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LONGITUDINAL PHYSICS PART II

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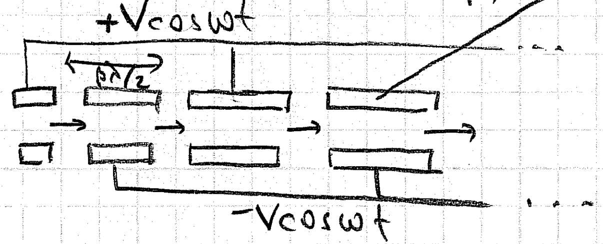
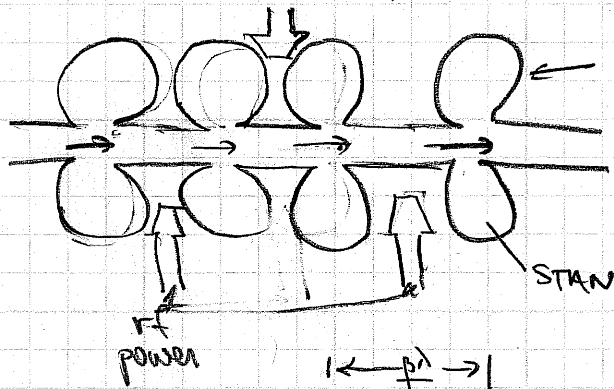
ACCELERATION

rf (radio-frequency)



TUBE SHIELDS BEAM

(Wideroe linac)
LOW FREQUENCIES (< 100 MHz)

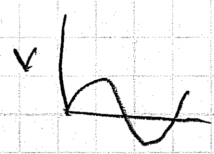


RESONANT CAVITY

(COUPLED CAVITY Linac)
$0.4 < \beta < 1.0$

STANDING EM wave

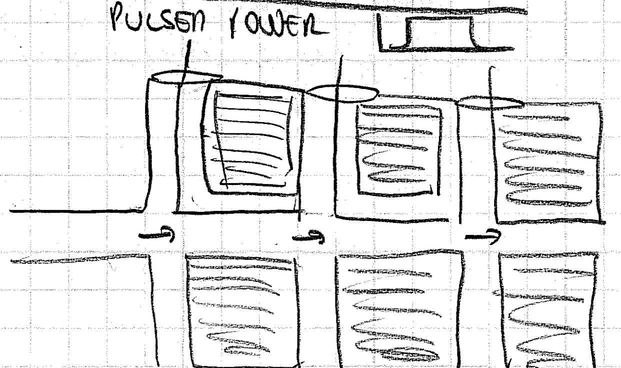
FREQUENCIES ~ 100's MHz - ~1 GHz



IN EACH GAP $E = E_m \sin \omega t$

induction acceleration

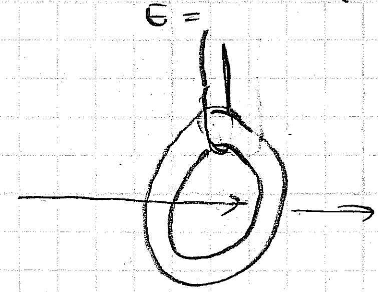
PULSED POWER



(INDUCTION LINAC)

$\nabla \times E = \partial B / \partial t$

$E =$

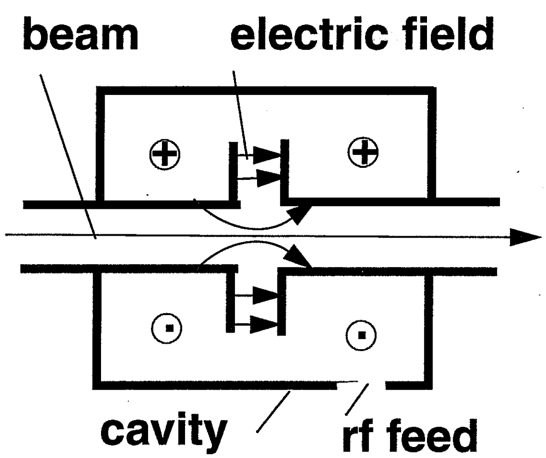


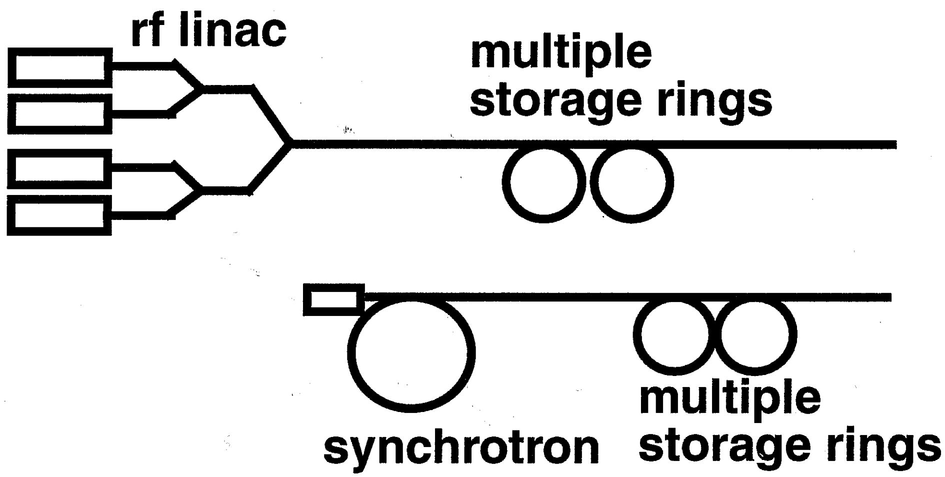
TRANSFORMER

IN EACH GAP $E =$ CONSTANT
(OR SOME PRESCRIBED FUNCTION)

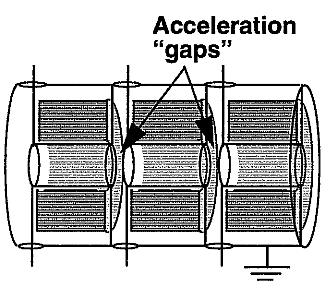
The principle methods of acceleration (for HIF)

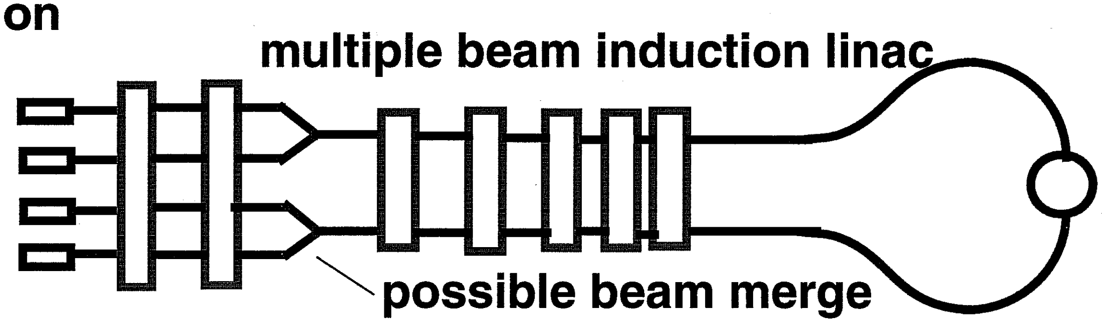
1. r.f. acceleration



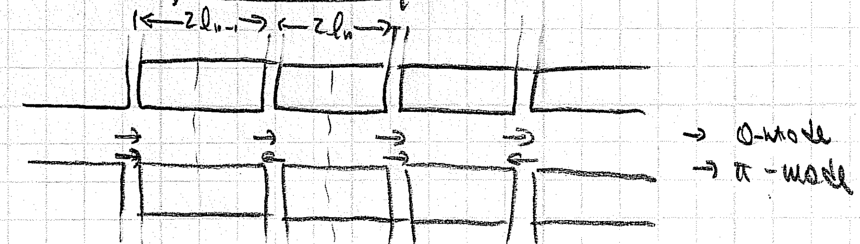


2. Induction acceleration





rf – longitudinal equation of motion



ϕ_{n-1} $\quad$ ϕ_n $\quad$ L_n

$$E_n = \begin{cases} E_0(-1)^n \cos\omega t & \leftarrow \pi\text{-mode} \\ E_0 \cos\omega t & \leftarrow 0\text{-mode} \end{cases}$$

