Wed Jan 16 Lecture Notes

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1 Transport and One-Turn Matrices

I will refer to Vladimir's morning lecture notes like he is, with the notation (initial-day-equation). For example, the definition of the **transport matrix** $M(s_0|s)$ is given by equation (V3-5):

$$X(s) = M(s_0|s)\, X_0 \tag{1}$$

Or, in one dimension (much to Vladimir's dismay)

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = M(s_0|s) \begin{pmatrix} x(0) \\ x'(0) \end{pmatrix} \tag{2}$$

where the prime denotes differentiation with respect to s.

Transport matrices (and, later, Courant-Snyder parameters) are the bread and butter of accelerator physics. This lecture will be devoted to outlining how some simple transport matrices look for the most common elements in separated-function synchrotrons: drift spaces, quadrupoles (thick and thin), bending dipoles and bending dipoles. We'll stick to one dimension for simplicity, and to give you a breather; nonetheless, we can calculate useful results even in one dimension. Remember from Vladimir's lecture that all the transport matrices M are **symplectic**, and it is easy to calculate its inverse (V3-21):

$$M(s_1|s_0) = M^{-1}(s_0|s_1) = -S\, \tilde{M}(s_0|s_1)\, S \tag{3}$$

where the tilde represents the transpose.

The trasport matrix describes the evolution of phase space coordinates (x, x'), so you can follow an evolution from s_0 to s_1 to s_2 by concatenating successive transport matrices:

$$X(s_3) = M(s_2|s_3)\, M(s_1|s_2)\, M(s_0|s_1)\, X(s_0) \tag{4}$$

We can then assemble a complete accelerator lattice by combining the transport matrices of individual elements.

For a synchrotron, the lattice clearly has a periodicity of its circumference C. (Most lattices are at least close to a higher-order periodicity; for example, the AGS is nearly 12-fold periodic and RHIC is nearly 3-fold periodic until we break that symmetry by focusing beams for collisions in 2 locations.) The **one-turn matrix** around an accelerator is the complete transport of a particle around the ring. Using the transport matrix,

$$X(s + C) = M(s|s + C)\, X(s) \tag{5}$$

The one-turn matrix can be used to generate a Poincare' plot, or a plot of phase space coordinates of a particle on successive circulations around the machine. For a static accelerator

lattice, the one-turn matrix has periodicity C. The solution for $X(s)$ that has the same periodicity is called the **closed orbit**, which ideally is the same as our reference orbit:

$$X_{\rm co}(s+nC) = M(s|s+C)^n\, X_{\rm co}(s) = X_{\rm co}(s) \quad \text{for all } n \tag{6}$$

One particularly powerful parameterization for the one-turn matrix is the exponential form:

$$M = I\cos(2\pi\nu) + J\sin(2\pi\nu) = e^{2\pi\nu J} \tag{7}$$

where ν is called the **betatron tune** for this plane,

$$J \equiv \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \qquad J^2 = -I \tag{8}$$

and β, $\alpha \equiv -\beta'/2$, and $\gamma \equiv \frac{1+\alpha^2}{\beta}$ are commonly referred to as the Courant-Snyder parameters.

2 Transport Matrices of Regular Elements

What are the transport matrices of some common accelerator elements? We will assume that the elements have negligible fringe fields compared to the wavelength of particle oscillations in the lattice, so we can treat each of our simple lattice elements individually.

2.1 Drift

The drift case (all fields are zero, $K = 0$, and torsion $\kappa = 0$) was treated in Vladimir's lecture, (V3-28/29). For the transverse dimensions (x, x') and (y, y') and a field-free drift space of length L,

$$M(s+L|s)_{\rm drift} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \tag{9}$$

This means that the angle x' does not change through a drift, but the position changes between the entrance and exit of the drift by $x(s+L) = x(s) + x'L$. From a visualization of the particle motion through a drift, it would be easy to write down the transport matrix even if you don't know any Hamiltonian dynamics.

Unfortunately it is hard to build an accelerator from just drifts and free space. We need to focus with quadrupoles, and we need to bend the beam with dipoles.

2.2 Thick and Thin Quadrupole

In yesterday's lecture and homework we took a look at the quadrupole field

$$B_y + iB_x = B_0 b_1 (x+iy) \quad \text{or} \quad B_y = B_0 b_1 x \quad B_x = B_0 b_1 y \tag{10}$$

We have not discussed the quadrupole length L, but we should assume that it is considerably longer than the aperture so we can neglect fringe fields. We also use the paraxial approximation, assuming that x' is small so we can neglect dL/dx' terms. Then we can write differential equations of the particle motion through the field:

$$x'' + Kx = 0 \quad y'' - Ky = 0 \quad \text{where} \quad K = \frac{q}{p}\frac{\partial B_y}{\partial x}\Big|_{x=0,y=x} \tag{11}$$

having used Maxwell's equations.

These differential equations can be solved to find the transport matrix for a thick quadrupole, which may look familiar:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos\sqrt{K}L & \frac{1}{\sqrt{K}}\sin\sqrt{K}L \\ -\sqrt{K}\sin\sqrt{K} & \cos\sqrt{K}L \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = M_{\text{focusing}} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} \tag{12}$$

$$\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} \cosh\sqrt{K}L & \frac{1}{\sqrt{K}}\sinh\sqrt{K}L \\ -\sqrt{K}\sinh\sqrt{K} & \cosh\sqrt{K}L \end{pmatrix} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix} = M_{\text{defocusing}} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix} \tag{13}$$

Here we have applied the boundary conditions that $M_{\text{focusing}} = M_{\text{defocusing}} = I$ for $K = 0$. Note that a particle displaced by a small position Δx receives a kick, or $\Delta x'$ towards $x = 0$, the reference orbit; this quadrupole is focusing in the horizontal plane and defocusing in the vertical plane. Reversing the current in the leads or rotating the quadrupole by 90 degrees makes it vertically focusing and horizontally defocusing.

Your first homework problem is to demonstrate that a net focusing can be achieved in both planes by alternating focusing and defocusing quadrupoles under certain circumstances.

In some quadrupoles, particularly in large accelerators, the focal length of the quadrupole is much longer than the quadrupole length itself. In this case we can approximate the effect of the quadrupole as a single kick and neglect the particle's change in position through the quadrupole. This is equivalent to keeping only first-order terms in the cos and sin expansions above. For $\sqrt{K}L << 1$ with $L \to 0$ and kL constant, we then have

$$M_{\text{focusing,defocusing}} = \begin{pmatrix} 1 & 0 \\ \mp KL & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \mp KL & 1 \end{pmatrix} \tag{14}$$

2.3 Dipole: Different Than A Drift

A pure dipole field bends the beam in an arc of a circle. We will neglect edge focusing and use the coordinate system that follows a reference particle through the center of the dipole. First consider a short dipole of length L that gives the particle a horizontal kick $\Delta x' << 1$. Here we cannot write the effect of this magnet as a transport matrix since the kick is not proportional to x or x' — even the reference orbit is affected! Indeed, dipole correctors are used to perform orbit correction (fixing the reference orbit to be through the centers of the quadrupoles).

Now consider a long sector dipole which bends the beam by angle θ with a radius ρ in the horizontal plane. A reference particle passes through the center of this dipole on a trajectory that is an arc of a circle, and indeed all particles travel on circular arcs through the dipole with constant field, so we can derive the transport matrix of the dipole geometrically.

