# Impedance Matching and Smith Charts

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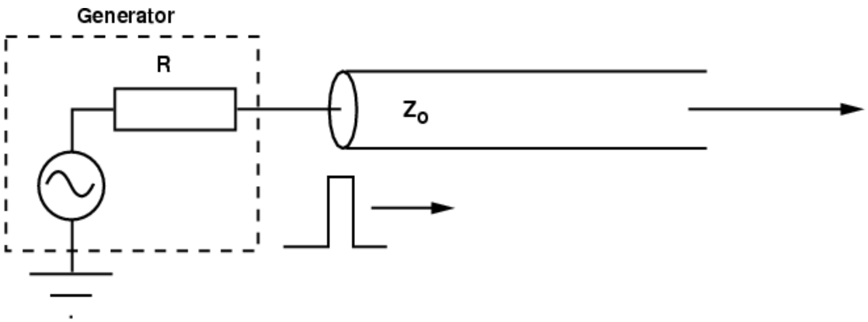
# Impedance of a Coaxial Transmission Line

A pulse generator with an internal impedance of R launches a pulse down an infinitely long coaxial transmission line.

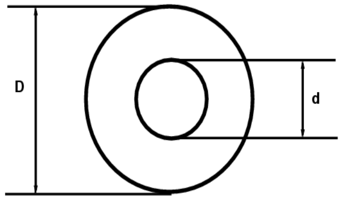



Even though the transmission line itself has no ohmic resistance, a definite current *I* is measured passing into the line by during the period of the pulse with voltage *V*.

The impedance of the coaxial line $Z_0$ is defined by $Z_0 = V / I$.

The impedance of a coaxial transmission line is determined by the ratio of the electric field *E* between the outer and inner conductor, and the induced magnetic induction *H* by the current in the conductors.

The **surge impedance** is, $Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu_0}{\epsilon_0}} \ln\left(\frac{D}{d}\right) = 60 \ln\left(\frac{D}{d}\right)$




where D is the diameter of the outer conductor, and d is the diameter of the inner conductor. For 50 ohm air-dielectric, D/d = 2.3.

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377 \ ohms$$ is the impedance of free space.

## Velocity of Propagation in a Coaxial Transmission Line

Typically, a coaxial cable will have a dielectric with relative dielectric constant $\epsilon_r$ between the inner and outer conductor, where $\epsilon_r$ = 1 for vacuum, and $\epsilon_r$ = 2.29 for a typical polyethylene-insulated cable.

The characteristic impedance of a coaxial cable with a dielectric is then

$$Z_0 = \frac{1}{\sqrt{\epsilon_r}} 60 \ln\left(\frac{D}{d}\right)$$

and the propagation velocity of a wave is, $v_p = \frac{c}{\sqrt{\epsilon_r}}$
where $c$ is the speed of light

In free space, the wavelength of a wave with frequency $f$ is

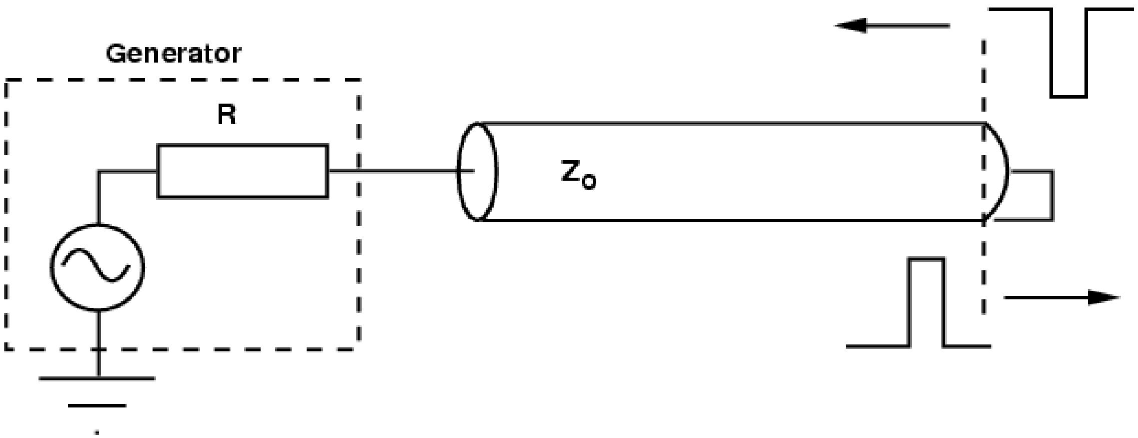
$$\lambda_{coax} = \frac{1}{\sqrt{\epsilon_r}} \frac{c}{f} = \frac{\lambda_{free-space}}{\sqrt{\epsilon_r}}$$

For a polyethylene-insulated coaxial cable, the propagation velocity is roughly 2/3 the speed of light.

# Reflection from End of a Shorted Transmission Line

Instead of an infinitely long transmission line, consider a finite length that is terminated by a **short** circuit. At the short, the **voltage** is zero.

This boundary condition can be satisfied by imagining a negative pulse coming from the right, overlapping the forward pulse as it encounters the short, and continues on to the left toward the generator.



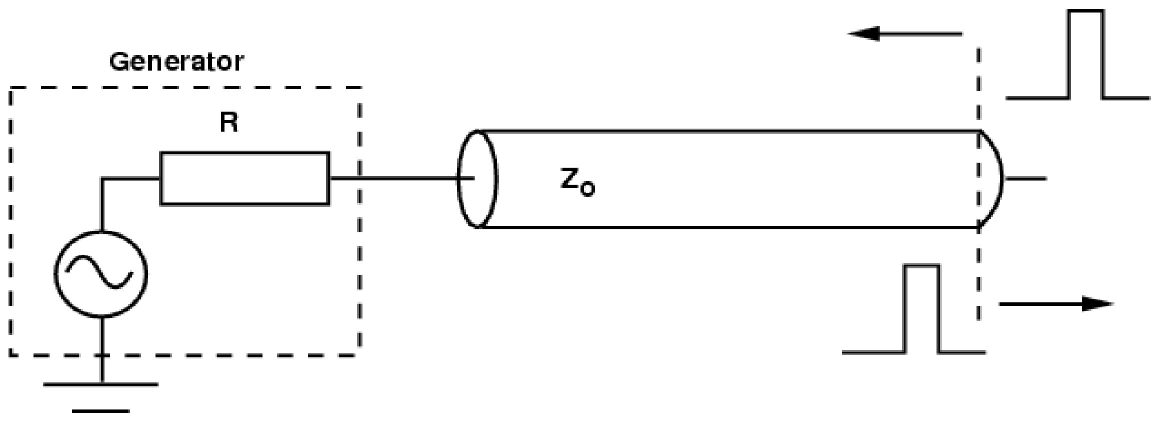

If the time it takes to propagate the pulse down the line is $t_p$, then the inverted pulse arrives back at the generator at time 2 $t_p$.

Note that the generator has an internal impedance $R$. If $R = Z_0$, the returning pulse is completely absorbed in the generator, as the *transmission line is terminated in its characteristic impedance* $Z_0$.

## Reflection from End of an Open Transmission Line

Instead of an infinitely long transmission line, consider a finite length that is terminated by an **open** circuit.   At the open, the **current** is zero.

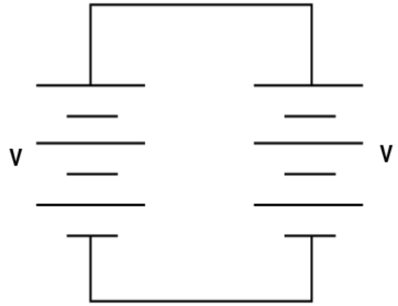
This boundary condition can be satisfied by imagining a positive pulse coming from the right, overlapping the forward pulse as it encounters the short, and continues on to the left toward the generator.




