',y'} \equiv \frac{\int d^2x' \;\cdots\cdots\; f_\perp}{\int d^2x' f_\perp}}$$

For any function $g(x,y)$ that excludes angle variables:

$$\boxed{\langle g(x,y) \rangle_\perp = \frac{\int d^2x \; g(x,y)\, n(x,y)}{\int d^2x \; n(x,y)}}$$

//

Schematically:



hyper-surface with ∞ density on shell will find that singularity provides free energy to drive high-order unstable modes.

Hard to believe that any reasonable source could produce such a structure to inject in an accelerator! However, we will find that low-order phase-space projections of the distribution have reasonable structure. Moreover, the KV envelope equation is well established as a reliable an invaluable design tool for accelerators with high space-charge intensity.

The previous analysis applies to any s-dependence in the focusing functions $\kappa_x(s)$ and $\kappa_y(s)$. We now analyze further the case of a periodic lattice with:

$$\boxed{\begin{aligned} \kappa_x(s+L_p) &= \kappa_x(s) \\ \kappa_y(s+L_p) &= \kappa_y(s) \end{aligned}}$$

For the case of a <u>matched</u> beam with the periodicity of the lattice:

$$\boxed{\begin{aligned} r_x(s+L_p) &= r_x(s) \\ r_y(s+L_p) &= r_y(s) \end{aligned}}$$

we may define depressed and undepressed phase advances of single particle oscillations using previous lecture results

<u>Undepressed Oscillations</u> (no space-charge, $Q \rightarrow 0$)

$$\sigma_{0x} = \psi_{0x}(s_i+L_p) - \psi_{0x}(s_i) = \int_{s_i}^{s_i+L_p} \frac{ds}{w_{0x}^2(s)} \quad ; \quad w_{0x}'' + \kappa_x w_{0x} - \frac{1}{w_{0x}^3} = 0$$

$$w_{0x}(s+L_p) = w_{0x}(s)$$
$$w_{0x} > 0$$

$$\sigma_{0y} = \psi_{0y}(s_i+L_p) - \psi_{0y}(s_i) = \int_{s_i}^{s_i+L_p} \frac{ds}{w_{0y}^2(s)} \quad ; \quad w_{0y}'' + \kappa_y w_{0y} - \frac{1}{w_{0y}^3} = 0$$

$$w_{0y}(s+L_p) = w_{0y}(s)$$
$$w_{0y} > 0$$

<u>Depressed Oscillations</u> (with space-charge)

$$\sigma_x = \psi_x(s_i+L_p) - \psi_x(s_i) = \varepsilon_x \int_{s_i}^{s_i+L_p} \frac{ds}{r_x^2(s)}$$

$$\sigma_y = \psi_y(s_i+L_p) - \psi_y(s_i) = \varepsilon_y \int_{s_i}^{s_i+L_p} \frac{ds}{r_y^2(s)}$$

$$r_x'' + \kappa_x r_x - \frac{2Q}{r_x+r_y} - \frac{\varepsilon_x^2}{r_x^3} = 0 \qquad r_x(s+L_p) = r_x(s), \quad r_x > 0$$

$$r_y'' + \kappa_y r_y - \frac{2Q}{r_x+r_y} - \frac{\varepsilon_y^2}{r_y^3} = 0 \qquad r_y(s+L_p) = r_x(s), \quad r_y > 0.$$

Note that <u>all particles have the same depressed oscillation frequency in a KV beam.</u>

- when instability exists will be extreme since all particles will be resonant.

Moments of the KV distribution

Some useful moments that can be calculated directly from the KV distribution are

$$\int d^2x'\, x' f_\perp = r_x' \frac{x}{r_x} n$$

$$\int d^2x'\, y' f_\perp = r_y' \frac{y}{r_y} n$$

$$\int d^2x'\, x'^2 f_\perp = \left[\frac{\varepsilon_x^2}{2r_x^2}\left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right) + r_x'^2 \frac{x^2}{r_x^2} \right] n$$

$$\int d^2x'\, y'^2 f_\perp = \left[\frac{\varepsilon_y^2}{2r_y^2}\left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right) + r_y'^2 \frac{y^2}{r_y^2} \right] n$$

$$\int d^2x'\, xx' f_\perp = \frac{r_x'}{r_x} x^2 n$$

$$\int d^2x'\, yy' f_\perp = \frac{r_y'}{r_y} y^2 n$$

See: Lund and Bukh, to appear PRSTAB, 2004, Appendix A

where: $n = \begin{cases} \dfrac{\lambda}{q\pi r_x r_y} & \dfrac{x^2}{r_x^2} + \dfrac{y^2}{r_y^2} \le 1 \\ 0 & \text{otherwise} \end{cases}$

All 2nd order terms not listed vanish, e.g.

$$\int d^2x'\, xy' f_\perp = 0.$$

Thus:

$$\int d^2x' (xy' - yx') f_\perp = 0 \Rightarrow$$ Angular momentum of distribution is zero.

Also,

$$\boxed{\begin{aligned}
\langle x^2 \rangle_\perp &= \frac{r_x^2}{4} \\
\langle y^2 \rangle_\perp &= \frac{r_y^2}{4} \\
\langle x'^2 \rangle_\perp &= \frac{r_x'^2}{4} + \frac{\varepsilon_x^2}{4 r_x^2} \\
\langle y'^2 \rangle_\perp &= \frac{r_y'^2}{4} + \frac{\varepsilon_y^2}{4 r_y^2} \\
\langle x x' \rangle_\perp &= \frac{r_x r_x'}{4} \\
\langle y y' \rangle_\perp &= \frac{r_y r_y'}{4}
\end{aligned}}$$

Again, all 2nd order terms not listed vanish, e.g.,

$$\langle x y' \rangle_\perp = 0$$

Note from these results that:

$$\varepsilon_x^2 = 16 \left[\langle x^2 \rangle_\perp \langle x'^2 \rangle_\perp - \langle x x' \rangle_\perp^2 \right]$$
$$\varepsilon_y^2 = 16 \left[\langle y^2 \rangle_\perp \langle y'^2 \rangle_\perp - \langle y y' \rangle_\perp^2 \right]$$

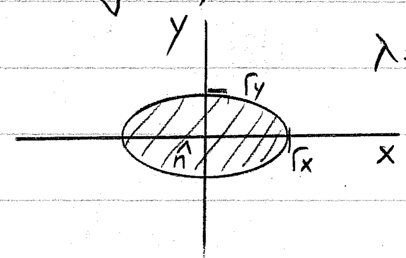
Properties of the KV Distribution

1) The KV distribution is a singular hyper-shell in 4D phase-space:

$$\boxed{f_\perp(x,y,x',y',s) = \frac{\lambda}{q\pi^2\varepsilon_x\varepsilon_y}\,\delta\left[\left(\frac{x}{r_x}\right)^2 + \left(\frac{r_x x' - r_x' x}{\varepsilon_x}\right)^2 + \left(\frac{y}{r_y}\right)^2 + \left(\frac{r_y y' - r_y' y}{\varepsilon_y}\right)^2 - 1\right]}$$

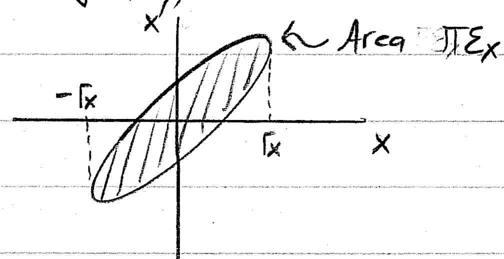
All 2D projections of the KV distribution are uniformly filled and elliptical in shape.

$\int dx'dy' f_\perp$

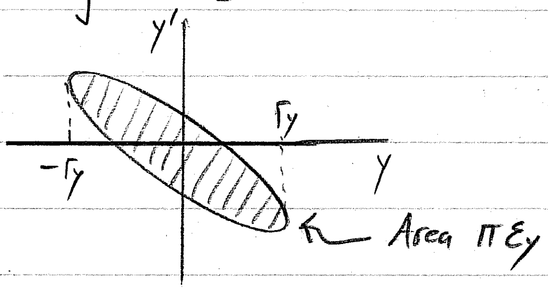


$\lambda = q\hat{n}\pi r_x r_y$

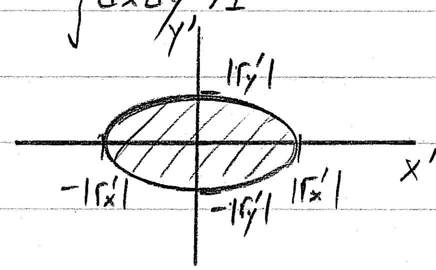
$\int dy'dy\, f_\perp$



$\int dx dx' f_\perp$



$\int dx dy\, f_\perp$



2) Statistical RMS measures are simply related to the edge measures

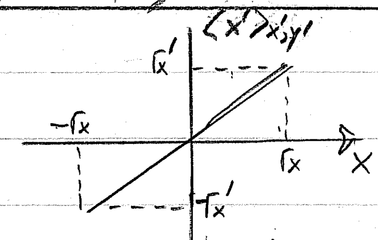
$$\langle \cdots \rangle_\perp \equiv \int dx \int dy \int dx' \int dy' \cdots f_\perp \Big/ \left(\int dx \int dy \int dx' \int dy'\, f_\perp\right)$$

$$\boxed{\begin{aligned} r_x &= 2\langle x^2\rangle_\perp^{1/2} & r_x' &= 2\langle xx'\rangle_\perp / \langle x^2\rangle_\perp^{1/2} \\ r_y &= 2\langle y^2\rangle_\perp^{1/2} & r_y' &= 2\langle yy'\rangle_\perp / \langle y^2\rangle_\perp^{1/2} \\ \varepsilon_x &= 4\left[\langle x^2\rangle_\perp \langle x'^2\rangle_\perp - \langle xx'\rangle_\perp^2\right]^{1/2} \\ \varepsilon_y &= 4\left[\langle y^2\rangle_\perp \langle y'^2\rangle_\perp - \langle yy'\rangle_\perp^2\right]^{1/2} \end{aligned}}$$

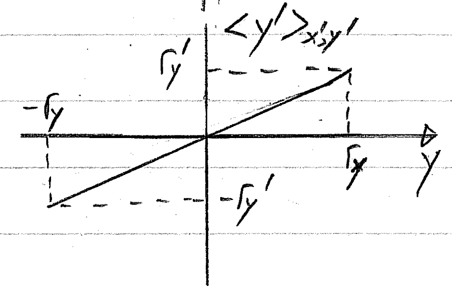
3) The KV distribution has coherent "velocity" spread (flow) associated with the envelope flutter motion. This is a fluid "flow" resulting from periodic focusing.

Within the beam: $\left(x^2/r_x^2 + y^2/r_y^2 < 1 \right)$

$$\langle x' \rangle_{x,y'} = \frac{\int dx' \int dy' \; x' f_\perp}{\int dx' \int dy' \; f_\perp} = r_x' \frac{x}{r_x}$$



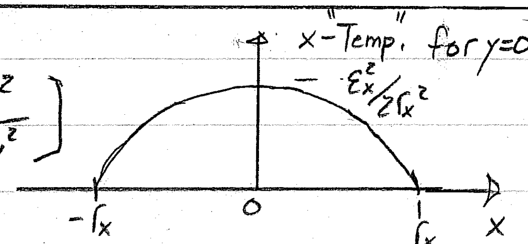
$$\langle y' \rangle_{x,y'} = \frac{\int dx' \int dy' \; y' f_\perp}{\int dx' \int dy' \; f_\perp} = r_y' \frac{y}{r_y}$$



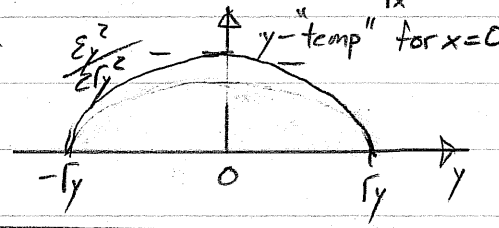
4) The KV distribution has incoherent "velocity" spread (temperature) that falls parabolically to zero at the outer edge of the beam:

Within the beam: $\left(\frac{x}{r_x}\right)^2 + \left(\frac{y}{r_y}\right)^2 < 1$

$$\langle (x' - r_x' x/r_x)^2 \rangle_{x,y} = \frac{\varepsilon_x^2}{2 r_x^2} \left[1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2} \right]$$



$$\langle (y' - r_y' y/r_y)^2 \rangle_{x,y} = \frac{\varepsilon_y^2}{2 r_y^2} \left[1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2} \right]$$



Note: (Heuristic)

$$P_x \sim \hat{n} T_x \qquad x\text{-pressure}$$

$$\frac{\partial}{\partial x} P_x \propto x \qquad x\text{-"thermal" force is linear in } x.$$

5) For a periodic lattice, the depressed phase advances

$$\boxed{\sigma_x = \varepsilon_x \int_{s_i}^{s_i+L_p} \frac{ds}{r_x^2(s)} \quad ; \quad \sigma_y = \varepsilon_y \int_{s_i}^{s_i+L_p} \frac{ds}{r_y^2(s)}}$$

become smaller for stronger space charge (Perveance Q larger) and in the limit of zero space-charge:

$$\boxed{\begin{array}{ll} \lim_{Q\to 0} \sigma_x = \sigma_{x0} & ; \quad \lim_{Q\to 0} \sigma_y = \sigma_{y0} \\ \sigma_x < \sigma_{x0} & \quad\quad \sigma_y < \sigma_{y0} \end{array}}$$

At the current limit of the transport channel ($Q \to Q_{max}$), the space-charge forces will cancel on-average the applied focusing forces along the axis with weaker focusing strength

$$\boxed{\lim_{Q\to Q_{max}^-} \mathrm{Min}[\sigma_x, \sigma_y] = 0}$$

This suggests the convenient normalized measures of space-charge strength ("tune depression")

$$\boxed{\begin{array}{c} 0 \le \dfrac{\sigma_x}{\sigma_{x0}} \le 1 \\ \\ 0 \le \dfrac{\sigma_y}{\sigma_{y0}} \le 1 \end{array}}$$

Transportable Space-charge limit $Q \to Q_{max}$

Zero space-charge limit $Q \to 0$

6) All particles in the KV distribution have the same undepressed (σ_{0x}, σ_{0y}) and depressed (σ_x, σ_y) oscillation "frequencies."

- If there is instability, all particles in the distribution can resonate, so the KV distribution can be violently unstable.
- The inverted population δ-function distribution has large free-energy to drive collective instabilities.

7) In spite of the pathologies of the KV distribution:

- Inverted, singular form in phase-space.
- Violently unstable to many perturbations.
- All particles oscillate with the same characteristic frequencies.
- Sharp edges in phase-space projections.