Vertical motion is unaffected, since the magnetic field is in this direction. The vertical transport matrix therefore looks like a drift with arc length $\rho\theta$:

$$\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} 1 & \rho\theta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix} \tag{15}$$

The horizontal motion is somewhat more complicated, and can be derived using Vladimir's methods or shown through geometric arguments:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{1}{\rho}\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} \tag{16}$$

The full dipole transport matrix can be derived from Vladimir's notes, including the

effects of longitudinal position and momentum offset relative to the reference particle:

$$M_{\text{dipole}} = \begin{pmatrix} \cos\theta & \rho\sin\theta & 0 & 0 & 0 & \rho(1-\cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & 0 & 0 & 0 & \sin\theta \\ 0 & 0 & 1 & \rho\theta & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\sin\theta & -\rho(1-\cos\theta) & 0 & 0 & 1 & -\rho(\theta-\sin\theta) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{17}$$

3 Thin Dipole Error

Consider a think dipole error $\Delta x'$ at a specific location in the lattice. What is its effect on the closed orbit $(x_{\text{co}}, x'_{\text{co}})$ at that location? We can write the motion using the one-turn map M as

$$M \begin{pmatrix} x_{\text{co}} \\ x'_{\text{co}} \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta x' \end{pmatrix} = \begin{pmatrix} x_{\text{co}} \\ x'_{\text{co}} \end{pmatrix} \tag{18}$$

to calculate the new closed orbit. This becomes

$$\begin{pmatrix} x_{\text{co}} \\ x'_{\text{co}} \end{pmatrix} = (I - M)^{-1} \begin{pmatrix} 0 \\ \Delta x' \end{pmatrix} \tag{19}$$

Now we can rewrite the matrix $(I - M)^{-1}$ using the exponential form for M:

$$\begin{aligned} (I - M)^{-1} &= \left[e^{\pi\nu J}\left(e^{-\pi\nu J} - e^{\pi\nu J}\right)\right] && (20) \\ &= -(2J\sin\pi\nu)^{-1}(e^{\pi\nu J})^{-1} && (21) \\ &= \frac{1}{2\sin\pi\nu} J e^{-\pi\nu J} && (22) \\ &= \frac{1}{2\sin\pi\nu}(J\cos\pi\nu + I\sin\pi\nu) && (23) \end{aligned}$$

Using the Courant-Snyder parameters from J, we find

$$\begin{pmatrix} x_{\text{co}} \\ x'_{\text{co}} \end{pmatrix} = \frac{\Delta x'}{2\sin\pi\nu} \begin{pmatrix} \beta\cos\pi\nu \\ \sin\pi\nu - \alpha\cos\pi\nu \end{pmatrix} \tag{24}$$

where β and α are defined at the point of the perturbation. We will see tomorrow how this error propagates around the ring to other locations.

Poisson Brackets and Lie Operators

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1 Symplecticity and Poisson Brackets

1.1 Symplecticity

Consider an n-dimensional (2n-dimensional phase space) linear system. Let the canonical coordinates of the system be

$$X = \begin{pmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \\ \cdots \\ q_n \\ p_n \end{pmatrix} \tag{1}$$

When we index X_i here, $0 < i < 2n$, including both q and p. Let M be the $2n \times 2n$ matrix that describes the map that brings the coordinates of the particles from the initial position $s = 0$ to the time of observation s in this linear dynamical system: $X = MX_0$. Then M must satisfy the symplecticity condition

$$M^{\mathrm{T}} S M = S \tag{2}$$

where the matrix S is the block-diagonal symplectic form:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_n \tag{3}$$

Note that all symplectic matrices are necessarily even dimensional. Since $S^2 = -I$, S may be thought of as the matrix equivalent of $i = \sqrt{-1}$. This analogy extends to the point that exponentials of symplectic matrices are similar to rotations.

What is remarkable is that the symplecticity condition, Eq. (2), applies also to a nonlinear system if we identify M to be the Jacobian matrix of the map, whose elements are defined as

$$X = MX_0 \qquad M_{ij} = \frac{\partial X_i}{\partial (X_0)_j} \tag{4}$$

where $(X_0)_j$ is the j^{th} component of the initial coordinates of a particle at $s = 0$ (including both coordinates and momenta!), and X_i is the i^{th} component of the final state X of the particle at arbitrary time s. In a linear system, the Jacobian is just the transformation matrix, and is independent of the particle coordinates. In a nonlinear system the Jacobian matrix M depends on the coordinates of X_0 and the symplecticity condition Eq. (2) must be satisfied for all X_0.

The symplecticity condition resembles a unitarity condition since the left hand side of Eq. (2) is quadratic in M, while the right hand side is almost a unit matrix. This imposes very strong constraints on M. Some things immediately follow from the symplecticity condition:

1. S and I are both symplectic.
2. If M is symplectic, then $\det(M) = \pm 1$. (We restrict ourselves to $\det(M) = +1$.)
3. M is invertible, with $M^{-1} = S^{-1}M^T S = S^T M^T S$.
4. If M is symplectic, so are M^{T} and M^{-1}.
5. If both M and N are symplectic, then MN is symplectic.
6. If λ is an eigenvalue of a symplectic matrix M, then so is $1/\lambda$.

This is already starting to look algebraic.

One bit of magic is that *all Hamiltonian systems are symplectic.* This includes both linear and nonlinear Hamiltonian systems, or even when the Jacobian depends on the coordinates! You proved this in your homework last week. For linear systems, the map is independent of X and X_0, so the symplectic condition only has to hold for all time s. However, for nonlinear systems where the map depends on X_0, the symplectic condition must hold for all s and X_0. That's a strong constraint!

The symplectic condition imposes a total of $n(2n-1)$ constraints since $M^{\mathrm{T}}SM = S$ is antisymmetric, so the $2n \times 2n$ matrix M therefore has $n(2n+1)$ independent elements. In the $n = 1$ case, there is only one constraint (unit determinant), and 3 independent elements; for $n = 2$ there are 6 constraints and 10 independent elements. When we get to treating the group of symplectic matrices as a Lie algebra, the independent elements give rise to the generators of the group.

1.2 Poisson Brackets

The Poisson bracket between functions $f(X;s)$ and $g(X;s)$ of canonical coordinates might be familiar from Hamiltonian mechanics:

$$[f,g] \equiv \sum_{i=0}^{2n} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right) \tag{5}$$

$$= \sum_{i,j} \frac{\partial f}{\partial X_i} S_{ij} \frac{\partial g}{\partial X_j} \tag{6}$$

where again the i, j range over all coordinates and momenta. Just like symplectic matrices, the Poisson bracket has some handy properties:

1. Antisymmetry: $[f,g] = -[g,f]$
2. Real distributivity: $[af + bg, h] = a[f,h] + b[g,h]$ for $\forall a, b \in \Re$
3. Functional distributivity: $[f, gh] = [f,g]h + g[f,h]$
4. Jacobi identity: $[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = 0$
5. Fundamental Poisson brackets: $[X_i, X_j] = S_{ij}$
6. $[f,g] = 0$ if f or g are constant with respect to X.

2 Lie Operators and Lie Algebras

2.1 Back To The Map

Recall that for the past week, we've been learning how to solve this in a different way, by writing down a linear Jacobian transport matrix that propagates the coordinates X_0 to a new coordinate $X(s)$ with a transport matrix:

$$X = MX_0 \qquad M_{ij} = \frac{\partial X_i}{\partial (X_0)_j} \tag{14}$$

The question is how to extend this to nonlinear systems, since accelerators are nonlinear. The Jacobian is the way that output conditions vary from small variations in input conditions. One natural way to express a nonlinear transport map is in terms of a truncated power series of the original coordinates:

$$X = F(X_0) \quad \text{(Truncated power series)} \tag{15}$$

where $F(X_0)$ is, say, a collection of N^{th} order power series in the components of X_0 where N is the order to which we are truncating in a perturbative expansion.