Imagine the two pulses overlap as two batteries of identical voltage:

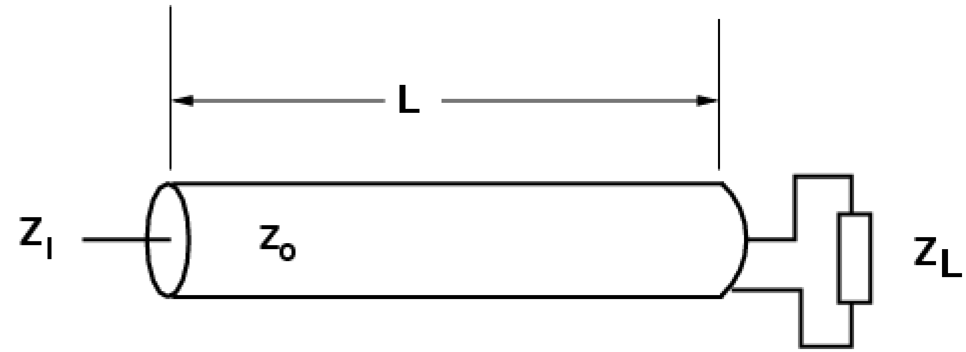
No current flows as the voltage on each battery is the same.

The pulse from the generator returns in time 2 $t_p$ with the same polarity as original.



## Input Impedance of a Transmission Line with Arbitrary Termination

The impedance at the entrance of a transmission line of length *L* and terminating impedance $Z_L$ is




$$Z_i = Z_0 \frac{Z_L + jZ_0 \tan\beta L}{Z_0 + jZ_L \tan\beta L}, \qquad j = \sqrt{-1}$$

where $\beta$ is the **propagation constant**

$$\beta = \frac{2\pi f}{c}\sqrt{\epsilon_r} = \frac{2\pi}{\lambda}\sqrt{\epsilon_r}$$

There are three special cases, where the end termination $Z_L$ is an open or a short circuit, or a termination resistance equal to the characteristic impedance $Z_0$ of the transmission line itself.

We will introduce the **Smith Chart** later, which simplifies the calculation of $Z_i$ for an arbitrary terminating impedance.

## Special Cases of Terminating Impedance

$Z_L = 0$ (short circuit) $Z_i = Z_0\, j \tan \beta L$ $= 0$, $L = \lambda/2,\ \lambda, \ldots$

$= \text{infinite}$, $L = \lambda/4,\ 3\lambda/4, \ldots$

$Z_L = \text{infinite}$ (open) $Z_i = -Z_0\, \mathrm{j} \cot \beta L$ $= \text{infinite}$, $L = \lambda/2,\ \lambda, \ldots$

$= 0$ , $L = \lambda/4,\ 3\lambda/4, \ldots$

$Z_L = Z_0$ (matched) $Z_i = Z_0$

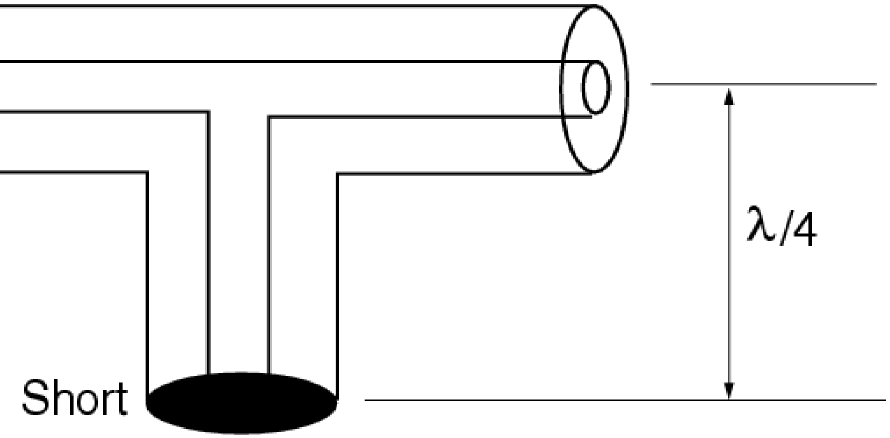
The shorted or open line is always reactive (like a capacitor or inductor), and reflects the terminating impedance for integrals of a half wavelength, and the conjugate of the terminating impedance for odd quarter-wavelengths.

The terminated line impedance is independent of the frequency or length of line (assuming a lossless transmission line).

## Special Use for a Quarter-Wave Stub

$Z_L = 0$ (short circuit) $\quad Z_i = Z_0\, j \tan \beta L$ = infinite, $L = \lambda/4, 3\lambda/4, \ldots$

A shorted stub transforms to an infinite impedance at odd multiples of a quarter wavelength.




A quarter-wave stub is a convenient way of supporting an inner conductor without an insulator, and accessing the inner conductor with water cooling, for example.

Stubs are also used in matching a load to a source by introducing an intentional reflection.

## Reflection Coefficient

These terms are used to describe the ratio of the voltage (at one frequency) launched down a transmission line and the voltage reflected back from the far end by a mismatched load.

Let $V_f$ be the forward voltage launched down the line, and $V_r$ the reflected voltage.

The **voltage reflection coefficient** is: $\Gamma_0 \equiv \dfrac{V_f}{V_r} = \dfrac{Z_L - Z_0}{Z_L + Z_0}$ ,

where $Z_L$ is the load impedance and $Z_0$ is the impedance of the transmission line.
If $Z_L = Z_0$, there is no reflected wave and $\Gamma_0 = 0$. The polarity of $V_r$ reverses for $Z_L < Z_0$.

The **return loss** *RL* is $\Gamma_0$ expressed as a logarithmic quantity:

$$RL = -20\log_{10}|\Gamma_0|$$

*RL* = -infinite dB for a matched load.

# Voltage Standing Wave Ratio

VSWR = Voltage Standing Wave Ratio (sometimes just called SWR)

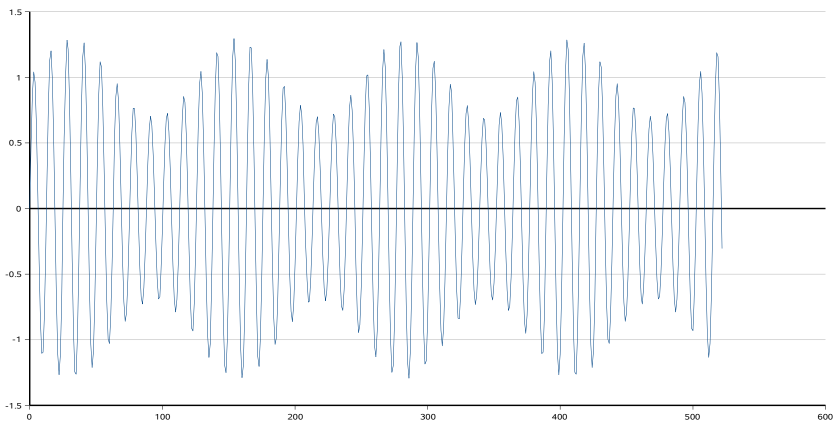
At a given frequency *f* with wavelength $\lambda = \frac{v_p}{f}$, $v_p = \frac{c}{\sqrt{\epsilon_r}}$

a standing wave will exist on the transmission line (if a reflected wave exists).

The forward and reverse waves will interfere with each other and produce a stationary pattern of the envelope.

The **Voltage Standing Wave Ratio** is the ratio of the maximum to the minimum amplitude of the standing wave

$$VSWR = \frac{V_{max}}{V_{min}} \geq 1$$



Standing wave with envelope variation

and is usually expressed as a ratio, such as VSWR = 3:1. In terms of the reflection coefficient:

$$VSWR = \frac{1 + |\Gamma_0|}{1 - |\Gamma_0|}$$

If VSWR = 1, $\Gamma_0 = 0$ (no reflection)

## Transformation of Arbitrary Impedance Load

Define a **normalized impedance**: $z_L = \dfrac{Z_L}{Z_0}$ $Z_L$ and $Z_0$ are the load and line impedance.