The KV envelope equation has in practice been a good approximation to the low-order evolution of more realistic non-equilibrium distribution functions with non-uniform space-charge forces when the envelope equation parameters are taken in an RMS equivalent beam sense:

See Reiser § 3.3.4 pg 358 - 364

KV Envelope	RMS Equivalent Beam
r_x = x - beam edge radius	$r_x \equiv 2\langle x^2\rangle_\perp^{1/2}$
r_y = y - beam edge radius	$r_y \equiv 2\langle y^2\rangle_\perp^{1/2}$
ε_x = x - edge emittance	$\varepsilon_x \equiv 4\left[\langle x^2\rangle_\perp\langle x'^2\rangle_\perp - \langle xx'\rangle_\perp^2\right]^{1/2}$
ε_y = y - edge emittance	$\varepsilon_y \equiv 4\left[\langle y^2\rangle_\perp\langle y'^2\rangle_\perp - \langle yy'\rangle_\perp^2\right]^{1/2}$
Q = perveance	$Q = q\lambda/[2\pi\varepsilon_0\gamma_b^3\beta_b^2c^2]$
κ_x = x - lattice focusing function	κ_x
κ_y = y - lattice focusing function	κ_y
σ_x, σ_y	σ_x, σ_y calculated from KV formulae

$$\langle \cdots \rangle_\perp = \frac{\int dx\,dy\,dx'\,dy' \cdots f_\perp}{\int dx\,dy\,dx'\,dy'\, f_\perp}$$

$f_\perp$ not KV

- KV envelope equation applied with this equivalence and $\varepsilon_x = \text{const}$, $\varepsilon_y = \text{const}$ to design transport lattices
- Higher order evolution is evaluated with theory and simulations to verify designs.

8) The KV distribution is the <u>only</u> known equilibrium distribution for lattices with s-varying focusing

$$\kappa_x(s) \neq \text{const}$$
$$\kappa_y(s) \neq \text{const}$$

and finite perveance $(Q \neq 0)$.

- Some approximate theories exist for distributions with nonlinear self-field forces, but these theories may have problems. may prove inaccurate.
- It is not clear if any solution even exists for nonlinear space-charge forces. Reliable transport may not even require such solutions to exist.
  - Construction of exact single particle invariants with nonlinear space-charge forces and s-varying focusing forces is likely to be very difficult ..... in 45 years nobody has done it!

Research Problem?

> Construct a valid Vlasov Equilibrium for a smooth, nonuniform density distribution in a periodic focusing channel with physical (low-order, hard-edge ok) applied focusing forces $\kappa_x(s)$ and $\kappa_y(s)$ and you will become famous in the physics of high space-charge intensity beams!

9) A hint of the low-order generality of the KV envelope equation beyond applying to just uniform beams can be found in the intro. lectures by J.J. Barnard where it was demonstrated [see F.J. Sacherer, IEEE Trans. Nucl. Sci. 18, 1101 (1971).] that for any beam charge density with elliptic symmetry:

$$\rho = \rho\left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} \right)$$

that

$$r_x'' - k_x r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} = 0$$

$$r_y'' - k_y r_y - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_y^2}{r_y^3} = 0$$

remains valid with

$$Q = \frac{q\lambda}{2\pi\epsilon_0 \gamma_b^3 \beta_b^2 c^2} = \text{const} \quad ; \quad \lambda = \int d^2x\, \rho = \text{const.}$$

and

$$r_x = 2\langle x^2 \rangle_\perp^{1/2}$$
$$r_y = 2\langle y^2 \rangle_\perp^{1/2}$$

is defined in terms of the density ρ:

$$\langle x^2 \rangle = \frac{\int d^2x\, x^2 \rho}{\int d^2x\, \rho} \quad \text{etc.}$$

These statements hold true provided that consistent emittance evolutions are employed for $\varepsilon_x(s)$ and $\varepsilon_y(s)$. Only for the uniform density KV beam do we have:

$$\varepsilon_x(s) = \text{const.}$$
$$\varepsilon_y(s) = \text{const.}$$

A subtle point is that the functional form in $\rho = \rho\left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2}\right)$ can evolve in s provided that elliptical symmetry is maintained in each slice in s.

Research Problem:

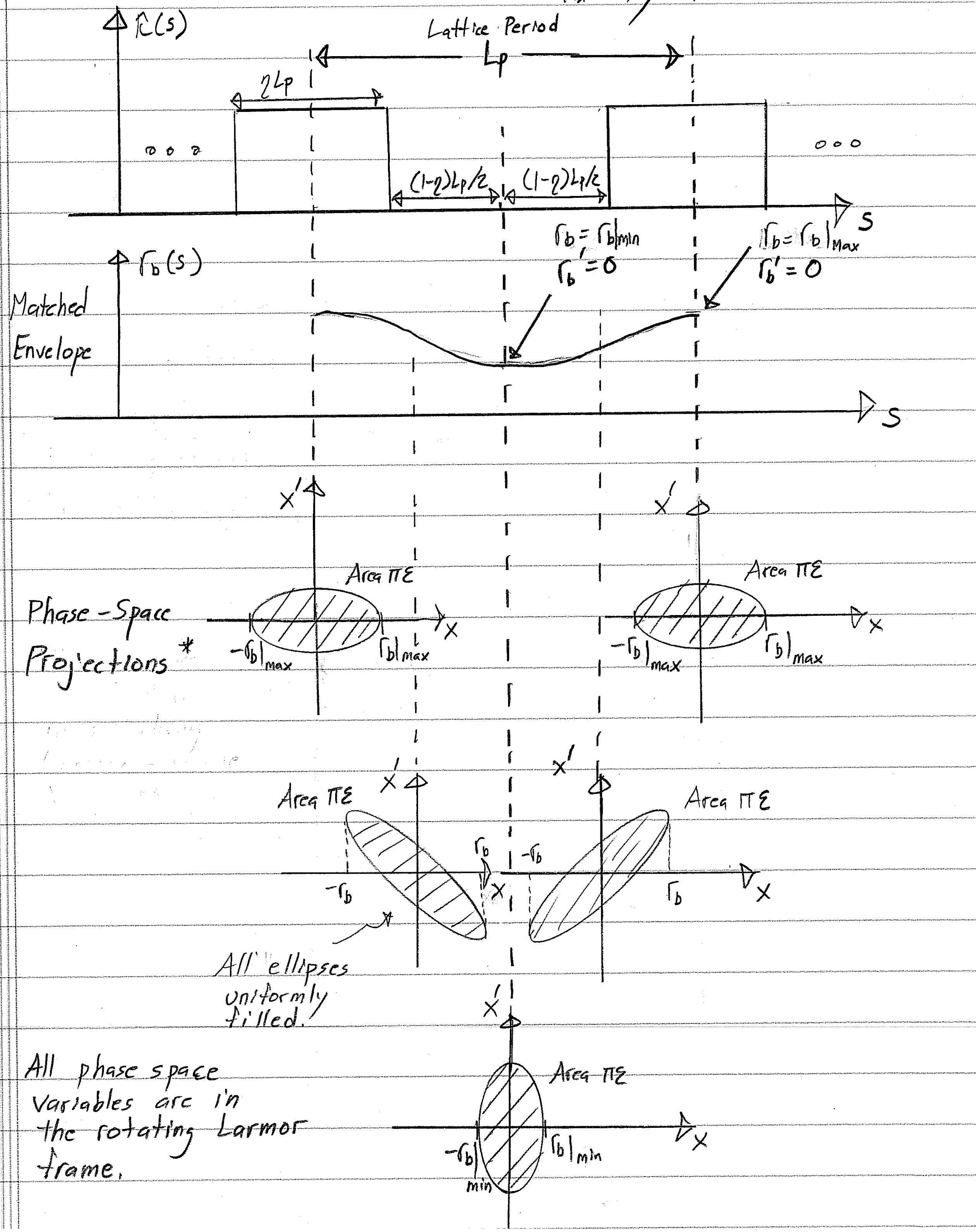
$\boxed{\text{An interesting problem (should be easier than the full self-consistency problem) is to prove whether Vlasov evolutions with initial elliptic symmetry in } \rho \text{ will maintain elliptic symmetry for all } s \text{ as the beam evolves from an initial condition. This does not require an equilibrium (weaker statement) if true.}}$

10) The KV distribution only "evolves" in the sense that lower dimensional projections evolve in s. $f_\perp$ itself is constructed from Courant-Snyder invariants of the motion in linear applied and self-field forces.
- Consider the examples on the next two pages to see this.

Example - KV Equilibrium in a Periodic Solenoidal Transport Channel

Axisymmetric Beam with $\varepsilon_x = \varepsilon_y = \varepsilon$ $\Rightarrow r_x = r_y \equiv r_b$

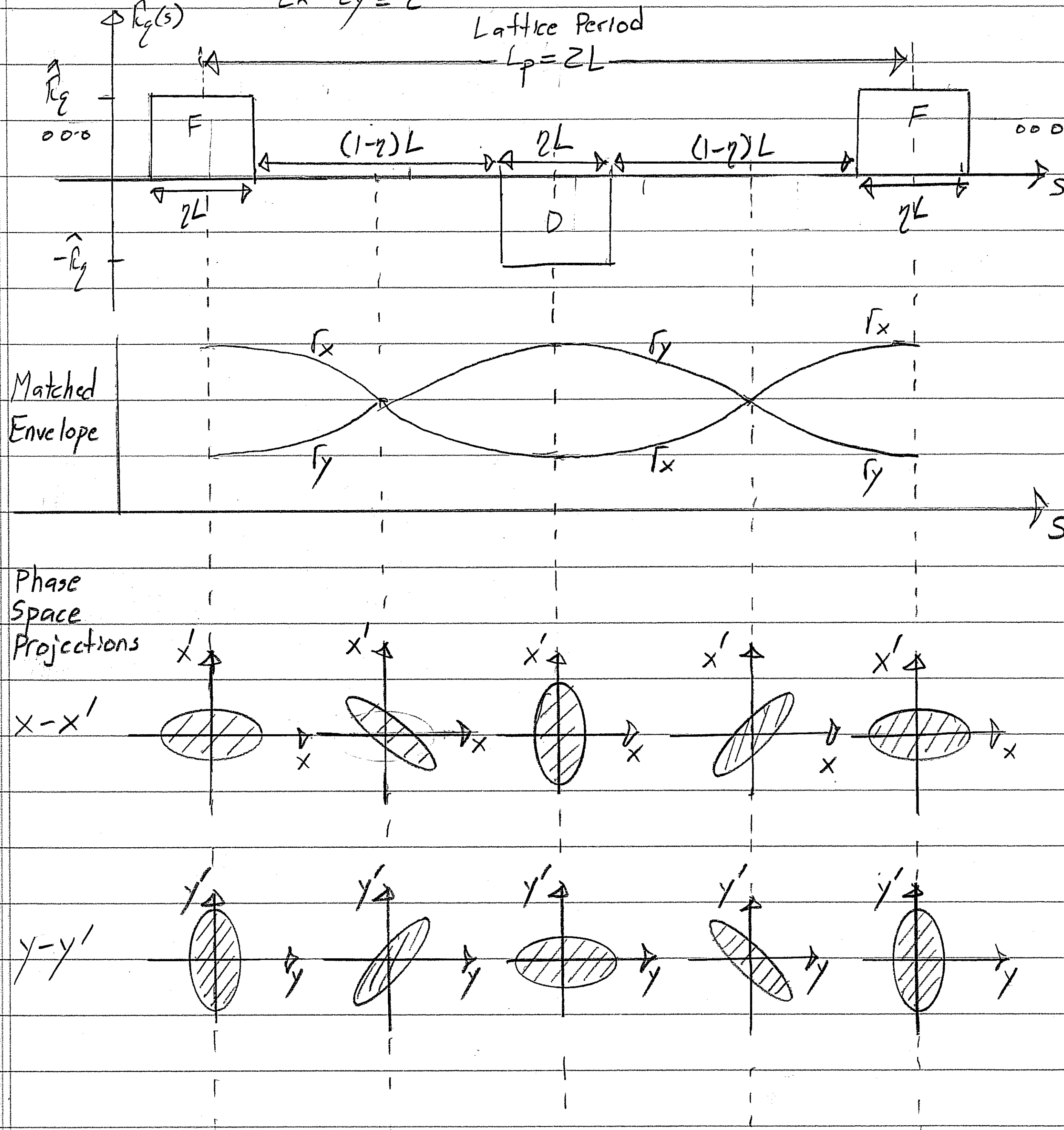
$\kappa_x = \kappa_y = \kappa$



* All phase space variables are in the rotating Larmor frame.

Example - KV Equilibrium in a FODO Periodic Quadrupole Transport Channel

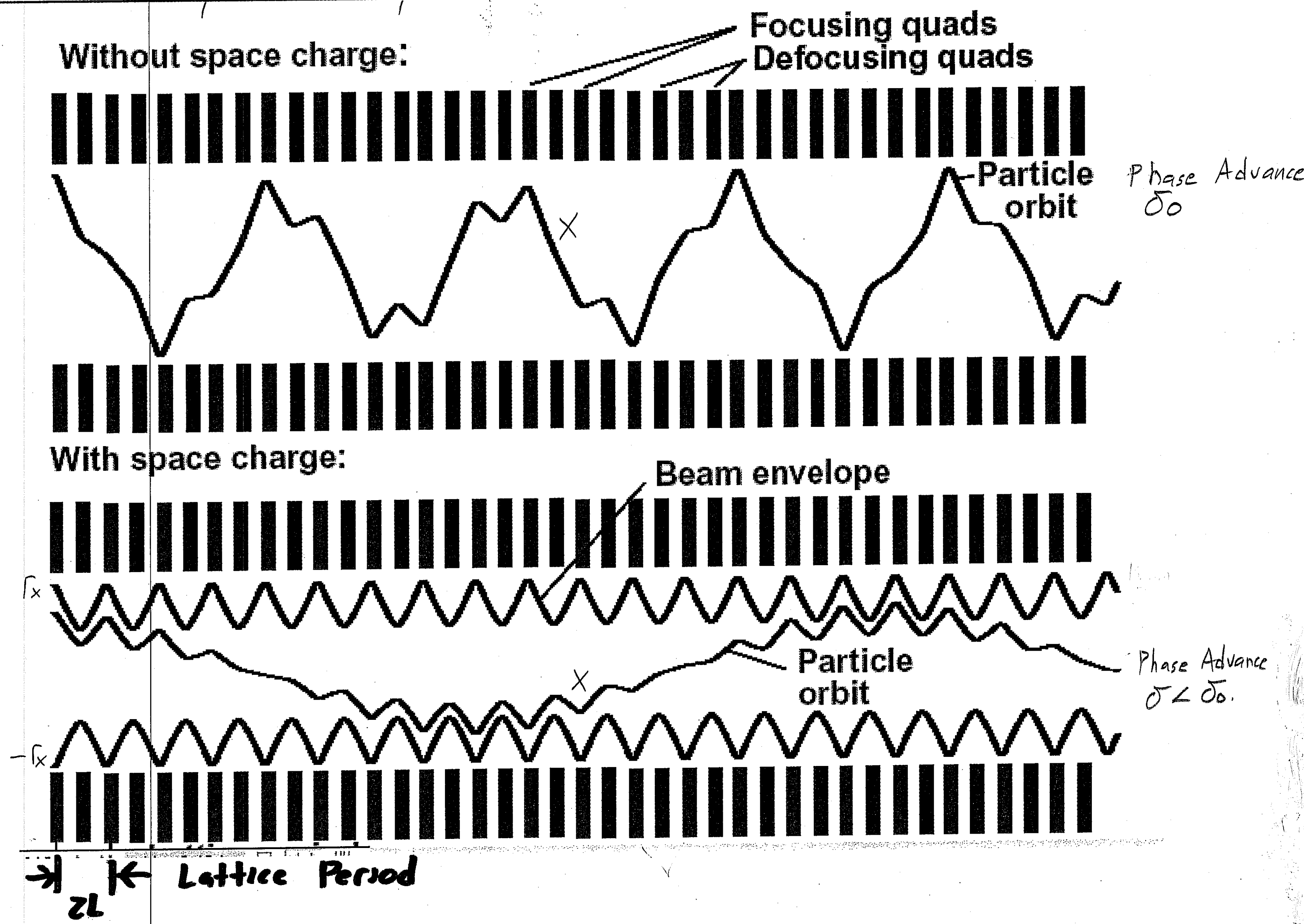
$\varepsilon_x = \varepsilon_y \equiv \varepsilon$