Recall that for periodic systems, we often wrote the transport matrix in terms of an exponential. This exponential depended on the details of our magnets and the lattice layout, but did not depend on the coordinates themselves. It turns out that another natural way to extend the Jacobian transport of Eq. (14) is with a Lie map, which is the exponential of a differential operator $:G(s):$ (which we'll define in a moment) that can depend on the coordinates:

$$X = \mathrm{e}^{:G(X):} X \Big|_{X=X_0} \quad \text{(Lie map)} \tag{16}$$

Note some subtle differences between the truncated power series and the Lie map:

- The Lie map is symplectic by definition, while the truncated power series is not!
- $G(X)$ is a function of the coordinates, while the truncated power series is only a function of the initial coordinates X_0.
- There are many functions F, one for each coordinate, while there is only one $G(X)$, called the generator of the Lie transformation. It is a linear combination of the generators we saw earlier.
- To get the same equivalent order, we need to write the Lie map to the $(N+1)^{\text{st}}$ order. (We'll see we need an extra term from the Poisson bracket differentiation.)

Some group-theoretic babble before we continue: a Lie group is a mathematical group which is also a finite-dimensional real smooth manifold, and in which the group operations of multiplication (or concatenation) and inversion are smooth maps. Generally Lie groups are generated by infinitesimal generators. Some examples of Lie groups are:

- $\Re^n$ is an abelian Lie group under addition
- The orthogonal group $\mathrm{O_n}(\Re)$, the group of all rotations and reflections of an n-dimensional vector space. The subgroup of elements of determinant one is called the $\mathrm{SO_n}(\Re)$ special orthogonal group, or rotation group.
- The group $\mathrm{Sp_{2n}}(\Re)$, or group of all symplectic matrices.
- U(1)×SU(2)×SU(3), the composition group of the Standard Model.

Why is the Poisson bracket useful? Consider $f(X(s);s)$: it changes in time s either because of explicit s dependence, or because it depends on $X(s)$. The total time derivative f' can then be written as:

$$f' = \frac{\partial f}{\partial s} + \sum_i \frac{\partial f}{\partial X_i} X_i' \tag{7}$$

$$= \frac{\partial f}{\partial s} + \sum_{i,j} \frac{\partial f}{\partial X_i} S_{ij} \frac{\partial H}{\partial X_j} \tag{8}$$

$$= \frac{\partial f}{\partial s} + [f, H] \tag{9}$$

So a quantity $f(X)$ is a constant of the motion described by H if it is not explicitly s-dependent, and if it has

$$[f, H] = 0 \tag{10}$$

Poisson brackets are therefore intimately related to the time evolution of phase space quantities. It looks like the only relevant Poisson brackets involve the Hamiltonian here, but we'll find out that Poisson brackets of other quantities are also useful. As we move from linear to nonlinear dynamics, we will see that Lie algebras in accelerator physics are basically a formalism to simplify calculations within the algebra of Poisson brackets.

Poisson brackets in differential algebras used in accelerator tracking are often computed between two Taylor series of X. We can see that if f is an n^{th} order and g is an m^{th} order Taylor series, their Poisson bracket is another Taylor series of order $m + n - 2$.

1.3 Example: Coupled harmonic oscillators

Consider a pair of simple degenerate harmonic oscillators described by the Hamiltonian

$$H = \frac{1}{2}(\omega^2 x^2 + p_x^2 + \omega^2 y^2 + p_y^2) \tag{11}$$

It's almost obvious (though we can show) that $f_1 = \omega^2 x^2 + p_x^2$ and $f_2 = \omega^2 y^2 + p_y^2$ are constants of the motion. For example,

$$[f_1, H] = [\omega^2 x^2 + p_x^2, \frac{1}{2}[\omega^2 x^2 + p_x^2]]$$

$$= \frac{\omega^2}{2}\left([x^2, p_x^2] + [p_x^2, x^2]\right) = 0 \tag{12}$$

However, another constant of the motion is $g = xp_y - yp_x$; this corresponds to the angular momentum:

$$[g, H] = \frac{1}{2}[xp_y - yp_x, \omega^2 x^2 + p_x^2 + \omega^2 y^2 + p_y^2]$$

$$= \frac{1}{2}\left([xp_y, p_x^2] + \omega^2[xp_y, y^2] - \omega^2[yp_x, x^2] - [yp_x, p_y^2]\right) = 0 \tag{13}$$

You can see here why this has to be a degenerate oscillator for the angular momentum to be a conserved quantity. By forming $[f_1, g]$ or $[f_2, g]$, we can also find that $h = \omega^2 xy + p_x p_y$ is a constant of the motion, but it's a combination of the other invariants: $\omega^2 g^2 + h^2 = f_1 f_2$.

2.2 Lie Operators

Since it's a pain to keep writing brackets all over the place, and because we're not confused enough yet, we'll rewrite the Poisson bracket in another notation that emphasizes its operator nature:

$$:f:g \equiv [f,g] = \sum_{i,j} \frac{\partial f}{\partial X_i} S_{ij} \frac{\partial g}{\partial X_j} \tag{17}$$

$f(X)$ is known as a Lie operator that operates on the function $g(X)$. Antisymmetry immediately follows: $:f:g = -:g:f$. One convenience of this notation is that powers of this operator are easier to write, so

$$(:f:)^2 g = [f,[f,g]] \qquad (:f:)^3 g = [f[f,[f,g]]] \qquad \ldots \tag{18}$$

$$(:f:)^k (gh) = \sum_{n=0}^{k} \frac{k!}{n!(k-n)!} [(:f:)^n g][(:f:)^{k-n} h] \tag{19}$$

The Jacobi identity also helps us find that the commutator of two Lie operators $:f:$ and $:g:$ is given by the Poisson brackets:

$$\{f,g\} \equiv :f::g: - :g::f: =: [f,g]: \tag{20}$$

This gives a cool variation of the Jacobi identity that can be used to simplify commutators of Lie operators:

$$\{f,g\}h =: h::g:f \tag{21}$$

Eq. (20) is the reason Poisson brackets play a prominent role in Lie algebra of operators. Commutators of operators occur often, and this equation states that they can be calculated in terms of Poisson brackets. So, for example, $:f:$ and $:g:$ commute if $[f,g]$ is constant.

Recall that we are expanding our map in higher order terms in a way similar to expanding a power series or Taylor series using the exponential of a differential operator:

$$e^{:f:} = \sum_{k=0}^{\infty} \frac{1}{k!} (:f:)^k \tag{22}$$

This exponential operator a Lie transformation, with $:f:$ as its generator. It is particularly useful when $:f:$ is nilpotent, i.e. $:f:^n = 0$ for some n.