The reflection coefficient is $\Gamma_0 = \dfrac{z_L - 1}{z_L + 1} = |\Gamma|\, e^{j\beta L}$ where we have expressed the reflection coefficient as a complex quantity.

and $\beta$ is the **propagation constant** of a transmission line.

$$\beta = \frac{2\pi f}{c}\sqrt{\epsilon_r} = \frac{2\pi\sqrt{\epsilon_r}}{\lambda}$$

The input impedance of a transmission line with arbitrary terminating impedance is

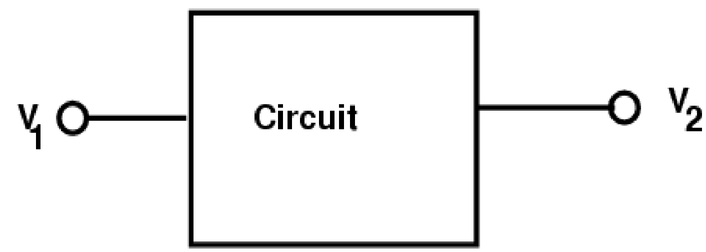
$$Z_{¿} = Z_0 \frac{1 + \Gamma e^{-2j\beta L}}{1 - \Gamma e^{-2j\beta L}}$$

We will use the **Smith Chart** to ease the calculation of this complex quantity.

# Scattering Matrix

In the RF (microwave) domain, it is difficult to probe the voltages and currents in various parts of a circuit, but easier to measure the voltages at given reference planes in a circuit. For a linear circuit, we can define a **scattering matrix** for the circuit in terms of the incident and reflected voltages at each of the circuit nodes.

$$\begin{vmatrix} V_1^- \\ V_2^- \end{vmatrix} = \begin{vmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{vmatrix} \begin{vmatrix} V_1^+ \\ V_2^+ \end{vmatrix}$$




where $V_i^+$ are the voltages incident on port *i*, $V_i^-$ are the voltages reflected from port *i*.

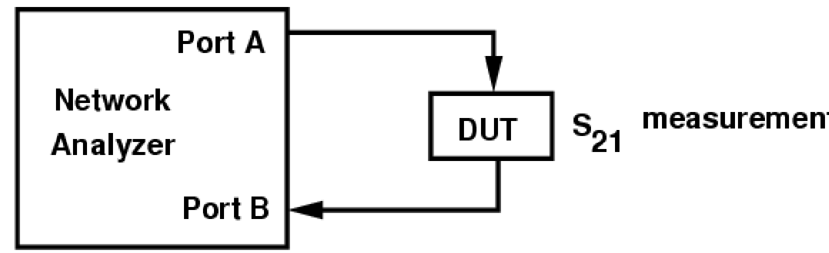
$S_{11}$ is the reflection looking into port 1, with all other ports terminated in matched loads. If $V_2^+ = 0$, then $S_{11} = \Gamma$, the reflection coefficient.

$S_{21}$ is the voltage transmission coefficient from port 1 to port 2. $S_{21} = \dfrac{V_2^-}{V_1^+}, \quad V_2^+ = 0$

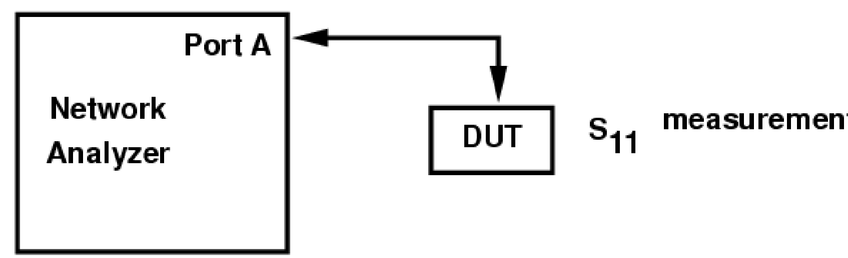
$$S_{11} = \frac{Z_i - Z_0}{Z_i + Z_0}, \qquad Z_i = Z_0 \frac{1 + \Gamma e^{-2j\beta L}}{1 - \Gamma e^{-2j\beta L}}$$

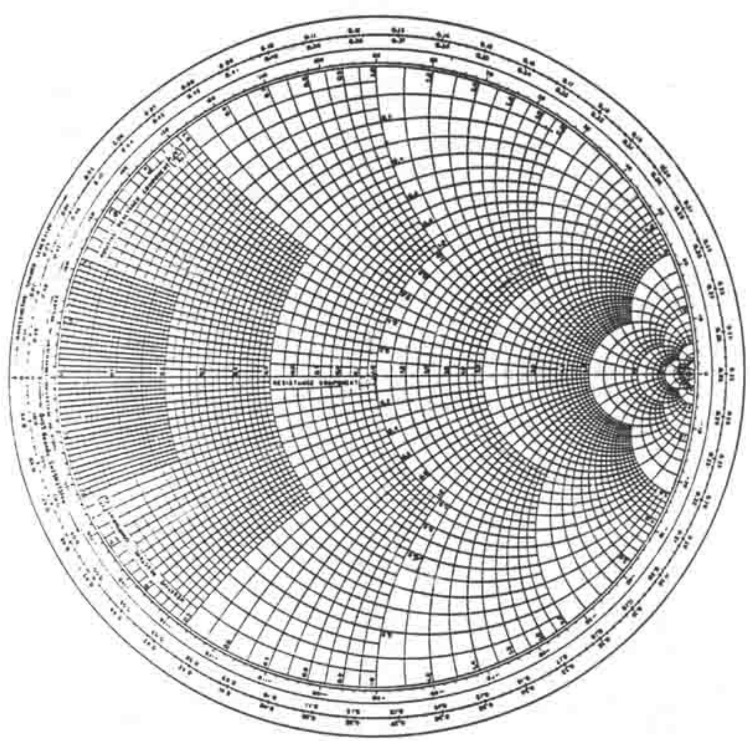
## S-Parameter Measurements with Network Analyzer

The **network analyzer** measures the scattering coefficients of a two-port device directly.




One type of display is the **Smith Chart**.



We will use the Smith Chart to calculate a single-stub tuner for a mismatched load.