All ellipses have the same area $\pi\varepsilon$

Space charge reduces betatron phase advance

FODO Quadrupole Transport Channel



S.M. Lund 21/

J. J. Barnard and S. M. Lund
USPAS Jan 2004

Appendix A

$$\boxed{\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = \begin{cases} \dfrac{-\lambda}{\pi\varepsilon_0 r_x r_y}\,; & \dfrac{x^2}{r_x^2}+\dfrac{y^2}{r_y^2}<1 \\ 0\,; & \dfrac{x^2}{r_x^2}+\dfrac{y^2}{r_y^2}>1 \end{cases} \qquad \frac{\partial\phi}{\partial r}\sim\frac{\lambda}{2\pi\varepsilon_0 r}\ \text{as}\ r\to\infty.}$$

The solution to this system to an arb. constant has been formally constructed by Landau & Lifshitz and others (gravitational field analog) as:

$$\boxed{\phi = \frac{-q\lambda}{4\pi\varepsilon_0}\left\{\int_0^{\xi}\frac{ds}{\left[(r_x^2+s)(r_y^2+s)\right]^{1/2}}+\int_{\xi}^{\infty}\frac{ds}{\left[(r_x^2+s)(r_y^2+s)\right]^{1/2}}\left(\frac{x^2}{r_x^2+s}+\frac{y^2}{r_y^2+s}\right)\right\} + \text{const.}}$$

where

$$\begin{cases} \xi = 0\,; & \text{when } (x/r_x)^2+(y/r_y)^2<1 \\ \xi:\ \text{root of } \dfrac{x^2}{r_x^2+\xi}+\dfrac{y^2}{r_y^2+\xi}=1\,, & \text{when } (x/r_x)^2+(y/r_y)^2>1 \end{cases}$$

Trivially for $x=y=0$

$$\phi(x=y=0) = \text{const.}$$

Calculate:

$$\frac{\partial\phi}{\partial x} = \frac{-q\lambda}{4\pi\varepsilon_0}\left\{\int_{\xi}^{\infty}\frac{ds}{\left[(r_x^2+s)(r_y^2+s)\right]^{1/2}}\frac{2x}{r_x^2+s} - \frac{1}{\left[(r_x^2+\xi)(r_y^2+\xi)\right]^{1/2}}\left[1-\frac{x^2}{r_x^2+\xi}-\frac{y^2}{r_y^2+\xi}\right]\frac{\partial\xi}{\partial x}\right\}$$

$$\left.\begin{array}{l} \text{If } \xi\neq 0 \ \Rightarrow\ 1=\dfrac{x^2}{r_x^2+\xi}+\dfrac{y^2}{r_y^2+\xi} \\ \xi=0 \ \Rightarrow\ \dfrac{\partial\xi}{\partial x}=0 \end{array}\right\} \Rightarrow \text{2nd term vanishes}$$

$$\boxed{\begin{aligned} \frac{\partial \phi}{\partial x} &= -\frac{\lambda}{2\pi\epsilon_0}\int_\xi^\infty \frac{ds}{[(r_x^2+s)(r_y^2+s)]^{1/2}}\,\frac{x}{r_x^2+s} \\ &\text{by symmetry} \\ \frac{\partial \phi}{\partial y} &= -\frac{\lambda}{2\pi\epsilon_0}\int_\xi^\infty \frac{ds}{[(r_x^2+s)(r_y^2+s)]^{1/2}}\,\frac{y}{r_y^2+s} \end{aligned}}$$

Differentiating again and using the chain rule:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = -\frac{\lambda}{2\pi\epsilon_0}\left\{ \int_\xi^\infty \frac{ds}{[(r_x^2+s)(r_y^2+s)]^{1/2}}\left[\frac{1}{r_x^2+s}+\frac{1}{r_y^2+s}\right] - \frac{1}{[(r_x^2+\xi)(r_y^2+\xi)]^{1/2}}\left[\frac{x\,\partial\xi/\partial x}{r_x^2+\xi}+\frac{y\,\partial\xi/\partial y}{r_y^2+\xi}\right]\right\}$$

Must show that the r.h.s. reduces to the needed forms for:

<u>case 1</u> exterior ξ satisfies: $\dfrac{x^2}{r_x^2+\xi}+\dfrac{y^2}{r_y^2+\xi}=1$

<u>case 2</u> interior $\xi = 0$

<u>case 1</u> (exterior: $x^2/r_x^2 + y^2/r_y^2 > 1$)

Differentiate $\dfrac{x^2}{r_x^2+\xi}+\dfrac{y^2}{r_y^2+\xi}=1$

$$\Rightarrow \quad \frac{\partial\xi}{\partial x} = \frac{2x}{(r_x^2+\xi)\left[\frac{x^2}{(r_x^2+\xi)^2}+\frac{y^2}{(r_y^2+\xi)^2}\right]}$$

$$\frac{\partial\xi}{\partial y} = \frac{2y}{(r_x^2+\xi)\left[\frac{x^2}{(r_x^2+\xi)^2}+\frac{y^2}{(r_y^2+\xi)^2}\right]}$$

$$\Rightarrow \frac{x\,\partial\xi/\partial x}{r_x^2+\xi}+\frac{y\,\partial\xi/\partial y}{r_y^2+\xi} = 2\left[\frac{x^2}{(r_x^2+\xi)^2}+\frac{y^2}{(r_y^2+\xi)^2}\right]\left[\frac{1}{\frac{x^2}{(r_x^2+\xi)^2}+\frac{y^2}{(r_y^2+\xi)^2}}\right]=2$$

Also need integrals like: $\quad w^2 \equiv s + r_y^2$

$$I_x(\xi) \equiv \int_\xi^\infty \frac{ds}{[(r_x^2+s)(r_y^2+s)]^{1/2}}\,\frac{1}{r_x^2+s} = 2\int_{\sqrt{r_x^2+\xi}}^\infty \frac{dw}{(r_x^2-r_y^2+w^2)^{3/2}}$$

This integral can be done using tables:

$$I_x(\xi) = \left.\frac{2w}{(r_x^2-r_y^2)\sqrt{r_x^2-r_y^2+w^2}}\right|_{w=\sqrt{r_x^2+\xi}}^{w\to\infty} = \frac{2}{r_x^2-r_y^2} - \frac{2\sqrt{r_y^2+\xi}}{(r_x^2-r_y^2)\sqrt{r_x^2+\xi}}$$

Similarly:

$$I_y(\xi) \equiv \int_\xi^\infty \frac{ds}{[(r_x^2+s)(r_y^2+s)]^{1/2}}\frac{1}{(r_y^2+s)} = \frac{2}{r_y^2-r_x^2} - \frac{2\sqrt{r_x^2+\xi}}{(r_y^2-r_x^2)\sqrt{r_y^2+\xi}}$$

$$\int_0^\infty \frac{ds}{[(r_x^2+s)(r_y^2+s)]^{1/2}}\left[\frac{1}{r_x^2+s}+\frac{1}{r_y^2+s}\right] = I_x(\xi)+I_y(\xi)$$

$$= \frac{2}{r_x^2-r_y^2}\left(\frac{\sqrt{r_x^2+\xi}}{\sqrt{r_y^2+\xi}} - \frac{\sqrt{r_y^2+\xi}}{\sqrt{r_x^2+\xi}}\right) = \frac{2}{[(r_x^2+\xi)(r_y^2+\xi)]^{1/2}}$$

Using these results:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = -\frac{\lambda}{2\pi\varepsilon_0}\left\{\frac{2}{[(r_x^2+\xi)(r_y^2+\xi)]^{1/2}} - \frac{2}{[(r_x^2+\xi)(r_y^2+\xi)]^{1/2}}\right\}$$

$$= 0 \qquad \text{checks.} \quad \checkmark$$

<u>Case 2</u> (Interior: $x^2/r_x^2 + y^2/r_y^2 < 1$)

$$\xi = 0 \quad \Rightarrow \quad \frac{x\,\partial\xi/\partial x}{r_x^2+\xi} + \frac{y\,\partial\xi/\partial y}{r_y^2+\xi} = 0$$

$$\Rightarrow \quad I_x(\xi=0) = \cancel{I_y(\xi=0)} = \frac{2}{(r_x+r_y)r_x} \quad \text{and} \quad I_y(\xi=0) = \frac{2}{(r_x+r_y)r_y}$$

Using these results: $\checkmark$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = -\frac{\lambda}{2\pi\varepsilon_0}\left\{\frac{2}{r_x r_y} - 0\right\} = \frac{-\lambda}{\varepsilon_0\pi r_x r_y} = -\frac{q\hat{n}}{\varepsilon_0} \quad \text{checks}$$

Finally, check the limiting form outside the beam for r large $\Rightarrow$ ξ large.

$$-\frac{\partial\phi}{\partial x} = \frac{\lambda}{2\pi\varepsilon_0}\,x\,I_x(\xi) \qquad \lim_{r\to\infty} I_x(\xi) = \frac{1}{\xi} = \frac{1}{r^2}$$

$$-\frac{\partial\phi}{\partial y} = \frac{\lambda}{2\pi\varepsilon_0}\,y\,I_y(\xi) \qquad \lim_{r\to\infty} I_y(\xi) = \frac{1}{\xi} = \frac{1}{r^2}$$

Thus:

$$\lim_{r\to\infty} -\frac{\partial \phi}{\partial x} = \frac{\lambda}{2\pi\varepsilon_0}\frac{x}{r^2}$$

$$\lim_{r\to\infty} -\frac{\partial \phi}{\partial y} = \frac{\lambda}{2\pi\varepsilon_0}\frac{y}{r^2}$$

These have the correct limiting forms for a line charge at the origin. Completing the verification of the general formula.

In the beam $\left(x^2/r_x^2 + y^2/r_y^2 < 1, \; \xi = 0 \right)$, the formula reduces to:

$$\phi = -\frac{\lambda}{4\pi\varepsilon_0}\left\{ x^2 I_x(\xi=0) + y^2 I_y(\xi=0) \right\} + \text{const.}$$

$$= -\frac{\lambda}{4\pi\varepsilon_0}\left\{ \frac{2x^2}{r_x(r_x+r_y)} + \frac{2y^2}{r_y(r_x+r_y)} \right\} + \text{const.}$$

$$\boxed{\phi = -\frac{\lambda}{2\pi\varepsilon_0}\left\{ \frac{x^2}{r_x(r_x+r_y)} + \frac{y^2}{r_y(r_x+r_y)} \right\} + \text{const.}}$$

The case of an axisymmetric beam with

$$r_x = r_y = r_b$$

is easy to construct explicitly and is included in the homework problems.

There is also an alternative way to do this field calculation, that is less direct but more efficient. We carry out this proof now since steps involved are useful for other purposes.

A density profile with elliptic symmetry can be expressed as:

$$n(x,y) = n\left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2}\right)$$

Here we do not assume a specific uniform density profile and we leave $n(x^2/r_x^2 + y^2/r_y^2)$ arbitrary outside of having elliptic symmetry. The solution to the 2D Poisson equation in free-space

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -\frac{qn}{\varepsilon_0}$$

is then given by

$$\phi = -\frac{q\, r_x r_y}{4\varepsilon_0}\int_0^\infty d\xi \, \frac{\eta(\mathcal{X})}{\sqrt{r_x^2+\xi}\,\sqrt{r_y^2+\xi}}$$

where $\eta(\mathcal{X})$ is a function defined such that:

$$n(x,y) = \left.\frac{d\eta(\mathcal{X})}{d\mathcal{X}}\right|_{\xi=0}$$

This choice for $\eta(\mathcal{X})$ can always be made.