Note that the Lie algebra is not the same as your typical algebra, particularly because operators do not necessarily commute. For example, $\exp(\ln(x)) = x$, but $\exp(:\ln(x):) \neq :x:$. In particular, the map $\exp(:\ln(x):)$ is symplectic: $\exp(:\ln(x):)x = x$, $\exp(:\ln(x):)p = p + 1/x$, so its Jacobian has unit determinant:

$$M = \frac{\partial X_i}{\partial (X_0)_j} = \begin{pmatrix} 1 & 0 \\ -1/x^2 & 1 \end{pmatrix} \tag{23}$$

while the map $:x:$ is nonsymplectic ($:x:x = 0$ and $:x:p = 1$). Some simple Lie operators in $2n$ dimensions are:

$$\begin{aligned}
:q_i: &= \frac{\partial}{\partial p_i} \qquad :p_i: = -\frac{\partial}{\partial q_i} \\
:q_i p_i: &= p_i \frac{\partial}{\partial q_i} - q_i \frac{\partial}{\partial p_i} \\
:q_i::p_i: &= :p_i::q_i: = -\frac{\partial^2}{\partial q_i \partial p_i} \qquad (:q_i: \text{ and } :p_i: \text{ commute}) \\
:q_i^2: &= 2q_i \frac{\partial}{\partial p_i} \qquad :p_i^2: = -2p_i \frac{\partial}{\partial q_i}
\end{aligned} \tag{24}$$

2.3 The One-Turn Map and the Hamilton-Cayley Theorem

How do we describe the general linear one-turn map of our "standard" Hamiltonian in this formalism? Consider the quadratic form

$$f_2 = -\frac{1}{2}X^T F X \tag{25}$$

where F is symmetric and positive-definite and F is linear, so it does not depend on X. Our linear Hamiltonians can be written this way, for example, since they are quadratic forms in x and p_x with symmetric terms. We can show that $:f_2: X = SFX$ —

$$\begin{aligned} :f_2: X_i &= \frac{\partial f_2}{X_j} S_{jk} \partial X_i \partial X_k \\ &= -\frac{\partial (F_{lm} X_l X_m)}{\partial X_j} S_{ji} \\ &= -(S_{ji} F_{kj} X_k) = -(SFX)_i \end{aligned}$$

This also gives

$$e^{:f_2:} X = e^{SF} X \quad or \quad e^{:f_2:} = e^{SF} \tag{26}$$

Recall that the matrix form of the one-turn map is $T = I\cos\mu + J\sin\mu = e^{\mu J}$ in the linear uncoupled case. From (26), $SF = \mu J$, and we can find F for the one-turn map in terms of the Courant-Snyder parameters:

$$F = \mu S^{-1} J = \mu S^T J = \mu \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} \tag{27}$$

Note that $\det(F) = \mu^2 \geq 0$ so this is positive-definite for a stable lattice. We can then write the one-turn Lie operator f_2 for the one-turn map from (25):

$$f_2 = -\frac{\mu}{2} X^T \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} X = -\frac{\mu}{2}\left(\gamma^2 x^2 + 2\alpha x p_x + \beta p_x^2\right) \tag{28}$$

More generally, we can find the Lie operator from the matrix and vice-versa for any map. (An example of doing this is above, and in your homework.) The above example had the advantage that the matrix form was already an exponential! For other simple cases, like the quadrupole, this is not the case. Then you will have to use the Cayley-Hamilton theorem: every square matrix satisfies its own characteristic equation. Another way of saying this is that if λ_i are the eigenvalues of an $N \times N$ matrix F, then any function

$$f(F) = \sum_{k=0}^{N-1} a_k F^k \tag{29}$$

where the a_k satisfy the $N-1$ equations

$$f(\lambda_i) = \sum_{k=0}^{N-1} a_k \lambda_i^k \tag{30}$$

We can use this to show that in the two-dimensional case (see homework again):

$$F = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad \mu \equiv \sqrt{\det F} \quad \Rightarrow e^{SF} = \cos\mu\, I + \frac{\sin\mu}{\mu} \begin{pmatrix} b & c \\ -a & -b \end{pmatrix} \tag{31}$$

2.4 Lie operators for other accelerator elements

The transport maps for accelerator elements can be represented as Lie transformations. For example, consider the one-dimensional drift. We know that it's usual map is

$$M_{\text{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \tag{32}$$

The Lie transformation corresponding to this is $\exp(:-\frac{1}{2}Lp^2:)$. We can see this by writing out a few terms:

$$\begin{aligned} :p^2:x &= -2p \qquad (:p^2:)^n x = 0 \quad \forall n > 1 \\ :p^2:p &= 0 \qquad (:p^2:)^n p = 0 \quad \forall n > 1 \end{aligned} \tag{33}$$

From this it's apparent that $\exp(:-\frac{1}{2}Lp^2:)x = x + Lp$, and $\exp(:-\frac{1}{2}Lp^2:)p = p$.

We can similarly establish Lie operators for other elements, including nonlinear terms such as thin-lens multipoles. We couldn't do this with the simple linear matrix formalism before, but now we can apply the full power of Lie operators and Lie algebras to concatenate these maps, simulate accelerator maps more efficiently, and solve nonlinear dynamics problems. Some examples of these elements are listed here in Table 1.

Table 1: Lie Operators for Common Accelerator Elements

Element	Map	Lie Operator
Drift space	$x = x_0 + Lp_0$ $p = p_0$	$\exp(:-\frac{1}{2}Lp^2:)$
Thin-lens quadrupole	$x = x_0$ $p = p_0 - \frac{1}{f}x_0$	$\exp(:-\frac{1}{2f}x^2:)$
Thin-lens kick	$x = x_0$ $p = p_0 + \lambda n x_0^{n-1}$	$\exp(:\lambda x^n:)$
Thick focusing quad	$x = x_0 \cos\sqrt{k}L + \frac{p_0}{\sqrt{k}} \sin\sqrt{k}L$ $p = -kx_0 \sin\sqrt{k}L + p_0 \cos\sqrt{k}L$	$\exp(:-\frac{1}{2}L(kx^2 + p^2):)$
Thick defocusing quad	$x = x_0 \cosh\sqrt{k}L + \frac{p_0}{\sqrt{k}} \sinh\sqrt{k}L$ $p = -kx_0 \sinh\sqrt{k}L + p_0 \cosh\sqrt{k}L$	$\exp(:-\frac{1}{2}L(kx^2 - p^2):)$
Coordinate shift	$x = x_0 - b$ $p = p_0 + a$	$\exp(:ax + bp:)$
Coordinate rotation (Phase advance μ)	$x = x_0 \cos\mu + p_0 \sin\mu$ $p = -x_0 \sin\mu + p_0 \cos\mu$	$\exp(:-\frac{\mu}{2}(x^2 + p^2):)$
Full-turn Hamiltonian	(lots of things)	$\exp(C : H_{\text{eff}}:)$ or $\exp(:-\frac{\mu}{2}(\gamma x^2 + 2\alpha xp + \beta p^2):)$

Note that Lie representations are really useful for generalizations to nonlinear systems, and for power series analysis when performed by computers. However, Lie operators like those listed in this table really aren't useful for simple linear accelerator problems. For example, consider the thin-lens FODO lattice: its Lie representation is given by the concatenation

$$\exp\left(:-\frac{1}{2f}x^2:\right) \exp\left(:-\frac{1}{2}Lp^2:\right) \exp\left(:\frac{1}{2f}x^2:\right) \exp\left(:-\frac{1}{2}Lp^2:\right) \tag{34}$$

Note the reverse ordering; these are operators, after all! Considering that these are infinite series before losing terms when they are applied to (x, p), expanding this is a complete headache compared to the simple 2×2 or 4×4 matrix approach.