# The Smith Chart

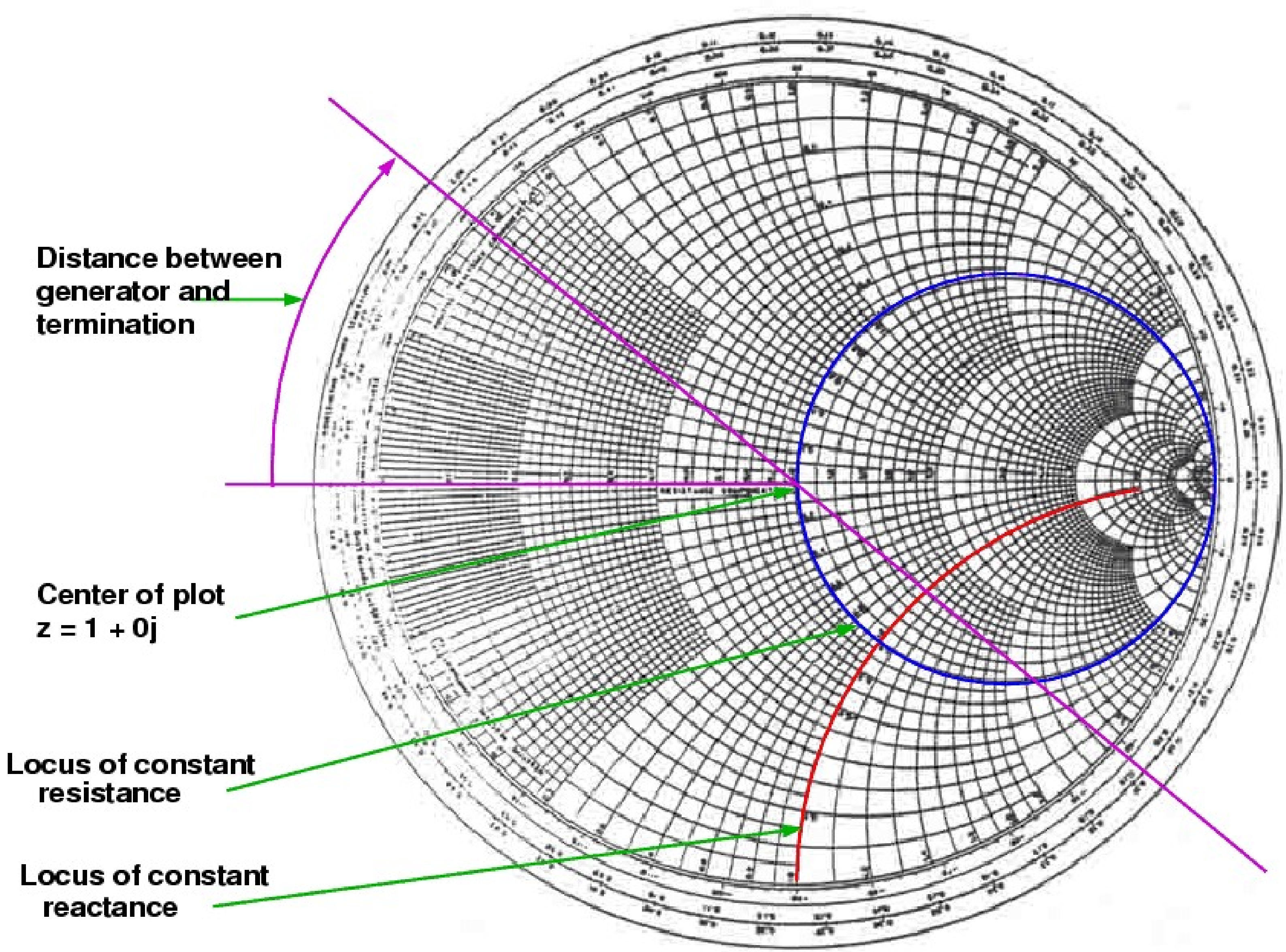
The Smith Chart allows easy calculation of the transformation of a complex load impedance through an arbitrary length of transmission line.

It also allows the calculation of the admittance  Y = 1/Z of an impedance.

The impedance is represented by a normalized impedance *z*.

$$z = \frac{Z}{Z_0}$$

Once around the circle is a line length of $\lambda/2$.



# Transform a Complex Impedance Through a Transmission Line

Start with an impedance $Z_i$ = 27 + 20j ohms

The normalized impedance for a 50 ohm line is

$$z_i = 0.54 + 0.4\,j$$

Plot this at point z1. Draw a circle through this point around the center. The radius of the circle is the reflection coefficient $\Gamma$, where the radius to the edge is 1.0.

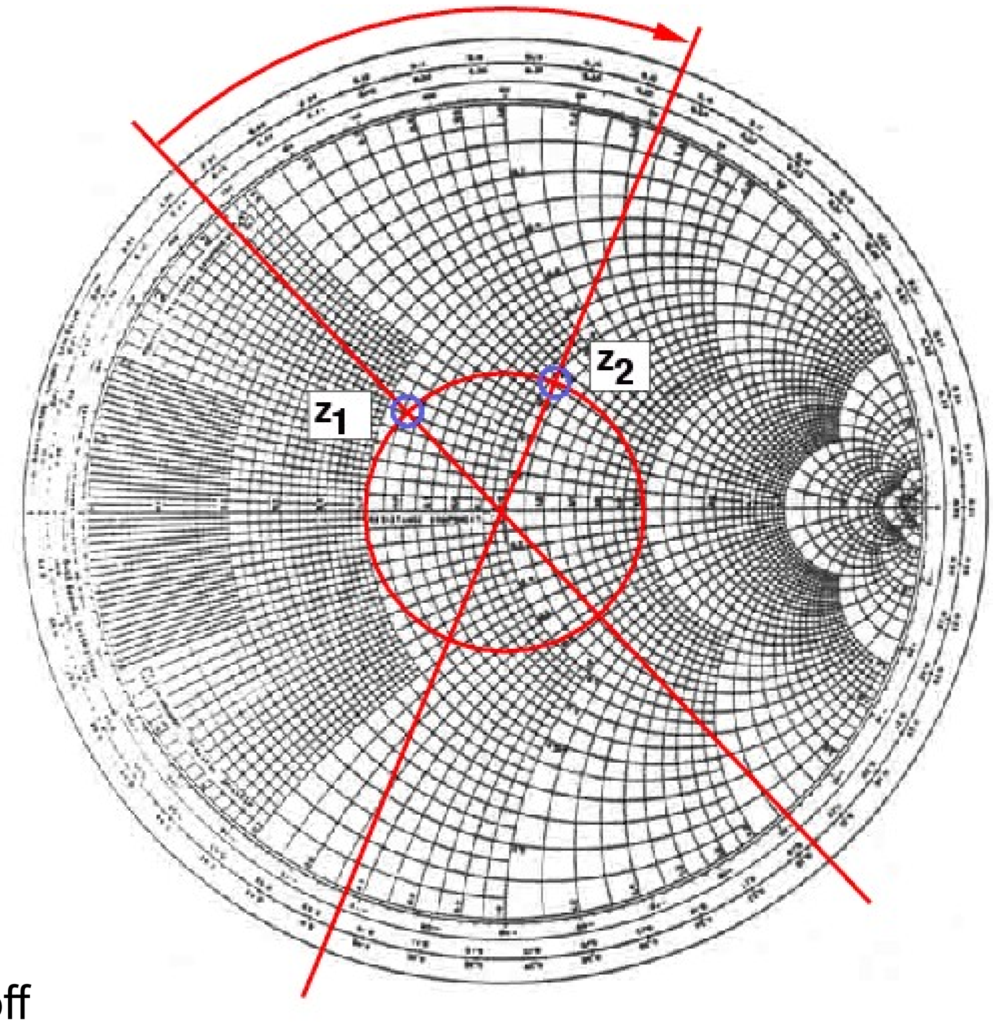
A transmission line is 0.085 wavelengths long. Draw a line through z1 and the center.

The line intersects the wavelength scale on the outer diameter at 0.07 $\lambda$.

Add 0.085 to 0.07 and draw a new line through the center to 0.155 $\lambda$.

Where the new line intersects the circle, read off the transformed normalized impedance

$z_f = 1.0 + 0.7\,j$ or $Z_f = 50 + 35\,j$ ohms.



