SURFACE IMPEDANCE

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Definitions

• The electromagnetic response of a metal, whether normal or superconducting, is described by a complex surface impedance, \( Z = R + iX \)

\[
R \, : \, \text{Surface resistance}
\]
\[
X \, : \, \text{Surface reactance}
\]

Both \( R \) and \( X \) are real
Definitions

For a semi-infinite slab:

\[ Z = \frac{E_x(0)}{\int_0^\infty J_x(z) \, dz} \]

Definition

\[ = \frac{E_x(0)}{H_y(0)} = i \omega \mu_0 \frac{E_x(0)}{\partial E_x(z)/\partial z\bigg|_{z=0^+}} \]

From Maxwell
Definitions

The surface resistance is also related to the power flow into the conductor

\[ Z = Z_0 \frac{\vec{S}(0_+)}{\vec{S}(0_-)} \]

\[ Z_0 = \left( \frac{\mu_0}{\varepsilon_0} \right)^{1/2} \approx 377\Omega \quad \text{Impedance of vacuum} \]

\[ \vec{S} = \vec{E} \times \vec{H} \quad \text{Poynting vector} \]

and to the power dissipated inside the conductor

\[ P = \frac{1}{2} R H^2(0_-) \]
Normal Conductors (local limit)

Maxwell equations are not sufficient to model the behavior of electromagnetic fields in materials. Need an additional equation to describe material properties

\[ \frac{\partial J}{\partial t} + \frac{J}{\tau} = \frac{\sigma}{\tau} E \]

\[ \Rightarrow \sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau} \]

For Cu at 300 K, \( \tau = 3 \times 10^{-14} \) sec
so for wavelengths longer than infrared \( J = \sigma E \)
Normal Conductors (local limit)

- In the local limit
  \[ \vec{J}(z) = \sigma \vec{E}(z) \]

- The fields decay with a characteristic length (skin depth)
  \[ \delta = \left( \frac{2}{\mu_0 \omega \sigma} \right)^{1/2} \]

\[
E_x(z) = E_x(0) e^{-z/\delta} e^{-iz/\delta}
\]

\[
H_y(z) = \frac{(1-i)}{\mu_0 \omega \delta} E_x(z)
\]

\[
Z = \frac{E_x(0)}{H_y(0)} = \frac{(1+i)}{2} \mu_0 \omega \delta = \frac{(1+i)}{\sigma \delta} = (1+i) \left( \frac{\mu_0 \omega}{2 \sigma} \right)^{1/2}
\]
Normal Conductors (anomalous limit)

• At low temperature, experiments show that the surface resistance becomes independent of the conductivity

• As the temperature decreases, the conductivity $\sigma$ increases
  - The skin depth decreases
    \[ \delta = \left( \frac{2}{\mu_0 \omega \sigma} \right)^{1/2} \]
  - The skin depth (the distance over which fields vary) can become less than the mean free path of the electrons (the distance they travel before being scattered)
  - The electrons do not experience a constant electric field over a mean free path
  - The local relationship between field and current is not valid
    \[ \vec{J}(z) \neq \sigma \vec{E}(z) \]
Normal Conductors (anomalous limit)

Introduce a new relationship where the current is related to the electric field over a volume of the size of the mean free path \( (l) \)

\[
\bar{J}(\vec{r},t) = \frac{3\sigma}{4\pi l} \int d\vec{r}' \frac{\overline{R} \cdot \overline{E}(\vec{r}', t - \overline{R}/v_F)}{R^4} e^{-R/l} \quad \text{with} \quad \overline{R} = \vec{r}' - \vec{r}
\]

Specular reflection: Boundaries act as perfect mirrors
Diffuse reflection: Electrons forget everything
Normal Conductors (anomalous limit)

- In the extreme anomalous limit

\[
\left( \frac{3l^2}{2\delta_{cl}^2} \gg 1 \right)
\]

\[
\frac{9}{8} Z_{p=1} = Z_{p=0} = \left( \frac{\sqrt{3} \mu_0^2 \omega^2 l}{16\pi \sigma} \right)^{1/3} \left(1 + i\sqrt{3} \right)
\]