$E_z = E_0 \cos(\phi_s)$ ⟵ Synchronous particle enters each gap at same phase

SYNCHRONOUS PARTICLE:

$$\Rightarrow \quad \ell_{n-1} = \frac{\beta_s \lambda}{2}\begin{Bmatrix} \frac{1}{2} \\ 1 \end{Bmatrix} \begin{matrix} \pi\text{-mode} \\ 0\text{-mode} \end{matrix} \qquad \lambda = \frac{2\pi}{\omega} c = \text{light travel distance in one of oscillation}$$

i.e. it takes $\begin{Bmatrix} 1/2 \\ 1 \end{Bmatrix}$ OSCILLATION PERIOD TO TRAVEL BETWEEN EACH GAP.

$$\phi_n = \phi_{n-1} + \omega \frac{2\ell_{n-1}}{\beta_{n-1} c} + \begin{cases} \pi & \pi\text{-mode} \\ 0 & 0\text{-mode} \end{cases}$$

FOR PARTICLES ARRIVING AT SLIGHTLY DIFFERENT PHASES; WITH SLIGHTLY DIFFERENT VELOCITIES

$$\Delta(\phi - \phi_s)_n = 2\pi \beta_{s,n-1}\left(\frac{1}{\beta_{n-1}} - \frac{1}{\beta_{s,n-1}}\right)\begin{Bmatrix} \frac{1}{2} \\ 1 \end{Bmatrix}$$

$$\frac{1}{\beta} - \frac{1}{\beta_s} \simeq \frac{-\delta\beta}{\beta_s^2} \qquad \cong -2\pi \frac{\delta\beta}{\beta_{s,n-1}}\begin{Bmatrix} 1/2 \\ 1 \end{Bmatrix}$$

$$\delta\beta \simeq \frac{\delta W}{\gamma_s^3 \beta_s m c^2} \qquad \cong -2\pi \frac{W_{n-1} - W_{s,n-1}}{mc^2 \gamma_{s,n-1}^3 \beta_{s,n-1}^2}\begin{Bmatrix} \frac{1}{2} \\ 1 \end{Bmatrix}$$

$$W = (\gamma - 1)mc^2$$

Similarly,

$$\Delta(W - W_s)_n = qE_0 L_n(\cos\varphi_n - \cos\varphi_{s,n})$$

where $L_n = \frac{(\beta_{s,n-1} + \beta_{s,n})\lambda}{2}\begin{Bmatrix} 1/2 \\ 1 \end{Bmatrix} \equiv$ center to center distance between drift tubes

$(\Delta W = qE_0 L \cos\phi_s)$

CONVERTING TO A CONTINUOUS VARIABLE:

$$\Delta(\phi-\phi_s) \to \frac{d\,\Delta\phi}{dn} \qquad \Delta(W-W_s) \to \frac{d\,\Delta W}{dn}$$

$$dn = \frac{ds}{\beta_s\lambda}\cdot\left\{\begin{matrix}2\\1\end{matrix}\right\}$$

$$\Rightarrow \boxed{\begin{aligned} \gamma_s^3\beta_s^3\,\frac{d\Delta\phi}{ds} &= -2\pi\,\frac{\Delta W}{mc^2\lambda} \\ \frac{d\Delta W}{ds} &= qE_0(\cos\phi-\cos\phi_s)\end{aligned}}$$

$$\frac{d}{ds}\left[\gamma_s^3\beta_s\,\frac{d\Delta\phi}{ds}\right] = -2\pi\,\frac{qE_0}{mc^2\lambda}\left[\cos\phi-\cos\phi_s\right] \qquad (\mathrm{I})$$

NOW THE SPATIAL SEPARATION IS GIVEN BY:

$$\Delta z \equiv z-z_s = -\frac{\beta_s\lambda}{2\pi}\,\Delta\phi$$

$\Rightarrow$ ALSO, LET $\cos\phi-\cos\phi_s \simeq -\sin\phi_s\,\Delta\phi$ $\left[\text{for } \frac{2\pi\,\Delta z}{\beta\lambda} = \Delta\phi \ll 1\right]$

$$\Rightarrow \frac{d}{ds}\left[\gamma_s^3\beta_s\frac{d}{ds}\left(\frac{\Delta z}{\beta_s}\right)\right] \cong -\frac{2\pi}{\lambda}\,\frac{qE_0}{mc^2}\sin\phi_s\,\frac{\Delta z}{\beta_s}$$

WHEN THE ACCELERATION RATE IS SMALL

$$\Rightarrow \frac{d^2}{ds^2}\Delta z \cong -\frac{2\pi}{\lambda}\,\frac{qE_0\sin\phi_s}{\gamma_s^3\beta_s mc^2}\,\Delta z$$

$$\equiv -k_{s0}^2\,\Delta z \qquad \text{(synchrotron oscillations)}$$

RETURNING TO $\Delta W - \phi$ NOTATION

Let $w = \dfrac{\Delta W}{mc^2}$ $\quad A = \dfrac{2\pi}{\beta_s^3 \gamma_s^3 \lambda}$ $\quad B = \dfrac{qE_0}{mc^2}$

$$\Rightarrow \quad w' = B(\cos\phi - \cos\phi_s)$$

$$\phi' = -Aw$$

$$\phi'' = -AB(\cos\phi - \cos\phi_s)$$

MULTIPLYING BY ϕ' AND INTEGRATING:

$$\frac{\phi'^2}{2} = -AB(\sin\phi - \phi\cos\phi_s) + \text{const}$$

USING $\phi' = -Aw$ (& DIVIDING BY A)

$$\Rightarrow \quad \underbrace{\frac{Aw^2}{2}}_{\text{kinetic energy}} + \underbrace{B(\sin\phi - \phi\cos\phi_s)}_{\text{potential energy}} = \text{CONST.}$$

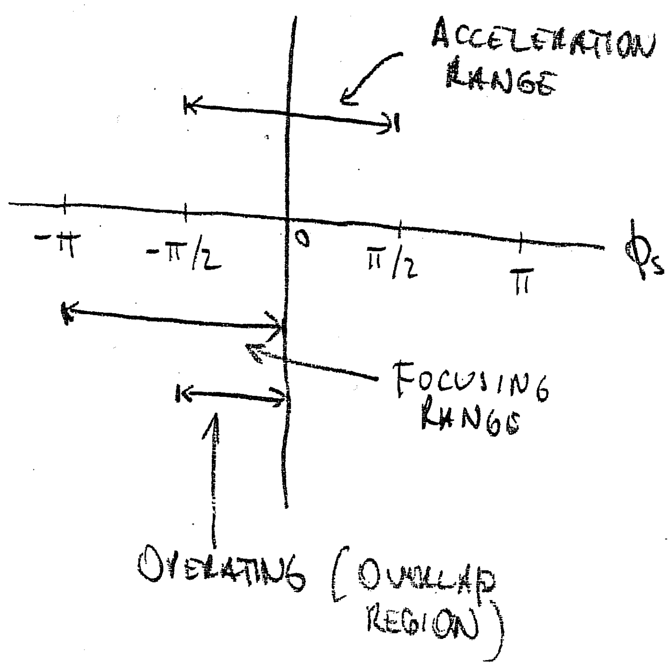
$$\frac{dW_s}{ds} \sim qE_0\cos\phi_s$$

$$V(\phi) = B(\sin\phi - \phi\cos\phi_s)$$

$$\frac{dV}{d\phi} = B(\cos\phi - \cos\phi_s)$$

$$\frac{d^2V}{d\phi^2} = -B\sin\phi$$

$$> 0 \Rightarrow -\pi < \phi_s < 0$$



FROM T. WANGLER'S
TEXT ON "rf linear accelerator"