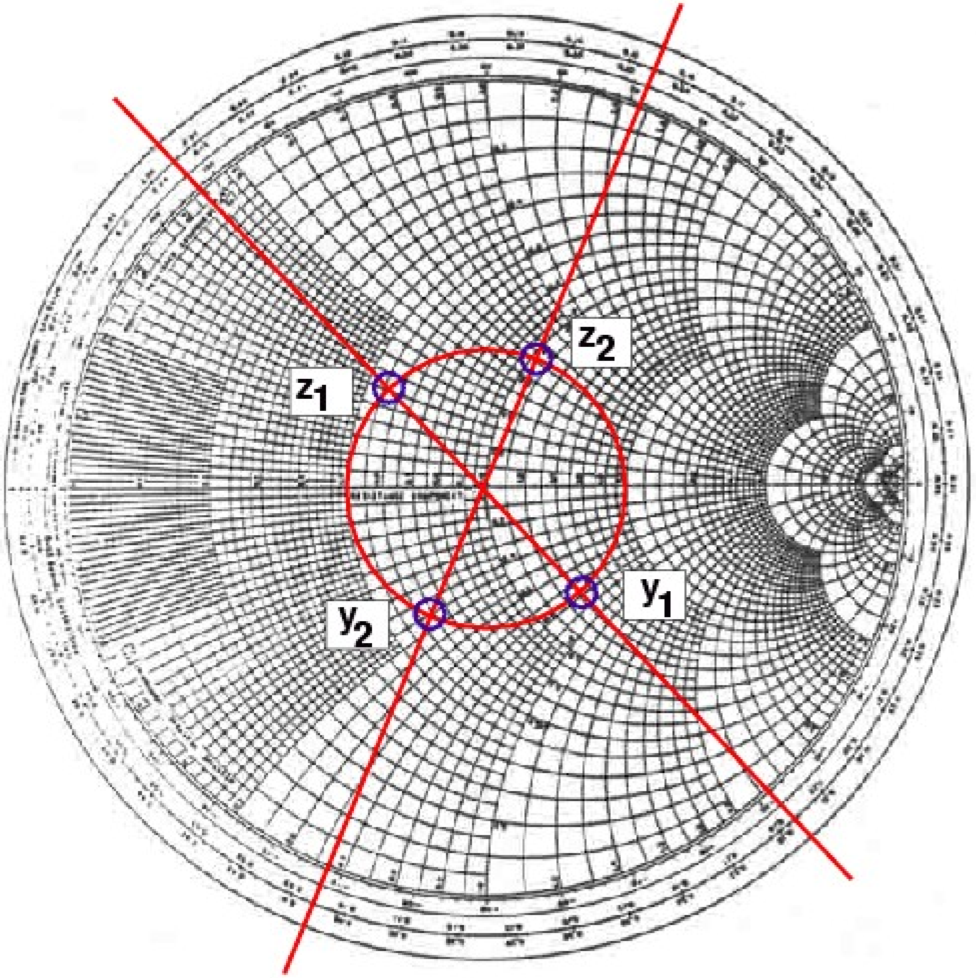
## Transforming an Impedance to an Admittance

It is often useful to find the admittance, the inverse of a given impedance.

$$y = 1/z$$

A line drawn through an impedance z to the opposite side of the red circle intersects the value of the admittance.

The Smith chart can be used to find the inverse of a complex quantity.



# Special Case: Shorted Line Stub

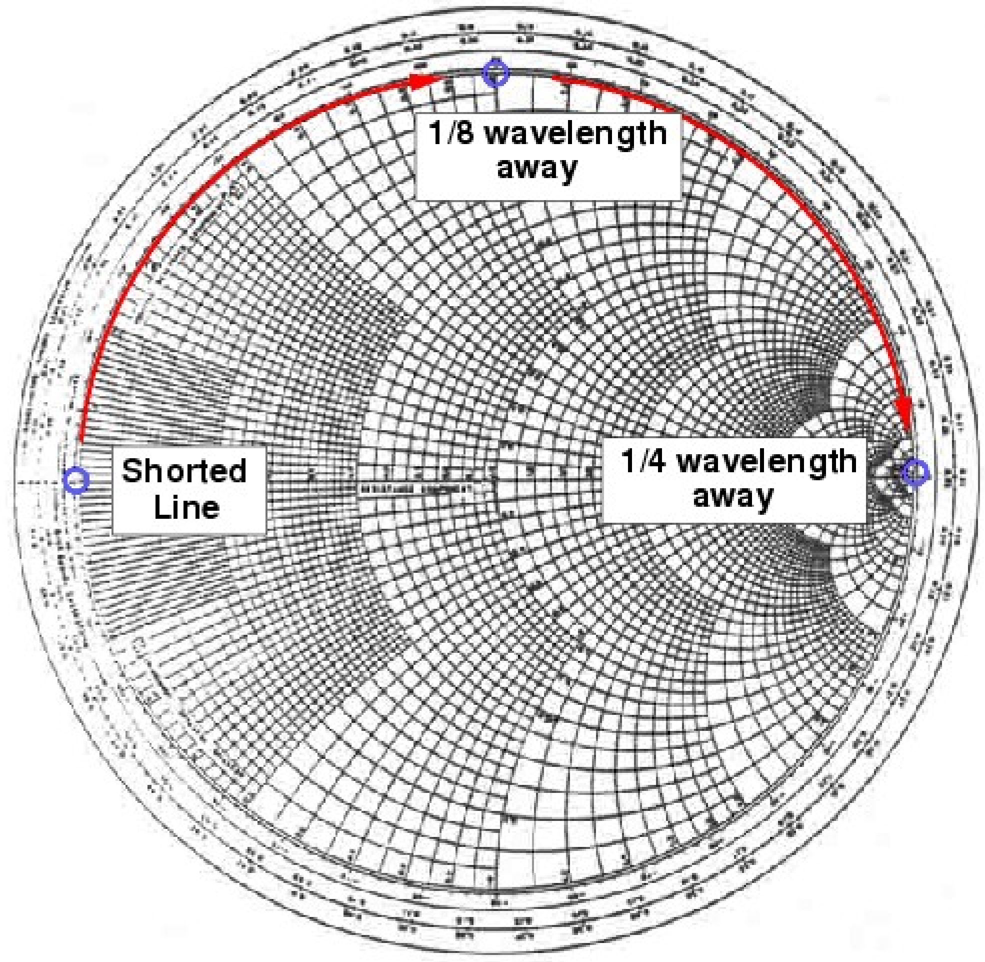
A shorted transmission line stub will have an impedance of $z = 0$.

1/8 $\lambda$ away it will have a normalized reactance of 0 +1 *j*. This is inductive reactance.

(The upper half of the diagram represents an inductive reactance.)

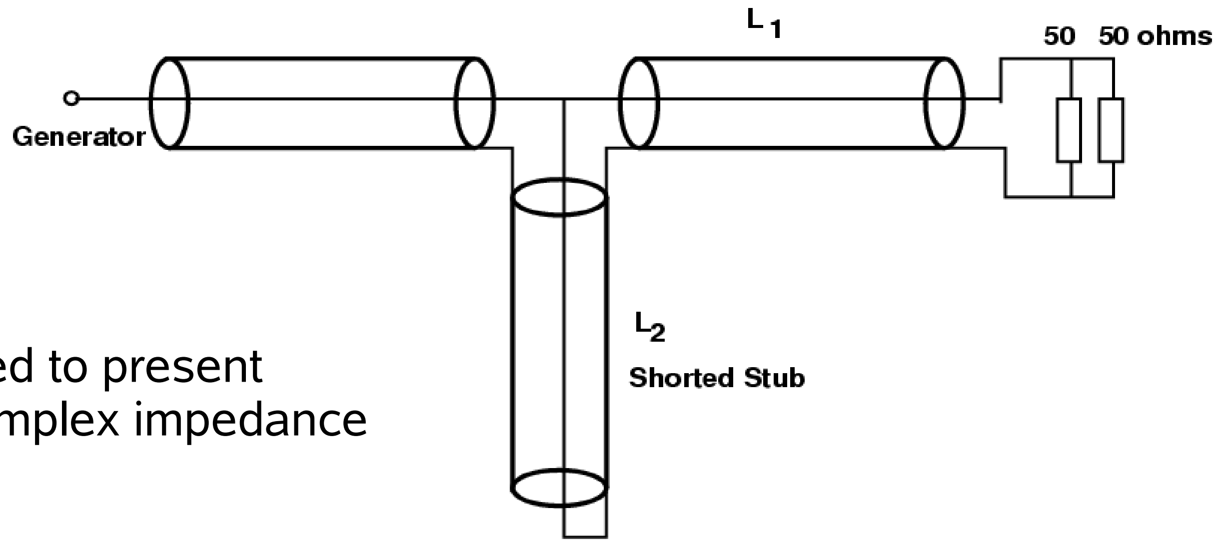
1/4 $\lambda$ away it will have a normalized reactance of infinity.