We first prove that this solution is valid by direct substitution:

$$\mathcal{V} = \frac{x^2}{r_x^2+\xi} + \frac{y^2}{r_y^2+\xi} \Rightarrow \frac{\partial \mathcal{V}}{\partial x} = \frac{2x}{r_x^2+\xi}\,;\ \frac{\partial^2 \mathcal{V}}{\partial x^2} = \frac{2}{r_x^2+\xi}$$

$$\frac{\partial \mathcal{V}}{\partial y} = \frac{2y}{r_y^2+\xi}\,;\ \frac{\partial^2 \mathcal{V}}{\partial y^2} = \frac{2}{r_y^2+\xi}$$

Substitute in Poisson's equation and use the chain rule and results above:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = \frac{-q r_x r_y}{4\varepsilon_0}\int_0^\infty d\xi\, \frac{\left(\frac{d^2\eta}{d\mathcal{V}^2}\right)\left(\frac{4x^2}{(r_x^2+\xi)^2}+\frac{4y^2}{(r_y^2+\xi)^2}\right)+\left(\frac{d\eta}{d\mathcal{V}}\right)\left(\frac{2}{r_x^2+\xi}+\frac{2}{r_y^2+\xi}\right)}{\sqrt{r_x^2+\xi}\,\sqrt{r_y^2+\xi}}$$

Note: $$d\mathcal{V} = -\left[\frac{x^2}{(r_x^2+\xi)^2}+\frac{y^2}{(r_y^2+\xi)}\right]d\xi$$

so the first integral can be simplified by partial integration:

$$\int_0^\infty d\xi\, \frac{\left(\frac{d^2\eta}{d\mathcal{V}^2}\right)\left(\frac{4x^2}{(r_x^2+\xi)^2}+\frac{4y^2}{(r_y^2+\xi)^2}\right)}{\sqrt{r_x^2+\xi}\,\sqrt{r_y^2+\xi}} = -4\int_0^\infty d\xi\, \frac{\frac{d^2\eta}{d\mathcal{V}^2}\frac{d\mathcal{V}}{d\xi}}{\sqrt{r_x^2+\xi}\,\sqrt{r_y^2+\xi}}$$

$$= -4\int_0^\infty d\xi\, \frac{\frac{d}{d\xi}\left(\frac{d\eta}{d\mathcal{V}}\right)}{\sqrt{r_x^2+\xi}\,\sqrt{r_y^2+\xi}} = -4\int_0^\infty d\xi\, \frac{d}{d\xi}\left[\frac{\frac{d\eta}{d\mathcal{V}}}{\sqrt{r_x^2+\xi}\,\sqrt{r_y^2+\xi}}\right] + 4\int_0^\infty d\xi\, \frac{d\eta}{d\mathcal{V}}\frac{d}{d\xi}\frac{1}{\sqrt{r_x^2+\xi}\,\sqrt{r_y^2+\xi}}$$

$$= \left.\frac{-4\frac{d\eta}{d\mathcal{V}}}{\sqrt{r_x^2+\xi}\,\sqrt{r_y^2+\xi}}\right|_{\xi\to 0}^{\xi\to\infty} - 2\int_0^\infty d\xi\, \frac{\frac{d\eta}{d\mathcal{V}}\left(\frac{1}{r_x^2+\xi}+\frac{1}{r_y^2+\xi}\right)}{\sqrt{r_x^2+\xi}\,\sqrt{r_y^2+\xi}}$$

$$= \frac{4}{r_x r_y}\left.\frac{d\eta}{d\mathcal{V}}\right|_{\xi=0} - 2\int_0^\infty d\xi\, \frac{\frac{d\eta}{d\mathcal{V}}\left(\frac{1}{r_x^2+\xi}+\frac{1}{r_y^2+\xi}\right)}{\sqrt{r_x^2+\xi}\,\sqrt{r_y^2+\xi}}$$

term will cancel 2nd Integral

Thus:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = -\frac{q\cancel{r_x r_y}}{\cancel{4}\varepsilon_0}\frac{\cancel{4}}{\cancel{r_x r_y}}\frac{d\eta(\mathcal{X})}{d\mathcal{X}}\bigg|_{\xi=0}$$

But $\displaystyle \frac{d\eta(\mathcal{X})}{d\mathcal{X}}\bigg|_{\xi=0} = n(x,y)$ by definition.

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = -\frac{q\,n(x,y)}{\varepsilon_0}$$ verifying the result.

For a uniform density ellipse we take:

$$\eta(\mathcal{X}) = \frac{\lambda}{\pi r_x r_y}\begin{cases}\mathcal{X} & ; \mathcal{X}<1 \\ 1 & ; \mathcal{X}>1\end{cases} \Rightarrow \frac{d\eta(\mathcal{X})}{d\mathcal{X}} = \begin{cases}\frac{\lambda}{q\pi r_x r_y} & ; \mathcal{X}<1 \\ 0 & ; \mathcal{X}>1\end{cases}$$

Thus

$$\frac{d\eta(\mathcal{X})}{d\mathcal{X}}\bigg|_{\xi=0} = \begin{cases}\frac{\lambda}{q\pi r_x r_y} & ; \mathcal{X}|_{\xi=0}<1 \\ 0 & ; \mathcal{X}|_{\xi=0}\end{cases} = \begin{cases}\frac{\lambda}{q\pi r_x r_y} & ; \frac{x^2}{r_x^2}+\frac{y^2}{r_y^2}<1 \\ 0 & ; \frac{x^2}{r_x^2}+\frac{y^2}{r_y^2}>1\end{cases}$$

$$\therefore \frac{d\eta(\mathcal{X})}{d\mathcal{X}}\bigg|_{\xi=0} = n(x,y)$$ for a uniform density elliptical beam with radii r_x, r_y and density $\lambda/(q\pi r_x r_y)$

Apply these results to calculate ϕ interior to a uniform density elliptical beam.

$$\phi = -\frac{q r_x r_y}{4\varepsilon_0}\int_0^\infty \frac{d\xi\,\eta(\mathcal{X})}{\sqrt{r_x^2+\xi}\sqrt{r_y^2+\xi}}$$

$$\mathcal{X} = \frac{x^2}{r_x^2+\xi}+\frac{y^2}{r_y^2+\xi}$$ if $\displaystyle \frac{x^2}{r_x^2}+\frac{y^2}{r_y^2}<1$, then

$$\mathcal{X} = \frac{x^2}{r_x^2+\xi}+\frac{y^2}{r_y^2+\xi}<1 \quad \text{for all} \quad 0\le\xi<\infty$$

Using this and the result above for $\eta(\mathcal{X})$, ϕ inside the elliptical beam is:

$$\phi = -\frac{q r_x r_y}{4\varepsilon_0}\int_0^\infty d\xi\,\frac{\lambda}{q\pi r_x r_y}\left[\frac{x^2}{(r_x^2+\xi)^{3/2}(r_y^2+\xi)^{1/2}}+\frac{y^2}{(r_x^2+\xi)^{1/2}(r_y^2+\xi)^{3/2}}\right]$$

$$\phi = \frac{-\lambda}{4\pi\varepsilon_0}\left\{ x^2\int_0^\infty d\xi \frac{1}{(r_x^2+\xi)^{3/2}(r_y^2+\xi)^{1/2}} + y^2\int_0^\infty d\xi \frac{1}{(r_x^2+\xi)^{1/2}(r_y^2+\xi)^{3/2}} \right\}$$

Using Mathematica or Integral tables:

$$\int_0^\infty d\xi \frac{1}{(r_x^2+\xi)^{3/2}(r_y^2+\xi)^{1/2}} = \frac{2}{r_x(r_x+r_y)}$$

$$\int_0^\infty d\xi \frac{1}{(r_x^2+\xi)^{1/2}(r_y^2+\xi)^{3/2}} = \frac{2}{r_y(r_x+r_y)}$$

Hence

$$\phi = \frac{-\lambda}{2\pi\varepsilon_0}\left\{ \frac{x^2}{r_x(r_x+r_y)} + \frac{y^2}{r_y(r_x+r_y)} \right\} + \text{const}$$

since an overall constant can always be added to ϕ (The integral has a reference choice $\phi(x=y=0)=0$ built in.).

The steps introduced in this proof can also be used to show that:

$$\left\langle x\frac{\partial\phi}{\partial x}\right\rangle_\perp = \frac{-\lambda}{4\pi\varepsilon_0}\frac{r_x}{r_x+r_y}$$

$$\lambda = q\int d^2x\, n$$

$$\left\langle y\frac{\partial\phi}{\partial y}\right\rangle_\perp = \frac{-\lambda}{4\pi\varepsilon_0}\frac{r_y}{r_x+r_y}$$

$$r_x \equiv 2\langle x^2\rangle^{1/2}$$
$$r_y \equiv 2\langle y^2\rangle^{1/2}$$

for any elliptic symmetry density profile $n(x,y) = n(x^2/r_x^2 + y^2/r_y^2)$. In the intro. lectures these results were employed to show that the KV envelope equations with evolving emittances can be applied to elliptic symmetry beams. This result was first demonstrate by Sacherer: [IEEE Trans Nucl. Sci. 18, 1105 (1971)]

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Ref: Davidson, *Physics of Nonneutral Plasmas*, Addison-Wesley, 1990.

Appendix B

The single-particle equations of motion can be derived from the Hamiltonian: $\left(\frac{d\vec{x}_\perp}{ds} = \frac{\partial H}{\partial \vec{x}_\perp'}, \frac{d\vec{x}_\perp'}{ds} = -\frac{\partial H}{\partial \vec{x}_\perp} \right)$

$$\boxed{\begin{aligned} H_\perp(x,y,x',y',s) = \frac{1}{2}x'^2 + \left\{ \kappa_x(s) - \frac{2Q}{r_x(s)[r_x(s)+r_y(s)]} \right\} \frac{x^2}{2} \\ + \frac{1}{2}y'^2 + \left\{ \kappa_y(s) - \frac{2Q}{r_y(s)[r_x(s)+r_y(s)]} \right\} \frac{y^2}{2} \end{aligned}}$$

Perform a canonical transform to new variables X, Y, X', Y' using the generating function

$$F_2(x,y,X',Y') = \frac{x}{w_x}\left[X' + \frac{x w_x'}{2} \right] + \frac{y}{w_y}\left[Y' + \frac{y w_y'}{2} \right]$$

Then:

$$\boxed{\begin{aligned} X &= \frac{\partial F_2}{\partial X'} = \frac{x}{w_x} \\ Y &= \frac{\partial F_2}{\partial Y'} = \frac{y}{w_y} \end{aligned}}$$

$$x' = \frac{\partial F_2}{\partial x} = \frac{1}{w_x}\left(X' + x w_x' \right)$$

$$y' = \frac{\partial F_2}{\partial y} = \frac{1}{w_y}\left(Y' + y w_y' \right)$$

and solving for X', Y':

$$\boxed{\begin{aligned} X' &= w_x x' - x w_x' \\ Y' &= w_y y' - y w_y' \end{aligned}}$$

The Courant – Snyder invariants are then simply expressed:

$$\boxed{\begin{aligned} \varepsilon_x &= X^2 + X'^2 \\ \varepsilon_y &= Y^2 + Y'^2 \end{aligned}}$$

One can show from the transformations that:

$$\boxed{\begin{aligned} dx\,dy &= w_x w_y \, dX\, dY \\ dx'dy' &= \frac{dX'dY'}{w_x w_y} \\ dx\,dy\,dx'dy' &= dX\,dY\,dX'dY' \;\; * \end{aligned}}$$

* Property of canonical transforms in general. — Results from structure of Generating Function

Therefore, the distribution in transformed phase space variables is the same as for the original variables:

$$\boxed{\begin{aligned} f_\perp(X,Y,X',Y',s) &= f_\perp(x,y,x',y',s) \\ &= \frac{\lambda}{q\pi^2 \varepsilon_x \varepsilon_y} \delta\left[\frac{X^2+X'^2}{\varepsilon_x} + \frac{Y^2+Y'^2}{\varepsilon_y} - 1 \right] \end{aligned}}$$

Now examine the density:

$$n(x,y) = \int dx'dy' \, f_\perp = \int \frac{dX'dY'}{w_x w_y} f_\perp =$$

$$U_x = X'/\sqrt{\varepsilon_x}, \quad U_y = Y'/\sqrt{\varepsilon_y}$$

$$dU_x dU_y = \frac{dX'dY'}{\sqrt{\varepsilon_x \varepsilon_y}}$$

$$r_x = \sqrt{\varepsilon_x}\, w_x, \quad r_y = \sqrt{\varepsilon_y}\, w_y$$

$$\boxed{n = \frac{\lambda}{q\pi^2 r_x r_y} \int dU_x dU_y \, \delta\left[U_x^2 + U_y^2 - \left(1 - \frac{X^2}{\varepsilon_x} - \frac{Y^2}{\varepsilon_y}\right) \right]}$$

Exploit the cylindrical symmetry:

$$U_\perp^2 = U_x^2 + U_y^2$$

$$dU_x dU_y = d\psi\, U_\perp dU_\perp = d\psi \frac{dU_\perp^2}{2}$$

$$n(x,y) = \frac{\lambda}{q\pi^2 r_x r_y}\int_0^{2\pi} d\psi \int_0^{\infty} \frac{dU_\perp^2}{2}\, \delta\left[U_\perp^2 - \left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right)\right]$$

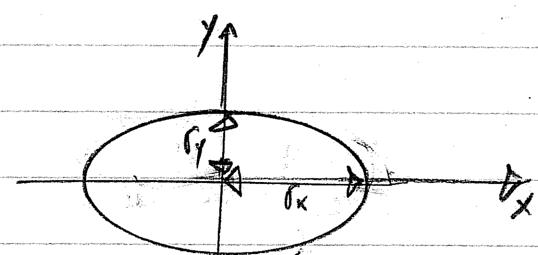
Thus:

$$\boxed{\begin{aligned} n(x,y) &= \frac{\lambda}{q\pi r_x r_y}\int_0^{\infty} dU_\perp^2\, \delta\left[U_\perp^2 - \left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right)\right] \\ &= \begin{cases} \dfrac{\lambda}{q\pi r_x r_y} = \hat{n} ; & \dfrac{x^2}{r_x^2} + \dfrac{y^2}{r_y^2} < 1 \\ 0 ; & \dfrac{x^2}{r_x^2} + \dfrac{y^2}{r_y^2} > 1 \end{cases} \end{aligned}}$$

Showing that the singular KV distribution yields the required uniform density beam of elliptical cross-section.

Note

$$\hat{n} = \frac{\lambda}{q\pi r_x r_y}$$



Area ellipse $= \pi r_x r_y$

$\lambda = q\hat{n}\pi r_x r_y$

for uniform density.

An interesting footnote to this appendix is that an infinity of Generating functions can be used to transform the KV distribution in standard quadratic form:

$$f_\perp \sim \delta\left[\underline{X}'^2 + \underline{Y}'^2 + \underline{X}^2 + \underline{Y}^2 - \mathrm{const} \right]$$

to other sets of variables. This will generate other distributions with KV form for skew coupling and other effects. It would not be logical to label such distributions as "new" as has been done in the literature. However, identifying physically relevant transforms has practical value.

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Reading: Reiser § 5.3.3 pgs. 347-351

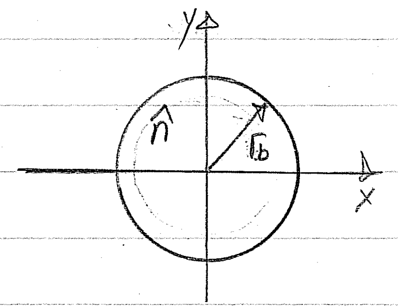
§3 KV Beams – Continuous Focusing Limit

The formulation developed to analyze the KV distribution with s-varying focusing can seem formal and abstract. Some insight into the structures presented can be gained when examining the most simple limit of a continuous focusing channel

$$\boxed{\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \text{const.}}$$

with a round KV beam with

$$\boxed{\begin{aligned} \varepsilon_x &= \varepsilon_y \equiv \varepsilon \\ r_x &= r_y \equiv r_b \end{aligned}}$$



Then the single-particle equations of motion reduce to

$$\boxed{x'' + k_{\beta 0}^2 x - \frac{Q}{r_b^2} x = 0}$$

and the envelope equation reduces to:

$$\boxed{r_b'' + k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon^2}{r_b^3} = 0}$$

In the limit of zero space-charge ($Q \to 0$) the particle equation of motion is solved as:

$$\boxed{\begin{aligned} &x'' + k_{\beta 0}^2 x = 0 \\ &x(s) = x_i \cos[k_{\beta 0}(s-s_i)] + \frac{x_i'}{k_{\beta 0}} \sin[k_{\beta 0}(s-s_i)] \\ &x(s=s_i) \equiv x_i \\ &x'(s=s_i) \equiv x_i' \end{aligned}}$$

Thus

$$\boxed{k_{\beta 0} = \text{wave-number of particle oscillations in the absence of space-charge } (Q=0)}$$

- $k_{\beta 0}$ is called the undepressed betatron wavenumber (spatial "frequency").