All of the elements above are exponentials of Poisson bracket operators, so they are Lie operators. Lie operators, like exponentials, have plenty of useful properties. Many are intuitive if $:f:$ and $:g:$ commute, such as $\exp(:f:)\exp(:g:) = \exp(:f+g:)$.

2.5 Example: Sextupole Taylor Map from Lie Operator Hamiltonian

As mentioned before, the one-turn Lie map is simply $\exp(:-CH:)$ where H is the Hamiltonian and C is the circumference of the accelerator. This can be extended to $\exp(:-LH:)$ where L is any length of integration, including multiple turns. Let's take the general sextupole Hamiltonian as an example, where

$$H = \frac{1}{3}S(x^3 - 3xy^2) + \frac{1}{2}(p_x^2 + p_y^2) \tag{35}$$

We can then calculate orders of the Hamiltonian:

$$\begin{aligned}
:H:x &= -\frac{\partial H}{\partial p_x} = -p_x \\
:H:^2 x &= -:H:p_x = -\frac{\partial H}{\partial x} = -S(x^2 - y^2) \\
:H:^3 x &= -S\left(\frac{\partial H}{\partial p_x}(2x) + \frac{\partial H}{\partial p_y}(2y)\right) = 2S(x\,p_x - y\,p_y) \\
:H:^4 x &= 2S\left(-\frac{\partial H}{\partial p_x}p_x + x\frac{\partial H}{\partial x} + \frac{\partial H}{\partial p_y}p_y - \frac{\partial H}{\partial y}y\right) \\
&= 2S\left[-p_x^2 + p_y^2 + Sx\left(x^2 + y^2\right)\right] \\
:H:^5 x &= O(S^2)
\end{aligned} \tag{36}$$

We can then obtain the Taylor map up to a modest order:

$$\begin{aligned}
\exp(-L:H:)x &= x + p_x L - \frac{1}{2}SL^2(x^2 - y^2) - \frac{1}{3}SL^3(xp_x - yp_y) \\
&\quad + \frac{1}{12}SL^4[-p_x^2 + p_y^2 + Sx(x^2 + y^2)] + O(S^2L^5)
\end{aligned} \tag{37}$$

where $O(S^2L^5)$ means terms same-or-higher order than S^2 in S and same-or-higher order than L^5 in L.

We can work through all the math (Mathematica *really* is your friend) to find the mappings of other coordinates as well:

$$\begin{aligned}
\exp(-L:H:)p_x &= p_x + SL(x^2 - y^2) - SL^2(xp_x - yp_y) - \frac{1}{3}SL^3[p_x^2 - p_y^2 - Sx(x^2 + y^2)] \\
&\quad + \frac{1}{12}S^2L^4(5x^2p_x - y^2p_x + 6xyp_y) + O(S^2L^5) \\
\exp(-L:H:)y &= y + Lp_y + SL^2xy + \frac{1}{3}SL^3(xp_y + yp_x) \\
&\quad + \frac{1}{12}SL^4[2p_xp_y + Sy(x^2 + y^2)] + O(S^2L^5) \\
\exp(-L:H:)p_y &= p_y + 2SLxy + SL^2(xp_y + yp_x) + \frac{1}{3}SL^3[2p_xp_y + Sy(x^2 + y^2)] \\
&\quad + \frac{1}{12}SL^4(x^2p_y - 6xyp_x - 5y^2p_y) + O(S^2L^5)
\end{aligned} \tag{38}$$

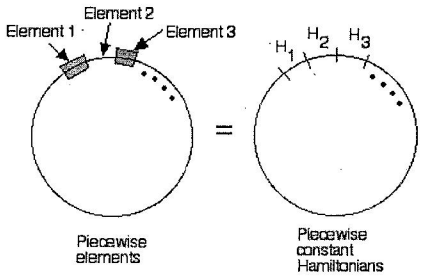
This is not too exciting, as we still would have to expand using the Floquet transformation to find resonance driving terms and strengths, but the real advantage here is that this expression can be explicitly calculated to any order in S. (Mathematica is your friend!) When you do this, you find that higher orders of sextupole powers drive higher order resonances, similar to octupoles, and even higher orders after that. The reason that accelerators still work despite an infinite number of resonance driving terms from nonlinearities is that these driving forces are perturbatively small — so small that the resonances are all tiny and isolated, and tend not to overlap according to the Chirikov resonance overlap criterion that Vladimir mentioned in the nonlinear dynamics lecture.

2.6 The Ring and the Baker-Campbell-Hausdorff Formula

How do we apply Lie techniques to piecewise continuous Hamiltonians in rings? We have individual elements (dipoles, quadrupoles, sextupoles, etc) that we chain together, so we end up with a Lie map that is their product:

$$\prod_{i=1} N e^{:-L_i H_i:} = e^{:-C H_{\rm eff}:} \tag{39}$$

This can be seen in the following figure, which in some sense expresses our desire to have a full-ring effective Hamiltonian that carries all the nonlinearity of the system. The goal is then to find the effective Hamiltonian $H_{\rm eff}$.



$$\text{Accelerator} = \prod_{j=1}^{N} e^{:-\ell_j H_j:} = e^{:-C H_{\rm eff}:}$$

Lie representation of the accelerator — Effective Hamiltonian

If $:f:$ and $:g:$ do not commute, how do we relate $\exp(:f:)\exp(:g:)$ to a single Lie operator $\exp(:h:)$ — that is, how do we concatenate Lie maps? The basic formula that allows concatenation of Lie operators is called the Baker-Campbell-Hausdorff formula. It comes in many useful forms, but we'll just state two here for convenience — it's already long enough! Given $\exp(:f:)\exp(:g:) = \exp(:h:)$, h is related to f and g by:

$$\begin{aligned} h \;=\; & f + g + \frac{1}{2}:f:g + \frac{1}{12}:f:^2 g + \frac{1}{12}:g:^2 f + \frac{1}{24}:f::g:^2 f \\ & - \frac{1}{720}:g:^4 f - \frac{1}{720}:f:^4 g + \frac{1}{360}:g::f:^3 g + \frac{1}{360}:f::g:^3 f \\ & + \frac{1}{120}:f:^2:g:^2 f + \frac{1}{120}:g:^2:f:^2 g + O((f,g)^6) \end{aligned} \tag{40}$$

We can rewrite this in terms of commutators, so we can see more of the functional nesting, but it really doesn't help much. The coefficients of this expansion of the original BCH formula don't seem to have a convenient pattern.

If one of the terms in the BCH expansion is perturbatively small, we can sum the infinite power series in the first form over the function f or g (whichever is NOT perturbative) to find

$$e^{:f:}e^{:g:} = \exp\left[:g + \left(\frac{:g:}{\exp(:g:)-1}\right) f + O(f^2):\right] \tag{41}$$

$$e^{:f:}e^{:g:} = \exp\left[:f + \left(\frac{:f:}{1-\exp(:f:)}\right) g + O(g^2):\right] \tag{42}$$

This and other forms of the BCH theorem, along with the Taylor expansion seen in the last section, allow us to calculate nonlinear accelerator maps to high order. These can be used for computer simulation, and for analysis. To determine dynamic aperture, or long-term beam stability, we often "track" for millions or tens of millions of turns around the accelerator. However, the maps that we've generated are not symplectic if they are just arbitrarily truncated! This leads to the field of "symplectification", where additional higher-order terms are added that make the map symplectic again, yet are high enough order that they do not dominate the dynamics.