(2)

simultaneous acceleration and a potential well when $-\pi/2 \leq \phi_s \leq 0$. The stable region for the phase motion extends from $\phi_2 < \phi < -\phi_s$, where the lower phase limit ϕ_2 can be obtained numerically by solving for ϕ_2 using $H_\phi(\phi_2) = H_\phi(-\phi_s)$. Figure 6.3 shows longitudinal phase space and the longitudinal potential well. At the potential maximum, where $\phi = -\phi_s$, we

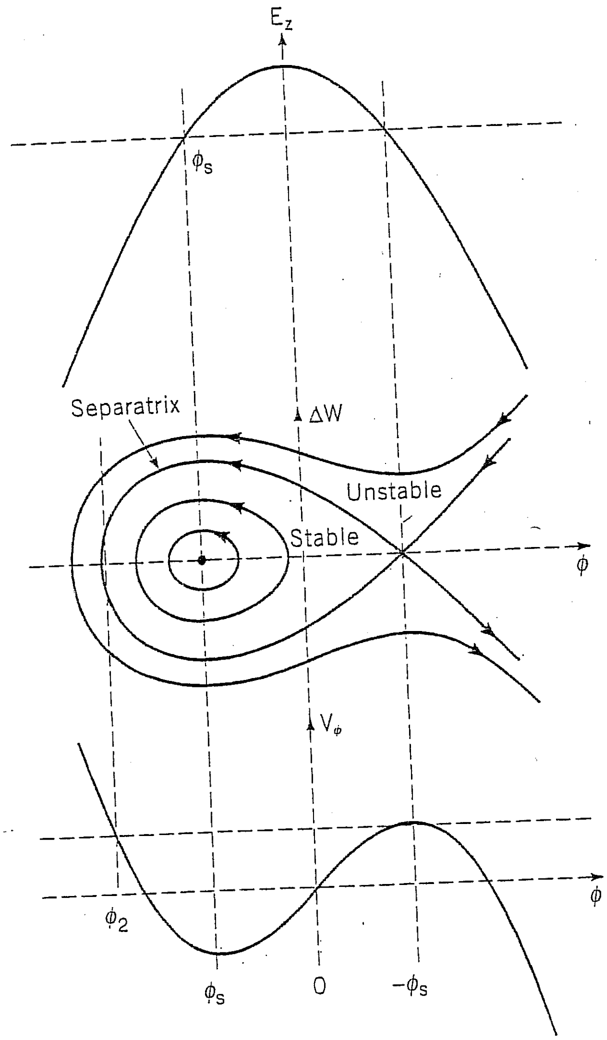
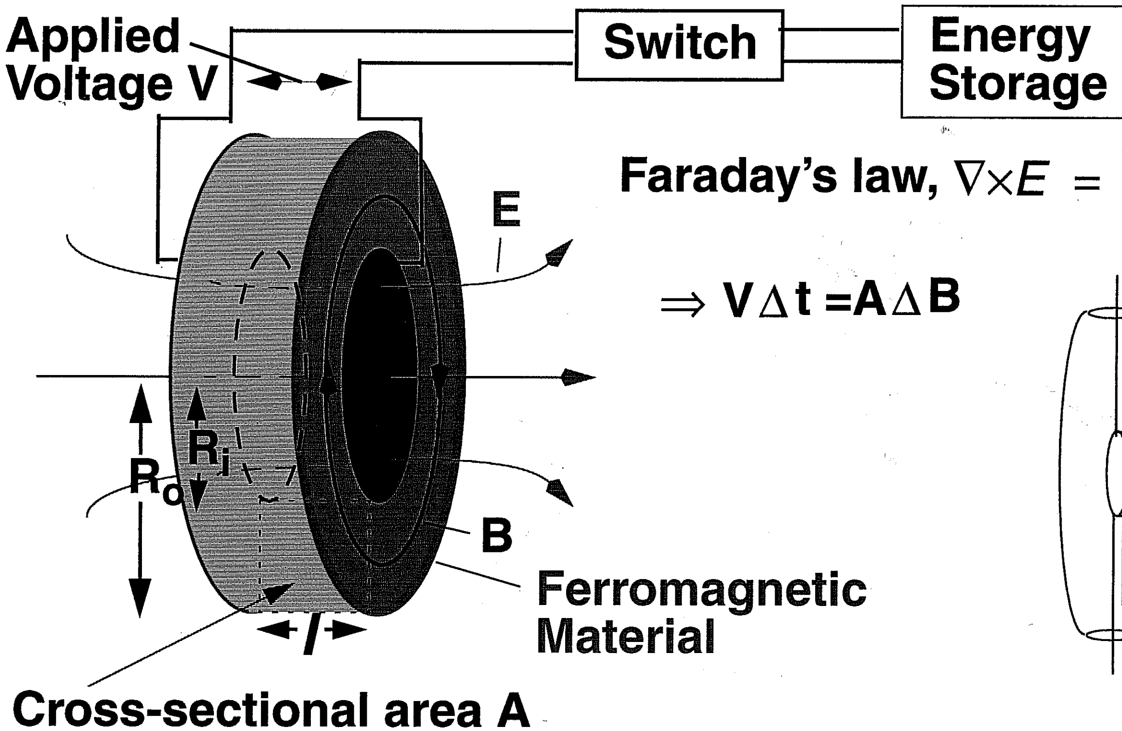


Figure 6.3. At the top, the accelerating field is shown as a cosine function of the phase; the synchronous phase ϕ_s is shown as a negative number, which lies earlier than the crest where the field is rising in time. The middle plot shows some longitudinal phase-space trajectories, including the separatrix, the limiting stable trajectory, which passes through the unstable fixed point at $\Delta W = 0$, and $\phi = -\phi_s$. The stable fixed point lies at $\Delta W = 0$ and $\phi = \phi_s$, where the longitudinal potential well has its minimum, as shown in the bottom plot.

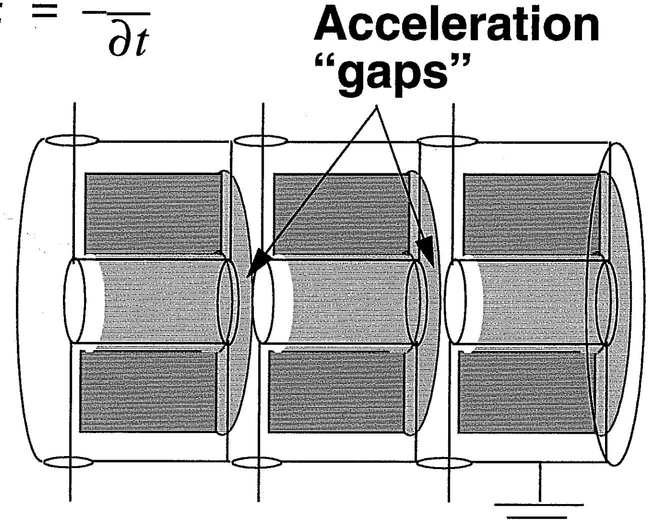
Induction acceleration



$A = (R_o - R_i)\, l$

Faraday's law, $\nabla \times E = -\dfrac{\partial B}{\partial t}$

$\Rightarrow V \Delta t = A \Delta B$

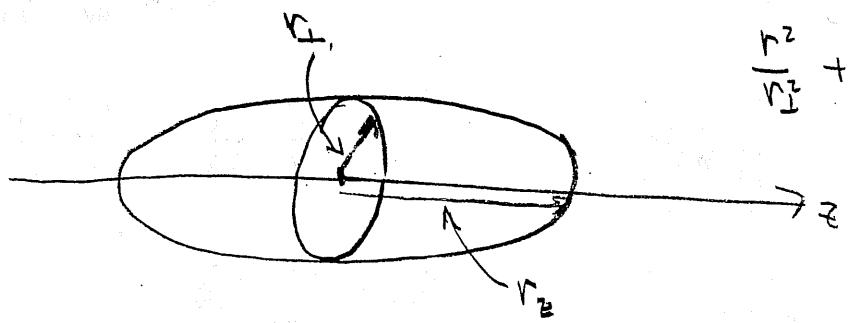


Volt-seconds per m: $(dV/dz)\,\Delta t = (R_o - R_i)\ \Delta B\ f_{radial}\ f_{longit.}$