For the case of finite space charge ($Q \neq 0$) a "matched" beam with r_b satisfying the envelope equation:

$$r_b'' + k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon^2}{r_b^3} = 0$$

$\Rightarrow r_b = \text{const.}$ solves equation

$$\boxed{k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon^2}{r_b^3} = 0}$$ Matched beam envelope equation

and the equation of motion for finite space-charge ($Q \neq 0$) can be expressed as:

$$\boxed{\begin{aligned} &x'' + \left(k_{\beta 0}^2 - \frac{Q}{r_b^2}\right)x = 0 \\ &k_\beta^2 \equiv k_{\beta 0}^2 - \frac{Q}{r_b^2} \\ &x(s) = x_i \cos[k_\beta(s-s_i)] + \frac{x_i'}{k_\beta}\sin[k_\beta(s-s_i)] \\ &x(s=s_i) \equiv x_i \\ &x'(s=s_i) \equiv x_i' \end{aligned}}$$

Thus

$$\boxed{k_\beta = \left(k_{\beta 0}^2 - \frac{Q}{r_b^2}\right)^{1/2} = \text{wavenumber of particle oscillations including space-charge } (Q \neq 0\text{, matched beam})}$$

- k_β is called the depressed betatron wavenumber

Note that:

$$\boxed{\begin{aligned} k_\beta^2 &= k_{\beta 0}^2 - \frac{Q}{r_b^2} = \frac{\varepsilon^2}{r_b^4} \quad ; \quad Q = \frac{q\lambda}{2\pi m \varepsilon_0 \gamma_b^3 \beta_b^2 c^2} = \frac{q^2 \hat{n} r_b^2}{2 m \varepsilon_0 \gamma_b^3 \beta_b^2 c^2} \\ &= k_{\beta 0}^2 - \frac{\hat{\omega}_p^2}{2\gamma_b^3 \beta_b^2 c^2} \qquad \hat{\omega}_p = \left(\frac{q\hat{n}}{m\varepsilon_0}\right)^{1/2} = \text{plasma Frequency} \\ &\hat{\omega}_p/(\beta_b c) = k_p = \text{Plasma wavenumber} \end{aligned}}$$

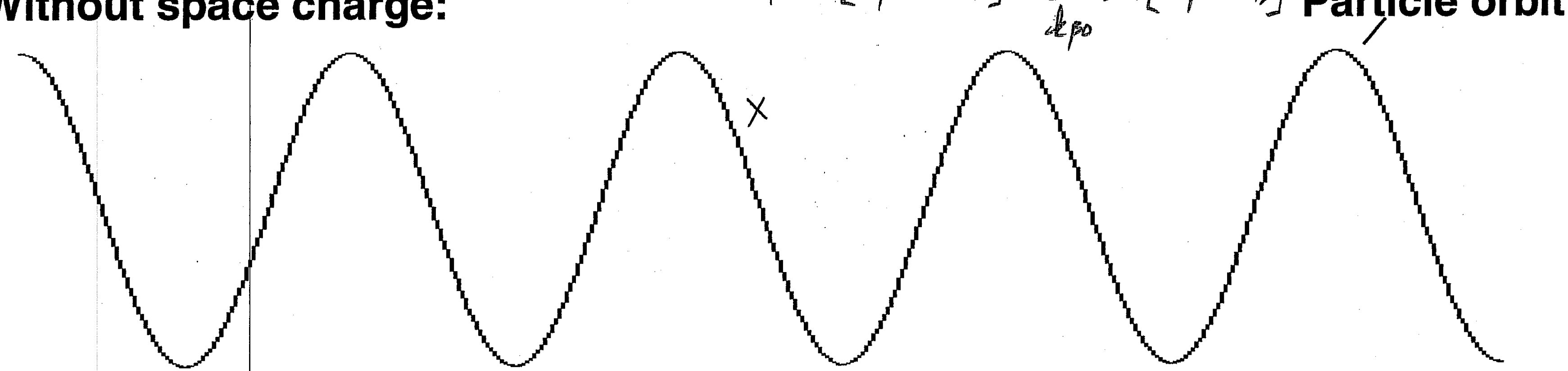
- $k_\beta^2 \leq k_{\beta 0}^2$ with $\lim_{Q \to 0} k_\beta^2 = k_{\beta 0}^2$
- $k_\beta^2 \geq 0$ with the cold beam limit $\lim_{\varepsilon \to 0} k_\beta^2 = 0$
- $0 \leq k_\beta / k_{\beta 0} \leq 1$ $\quad k_\beta/k_{\beta 0} \to 0$ cold limit; $\quad k_\beta/k_{\beta 0} = 1$ zero space charge limit.

Space charge reduces betatron phase advance

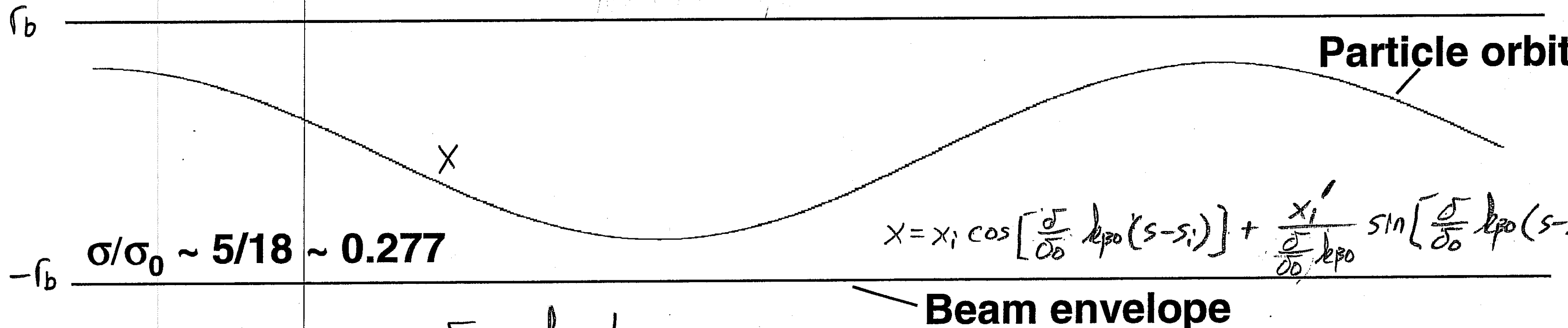
Continuous Focusing Channel

Without space charge:

$$x = x_i \cos[k_{\beta 0}(s-s_i)] + \frac{x_i'}{k_{\beta 0}} \sin[k_{\beta 0}(s-s_i)]$$



With space charge:



$$x = x_i \cos\left[\frac{\sigma}{\sigma_0} k_{\beta 0}(s-s_i)\right] + \frac{x_i'}{\frac{\sigma}{\sigma_0} k_{\beta 0}} \sin\left[\frac{\sigma}{\sigma_0} k_{\beta 0}(s-s_i)\right]$$

$\sigma/\sigma_0 \sim 5/18 \sim 0.277$

$\sigma_0 = k_{\beta 0} L_p$

$\sigma = k_{\beta 0} L_p$

L_p arbitrary "period" length to define phase

$\Rightarrow \sigma/\sigma_0 = k_\beta / k_{\beta 0}$

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For the matched beam case it is also instructive to examine the form of the KV distribution function:

$$r_x = r_y = r_b = \text{const.}$$
$$r_x' = r_y' = 0$$
$$\varepsilon_x = \varepsilon_y \equiv \varepsilon^2$$

$$f_\perp(x,y,x',y') = \frac{\lambda}{q\pi^2\varepsilon^2}\,\delta\left[\frac{x^2}{r_b^2}+\frac{y^2}{r_b^2}+\frac{r_b^2}{\varepsilon^2}x'^2+\frac{r_b^2}{\varepsilon^2}y'^2-1\right]$$

But: $\lambda = q\pi\hat{n}r_b^2$ and

$$\delta[cx] = \frac{\delta[x]}{c} \quad ; \quad c = \text{const}$$

$$\Rightarrow \quad f_\perp(x,y,x',y') = \frac{\hat{n}}{2\pi}\,\delta\left[\frac{1}{2}(x'^2+y'^2)+\frac{\varepsilon^2}{2r_b^4}(x^2+y^2)-\frac{\varepsilon^2}{2r_b^2}\right]$$

Note that the single-particle Hamiltonian (see pg 3) reduces to

$$H_\perp = \frac{x'^2}{2}+\frac{y'^2}{2}+\frac{1}{2}\left(k_{\beta 0}^2-\frac{Q}{r_b^2}\right)(x^2+y^2)$$

But: $$k_{\beta 0}^2 - \frac{Q}{r_b^2} = \frac{\varepsilon^2}{r_b^4}$$

$$\Rightarrow \quad \boxed{H_\perp = \frac{1}{2}(x'^2+y'^2)+\frac{\varepsilon^2}{2r_b^4}(x^2+y^2)}$$

At the beam edge, the particles must turn $\Rightarrow x'=y'=0$ at $r=r_b$ and then we may identify the edge value of $H_\perp$:

$$\boxed{H_b \equiv H_\perp\big|_{r=r_b} = \frac{\varepsilon^2}{2r_b^2}}$$

and the matched KV distribution reduces to:

$$\boxed{f_\perp(x,y,x',y') = \frac{\hat{n}}{2\pi}\,\delta[H_\perp - H_b]}$$

In summary,
The matched KV distribution for a continuous focusing channel is given by

$$\boxed{\begin{aligned} f_\perp(x,y,x',y') &= \frac{\hat{n}}{2\pi}\,\delta\left[H_\perp - H_b\right] \\ H_\perp &= \frac{1}{2}(x'^2+y'^2) + \frac{\varepsilon_x^2}{2r_b^4}(x^2+y^2) \\ H_b &\equiv \frac{\varepsilon_x^2}{2r_b^2} \end{aligned}}$$

- For continuous focusing, $H_\perp$ is a single-particle constant of the motion (Problem #1)
  - For s-varying focusing channels, the relation of the Courant-Snyder invariants to "energies" is not so simple. *
- Since $f_\perp = 0$ when $H_\perp \neq H_b$ all particles have the same transverse energy. $H = H_b$.

* Comments made under the discussions of particle equations of motion apply here - only for continuous focusing can $H_\perp$ be regarded as interchangeable with the Courant-Snyder invariant.

In the homework problems it will be shown from

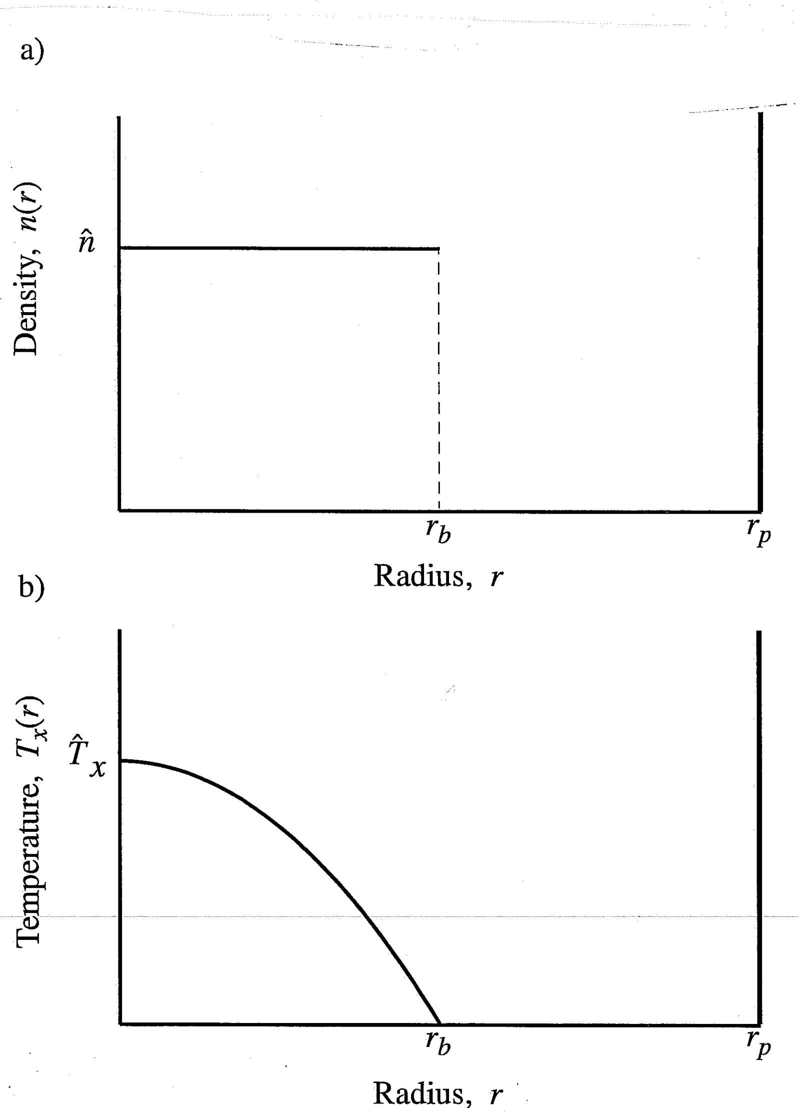
$$f_\perp = \frac{\hat{n}}{2\pi}\,\delta(H_\perp - H_b) \qquad \hat{n} = n(r=0)$$
$$H_b = \varepsilon_x^2/(2r_b^2)$$

that ($r_p \to \infty$ for free space):

Flat → <u>Density</u>: $n = \int d^2x'\, f_\perp = \begin{cases} \hat{n}\ ; & 0 \le r < r_b, \\ 0\ ; & r_b < r \le r_p \end{cases}$

Parabolic → <u>Temperature</u>: $T_x = \gamma_b m \beta_b^2 c^2 \dfrac{\int d^2x'\, x'^2 f_\perp}{\int d^2x'\, f_\perp} = \begin{cases} \hat{T}_x\left(1-\dfrac{r^2}{r_b^2}\right); & 0 \le r < r_b, \\ 0\ ; & r_b < r \le r_p \end{cases}$

$$\hat{T}_x \equiv \frac{\gamma_b m \beta_b^2 c^2 \varepsilon_x^2}{2 r_b^2} = \text{const.}$$



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§4 Equilibrium Distributions in Continuous Focusing Channels

Reading: Reiser §5.3.3 pgs 347-358

Most real transport channels have s-varying focusing with

$$\frac{d}{ds}\kappa_x(s) \neq 0$$

$$\frac{d}{ds}\kappa_y(s) \neq 0$$

However, only the KV equilibrium is known in this case. Due to the pathologies of the KV distribution it is insightful to analyze continuous focusing channels where other equilibria can be constructed with more physically reasonable smooth distribution functions.

- Physical distributions are expected to be smooth with phase-space densities falling smoothly to zero at the beam edge.
  - Falloff may be rapid for cold beams, but should be smooth.