Colliders and Beam-Beam Effects

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January 23, 2008

1 Luminosity, Collider Rings, and Emittance

Recalling homework one (lo these many years ago), the center of mass energy available for collisions is much greater in a collider than a fixed target experiment. Some of the first machines to collide beams were done with head-on collisions at Adone (Frascati), ACO, and VEPP-2, and nearly head-on collisions of protons in the ISR at CERN. Before discussing the beam-beam force, we'll review the parameters of collider luminosity, since beam-beam effects ultimately are one of the primary limitations of achievable luminosity.

The interaction rate for a fixed target experiment is

$$R = \sigma \dot{N} n l \tag{1.1}$$

where σ is the **interaction cross section** (usually measured in units of 1 barn $\equiv 10^{24}$ cm^{-2}), $\dot{N}$ is the number of particles incident on the target every ¿second, n is the number density of the target, and l is the target length. Note that we also use σ for the beam size, but generally the interpretation is obvious from context. **Luminosity** is defined as the rate per unit cross section, so we can define the luminosity for a fixed target experiment as

$$\mathcal{L} = \frac{R}{\sigma} = \dot{N} n l = \dot{N} l \rho \frac{N_A}{M} \tag{1.2}$$

where N_A is Avagadro's number and M is the molecular weight. For a liquid hydrogen target with density $\rho = 0.07$ g/cm^3 and a beam flux of 10^{13} particles per second, this gives a luminosity of about $\mathcal{L} = 2 \times 10^{35}$ cm^{-2} s^{-1}.

One example of a collider is a ring where a single bunch of particles and a single bunch of antiparticles circulate in opposite directions. The beams cross every half revolution at two opposite points. For your homework (problem 1 in homework 2, also Lee 1.7(b) p. 27), you calculated the luminosity of two round equal-size Gaussian beams colliding head-on:

$$\mathcal{L} = \frac{f N_1 N_2}{4\pi \sigma_x \sigma_y} \tag{1.3}$$

where $\sigma_{x,y}$ are the rms transverse beam sizes, f is the frequency of crossings or collision frequency, and $N_{1,2}$ are the number of particles in each bunch. For a collider, typical luminosities can range from $\mathcal{L} = 10^{30}$ cm^{-2} s^{-1} up to 10^{32} cm^{-2} s^{-1}. The online collision pattern for RHIC is shown in Fig. 1 at the top of the next page.

One way to maximize collider luminosity is by minimizing the beam size at the collider crossing. As you might expect, the beam sizes σ_x and σ_y are related to the beta function (or square of the envelope function), though they are also related to the dispersion $D_{x,y}$ through the longitudinal momentum spread σ_p:

$$\sigma_x^2 = \beta_x^\star \epsilon_x + \left(D_x^\star \frac{\sigma_p}{p_0} \right)^2 \qquad \sigma_y^2 = \beta_y^\star \epsilon_y + \left(D_y^\star \frac{\sigma_p}{p_0} \right)^2 \tag{1.4}$$

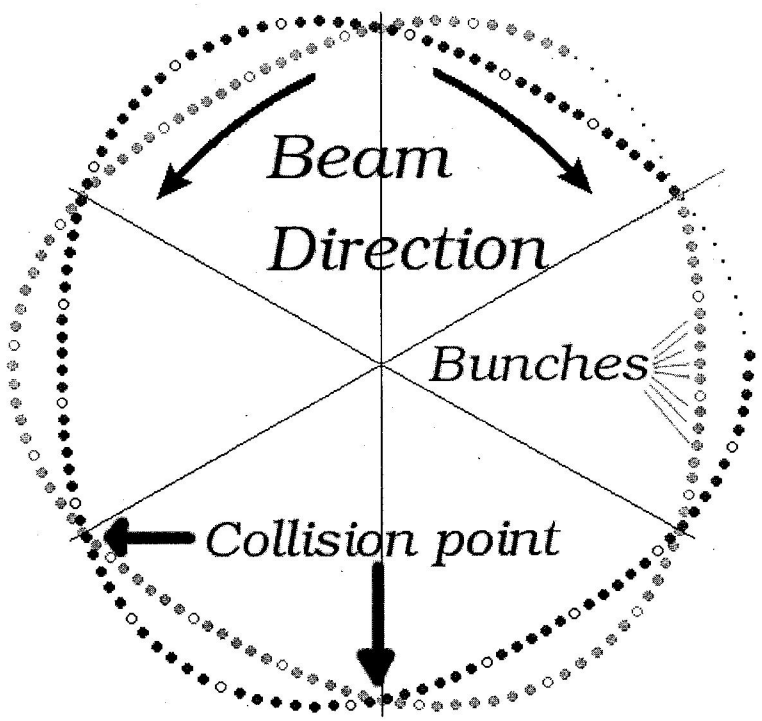

Figure 1: The collision pattern for RHIC. There are two counter-rotating beams in two separate rings, with the ability to collide at 6 separate experimental areas. The RHIC blue ring goes clockwise; the RHIC yellow ring goes counter-clockwise. There are up to about 110 buckets in each ring; some buckets are not populated with beam bunches to avoid the buildup of electron clouds and electron cloud instabilities, which will be discussed on Wednesday afternoon.

Here starred values indicate values at the interaction point (IP), or crossing point. Usually colliders are designed with $\beta_{x,y}$ as small as possible at the IP, and with zero dispersion to maximize the luminosity. Some e^+e^- collider parameters are listed on p. 514 of Lee.

There are two new quantities in Eq. (1.4), $\epsilon_{x,y}$, known as the **emittances**. This is simply the scale factor for the beam size related to the standard parameterization of motion:

$$x(s) = \sqrt{\beta(s)\epsilon_x}\cos(Q_x\psi_x(s) + \delta) \tag{1.5}$$

in a region where the dispersion is zero. Emittance can also be viewed as a measure of the total phase space occupied by a certain percentage of the beam, generally a conserved quantity according to Liouville. However, as the beam momentum p_0 increases, the transverse angles $(x', y') = (p_x/p_0, p_y/p_0)$ get smaller by a relativistic factor of $\beta_r\gamma_r$ where the r subscript indicates relativistic factors. This leads us to an expression for the **normalized emittance**

$$\epsilon_{N,x} = \beta_r\gamma_r\epsilon_x \tag{1.6}$$

which **is** invariant under acceleration of the beam. A typical RHIC Au beam normalized emittance is 10-20 π mm-mrad in both horizontal and vertical planes. One comment of confusion to the uninitiated: the emittance is commonly quoted in units of "π mm-mrad" with the π being part of the units and therefore not used in calculation! Conventions also vary from lab to lab — some labs use 95% emittance, while others use rms emittance.

2 The Beam-Beam Force

When our Gaussian beams collide head-on, there is a space-charge force arising from the fields of one beam acting on the other beam. For computational simplicity, we will assume that the fields of one beam are much stronger than the other. This is the case in $p\bar{p}$ colliders like the Tevatron, and is known as the **weak-strong approximation**. At RHIC, both beams have approximately the same number of charges and transverse size, so the beam-beam forces are in the **strong-strong** regime. For now, we'll examine the forces encountered by a single particle encountering the electromagnetic field of an opposing Gaussian beam.

The radial force is similar to the space charge force discussed earlier, but replacing $(1-\beta_r^2)$ with $(1+\beta_r^2)$ since the beams are colliding:

$$F_\perp(x) = \frac{Nq^2}{2\pi l x}(1+\beta_r^2)\left[1-\exp\left(-\frac{x^2}{\sigma_x(\sigma_x+\sigma_y)}\right)\right] \tag{2.1}$$

This function is plotted in Fig. (2) for normalized quantities. It is roughly linear in the range $|x| < \sigma_x/2$, then curls over strongly and asymptotically decays for large x as $1/x$.