1/2 $\lambda$ away it again has an impedance of 0.




(What happens with an open transmission line segment?)

# Single-Stub Tuner




A mismatched load (2-50 ohm resistors in parallel) is to be matched to a pure 50 ohms at the generator port.

The length of line L1 will be varied to present a 50 ohm resistive part of the complex impedance at the junction.

The remainder of the imaginary part will be removed by adjusting the length of line L2 to cancel out the imaginary part.

Note that L1 and L2 form a parallel circuit. The formula for adding impedances in parallel is

$$\frac{1}{z_{tot}} = \frac{1}{z_1} + \frac{1}{z_2}$$

However, the formula for adding admittances is simpler, where $y = 1/z$

$$y_{tot} = y_1 + y_2$$

We will use the ability of the Smith chart to calculate admittances and add them.

## Calculation of a Single-Stub Tuner

We will use the Smith Chart to calculate a **tuner** that matches a 25 ohm resistive load to a 50 ohm transmission line at 100 MHz.

The normalized impedance $z_L = Z_L/50$ ohms = $0.5 + 0\,j$. We can calculate the reflection coefficient $\Gamma$.

$$\Gamma = \frac{z_L - 1}{z_L + 1} = 0.333$$

When this termination is attached to the far end of a 50 ohm transmission line of length L, the impedance, looking into the near end, is

$$Z_i = Z_0 \frac{1 + \Gamma e^{\frac{-4\pi L}{\lambda}}}{1 - \Gamma e^{-4\pi \frac{L}{\lambda}}}$$

Strategy: first add some line so the real part of the complex impedance is 50 ohms (normalized $z_i = 1.0$). Then add a shorted stub, which provides only reactance, to cancel out the imaginary part of the complex impedance, leaving a pure 50 ohm resistive load.

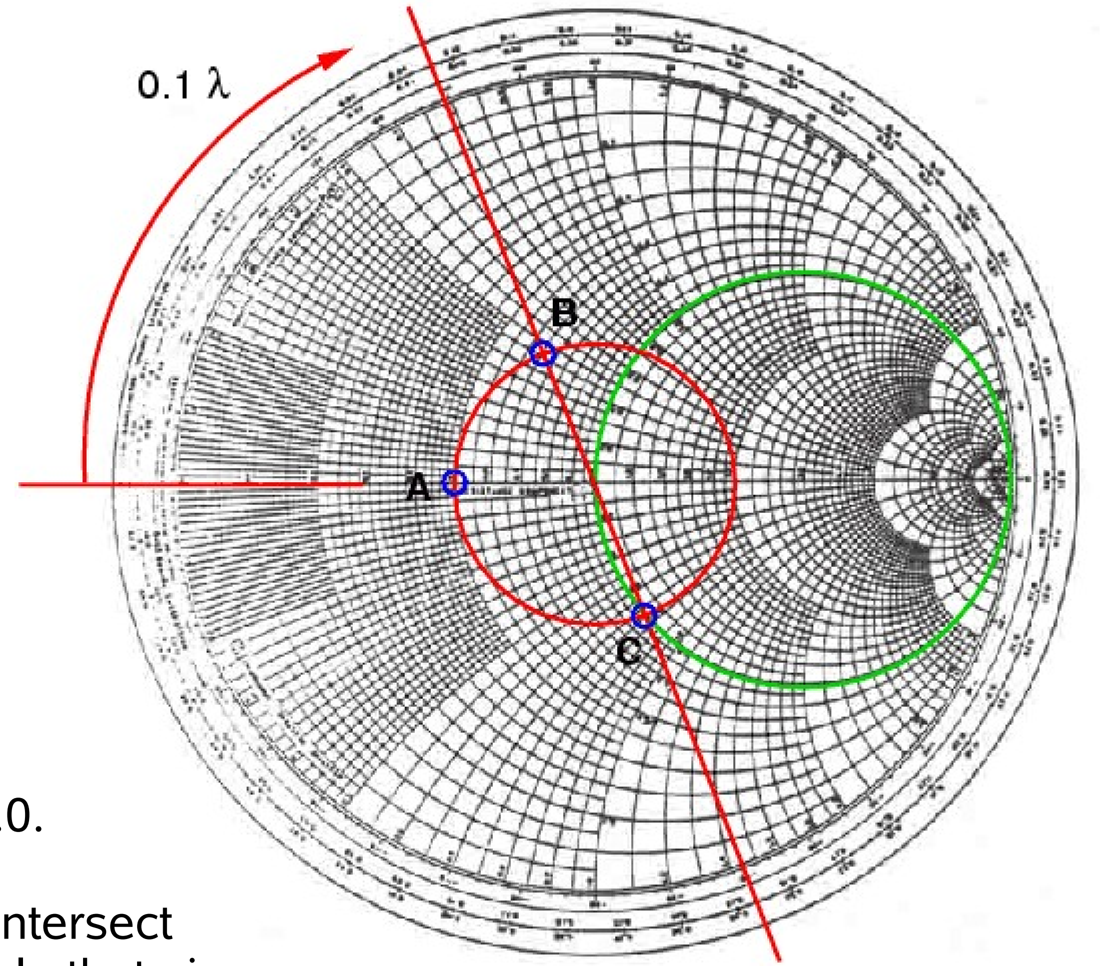
## Step 1: Transforming to z = 1.0 + 0 j : get the real part right

The normalized impedance of the 25 ohm resistive load is $z = 0.5 + 0$ j.

This is point A on the plot. Draw a circle through this point with the center of the plot as the center. The radius of the circle, $\Gamma = 0.333$ is the reflection coefficient for this load.

The real part of the complex impedance = 1.0 on the green circle.

The tricky part: we are going to add admittances, the inverse of impedance, so we want to find the length of $L_1$ that gives the real part of the admittance = 1.0.




That is where the red and green circles intersect at point C, and the length of $L_1$ is the angle that gives the impedance at point B. We can read off the outer labels that the required length of $L_1$ is about 0.1 wavelengths.

There is still a non-zero imaginary part of the impedance that can be read off point C.