- Space-charge forces will be nonlinear for smooth distributions with nonuniform density projections. Nonlinear terms will be largest at the edge where the density falls off rapidly.

- Stability properties may differ strongly for smooth distributions and non-smooth distributions (e.g. KV beam). These will be analyzed later. Intuitively expect smooth forms to have less free energy to drive instabilities.

Consider a symmetric continuous focusing channel:

$$\boxed{\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \text{const.}}$$

In this case, the single-particle Hamiltonian

$$\boxed{H_\perp = \frac{1}{2}\vec{x}_\perp'^2 + \frac{k_{\beta 0}^2}{2}\vec{x}_\perp^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2 c^2}}$$

is a single-particle constant of the motion (See problem sets). Thus:

$$\boxed{f_\perp = f_\perp(H_\perp)}$$

is a valid equilibrium distribution for any differentiable function $f_\perp(H_\perp)$. The solutions generated will be matched and ϕ must be solved for self-consistently from the Poisson Equation:

$$\boxed{\nabla_\perp^2\phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -\frac{q}{\epsilon_0}\int dx'dy'\, f_\perp(H_\perp)}$$

- It can be shown that in the cases:
  1) $\lim_{r\to\infty} \frac{\partial\phi}{\partial r} \sim \frac{\lambda}{2\pi\epsilon_0 r}$ (free space limit)
  2) $\phi(r = r_p) = \text{const.}$ (conducting pipe)

  that the Poisson equation only admits axisymmetric ($\partial/\partial\theta = 0$) solutions $\Rightarrow \phi = \phi(r)$ [Prove by contradiction]

$$\boxed{\nabla_\perp^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = -\frac{q}{\epsilon_0}\int dx'dy'\, f_\perp(H_\perp)}$$

- The solutions generated will be RMS matched (by construction) and will satisfy the envelope equation:

$$\boxed{k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon_x^2}{r_b^3} = 0}$$

Provided that

$$Q = \frac{q\lambda}{2\pi\varepsilon_0 m\gamma_b^3\beta_b^2 c^2} \qquad \lambda = q\int dx\,dy \int dx'\,dy'\, f_\perp(H_\perp)$$

and statistical definitions of r_b and ε_x^2 are employed

$$r_b = 2\langle x^2\rangle_\perp^{1/2} = 2\langle y^2\rangle_\perp^{1/2} = \sqrt{2}\,\langle r^2\rangle_\perp^{1/2}$$

$$\varepsilon_x = 4\left[\langle x^2\rangle_\perp\langle x'^2\rangle_\perp - \langle xx'\rangle_\perp^2\right]^{1/2} = 4\left[\langle x^2\rangle_\perp\langle x'^2\rangle_\perp\right]^{1/2}$$

$\varepsilon_x = \varepsilon_y$ axisymmetric,

with

$$\langle \cdots \rangle_\perp \equiv \frac{\int dx\,dy \int dx'\,dy' \cdots f_\perp(H_\perp)}{\int dx\,dy \int dx'\,dy'\, f_\perp(H_\perp)}$$

// Example

We showed in the previous section that the KV equilibrium distribution reduces in continuous focusing to

$$f_\perp = \frac{\hat{n}}{2\pi}\,\delta[H_\perp - H_b]$$

$$H_b = \text{const} \quad , \quad \text{fixing the beam edge}$$

$\Rightarrow f_\perp$ = function of $H_\perp$ is an equilibrium since $H_\perp$ is a single-particle constant. //

Since the solutions for ϕ are axisymmetric ($\partial/\partial\theta = 0$) it is convenient even if there is a conducting pipe to choose referance such that

$$\boxed{\begin{aligned} \phi(r=0) &\equiv 0 \\ H_\perp(\vec{x}=0, \vec{x}'=0) &= 0 \end{aligned}}$$

* choice can be made without loss in generality since $\partial/\partial\theta = 0$.

Examine the Poisson equation:

$$\boxed{\nabla_\perp^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = \frac{-q}{\epsilon_0}\int dx'dy'\, f(H_\perp)}$$

$$H_\perp = \frac{1}{2}(x'^2 + y'^2) + \psi(r)$$

$$\psi(r) = \frac{k_{\beta 0}^2}{2} r^2 + \frac{q\phi(r)}{m\gamma_b^3\beta_b^2 c^2}$$

cylindrical element:

$$dx'dy' = R'dR'd\theta'$$

$$U = \frac{R'^2}{2} = \frac{(x'^2+y'^2)}{2}$$

↓ see other notes for steps!

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = \frac{-q 2\pi}{\epsilon_0}\int_0^\infty dU\, f_\perp(U+\psi)$$

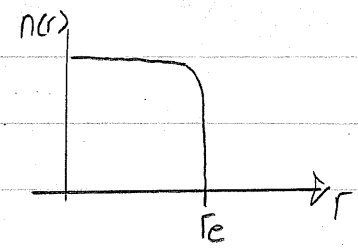
change variables:

$$H_\perp = U + \psi(r)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = \frac{-2\pi q}{\epsilon_0}\int_{\psi(r)}^\infty dH_\perp\, f_\perp(H_\perp)$$

$$\boxed{\begin{gathered} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) = 2k_{\beta 0}^2 - \frac{2\pi q^2}{m\epsilon_0\gamma_b^3\beta_b^2c^2}\int_{\psi(r)}^\infty dH_\perp\, f_\perp(H_\perp) \\ \psi(r=0) \equiv 0 \end{gathered}}$$

For case beams with a sharp edge:



$r = r_e$ = edge radius

(do not confuse with $r_b = 2\langle x^2 \rangle_\perp^{1/2}$)

there will be turning point restrictions on the particles:

$$\vec{x}_\perp' = 0 \quad \text{at} \quad r = r_e$$

$$\Rightarrow \quad H_\perp \Big|_{r=r_e} = \psi(r_e)$$

and since $H_\perp$ = const. for all particles, a particle that is at one point on the edge of the beam will always have this max-value of $H_\perp$.
This feature can be used to restrict integration ranges if needed in explicit evaluations of ψ

For max $H_\perp$ particle

At edge: $r = r_e$ $\quad H_\perp = \psi(r_e) = \cancel{\frac{1}{2}\vec{x}_\perp'^2}^{\,0} + \psi(r_e)$

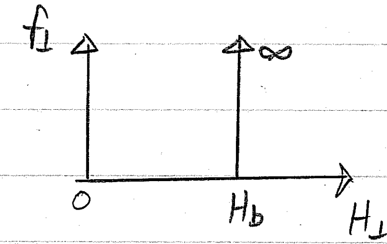
Interior $r < r_e$ $\quad H_\perp = \psi(r_e) = \frac{1}{2}\vec{x}_\perp'^2 + \psi(r)$

$$\hookrightarrow \quad \frac{1}{2}\vec{x}_\perp'^2 = \psi(r_e) - \psi(r)$$

- See Reiser for explicit examples.
  The edge constraint can be cast into an auxiliary constraint equation fixing the edge parameter.

There are many examples of $f_\perp(H_\perp)$ analyzed in the literature. Some common examples:

1) <u>KV</u> (covered)

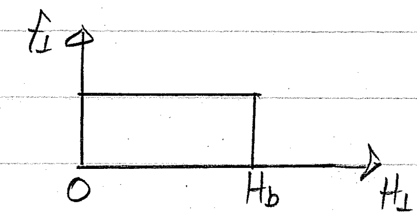


$$f_\perp \propto \delta[H_\perp - H_b]$$

$H_b = \text{const.}$

2) <u>Waterbag</u> (Discussed in detail in Reiser).

Another analytic example:

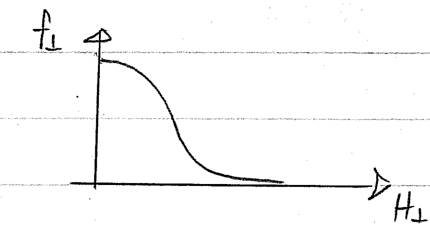


$$f_\perp \propto \Theta[H_b - H_\perp]$$

$$\Theta[x] = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$$

3) <u>Thermal</u> (Boltzmann) to be covered in class.
(Discussed in detail in Reiser)

$$f_\perp \propto \exp[-H_\perp/T] \qquad T = \text{const} > 0.$$



∞ of choices so you can generate an ∞ of papers ..... a physicists dream ..... !

However, you can understand range of behavior with just a few choices.

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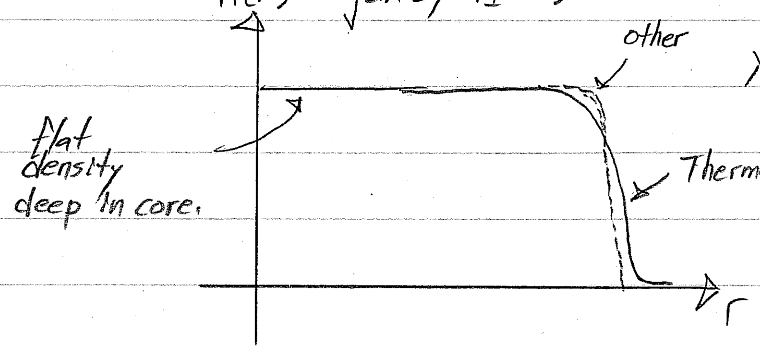
General Comment.

For any smooth, monotonic decreasing choice of $f_\perp(H_\perp)$ one finds that near the space-charge limit of the applied focusing that the density projection is flat and falls rapidly (smoothly) to zero near the beam edge.

- Specific rate of falloff depends on the choice of distribution.

$$\frac{\varepsilon_x^2}{r_b^3} << \frac{Q}{r_b}$$ strong space-charge (thermal forces weak)

$n(r) = \int dx'dy' f_\perp(H_\perp)$



The KV model represents the bulk features of the core well but misses "edge" physics.

- Not surprising statistical envelope models based on KV envelope equations work well for strong space-charge since most particles are in the core where the density is flat.

We next analyze the case of a smooth, thermal equilibrium distribution in detail.

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USPAS Jan 2004

Reading: Reiser §5.4
pgs. 364 – 390.

§5 The Thermal Equilibrium Distribution in a Continuous Focusing Channel

In an infinite propogation in a continuous focusing channel, collisions will relax any initial distribution to thermal equilibrium. The Fockker – Planck equation for the single particle distribution f:

$$\underbrace{\frac{df}{dt}}_{\text{Vlasov Form}} = \left.\frac{df}{dt}\right|_{\substack{\text{collision}\\ \text{operator}}}$$

Fokker – Planck collision operator, See Reiser.

predicts that the unique distribution that any initial f will relax to as $t \to \infty$ is the Maxwell-Boltzmann distribution:

$$f \propto \exp\left\{-\frac{H_{rest}}{T}\right\}$$

H_{rest} = Beam single particle Hamiltonian in the rest-frame of the beam. (energy units)

T = const. Thermodynamic temp. (energy units)

One caution in taking this result too literally is that
- The residence time of most intense beams in the transport channel is short.
  - Especially true in Linacs
  - Can be longer in rings

Collective waves launched by nonequilibrium distributions and instabilities can result in phase mixing and and nonlinear interactions that can induce collective interactions that rapidly drive the beam closer to thermal equilibrium form.

- Will show later that thermal equilibrium is both linearly and nonlinearly stable to perturbations
- From these features thermal equilibrium may be regarded as a perferred equilibrium state of the system.

To simplify discussions, we take a nonrelativistic limit of an unbunched beam ($\partial/\partial z = 0$) with:

$$\boxed{\begin{aligned} &\gamma_b \to 1 \\ &\beta_b c = v_b = \text{const.}, \quad \text{beam axial velocity} \\ &\langle p_z \rangle = m v_b \end{aligned}}$$

- In relativistic formulations some care must be taken to clearly define the temperature. The <u>correct</u> way to do this is as a Lorentz scalar since the distribution f must be Lorentz invariant. ($f d^3x\, d^3p$ = # particles in volume, which must be invariant). Reiser addresses some of these issues in his analysis.

In $\vec{x}, \vec{p}$ phase-space, the nonrelativistic Hamiltonian is:

$$H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + \frac{k_{\beta 0}^2}{2}(x^2+y^2) + q\phi$$

$$\boxed{\begin{aligned} H_{rest} &= \frac{1}{2m}(p_x^2 + p_y^2 + \tilde{p}_z^2) + m v_b^2 \frac{k_{\beta 0}^2}{2}(x^2+y^2) + q\phi \\ \tilde{p}_z &\equiv p_z - m v_b \end{aligned}}$$

Then we have the Maxwell-Boltzmann form:

$$f \propto e^{-H_{rest}/T} \qquad T = \text{temperature (energy units)}$$

For the unbunched beam:

$$f_\perp \propto \int d\tilde{p}_z \, f \propto \underbrace{\left(\int_{-\infty}^{\infty} d\tilde{p}_z \, e^{-\frac{\tilde{p}_z^2}{2mT}} \right)}_{\text{constant.}} e^{-H_{\perp r}/T}$$

$$\begin{aligned} H_{\perp r} &= \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + m v_b^2 \frac{k_{\beta 0}^2}{2}(x^2+y^2) + q\phi \\ &= m v_b^2 \left(\frac{1}{2}(x'^2+y'^2) + \frac{k_{\beta 0}^2}{2}(x^2+y^2) + \frac{q\phi}{m v_b^2} \right) \\ &= m v_b^2 H_\perp \end{aligned}$$

Thus we may take for 2D $\perp$ thermal equilibrium:

$$\boxed{\begin{aligned} &f_\perp(x, y, x', y') = \hat{f} \exp\left\{ -\frac{m v_b^2 H_\perp}{T} \right\} \\ &H_\perp = \frac{1}{2} \vec{x}_\perp'^2 + \frac{k_{\beta 0}^2}{2} \vec{x}_\perp^2 + \frac{q\phi}{m v_b^2} \\ &T = \text{const; temperature (energy units)} \\ &\hat{f} = \text{const; distribution normalization} \end{aligned}}$$

To fix f_0, we note that reference is taken such that $\phi(r=0) = 0$, and from the form of $f_\perp(H_\perp)$ it is clear that the density must decrease in radius.

$$\boxed{n(r=0) \equiv \hat{n} = \text{peak on-axis density}}$$

Then we have:

$$\hat{n}_0 = \hat{f}_0 \int_{-\infty}^{\infty} dx' \, e^{-\frac{m v_b^2}{2T} x'^2} \int_{-\infty}^{\infty} dy' \, e^{-\frac{m v_b^2}{2T} y'^2}$$