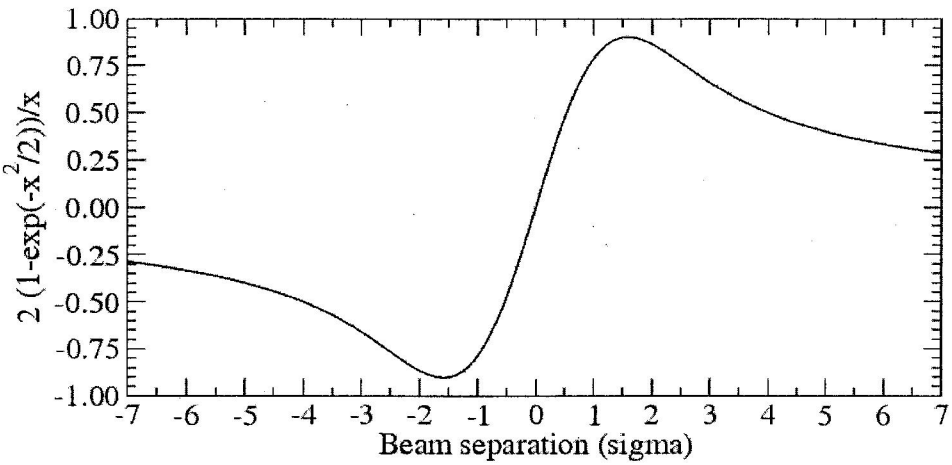

Figure 2: The beam-beam force given by Eqn. (2.1), with the horizontal axis units normalized to the beam size $x = \sigma/\sigma_x$. This is a defocusing force for an oncoming Gaussian beam of the same charge; this force is reversed for beams of opposite charge.

For ultrarelativistic particles where $\beta_r \to 1$ and for small amplitude displacements from the opposing beam center, Eqn. (2.1) becomes linear:

$$F_\perp(x) \approx \frac{Nq^2 x}{2\pi l \sigma_x(\sigma_x+\sigma_y)} \tag{2.2}$$

And the effective angular kick from this force is

$$\Delta x' = \frac{\Delta p_x}{p} = \frac{F_\perp \delta t}{p} = \frac{Nr_0}{\gamma\sigma_x(\sigma_x+\sigma_y)}x \tag{2.3}$$

where $r_0 = q^2/4\pi mc^2$ is the classical radius of the particle and the time of flight $\delta t = l/2c$. This looks like the defocusing from a thin defocusing quadrupole; Vladimir calculated the tune shift from a similar term yesterday:

$$\Delta Q_x = -\frac{\beta_x^\star \Delta k}{4\pi} = -\frac{\beta_x^\star N r_0}{2\pi\gamma\sigma_x(\sigma_x+\sigma_y)} = -\frac{Nr_0}{4\pi\epsilon_{rms,N}} \tag{2.4}$$

Since the beams are rotationally symmetric, one finds this focusing or defocusing in both planes, depending on the relative sign of the colliding particles. We have used

$$\sigma_x \approx \sigma_y = \sqrt{\epsilon_{\mathrm{rms,x}}\beta_x^\star} = \sqrt{\frac{\epsilon_{\mathrm{rms,x,N}}\beta_x^\star}{\beta_r\gamma_r}}$$

and $\beta_r \approx 1$ in the last expression to demonstrate that the beam-beam tune shift is independent of $\beta^\star$ and only depends on the bunch intensity and emittance for relativistic round beams. This parameter is so commonly used that it gets its own letter and name in the common literature, the linear beam-beam tune shift or beam-beam parameter ξ:

$$\xi = -\frac{Nr_0}{4\pi\epsilon_{rms,N}} \tag{2.5}$$

where again we have assumed round beams.

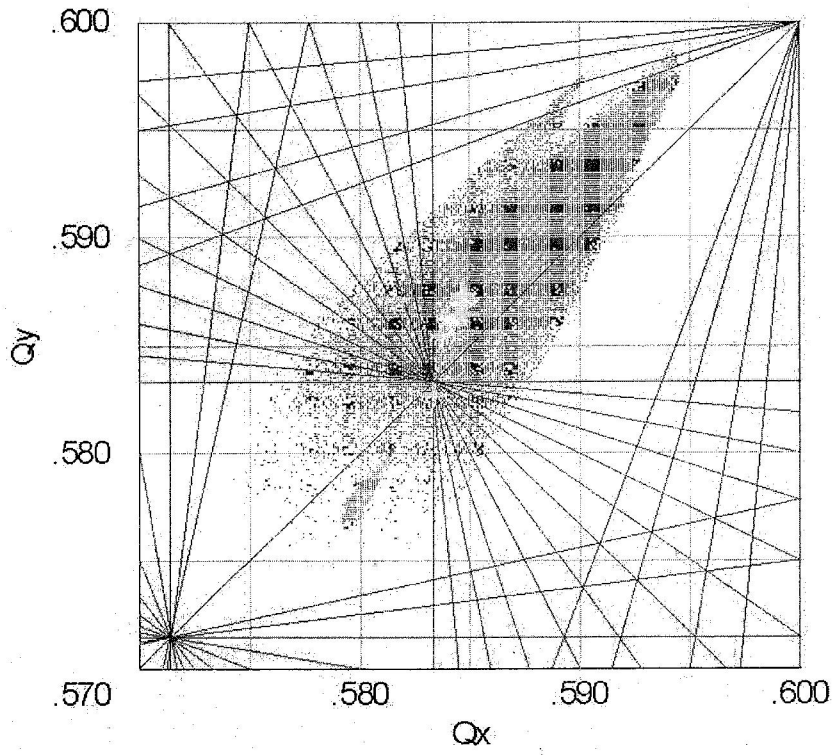

Figure 3: Measured beam-beam tune footprint from the Fermilab Tevatron. The resonance shown at the center of the diagram is 13^{th} order! Here the total antiproton beam-beam parameter is about $\xi = 0.02$ since the beams are in the weak-strong regime. These are oppositely charged beams, so the beam-beam tuneshift is positive (additional focusing) and the majority of low-amplitude particles lie in the resonance free area near the diagonal in the upper right quadrant. The unperturbed antiproton tunes here are approximately (0.576,0.576). (Alexahin, "Theory and Reality of Beam-Beam Effects at Hadron Colliders", PAC 2005)

ξ serves as a natural scaling parameter for the beam-beam interaction, and it is usually conveniently perturbatively small. Since the beam-beam interaction is strongly nonlinear for larger amplitudes (as we'll investigate more in the next section), the maximum total tune shift from all beam crossings from beam-beam is quite modest, about $\leq 5 \times 10^{-3}$ at the ISR, about $\leq 8 \times 10^{-3}$ at RHIC (both strong-strong machines dominated by coherent beam-beam modes), and $\leq 3 \times 10^{-2}$ at CESR for e^+e^- storage rings, where radiation damping helps stabilize the beams.

3 Nonlinear Beam-Beam Interactions

3.1 Nonlinear Hamiltonian

Here we will examine one-dimensional horizontal resonances driven by the beam-beam interaction, and use this as a springboard to calculate some measurable quantities about resonances. There is a computer simulation homework at the end of this section that will help (literally) illustrate the topology of resonant phase space.