# Step 2: Adding a Shorted Stub

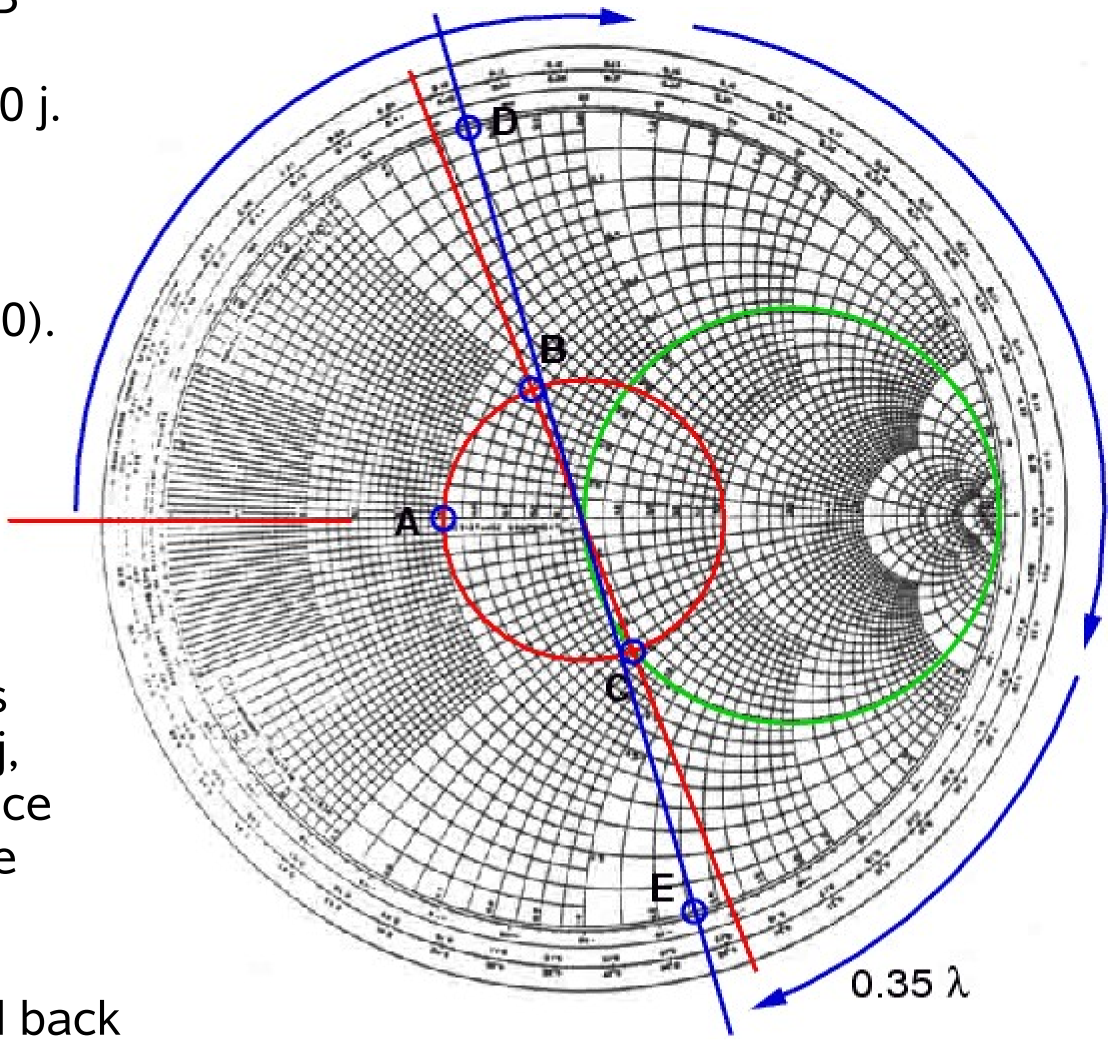
The normalized impedance z1 of point B is 0.67 + 0.471j. The corresponding admittance, found at point C is 1.0 - 0.70 j.

A shorted stub with admittance 0 + 0.7 j will be added to this, resulting in an admittance of 1.0 (also impedance of 1.0).

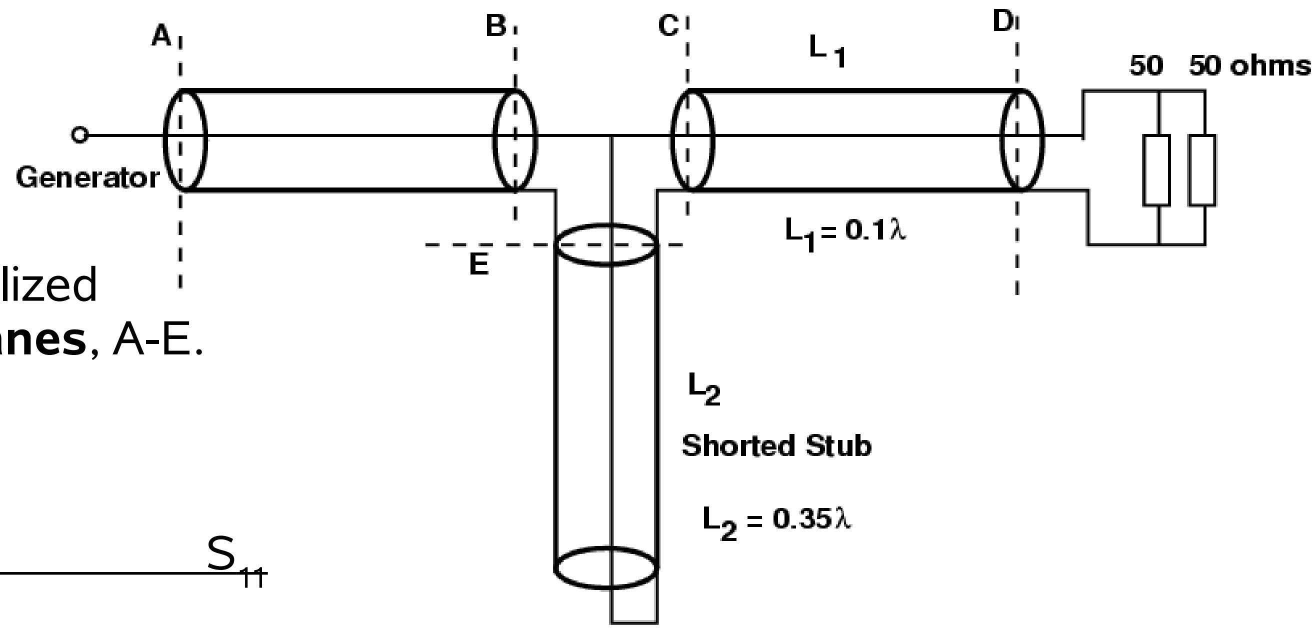
The length of the shorted stub is 0.35 $\lambda$, and its impedance, at point E is 0 - 1.43 j. The admittance at point D is 0 + 0.7 j.

The sum of the two parallel admittances is $y$ = (1.0 - 0.7 j) + (0 + 0.7 j) = 1.0 + 0 j, corresponding to a normalized admittance of 1.0 + 0 j. The normalized impedance $z$ is also 1.0 + 0 j.

This represents a matched 50 ohm load back to the generator.



## Reference Planes



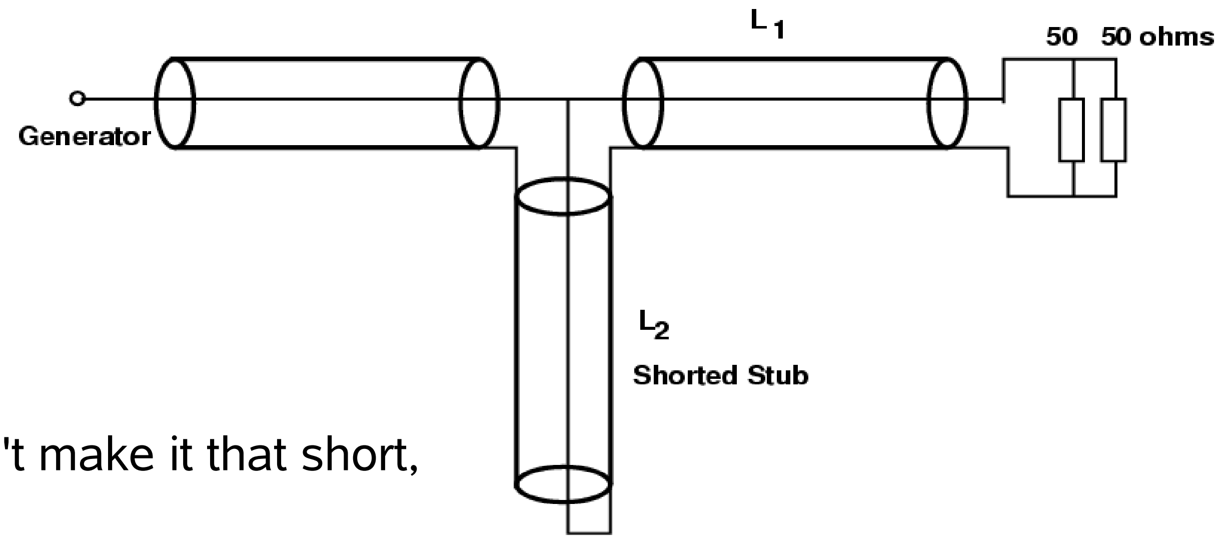

This problem can be broken down to calculating the normalized impedance at 5 **reference planes**, A-E.

| Plane | z | y | $S_{11}$ |
|---|---|---|---|
| A | 1.0 + 0.0 j | 1.0 + 0.0 j | 0.0 + 0.0 j |
| B | 1.0 + 0.0 j | 1.0 + 0.0 j | 0.0 + 0.0 j |
| C | 0.67 + .47 j | 1.0 - 0.7 j | -0.11 + .31 j |
| D | 0.5 + 0.0 j | 2.0 + 0.0 j | -0.33 + 0.0 j |
| E | 0.0 - 1.43 j | 0.0 + 0.7 j | 0.34 - 0.94 j |

$$S_{11} = \frac{z - 1}{z + 1}$$

The $S_{11}$ could be measured by a network analyzer, looking into each reference plane.