~ 1 m ~ 2.5T ~ 0.8 ~ 0.8

$\boxed{(dV/dz)\,\Delta t <\sim 1.6 \text{ V-s/m}}$

SPACE-CHARGE FIELD OF BUNCHED BEAMS



$$\frac{r^2}{r_\perp^2} + \frac{z^2}{r_z^2} = 1$$

BOUNDARY OF ELLIPSOID

THE POTENTIAL OF A UNIFORM DENSITY BUNCH IN FREE SPACE (A MACLAURIN SPHEROID) IS GIVEN BY: (cf Landau & Lifshitz, Classical Theory of Fluids, p 297)

$$\varphi = \frac{\rho}{4\epsilon_0}\left(\alpha_\perp r^2 + \alpha_{\parallel} z^2 - \delta \right)$$

where $\alpha_\perp = r_\perp^2 r_z \int_0^\infty \frac{ds}{(r_\perp^2 + s)\,\Delta}$

$$\alpha_{\parallel} = r_\perp^2 r_z \int_0^\infty \frac{ds}{(r_z^2 + s)\,\Delta}$$

$$\delta = r_\perp^2 r_z \int_0^\infty \frac{ds}{\Delta}$$

where $\Delta^2 = (r_\perp^2 + s)^2 (r_z^2 + s)$

FOR NON-RELATIVISTIC BEAM:

$$E_z = -\frac{\partial\phi}{\partial z} = f \frac{\rho}{\epsilon_0} z$$

$$E_r = -\frac{\partial\phi}{\partial r} = \frac{(1-f)}{2} \frac{\rho}{\epsilon_0} r$$

$$f = f(\alpha) = \begin{cases} \frac{\alpha^2}{1-\alpha^2}\left[\frac{1}{\sqrt{1-\alpha^2}} \tanh^{-1}\sqrt{1-\alpha^2} - 1\right] & \alpha < 1 \\ \frac{1}{3} & \alpha = 1 \\ \frac{\alpha^2}{\alpha^2-1}\left[1 - \frac{1}{\sqrt{\alpha^2-1}} \tan^{-1}\sqrt{\alpha^2-1}\right] & \alpha > 1 \end{cases} \qquad \alpha \equiv \frac{r_\perp}{r_z}$$

FOR RELATIVISTIC BEAM

(cf. Lund & Barnard 1997
PAC 97 Conf Proceedings)

$$\frac{d^2 x_\perp}{ds^2} = \frac{F_\perp}{\gamma_s \beta_s^2 m c^2} \qquad F_{\perp s} = \frac{-q}{\gamma_s^2}\frac{\partial \phi}{\partial x_\perp} = \frac{q\rho}{2\gamma_s^2 \epsilon_0}\left[1 - f(\alpha)\right] x_\perp$$

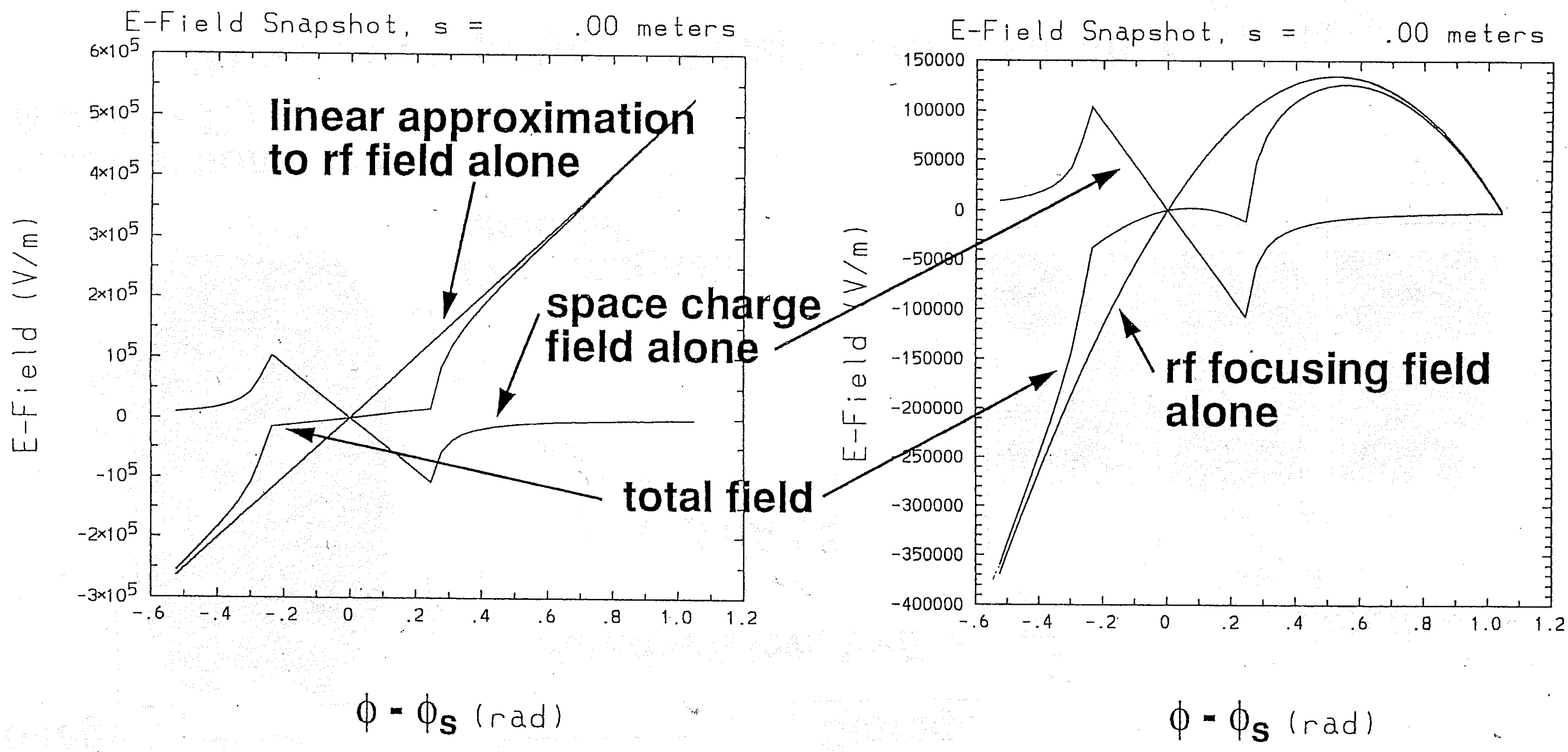
$$\frac{d^2 \Delta z}{ds^2} = \frac{F_z}{\gamma_s^3 \beta_s^2 m c^2} \qquad F_{zs} = \frac{-q}{\gamma_s^2}\frac{\partial \phi}{\partial z} = \frac{q\rho}{\epsilon_0} f(\alpha)\,\Delta z$$

$$\alpha = \frac{r_\perp}{\gamma r_z} \qquad \left[\alpha = \frac{r_\perp}{(r_z \text{ in comoving frame})}\right]$$

COMBINING FOCUSING + SELF FIELDS

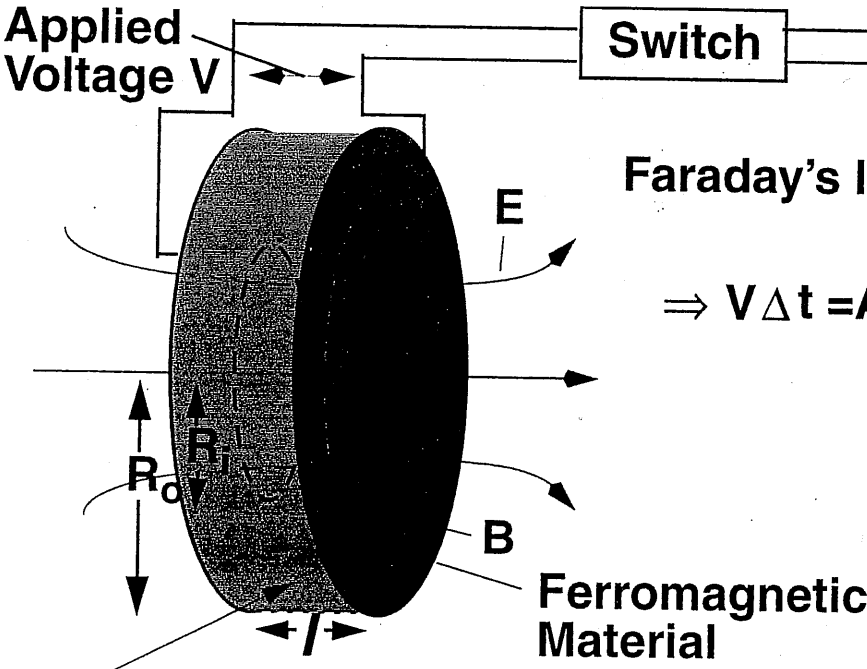
$$\frac{d^2}{ds^2}\Delta z = -k_{s0}^2 \Delta z + \frac{q\rho\, f(\alpha)}{\gamma_s^3 \beta_s^2 m c^2 \epsilon_0}\Delta z \qquad \text{(LINEAR rf)}$$