**Fig. 2** Anomalous skin effect in a 500 MHz Cu cavity

- $p$: fraction of electrons specularly scattered at surface
- $1 - p$: fraction of electrons diffusively scattered
Normal Conductors (anomalous limit)

\[ R(l \to \infty) = 3.79 \times 10^{-5} \omega^{2/3} \left( \frac{l}{\sigma} \right)^{1/3} \]

For Cu: \( \frac{l}{\sigma} = 6.8 \times 10^{-16} \text{ } \Omega \cdot \text{m}^2 \)

\[
\frac{R(4.2 \text{ K}, 500 \text{ MHz})}{R(273 \text{ K}, 500 \text{ MHz})} = \frac{3.79 \times 10^{-5} \omega^{2/3} \left( \frac{l}{\sigma} \right)^{1/3}}{\sqrt{\frac{\mu_0 \omega}{2\sigma}}} \approx 0.12
\]

Does not compensate for the Carnot efficiency
Surface Resistance of Superconductors

Superconductors are free of power dissipation in static fields.

In microwave fields, the time-dependent magnetic field in the penetration depth will generate an electric field:

\[ \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]

The electric field will induce oscillations in the normal electrons, which will lead to power dissipation.
Surface Impedance in the Two-Fluid Model

In a superconductor, a time-dependent current will be carried by the Copper pairs (superfluid component) and by the unpaired electrons (normal component)

\[ J = J_n + J_s \]

\[ J_n = \sigma_n E_0 e^{-i\omega t} \]  
(Ohm's law for normal electrons)

\[ J_s = i \frac{2n_c e^2}{m_e \omega} E_0 e^{-i\omega t} \]  
\( (m_e \dot{v}_c = -eE_0 e^{-i\omega t}) \)

\[ J = \sigma E_0 e^{-i\omega t} \]

\[ \sigma = \sigma_n + i\sigma_s \]  
with \[ \sigma_s = \frac{2n_c e^2}{m_e \omega} = \frac{1}{\mu_0 \lambda_L^2 \omega} \]
Surface Impedance in the Two-Fluid Model

For normal conductors

\[ R_s = \frac{1}{\sigma\delta} \]

For superconductors

\[ R_s = \Re \left[ \frac{1}{\lambda_L \left( \sigma_n + i\sigma_s \right)} \right] = \frac{1}{\lambda_L} \frac{\sigma_n}{\sigma_n^2 + \sigma_s^2} \approx \frac{1}{\lambda_L} \frac{\sigma_n}{\sigma_s^2} \]

The superconducting state surface resistance is proportional to the normal state conductivity
Surface Impedance in the Two-Fluid Model

\[ R_s \approx \frac{1}{\lambda_L} \frac{\sigma_n}{\sigma_s^2} \]

\[ \sigma_n = \frac{n_n e^2 l}{m_e v_F} \propto l \exp \left[ -\frac{\Delta(T)}{kT} \right] \quad \sigma_s = \frac{1}{\mu_0 \lambda_L^2 \omega} \]

\[ R_s \propto \lambda_L^3 \omega^2 l \exp \left[ -\frac{\Delta(T)}{kT} \right] \]

This assumes that the mean free path is much larger than the coherence length
Surface Impedance in the Two-Fluid Model

For niobium we need to replace the London penetration depth with

\[ \Lambda = \lambda_L \sqrt{1 + \frac{\xi}{l}} \]

As a result, the surface resistance shows a minimum when

\[ \xi \approx l \]
Surface Resistance of Niobium
Electrodynamics and Surface Impedance in BCS Model

\[ H_0 \phi + H_{ex} \phi = i\hbar \frac{\partial \phi}{\partial t} \]

\[ H_{ex} = \frac{e}{mc} \sum A(r_i, t) p_i \]

\( H_{ex} \) is treated as a small perturbation \( H_{rf} \ll H_c \)

There is, at present, no model for superconducting surface resistance at high rf field

\[ J \propto \int \frac{R[R \cdot A] I(\omega, R, T) e^{-\frac{R}{T}}}{R^4} dr \]

\[ J(k) = -\frac{c}{4\pi} K(k) A(k) \]