Using Gaussian integral formulas:

$$\int_{-\infty}^{\infty} dx' \, e^{-\frac{m v_b^2}{2T} x'} = \sqrt{\frac{2\pi T}{m v_b^2}}$$

$$\Rightarrow \quad \hat{f}_0 = \frac{m v_b^2 \hat{n}_0}{2\pi T}$$

Thus:

$$\boxed{\begin{aligned} f_\perp(x, y, x', y') &= \frac{m v_b^2 \hat{n}_0}{2\pi T} \exp\left\{ \frac{-m v_b^2 H_\perp}{T} \right\} \\ n(r) &= \int dx' dy' \, f_\perp = \hat{n}_0 \, e^{-\tilde{\psi}(r)} \\ \text{where: } \quad \tilde{\psi}(r) &\equiv \frac{m v_b^2}{T} \left\{ \frac{k_{\beta 0}^2}{2} r^2 + \frac{q\phi}{m v_b^2} \right\} \end{aligned}} \; *$$

Note:

$\tilde{\psi} = \frac{m v_b^2}{T} \psi$ where ψ is defined in the previous section on continuously focused equilibria. (nonrel limit of ψ)

We will now put back in relevant relativistic factors (without proof.... beyond the scope of this class) γ_b, β_b:

$$\boxed{\begin{gathered} f_\perp = \frac{\gamma_b m \beta_b^2 c^2 \hat{n}}{2\pi T} \exp\left\{ \frac{-\gamma_b m \beta_b^2 c^2 H_\perp}{T} \right\} \\ T = \text{const. Temp (lab frame).} \\ H_\perp = \frac{1}{2}\vec{x}_\perp'^2 + \frac{k_{\beta 0}^2}{2}\vec{x}_\perp^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2} \\ \hat{n} = n(r=0) \qquad \phi(r=0) = 0 \end{gathered}}$$

Then:

x-Temperature

$$\boxed{T_x = \gamma_b m \beta_b^2 c^2 \frac{\int d^2x' \cdot x'^2 f_\perp}{\int d^2x' f_\perp} = T = \text{const}}$$

Density

$$\boxed{\begin{gathered} n(r) = \hat{n} \exp\{-\tilde{\psi}\} \\ \tilde{\psi} \equiv \frac{1}{T}\left(\frac{\gamma_b m \beta_b^2 c^2 k_{\beta 0}^2}{2} r^2 + \frac{q\phi}{\gamma_b^2} \right) \end{gathered}}$$

Using these results the Poisson equation for thermal equilibrium

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r} \right) = -q\frac{n(r)}{\varepsilon_0}$$

can be recast into a more convenient form for analysis:

Take:

$$\lambda_D \equiv \left(\frac{\varepsilon_0 T}{q^2 \hat{n}} \right)^{1/2}$$ = Debye length formed from the on-axis ($r=0$) peak beam density.

$$\rho \equiv \frac{r}{\gamma_b \lambda_D}$$ = Scaled radial coordinate.

$$\hat{\omega}_p \equiv \left(\frac{q^2 \hat{n}}{\varepsilon_0 m} \right)^{1/2}$$ = Plasma frequency formed from the peak on-axis ($r=0$) beam density

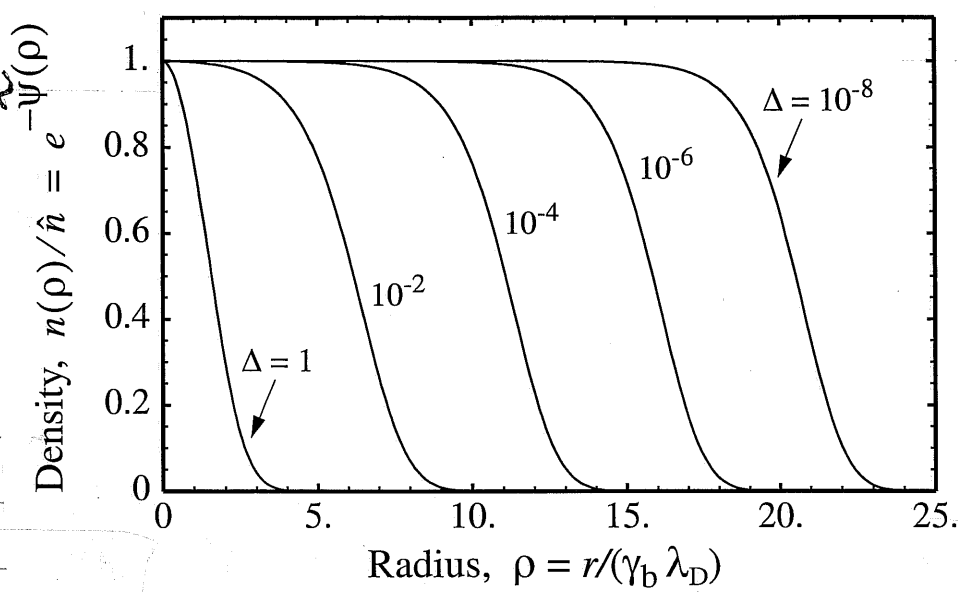
Note: $\lambda_D = \left(\dfrac{T}{\hat{\omega}_p^2} \right)^{1/2}$

$$1 + \Delta \equiv \frac{2 \gamma_b^3 \beta_b^2 c^2 k_{\beta 0}^2}{\hat{\omega}_p^2}$$ $\Rightarrow$ Δ is a dimensionless parameter relating the ratio of applied to space-charge defocusing forces.

Then the Poisson equation (see problem set) becomes:

$$\boxed{\begin{gathered} \frac{1}{\rho} \frac{\partial}{\partial \rho} \left[\rho \frac{\partial \tilde{\varphi}(\rho)}{\partial \rho} \right] = 1 + \Delta - e^{-\tilde{\varphi}(\rho)} \\ \tilde{\varphi}(\rho = 0) = 0 \\ n(\rho) = \hat{n}\, e^{-\tilde{\varphi}(\rho)} \end{gathered}}$$

This equation is highly nonlinear and must in general be solved numerically.



Note:
- $n/\hat{n}$ plotted vs $r/(\gamma_b \lambda_D)$ for values of Δ to illustrate solution
- Δ is only parameter of scaled solution
- Dependance on Δ is very sensitive — For $\Delta << 1$, the beam is nearly uniform in the core for many Debye radii $(\gamma_b \lambda_D)$.
- The fall-off at the edge is always for in a few units of $\gamma_b \lambda_D$ for $\Delta << 1$ regardless of the value of Δ. The edge becomes very sharp at fixed λ for $\gamma_b \lambda_D$ small.

The parameters in $f(H_\perp)$ that we have employed to fix the equilibrium:

$$\hat{n}, \; T, \; \Delta$$

must be related to

$k_{\beta 0}$, Q, ε_x, and kinematic parameters $(q, m, \gamma_b, \beta_b)$

to make physical connection to the equilibrium. To do this we calculate:

$$\boxed{\begin{aligned} \lambda &= \frac{\gamma_b^2 T}{2q} \int_0^\infty d\rho \, \rho \, e^{-\tilde{\psi}} \\ r_b^2 &= 4\langle x^2 \rangle_\perp = 2\gamma_b^2 \lambda_D^2 \frac{\int_0^\infty d\rho \, \rho^3 \, e^{-\tilde{\psi}}}{\int_0^\infty d\rho \, \rho \, e^{-\tilde{\psi}}} \end{aligned}} \quad -(1)$$

These integrals depend only on Δ.

Next we calculate

$$\varepsilon_x^2 = 16\left[\langle x^2 \rangle_\perp \langle x'^2 \rangle_\perp - \langle x x' \rangle_\perp^2\right]$$

where $\langle x x' \rangle_\perp^2 = 0$.

$$\rightarrow \boxed{\varepsilon_x^2 = 16 \frac{T}{\gamma_b m \beta_b^2 c^2} \langle x^2 \rangle_\perp = 4\left(\frac{T}{\gamma_b m \beta_b^2 c^2}\right) r_b^2} \quad -(2)$$

The matched envelope equation

$$\cancel{r_b''}^{\,0} + k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon_x^2}{r_b^3} = 0$$

can then be used to equivalently express r_b as:

$$\boxed{r_b^2 = 4\langle x^2\rangle_\perp = \frac{1}{k_{\beta 0}^2}\left[4\left(\frac{T}{\gamma_b m \beta_b^2 c^2}\right) + Q\right]} \quad (3)$$

Eqs. (1) – (3) obtain the needed constraints to parameterize the solutions in terms of usual accelerator variables. They can be written as

$$\boxed{\begin{aligned} Q &= \left(\frac{T}{\gamma_b m \beta_b^2 c^2}\right)\int_0^\infty d\rho\, \rho\, e^{-\tilde{\psi}(\rho)} \\ k_{\beta 0}^2 \varepsilon_x^2 &= 4\left(\frac{T}{\gamma_b m \beta_b^2 c^2}\right)\left[4\left(\frac{T}{\gamma_b m \beta_b^2 c^2}\right) + Q\right] \\ k_{\beta 0}^2 &= \left(\frac{T}{\gamma_b m \beta_b^2 c^2}\right)\frac{(1+\Delta)}{2(\gamma_b \lambda_D)^2} \end{aligned}}$$

- Integrals depend only on Δ via solution to

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \tilde{\psi}}{\partial \rho}\right) = 1 + \Delta - e^{-\tilde{\psi}}$$

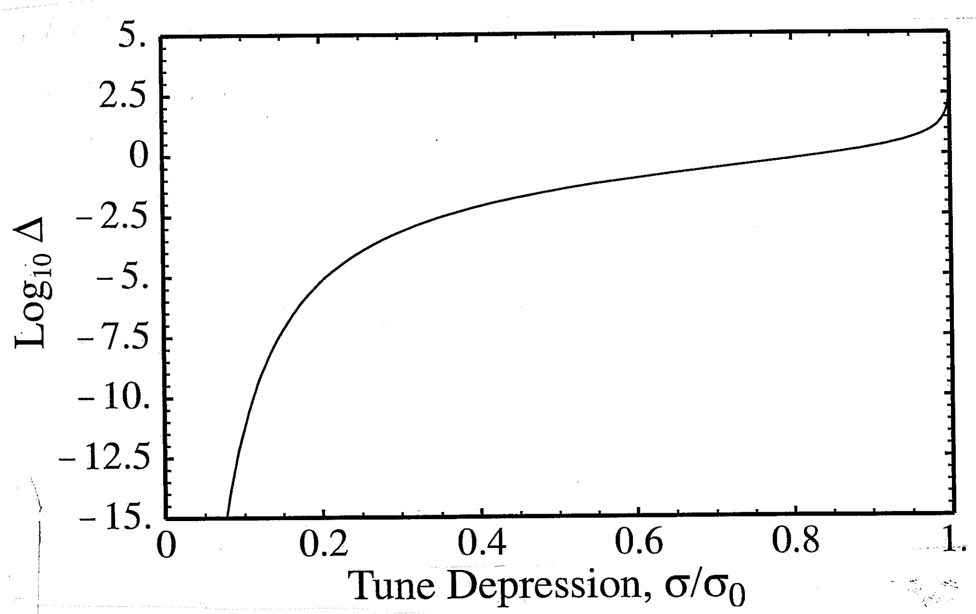
$$\tilde{\psi}(\rho = 0) = 0$$

- Eqns fix $(\gamma_b \lambda_D)$, $T/(\gamma_b m \beta_b^2 c^2)$, and Δ in terms of Q, ε_x and $k_{\beta 0}$.

Using an rms equivalent beam model for Thermal equilibrium we can define an effective tune depression:

$$\boxed{\begin{aligned}\frac{\sigma}{\sigma_0} &= \sqrt{1 - \frac{Q}{k_{\beta 0}^2 r_b^2}} \\ &= \left\{ 1 - \frac{\left[\int_0^\infty d\rho\, \rho\, e^{-\tilde{\psi}(\rho)}\right]^2}{(1+\Delta)\int_0^\infty d\rho\, \rho^3 e^{-\tilde{\psi}(\rho)}} \right\}^{1/2}\end{aligned}}$$

Numerical solution obtains:

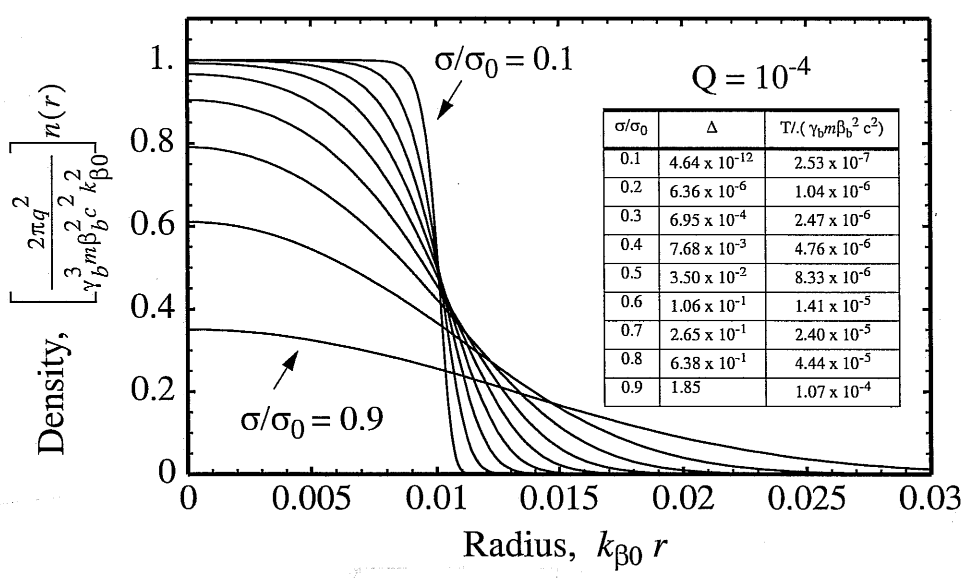


Note that small values of σ/σ_0 correspond to extremely small values of Δ. Special methods must be used to deal with this numerically.

Finally, to better visualize thermal equilibrium structure, we use the equations of constraint to calculate the changes in Thermal equilibrium structure with T for fixed:

$$k_{\beta 0}^2 = \text{const} \quad \Rightarrow \text{Fixed focusing}$$
$$Q = \text{const} \quad \Rightarrow \text{Fixed } \lambda \text{ (charge)}$$

Numerical calculations yield:



Note how the distribution structure changes as T decreases from a broad Gaussian-like structure to a profile with a sharp edge.

From these results it is not surprising that the KV model works well for strong space-charge since the charge distribution becomes sharp edge and flat for $(\delta\sigma/\sigma_0)_{rms}$ small.

- Thermal equilibrium may also overestimate the "edge" width since $T = \text{const}$.
- Edge contains strongest nonlinear terms and deviations from KV model. This can and does change the physics!
- More realistic smooth $f_\perp(H_\perp)$ will always have a spectrum of particle oscillation frequencies that are amplitude dependent.

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USPAS Jan. 2004

Ref: R.C. Davidson "Theory of Nonneutral Plasmas" Addison-Wesley, 1990 §3.4

§6 Debye Screening in Thermal Equilibrium.