Total field seen by particle is sum of rf and spacecharge



here $\phi - \phi_s = -(2\pi/\beta_s\lambda)\,\Delta z$, where $\beta_s c$ is the longitudinal velocity of the synchronous particle and $\lambda = c/\nu$ is the rf vacuum wavelength

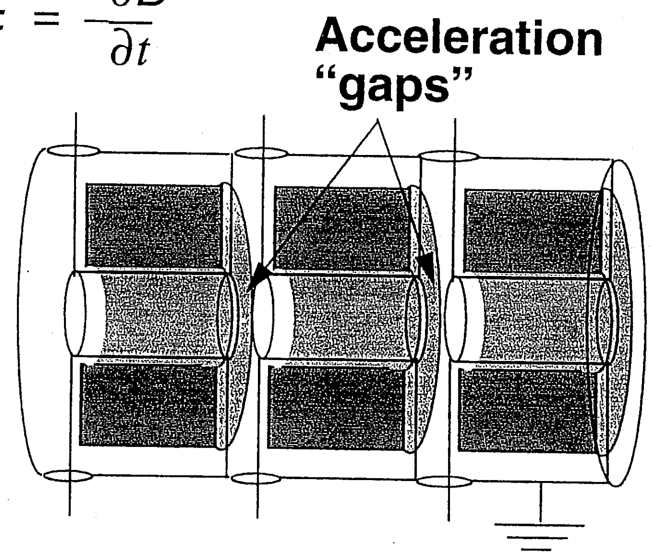
Induction acceleration



Cross-sectional area A
$A = (R_o - R_i)\ l$

Faraday's law, $\nabla \times E = -\dfrac{\partial B}{\partial t}$

$\Rightarrow V \Delta t = A \Delta B$

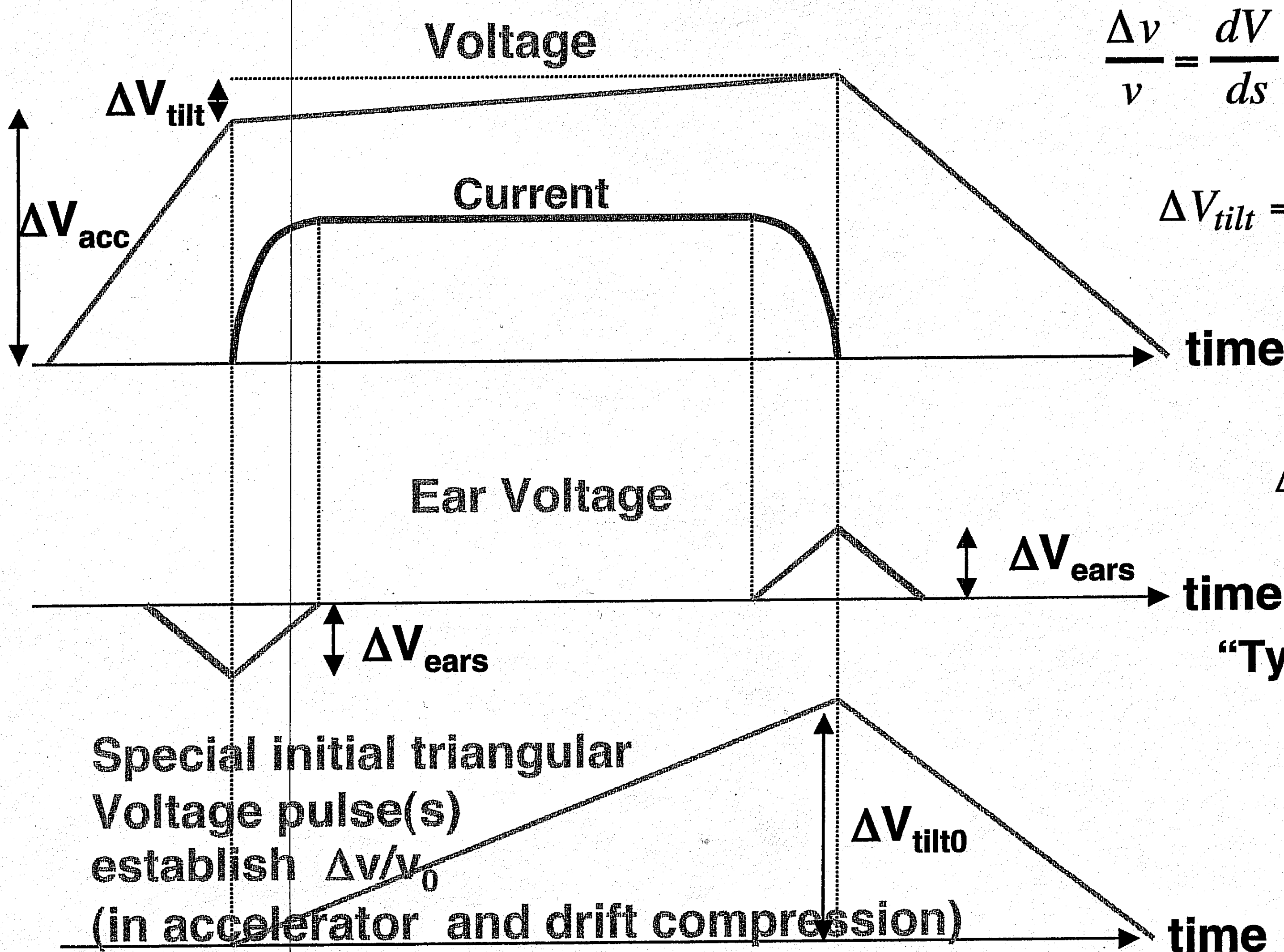


Volt-seconds per m: $(dV/dz)\,\Delta t = (R_o - R_i)\ \Delta B\ f_{radial}\ f_{longit.}$

~ 1 m ~ 2.5T ~ 0.8 ~ 0.8

$\boxed{(dV/dz)\,\Delta t \lesssim 1.6 \text{ V-s/m}}$

Several types of waveform are needed to accelerate, compress, and confine the beam



$$\frac{\Delta v}{v} = \frac{dV}{ds}\left(\frac{\ell}{V}\right)\left(\frac{1}{2} - \frac{V}{\ell}\frac{d\ell}{dV}\right)$$

$$\Delta V_{tilt} = \frac{d(2V\Delta v/v)}{ds} L$$

$$\Delta V_{ear} = \frac{2g\lambda_{flat} L}{4\pi\varepsilon_0 \beta c \Delta t}$$

"Typical" numbers:

$\Delta V_{tilt0} \sim$ 100 kV
$\Delta V_{tilt} \sim$ 1 kV
$\Delta V_{ears} \sim$ 14 kV
$\Delta V_{acc} \sim$ 100 kV