Substituting the general beam-beam kick, Eqn. (2.1), into the beam-beam kick equation, Eqn. (2.3), and using the expression for the beam-beam parameter ξ, we get

$$\Delta x' = -\frac{4\pi\xi}{\beta^\star}\frac{\sigma_x^2}{x}\left[1-\exp\left(-\frac{x^2}{2\sigma_x^2}\right)\right] \tag{3.1}$$

Here it is clear that the problem scales with x/σ_x, so change variables via $u = x/\sigma_x$ and $u' = x'/\sigma_x = p_x'/(\sigma_x p_0)$ (a canonical transformation) to give

$$\Delta u' = -\frac{4\pi\xi}{\beta^\star u}\left[1-\exp\left(-\frac{u^2}{2}\right)\right] = -\frac{2\pi\xi}{\beta^\star}\left[u-\frac{u^3}{4}+\frac{u^5}{12}+\ldots\right] \tag{3.2}$$

The force is antisymmetric, so the kick is antisymmetric (or odd) as well, to all higher orders. The first order term reproduces the linear beam-beam tune shift for small-amplitude motion.

Eq. (3.2 has the form of a Hamilton's equation, so we can integrate to produce a perturbative thin-kick nonlinear beam-beam Hamiltonian:

$$H_p(u,u') = \frac{2\pi\xi}{\beta^\star}\left[\frac{u^2}{2}-\frac{u^4}{16}+\frac{u^6}{72}+\ldots\right] \tag{3.3}$$

(Remember that this is a perturbation on top of the normal beam motion.) This is a combination of quadrupole, octupole, dodecapole, and higher order terms that are now all even, so the beam-beam force drives all even-order resonances, or resonances of the form $2nQ_x = l$. The generalization to two dimensional motion will drive all even order coupling resonances as well, since the system is radially symmetric in the (x,y) plane. Also note that it drives all these resonances to first order in the beam-beam parameter ξ! Fortunately higher order resonance driving terms decay with the terms of the exponential.

3.2 Resonance Analysis

As an example, let's look at the $4Q_x$ resonance driven by the octupole term in Eq. (3.3), assuming $4Q_x = l + \delta Q_x$ where $\delta Q_x << 1$. Detailed steps to eliminate non-resonant terms are in Lee pp. 203-205, or in class notes given by Eduard and Vladimir; our primary objective here is to write a single-resonance Hamiltonian from the beam-beam force and learn about its resonant topology.

Expanding $u = \sqrt{2I}\cos(\varphi)$ and writing the full Hamiltonian in action-angle coordinates, we have

$$H_1(\varphi, I) = 2\pi Q_x I + AI^2\cos^4(\varphi) \tag{3.4}$$

where constant terms have been absorbed into $A = \pi\xi/2\beta^\star$ (which is first order in ξ), and we have disregarded all non-octupole terms. This is a so-called "one-turn" Hamiltonian; without the resonance, $\dot{\phi} = 2\pi Q_x$ and $\dot{I} = 0$. The cosine can be expanded to yield

$$H_1(\varphi, I) = 2\pi Q_x I + \frac{AI^2}{16}\left[\cos(4\varphi) + 4\cos(2\varphi) + 6\right] \tag{3.5}$$

This has resonance terms for the $4Q_x$ and $2Q_x$ resonances, and also has a term that gives action-dependent (or amplitude-dependent) tune variation, or detuning:

$$\dot{\varphi} = \frac{\partial H_1}{\partial I} = 2\pi Q + \frac{3}{4}AI + (\text{terms that vary with } \varphi \tag{3.6}$$

So the phase changes by $2\pi Q_x$ every turn, plus a term that depends on the action or square of particle amplitude.

We'll consider only the fourth-order resonance, $4Q_x = l + \delta Q$ where $\delta Q << 1$, and assume l is odd so we are not also near the second order resonance! Then the Hamiltonian can be "integrated" to find a four-turn Hamiltonian

$$H_4(\bar{\varphi}, \bar{I}) = 2\pi(4Q_x)\bar{I} + \frac{3}{2}A\bar{I}^2 + \frac{A\bar{I}^2}{4}\cos(4\bar{\varphi}) \tag{3.7}$$

$$= 8\pi\delta Q\bar{I} + \frac{3}{2}A\bar{I}^2 + \frac{A\bar{I}^2}{4}\cos(4\bar{\varphi}) \tag{3.8}$$

Hamilton's equations give for this motion:

$$\dot{\bar{\varphi}} = 8\pi\delta Q + 3A\bar{I} + \frac{A\bar{I}}{2}\cos(4\bar{\varphi}) \tag{3.9}$$

$$\dot{\bar{I}} = A\bar{I}^2\sin(4\bar{\varphi}) \tag{3.10}$$

Fixed points are found where $\dot{\bar{\varphi}} = \dot{\bar{I}} = 0$. This immediately gives

$$\bar{\varphi}_{\text{FP}} = n\pi/4 \tag{3.11}$$

$$\bar{I}_{\text{FP}} = \frac{8\pi\delta Q}{A(3 \pm 1/2)} \tag{3.12}$$

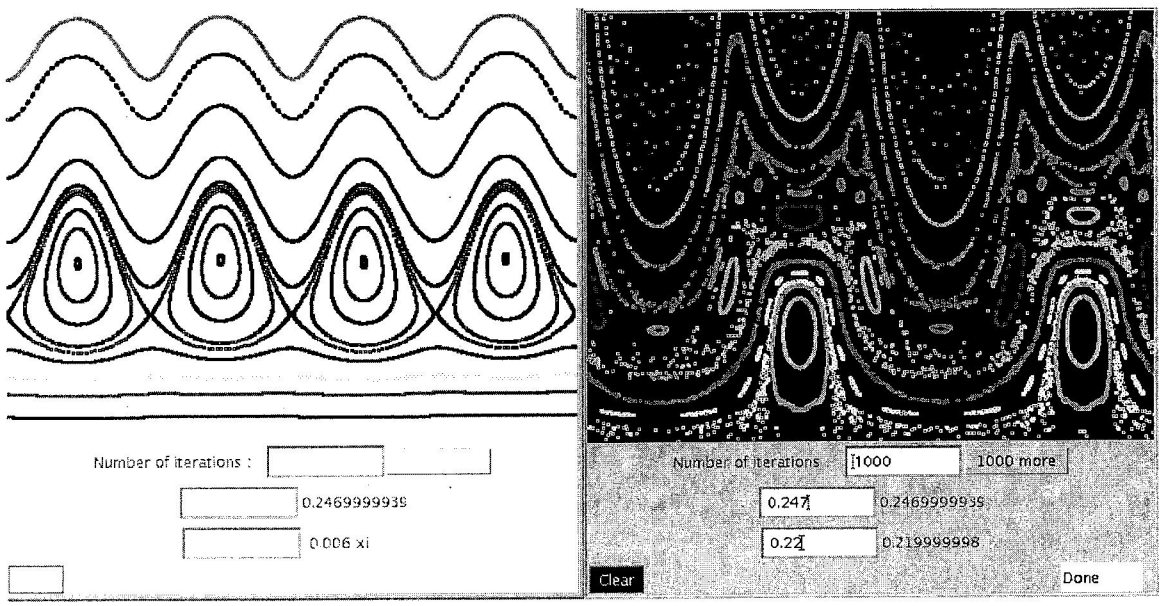

Figure 4: Examples of regular and chaotic phase space driven by beam-beam forces. $\xi = 6 \times 10^{-3}$ in the left plot, while $\xi = 0.22$ in the right plot