Nonrel. limit in this section

$\gamma_b = 1$

$\beta_b c = v_b$

Consider a test line-charge λ_T placed at $r=0$ in cylindrical geometry. In free-space the Poisson equation describing this field is:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi_T}{\partial r}\right) = \frac{-\lambda_T}{2\pi\varepsilon_0}\frac{\delta(r)}{r}$$

$$\boxed{\begin{aligned} E_r = -\frac{\partial \phi_T}{\partial r} &= \frac{\lambda_T}{2\pi\varepsilon_0 r} \\ \phi_T &= -\frac{\lambda_T}{2\pi\varepsilon_0}\ln r + \text{const.} \end{aligned}}$$

Now consider the test line-charge placed in the center ($r=0$) of a thermal equilibrium beam.

$$\boxed{\phi = \phi_0 + \delta\phi}$$

ϕ_0 = thermal eq. potential

$\delta\phi$ = perturbed potential due to test line-charge

The Poisson equation for ϕ is:

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) = \frac{-q}{\varepsilon_0}\int d^2x' \, f_\perp(H_\perp) - \frac{\lambda_T}{\varepsilon_0}\frac{\delta(r)}{r}}$$

Assume that the test line-charge is a small perturbation on a thermal equilibrium core and that the equilibrium adapts adiabatically to the test charge, then:

$n_0 \equiv \hat{n}$ in §5

$$\boxed{\begin{aligned} n(r) &= \int d^2x' f_\perp(H_\perp) \simeq n_0\, e^{-\tilde{\psi}_0(r)}\, e^{-q\delta\phi/T} \\ \tilde{\psi}_0 &\equiv \frac{m v_b^2}{T}\left\{ \frac{k_{\beta 0}^2}{2} r^2 + \frac{q\phi_0}{m v_b^2} \right\} \end{aligned}}$$

Consistently, to leading order: $\left(|q\delta\phi| \ll T \right)$

$$\boxed{n(r) \simeq n_0\, e^{-\tilde{\psi}_0(r)} \left[1 - \frac{q\delta\phi}{T} \right]}$$

Then:

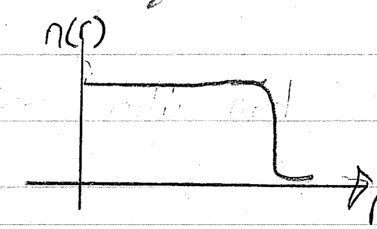
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \phi_0}{\partial r} \right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \delta\phi}{\partial r} \right) = -q n_0\, e^{-\tilde{\psi}_0(r)} + \frac{q^2 \delta\phi}{\epsilon_0 T} n_0\, e^{-\tilde{\psi}_0(r)} - \frac{\lambda_T}{2\pi\epsilon_0}\frac{\delta(r)}{r}$$

cancels

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \delta\phi}{\partial r} \right) = \frac{q^2 \delta\phi}{\epsilon_0 T} \underbrace{n_0\, e^{-\hat{\psi}_0(r)}}_{\equiv n_0(r)} - \frac{\lambda_T}{2\pi\epsilon_0}\frac{\delta(r)}{r}}$$

We assume further that the background thermal equilibrium is relatively cold so that:

$$\boxed{n_0\, e^{-\tilde{\psi}_0(r)} \simeq n_0}$$



for radii not near the beam edge.

Then the Poisson equation becomes:

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \delta\phi}{\partial r}\right) - \frac{\delta\phi}{\lambda_D^2} = -\frac{\lambda_T}{2\pi\varepsilon_0}\frac{\delta(r)}{r}}$$

where: $\lambda_D = \left(\frac{\varepsilon_0 T}{q^2 n_0}\right)^{1/2}$ = Debye radius formed from the on-axis, peak density at $r=0$.

Close to the test charge, the $\delta\phi/\lambda_D^2$ term is negligible and the Poisson equation reduces to:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \delta\phi}{\partial r}\right) = -\frac{\lambda_T}{2\pi\varepsilon_0}\frac{\delta(r)}{r}$$

$$\boxed{r \to 0 \Rightarrow \quad \delta\phi \simeq -\frac{\lambda_T}{2\pi\varepsilon_0}\ln r + \text{const.}}$$

On the other hand, for $r \neq 0$, the Poisson equation reduces to:

$$\left\{\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\right\}\delta\phi - \delta\phi = 0$$

$$\rho = \frac{r}{\lambda_D}$$

This is a modified Bessel equation of order 0 with a general solution:

$$\boxed{r \neq 0: \quad \delta\phi = C_1 I_0(\rho) + C_2 K_0(\rho) \qquad C_1, C_2 \text{ constants.}}$$

From the limiting forms:

$\rho \ll 1$

$$I_0(\rho) \to 1 + \mathcal{O}(\rho^2)$$
$$K_0(\rho) \to -\left[\ln\left(\frac{\rho}{2}\right) + 0.5772\ldots\right] + \mathcal{O}(\rho^2)$$

$\rho \gg 1$

$$I_0(\rho) \to \frac{1}{\sqrt{2\pi\rho}}\, e^{\rho}\left[1 + \mathcal{O}\left(\frac{1}{\rho}\right)\right]$$
$$K_0(\rho) \to \sqrt{\frac{\pi}{2\rho}}\, e^{-\rho}\left[1 + \mathcal{O}\left(\frac{1}{\rho}\right)\right]$$

It is clear for a physical solution with bounded field as $\rho \to \infty$ and connecting to the limiting form:

$$\lim_{r \to 0} \delta\phi \simeq -\frac{\lambda_T}{2\pi\epsilon_0} \ln \rho + \text{const}$$

we must choose:

$$C_1 = 0$$
$$C_2 = \frac{\lambda_T}{2\pi\epsilon_0}$$

and the solution for all r can be expressed as:

$$\boxed{\delta\phi = \frac{\lambda_T}{2\pi\epsilon_0} K_0\left(\frac{r}{\lambda_D}\right)}$$

Then for $r \gg \lambda_D$ we have:

$$\boxed{\delta\phi = \frac{\lambda_T}{2\sqrt{2\pi}\,\epsilon_0 \sqrt{r/\lambda_D}}\, e^{-r/\lambda_D} \qquad r \gg \lambda_D.}$$

- Distant interactions are screened by the factor e^{-r/λ_D}
- Note that this screening occurs regardless of the sign of the test line-charge λ_T.

We have shown that "particles" surrounding a test charge in an intense beam will tend to redistribute so as to screen the bare interaction for distances $r > \lambda_D$. This plasma property does not require overall charge neutrality! The main difference from the neutral plasma case is that the interaction is superimposed on the intense beam self-fields of the thermal equilibrium core:

$$\phi_0 \simeq -\frac{q n_0}{4\epsilon_0} r^2 \qquad r \text{ far from edge}$$

- In the homework problems a solution is found for the 3D case and a similar screening result holds.
  - Analysis simpler in 3D!

This is part of the reason why lower dimension numerical simulations can often predict collective plasma effects reliably that are really 3D effects .... because the screening form is ~ the same, the reduction in dimensionality does not change the essential physics.

More on this later in simulation lectures.

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USPAS Jan. 2004

§7 Density Inversion Theorem.

In a continuous focusing channel, the dependance of an equilibrium distribution on x, x' are strongly interconnected. We will show that a macroscopic specification of a monotonic density profile sufficient to completely fix the full distribution function.

$$f = f_\perp(H_\perp) \quad ; \quad H_\perp = \frac{1}{2}\vec{x}_\perp'^2 + \frac{k_{\beta 0}^2}{2}\vec{x}_\perp^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2}$$

$$\equiv \frac{1}{2}\vec{x}_\perp'^2 + \psi(\vec{x}_\perp)$$

Density

$$\boxed{n(\vec{x}_\perp) = \int d^2x' \, f_\perp(H_\perp) = \int d^2x' \, f_\perp\left(\frac{\vec{x}_\perp'^2}{2} + \psi(\vec{x}_\perp)\right)}$$

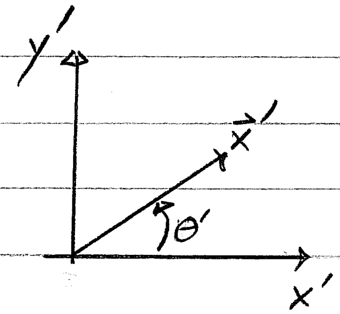
$$U \equiv \frac{\vec{x}_\perp'^2}{2} \quad ; \quad \vec{x}_\perp'^2 \text{ independent of "angle"}$$

$$\Rightarrow \quad n(\vec{x}_\perp) = 2\pi \int_0^\infty dU \, f_\perp(U + \psi(\vec{x}_\perp))$$

$$\frac{\partial n}{\partial \psi} = 2\pi \int_0^\infty dU \, \frac{\partial f_\perp}{\partial \psi}(U + \psi) = 2\pi \int_0^\infty dU \, \frac{\partial f_\perp(U+\psi)}{\partial U}$$

$$= 2\pi \underbrace{\lim_{U\to\infty} \overbrace{f_\perp(U+\psi)}^{\to 0}}_{\text{Bounded dist.}} - 2\pi f_\perp(\psi) = -2\pi f_\perp(\psi)$$

// **Aside**



$$\vec{x}' = R' \hat{e}_{R'} \qquad ; \qquad \vec{x}'^2 = R'^2$$

$$n(\vec{x}_\perp) = \int d^2x' \, f_\perp\left(\frac{\vec{x}'^2}{2} + \psi\right)$$

$$= \int_0^\infty dR' R' \int_0^{2\pi} d\theta' \, f_\perp\left(\frac{R'^2}{2} + \psi\right)$$

$$= 2\pi \int_0^\infty dR' R' \, f_\perp\left(\frac{R'^2}{2} + \psi\right)$$

$$\mathcal{T} \equiv \frac{R'^2}{2} \qquad d\mathcal{T} = R' dR'$$

$$n(\vec{x}_\perp) = 2\pi \int_0^\infty d\mathcal{T} \, f_\perp(\mathcal{T} + \psi)$$

//

Obtain the inversion theorem:

$$\boxed{\begin{aligned} f_\perp(H_\perp) &= -\frac{1}{2\pi}\frac{\partial n}{\partial \psi}\bigg|_{\psi = H_\perp} \\ \psi(\vec{x}_\perp) &= \frac{k_{\beta 0}^2}{2}\vec{x}_\perp^2 + \frac{q}{m\gamma_b^3\beta_b^2 c^2}\phi \end{aligned}}$$

This inversion theorem can be used to construct equilibrium distribution functions for a continuous focusing channel. To see this, take

$n(r)$ = density, specified (measured) function of r

The Poisson equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) = \frac{-q}{\epsilon_0} n(r)$$

can then be integrated to show that

$$\psi(r) - \frac{q\phi(0)}{m\gamma_b^3\beta_b^2 c^2 \epsilon_0} = \frac{k_{\beta 0}^2}{2} r^2 - \frac{q^2}{m\gamma_b^3\beta_b^2 c^2 \epsilon_0}\int_0^r \frac{d\tilde{r}}{\tilde{r}}\int_0^{\tilde{r}} d\tilde{\tilde{r}}\, \tilde{\tilde{r}}\, n(\tilde{\tilde{r}})$$

For $n(r) = \text{const}$, note that:

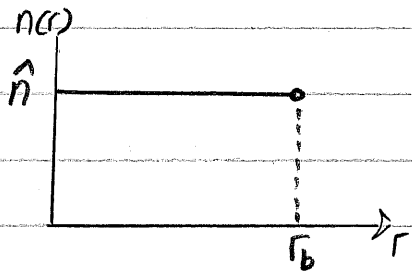
$$\int_0^r \frac{d\tilde{r}}{\tilde{r}}\int_0^{\tilde{r}} d\tilde{\tilde{r}}\, \tilde{\tilde{r}}\, n(\tilde{\tilde{r}}) \propto r^2$$

Thus, expect $\psi(r)$ monotonic in r when dn/dr is monotonic.

In this case of monotonic $\psi(r)$:

$$\boxed{f_\perp(H_\perp) = \frac{-1}{2\pi}\left.\frac{\partial n}{\partial \psi}\right|_{\psi=H_\perp} = \frac{-1}{2\pi}\left.\frac{\frac{\partial n(r)}{\partial r}}{\frac{\partial \psi(r)}{\partial r}}\right|_{\psi=H_\perp}}$$

// **Example:** KV equilibrium distribution: (uniform charge)



$$n(r) = \begin{cases} \hat{n} = \text{const} & 0 < r < r_b \\ 0 & r_b < r \end{cases}$$

$$\frac{\partial n}{\partial \psi} = \frac{\partial n}{\partial r}\frac{\partial r}{\partial \psi}$$

$$\frac{\partial n}{\partial r} = -\hat{n}\,\delta(r - r_b) \qquad \delta(x) = \text{Dirac } \delta\text{-function}$$

$$\frac{\partial n}{\partial \psi} = -\frac{\hat{n}\,\delta(r-r_b)}{\left.\frac{\partial \psi}{\partial r}\right|_{r=r_b}} = -\hat{n}\,\delta\big(\psi(r) - \psi(r=r_b)\big)$$

$$f_\perp(H_\perp) = \frac{-1}{2\pi}\left.\frac{\partial n}{\partial \psi}\right|_{\psi=H_\perp} = \frac{\hat{n}}{2\pi}\delta\big(H_\perp - \psi(r=r_b)\big)$$
$$= \frac{\hat{n}}{2\pi}\delta\big(H_\perp - H_b\big)$$

$$\psi(r=r_b) = \frac{k_{\beta 0}^2}{2} r_b^2 + \frac{q\,\phi(r=r_b)}{m\gamma_b^3\beta_b^2 c^2} = H_\perp(\vec{x}\,', r_b)\Big|_{\vec{x}\,'=0} = H_b$$

This treatment is simple but not rigorous. Application is conceptually easier (but more sloppy) for strictly monotonic profiles with $dn/dr < 0$. //

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USPAS, Jan. 2004

References

The class text Reiser contains part of the material presented in these lectures. Some parts of the treatment in Reiser are not complete (self-field forces are not derived from the KV distribution, etc.) The class notes are fairly complete. For more information see:

KV equilibrium, Thermal equilibrium, Debye screening, density inversion theorem, ...

R.C. Davidson, "Theory of Nonneutral Plasmas," (Addison-Wesley, New York 1990).

-and (updated material)

R.C. Davidson and H. Qin, "Physics of Intense Charged Particle Beams in High Energy Accelerators" (World Scientific, 2001)

Original Paper of KV equilibrium (for historic value):

I Kapchinskij and V. Vladimirskij, in "Proceedings of the International Conference on High Energy Accelerators and Instrumentation" (CERN Scientific Information Service, Geneva, 1959) p. 274.

Envelope mode review paper that contains much information on self-consistent KV distributions (Appendix A) (Handed out in class.)

S.M. Lund and B. Bukh, "Stability Properties of the Transverse Envelope Equations Describing Intense Ion Beam Transport," Phys. Rev. Special Topics — Accel. and Beams, to appear 2004.

Other useful information (hard to find) on KV
beam equilibrium structure can be found in:
F.J. Sacherer, Ph.D thesis, University of California
at Berkeley (1968).