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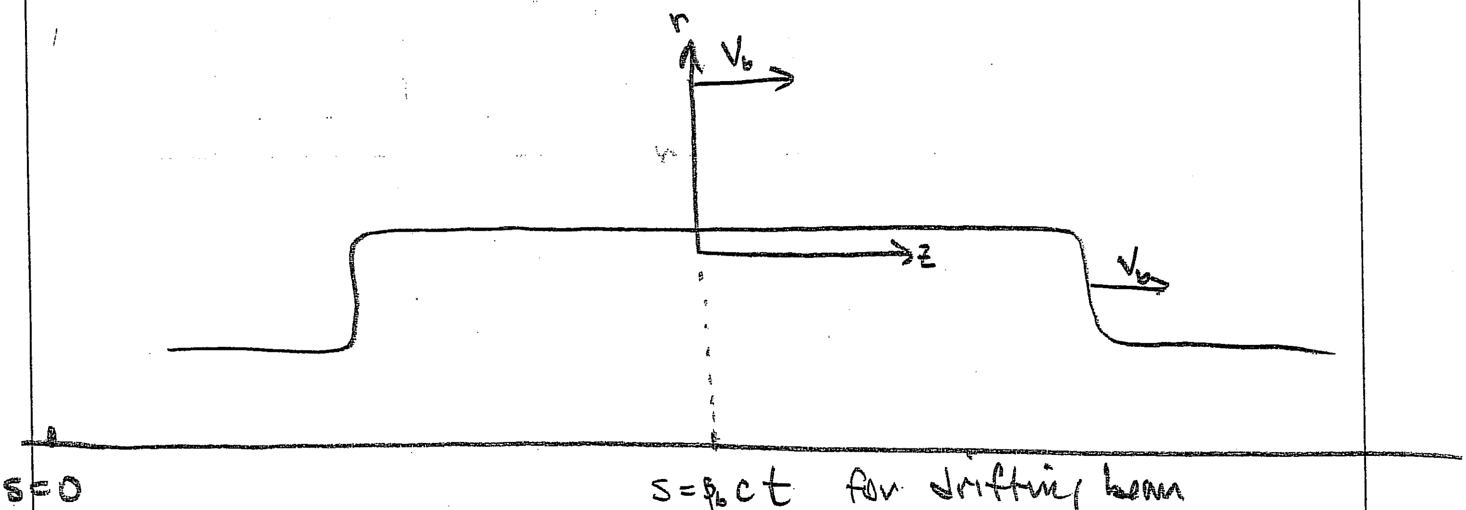
COORDINATE SYSTEM



$s = \beta_b c t$ for drifting beam
= position of beam center in lab frame

$s \Longleftrightarrow t$ are related by $\beta_b c$ for drifting beam

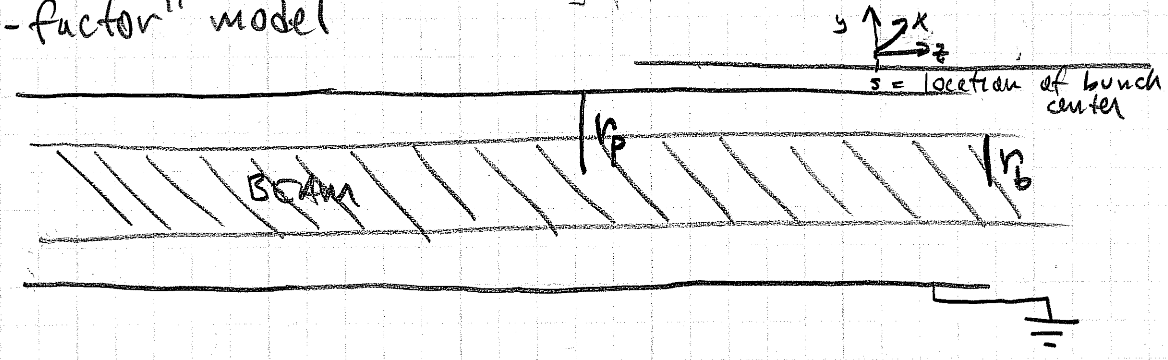
z = longitudinal coordinate in beam frame ($z=0$ = beam center)

r = radial coordinate in beam frame (or lab frame).

(This class will assume non-relativistic dynamics)
These are ions with $\beta < 0.2$).

LONGITUDINAL PHYSICS OF LONG PULSES (BUNCH LENGTH >> r_{pipe})

"g-factor" model



If $\dfrac{\partial^2\varphi}{\partial z^2} \ll \dfrac{1}{r}\left(\dfrac{\partial}{\partial r} r \dfrac{\partial\varphi}{\partial r}\right) \Rightarrow \dfrac{\partial\varphi}{\partial r} = \dfrac{-\lambda(r)}{2\pi\varepsilon_0 r}$

Let $\rho = \begin{cases} \rho_0 & 0 < r < r_b \\ 0 & r_b < r < r_p \end{cases} \Rightarrow \lambda = \lambda_0\left(\dfrac{r}{r_b}\right)^2$

$$\varphi = \int \frac{\partial\varphi}{\partial r}\,dr = \begin{cases} \dfrac{\lambda}{2\pi\varepsilon_0}\left[\dfrac{1}{2}\left(1 - \dfrac{r^2}{r_b^2}\right) + \ln\dfrac{r_p}{r_b}\right] & 0 < r < r_b \\ \dfrac{\lambda}{2\pi\varepsilon_0}\ln\left(\dfrac{r_p}{r}\right) & r_b < r < r_p \end{cases}$$

$$\frac{\partial\phi}{\partial z} = \frac{1}{2\pi\varepsilon_0}\left[\frac{1}{2}\left(1-\frac{r^2}{r_b^2}\right) + \ln\frac{r_p}{r_b}\right]\frac{d\lambda}{dz} - \frac{1}{2\pi\varepsilon_0}\left[1 - \frac{r^2}{r_b^2}\right]\frac{\lambda}{r_b}\frac{dr_b}{dz}$$

<u>If</u> $\rho = \text{const} \Rightarrow \dfrac{\lambda}{r_b^2} = \text{const} \qquad \dfrac{d\lambda}{dz} = \dfrac{-2\lambda}{r_b}\dfrac{dr_b}{dz}$

[Example of $\rho = \text{const}$.
Magnetic Quad focusing
$\dfrac{\lambda}{4\pi\varepsilon_0 V a} \approx k_{\beta 0}^2 a$
$\Rightarrow \rho \sim V k_{\beta 0}^2 \approx \text{const}$]

$$\Rightarrow \frac{\partial\phi}{\partial z} = \frac{1}{2\pi\varepsilon_0}\ln\left(\frac{r_p}{r_b}\right)\frac{\partial\lambda}{\partial z}$$

$$E_z = \frac{-g}{4\pi\varepsilon_0}\frac{\partial\lambda}{\partial z}$$

where $g = 2\ln\left(\dfrac{r_p}{r_b}\right)$

Vlasov-equation for a drifting beam:

$$\frac{\partial f}{\partial s} + x'\frac{\partial f}{\partial x} + x''\frac{\partial f}{\partial x'} + y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} + z'\frac{\partial f}{\partial z} + z''\frac{\partial f}{\partial z'} = 0$$

let $\tilde{f}(z,z',s) \equiv \iiiint f \, dx\, dx'\, dy\, dy'$

INTEGRATING VLASOV EQUATION: If $z'' \neq f(x,x',y,y')$:

$$\Rightarrow \frac{\partial \tilde{f}}{\partial s} + \iiint x' \underbrace{\int \frac{\partial f}{\partial x}\, dx}_{= f\big|_{-\infty}^{\infty} = 0}\, dx'\, dy\, dy' + \cdots \; z'\frac{\partial \tilde{f}}{\partial z} + z''\frac{\partial \tilde{f}}{\partial z'} = 0$$

$$\Rightarrow \boxed{\frac{\partial \tilde{f}}{\partial s} + z'\frac{\partial \tilde{f}}{\partial z} + z''\frac{\partial \tilde{f}}{\partial z'} = 0} \qquad \text{1 D Vlasov}$$

Now let $\lambda \equiv q\int \tilde{f}\, dz'$; $\lambda \overline{z'} = \int \tilde{f} z'\, dz'$; $\lambda \overline{z'^2} \equiv \int \tilde{f} z'^2\, dz'$