\( K(0) \neq 0 \): Meissner effect
Surface Resistance of Superconductors

Temperature dependence

- close to $T_c$:
  
  dominated by change in $\lambda(t) \frac{t^4}{(1-t^2)^{3/2}}$

- for $T < \frac{T_c}{2}$:
  
  dominated by density of excited states $\sim e^{-\Delta/\kappa T}$
  
  $R_s \sim \frac{A}{T} \omega^2 \exp\left(-\frac{\Delta}{\kappa T}\right)$

Frequency dependence

$\omega^2$ is a good approximation

Figure 4.5: Theoretical surface resistance at 1.5 GHz of lead, niobium and Nb$_3$Sn as calculated from program [94]. The values given in Table 4.1 were used for the material parameters.
Surface Resistance of Superconductors

• The surface resistance of superconductors depends on the frequency, the temperature, and a few material parameters
  – Transition temperature
  – Energy gap
  – Coherence length
  – Penetration depth
  – Mean free path

• A good approximation for $T<T_c/2$ and $\omega<<\Delta/h$ is

$$R_s \sim \frac{A}{T} \omega^2 \exp \left( -\frac{\Delta}{kT} \right) + R_{res}$$
Surface Resistance of Superconductors

\[ R_s \sim \frac{A}{T} \omega^2 \exp \left( -\frac{\Delta}{kT} \right) + R_{\text{res}} \]

In the dirty limit \( l \ll \xi_0 \) \[ R_{\text{BCS}} \propto l^{-1/2} \]

In the clean limit \( l \gg \xi_0 \) \[ R_{\text{BCS}} \propto l \]

\[ R_{\text{res}} \]
Residual surface resistance
No clear temperature dependence
No clear frequency dependence
Depends on trapped flux, impurities, grain boundaries, …
Surface Resistance of Superconductors

Fig. 2. Temperature dependence of surface resistance of niobium at 3.7 GHz measured in the $TE_{011}$ mode at $H_{c2} \approx 10$ G. The values computed with the BCS theory used the following material parameters:

- $T_c = 9.25$ K;
- $\lambda_L(T = 0, l = \infty) = 320$ Å;
- $\Delta(0)/kT = 1.85$;
- $\xi_{\parallel}(T = 0, l = \infty) = 620$ Å;
- $l = 1000$ Å or 80 Å.

Fig. 5. The surface resistance of Nb at 4.2 K as a function of frequency [62,63]. Whereas the isotropic BCS surface resistance $\sim \omega^{1.8}$ around 1 GHz, the measurements fit better to $\omega^{2}$ $(-\cdots-\cdot)$. The solid curve, which fits the data over the entire range, is a calculation based on the smearing of the BCS density-of-states singularity by the energy gap anisotropy in the presence of impurity scattering [61]. The authors thank G. Müller for providing this figure.
Surface Resistance of Niobium

Surface Resistance of Niobium
at \( F = 700 \text{ MHz} \)

Transition Temperature
\( T_c = 9.25 \text{ K} \)

Residual Resistance

Temperature (K)
Surface Resistance of Niobium

![Graph showing the surface resistance of niobium over a range of mean free paths and frequencies.](image)

- **Mean free path [Å]**
- **BCS surface resistance [Ohm]**
- **1500 MHz**

Legend:
- Green line: Diffuse ref. 4.25K
- Red line: Specular ref. 4.25K
- Green line: Diffuse ref. 2.0K
- Red line: Specular ref. 2.0K

Thomas Jefferson National Accelerator Facility

Jefferson Lab
Super and Normal Conductors

- Normal Conductors
  - Skin depth proportional to $\omega^{-1/2}$
  - Surface resistance proportional to $\omega^{1/2} \rightarrow 2/3$
  - Surface resistance independent of temperature (at low T)
  - For Cu at 300K and 1 GHz, $R_s = 8.3$ m$\Omega$

- Superconductors
  - Penetration depth independent of $\omega$
  - Surface resistance proportional to $\omega^2$
  - Surface resistance strongly dependent of temperature
  - For Nb at 2 K and 1 GHz, $R_s \approx 7$ n$\Omega$

However: do not forget Carnot