# Single-Stub Tuner Questions




L1 is 0.1 wavelength. If we can't make it that short, what could we do?

If L2 is an open stub, what would be its length. The shorted stub length is 0.35 wavelength.

If the load is 100 ohms, not 25 ohms, what would the line lengths need to be? What is the reflection coefficient of a 100 ohm load in a 50 ohm system?

If the shorted stub is placed in *series* with the center conductor of the transmission line, rather in parallel with it, how would the lengths of the lines change?

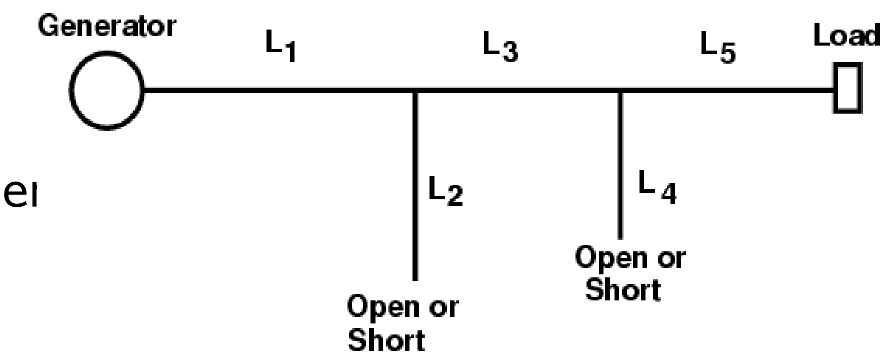
## Other Tuner Configurations

The major difficulty with the single-stub tuner is that it is not always possible to have a variable length of transmission line to the load.

Since two variables are involved, a **double-stub** tuner is an alternate configuration.

L3 and L5 are fixed lengths.

L2 and L4 are adjustable, open or shorted e




What are the constraints on L3?

The tuning procedure is more complex, requiring more operations with the Smith Chart.

The match is correct only for the design wavelength.

What happens if a multiple of a half-wave line is added to any of the matching sections?

## Single-Stub Tuner Lab Experiment

Calibrate a network analyzer with a 50 ohm terminator at over the range of 100 to 500 MHz and show that $S_{11} = 0$.

Using a single trombone with a 25 ohm load, measure S11 for various lengths of the trombone.   You may have to include some sections of coaxial cable to increase the length to 1 wavelength.   Pay attention to the velocity factor in the trombone and in any piece of polyethylene coaxial cable you may use.   The load can be two 50-ohm terminators on a coaxial “T”.

Using two trombones (variable-length coaxial line segments), assemble a single-stub tuner with a 25 ohm load and attach to port A of the NA.

Reduce the sweep of the NA to 300 to 350 MHz.   Set the NA to the Smith Chart display.   Set the marker on the NA to 320 MHz.

Adjust the length of the two trombones so that S11 = 0 (crosses the center of the Smith chart) at 320 MHz.    Measure the lengths of all the line sections, noting that the velocity factor in the trombones is the speed of light, but is 0.66c in the connectors and the flexible coaxial cables.

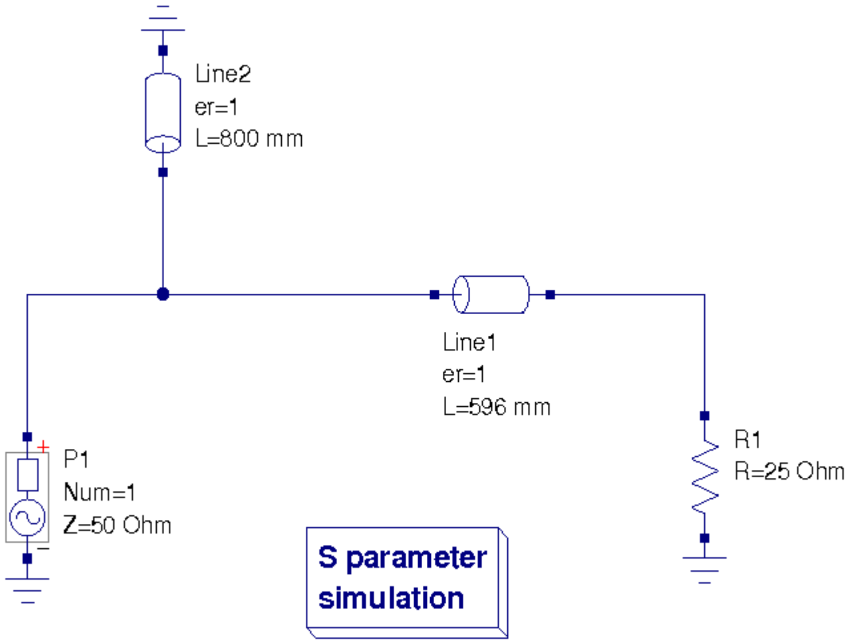
# Analysis

Convert the physical length of the two branches of the single-stub tuner after the "T" to fractions of a wavelength of 320 MHz.

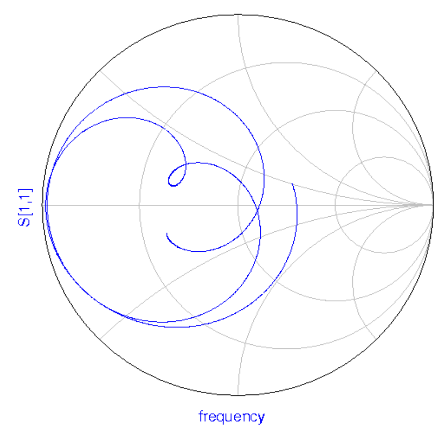
Plot the transformed impedance and admittance of each transmission line section, the line to the 25 ohm load, and the shorted stub, on the Smith chart. Do the lengths agree with the lengths calculated in the example above in the writeup? Why not?

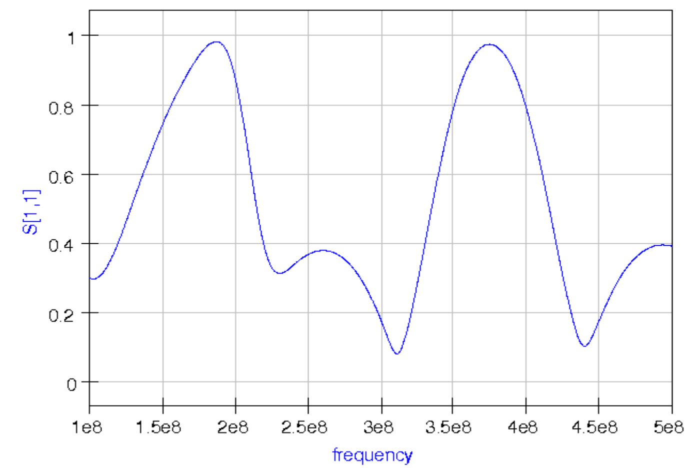
What are the differences?

# Single-Stub Tuner Simulation and Experiment




The **qucs** program has a very useful module for simulating S-parameters in many system.




Here is the qucs simulation for this problem, along with the actual NA display.

To get the right Smith chart plot, an electrical delay must be included to establish the reference plane at the “T”.