Also, let $\Delta z'^2 \equiv \overline{z'^2} - (\overline{z'})^2$

FLUID EQUATIONS

INTEGRATING 1D VLASOV OVER z':

$$\boxed{\frac{\partial \lambda}{\partial s} + \frac{\partial}{\partial z}(\lambda \overline{z'}) = 0} \qquad \text{(CONTINUITY EQUATION)}$$

MULTIPLYING BY z' & INTEGRATING VLASOV OR z':

$$\frac{\partial}{\partial s}\lambda \overline{z'} + \frac{\partial}{\partial z}\lambda \overline{z'^2} - \lambda z'' = 0$$

DIVIDING BY λ, USING CONTINUITY EQUATION & DEFINITION OF $\Delta z'^2$:

$$\boxed{\underbrace{\frac{\partial \overline{z'}}{\partial s} + \overline{z'}\frac{\partial}{\partial z}\overline{z'}}_{\text{INERTIAL}} + \underbrace{\frac{1}{\lambda}\frac{\partial}{\partial z}(\lambda \Delta z'^2)}_{\text{PRESSURE TERM}} = \underbrace{z''}_{\text{FORCE}}} \qquad \text{(MOMENTUM EQUATION)}$$

COMBINING g-factor model with fluid equations

$$\frac{\partial \lambda}{\partial s} + \frac{\partial}{\partial z}\left(\lambda \bar{z}'\right) = 0$$

$$\frac{\partial \bar{z}'}{\partial s} + \bar{z}'\frac{\partial \bar{z}'}{\partial z} + \underbrace{\frac{1}{\lambda}\frac{\partial}{\partial z}\left(\lambda \Delta z'^2\right)}_{\text{PRESSURE TERM}} = \underbrace{\frac{-gq}{4\pi\epsilon_0 m V_0^2}\frac{\partial \lambda}{\partial z}}_{\text{SPACE CHARGE TERM}}$$

WHEN PRESSURE TERM $\ll$ SPACE CHARGE TERM,

(& LET $C_s^2 \equiv \frac{q g \lambda_0}{4\pi\epsilon_0 m}$ = "(SPACE CHARGE WAVE SPEED)^2"

~~& FROM HERE ON LET $\bar{z}' \to z'$, FOR NOTATIONAL SIMPLICITY~~)

$$\Longrightarrow \boxed{\begin{aligned} &\frac{\partial \lambda}{\partial s} + \lambda\frac{\partial \bar{z}'}{\partial z} + z'\frac{\partial \lambda}{\partial z} = 0 \\ \& \quad &\frac{\partial z'}{\partial s} + z'\frac{\partial \bar{z}'}{\partial z} + \frac{C_s^2}{\lambda_0 V_0^2}\frac{\partial \lambda}{\partial z} = 0 \end{aligned}} \qquad (g1)$$

LINEARIZING g1

LET $\lambda = \lambda_0 + \lambda_1 \qquad z' = z_0' + z_1'$

EQUILIBRIUM λ_0 = CONSTANT

$z_0' = 0$

LINEARIZING

$$\frac{\partial \lambda_1}{\partial s} + \lambda_0\frac{\partial \bar{z}_1'}{\partial z} = 0 \qquad (g2a)$$

$$\frac{\partial \bar{z}_1'}{\partial s} + \frac{C_s^2}{\lambda_0 V_0^2}\frac{\partial \lambda_1}{\partial z} = 0 \qquad (g2b)$$

TAKING $\frac{\partial}{\partial s}$ of (g2a) & $\frac{\partial}{\partial z}$ of g2b and combining:

$$\Longrightarrow \boxed{\frac{\partial^2 \lambda_1}{\partial s^2} - \frac{C_s^2}{V_0^2}\frac{\partial^2 \lambda_1}{\partial z^2} = 0 \quad \Longrightarrow \text{WAVE EQUATION}}$$

SOLVING WAVE EQUATION

$$\frac{\partial^2 \lambda_1}{\partial s^2} - \frac{c_s^2}{V_0^2}\frac{\partial^2 \lambda_1}{\partial z^2} = 0$$

Let $\lambda_1 = \tilde{\lambda}_1 \exp\left[\frac{i\omega}{V_0}s \pm ikz\right]$

$$-\frac{\omega^2}{V_0^2} + \frac{k^2 c_s^2}{V_0^2} = 0 \Rightarrow \omega = c_s k$$

$\Rightarrow$ PHASE & GROUP VELOCITY OF WAVES $= C_s$ (in beam frame)

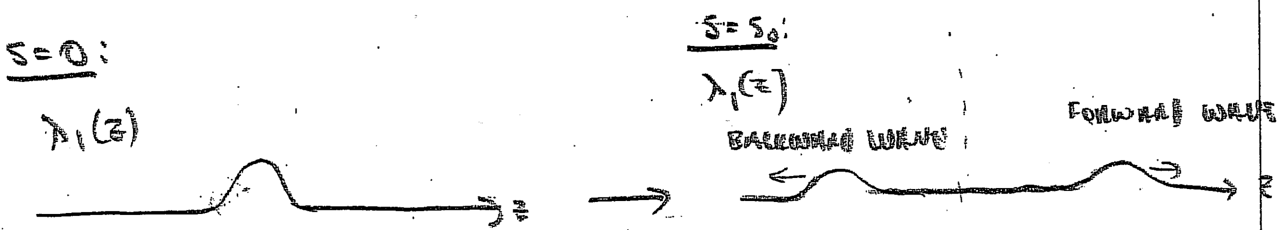
GENERAL SOLUTION

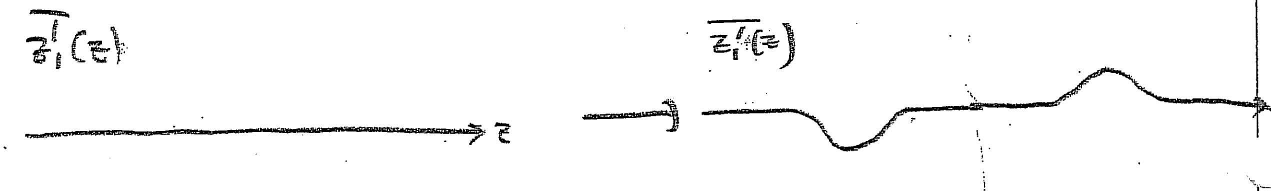
$$\lambda_1 = \lambda_0 f_+[u_+] + \lambda_0 f_-[u_-]$$

where $u_+ = z + \frac{c_s s}{V_0} + C_0$ & $u_- = z - \frac{c_s s}{V_0} + C_0$

& $f_+[u]$ & $f_-[u]$ are any functions of the argument. & C_0 is an arbitrary constant

$$\overline{z_1'} = \frac{c_s}{V_0}\left[-f_+[u_+] + f_-[u_-]\right]$$

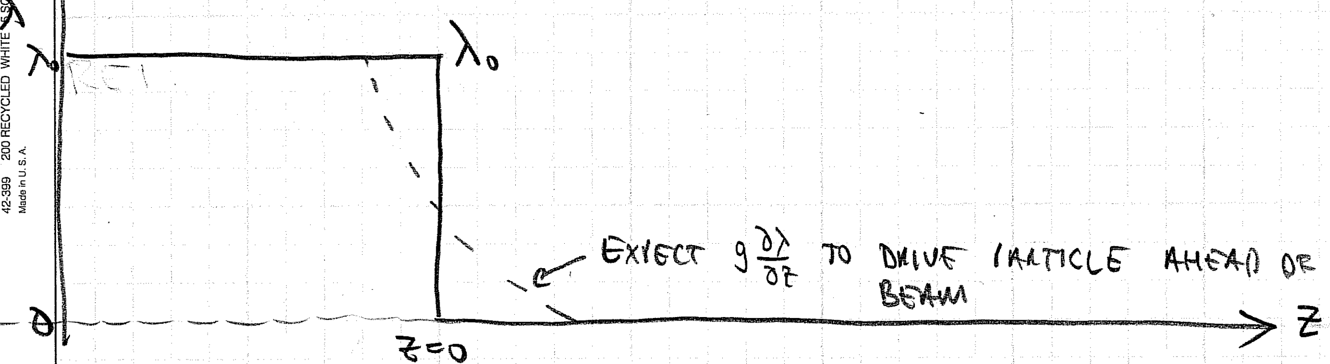




BEAM ENDS & RAREFACTION WAVES

(FALTENS & LEE, J. App. Phys. 61, 5219)
(Also Landau & Lifshitz, Fluid Mechanics)

SUPPOSE YOU START WITH A PULSE THAT ENDS WITH A STEP FUNCTION IN λ. WHAT HAPPENS TO THE END?



TO ANALYZE: RETURN TO NON-LINEAR FLUID EQUATIONS (SINCE $\delta\lambda \sim \lambda_0$) (g1):

$$\frac{\partial \lambda}{\partial s} + \lambda \frac{\partial \bar{z}'}{\partial z} + \bar{z}' \frac{\partial \lambda}{\partial z} = 0 \qquad \text{(CONTINUITY)}$$

$$\frac{\partial \bar{z}'}{\partial s} + \bar{z}' \frac{\partial \bar{z}'}{\partial z} + \frac{c_s^2}{\lambda_0 v_0^2} \frac{\partial \lambda}{\partial z} = 0 \qquad \text{(momentum)}$$

1st IT IS CONVENIENT TO DEFINE: $\Lambda \equiv \lambda/\lambda_0$

$$V \equiv \frac{v_0}{c_s} \bar{z}'$$

$$\left(c_s^2 \equiv \frac{q}{m} \frac{g \lambda_0}{4\pi\varepsilon_0} \right) \qquad \zeta \equiv \frac{v_0}{c_s} z$$

$$\Rightarrow \quad \frac{\partial \Lambda}{\partial s} + \Lambda \frac{\partial V}{\partial \zeta} + V \frac{\partial \Lambda}{\partial \zeta} = 0 \qquad \text{(CONTINUITY)}$$

$$\frac{\partial V}{\partial s} + V \frac{\partial V}{\partial \zeta} + \frac{\partial \Lambda}{\partial \zeta} = 0 \qquad \text{(momentum)} \qquad (r1)$$

TRY A SIMILARITY SOLUTION: $\Lambda = \Lambda(x)$ & $V = V(x)$

WHERE $x = \frac{z}{s} = \left(\frac{V_0 z}{C_s s}\right)$

$$\frac{\partial x}{\partial s} = -\frac{x}{s} \qquad \frac{\partial x}{\partial z} = \frac{x}{z}$$

$$\frac{\partial \Lambda}{\partial s} = -\frac{d\Lambda}{dx}\frac{x}{s} \qquad \frac{\partial \Lambda}{\partial z} = \frac{d\Lambda}{dx}\frac{x}{z} \qquad \frac{\partial V}{\partial s} = -\frac{dV}{dx}\frac{x}{s} \qquad \frac{\partial V}{\partial z} = \frac{dV}{dx}\frac{x}{z}$$

$$\left[-\frac{d\Lambda}{dx}\frac{x}{s} + \Lambda\frac{dV}{dx}\frac{x}{z} + V\frac{d\Lambda}{dx}\frac{x}{z}\right] = 0 \qquad \text{(CONTINUITY)}$$

$$\left[-\frac{dV}{dx}\frac{x}{s} + V\frac{dV}{dx}\frac{x}{z} + \frac{d\Lambda}{dx}\frac{x}{z}\right] = 0 \qquad \text{(momentum)}$$

MULTIPLY BY z/s & gather terms):

$$\Rightarrow \begin{bmatrix} V - x & \Lambda \\ 1 & V - x \end{bmatrix}\begin{bmatrix} d\Lambda/dx \\ dV/dx \end{bmatrix} = 0$$

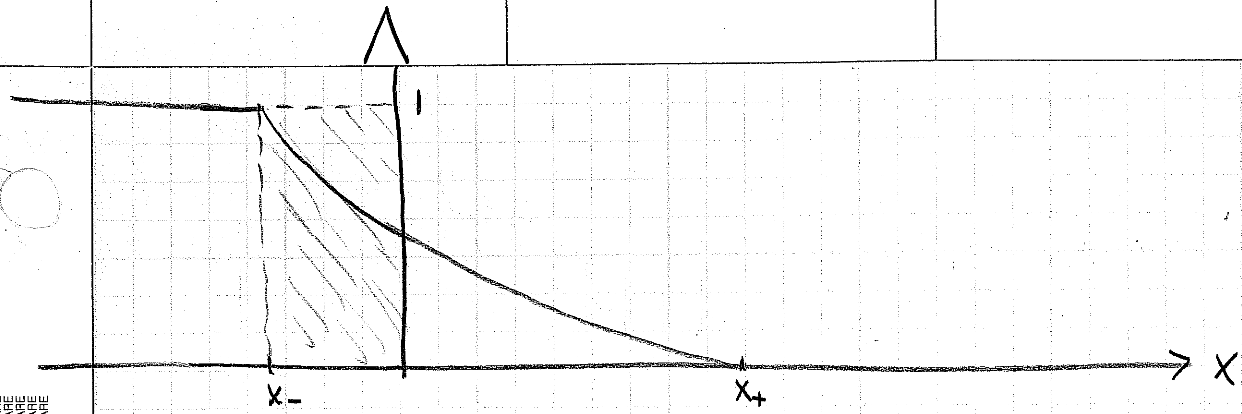
FOR NON-TRIVIAL SOLUTION DETERMINANT MUST VANISH:

$$\Lambda = [V - x]^2$$

$$\Rightarrow \frac{d\Lambda}{dx} = 2[V - x]\left[\frac{dV}{dx} - 1\right] \quad \& \quad \frac{d\Lambda}{dx} = -[V - x]\frac{dV}{dx}$$

$$\Rightarrow -\frac{dV}{dx} = 2\frac{dV}{dx} - 2 \Rightarrow \frac{dV}{dx} = \frac{2}{3}$$

$$\Rightarrow \boxed{\begin{aligned} V &= \tfrac{2}{3}x + C \\ \Lambda &= \left[-\tfrac{1}{3}x + C\right]^2 \end{aligned}}$$



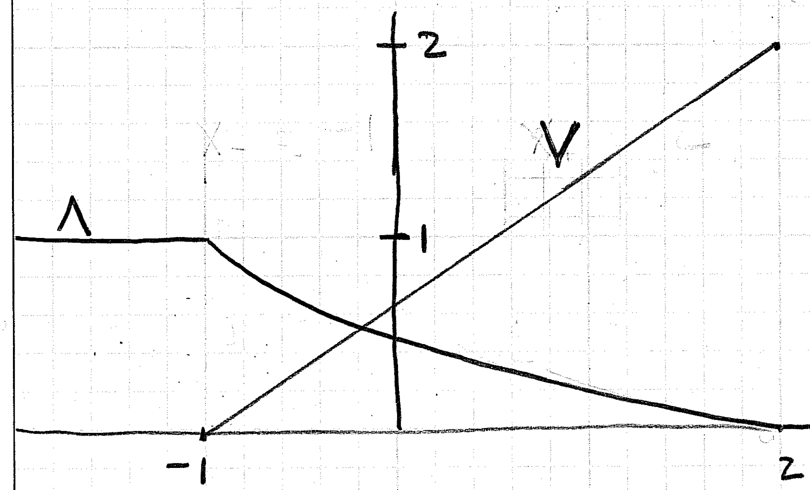
At x_+: $\Lambda = 0 \Rightarrow C = \frac{1}{3}x \qquad x_+ = 3C$

At x_-: $\Lambda = 1 \Rightarrow C = \frac{1}{3}x + 1 \Rightarrow x_- = 3C - 3$

MASS CONSERVATION $\Rightarrow (-x_-) = -(3C-3) = \int_{3C-3}^{3C} \left[C - \frac{1}{3}x\right]^2 dx = -3\left.\frac{\left[C - \frac{1}{3}x\right]^3}{3}\right|_{3C-3}^{3C}$

$-(3C-3) = 1$

$\Rightarrow \boxed{C = \frac{2}{3}} \Rightarrow \begin{array}{l} x_- = -1 \\ x_+ = 2 \end{array}$



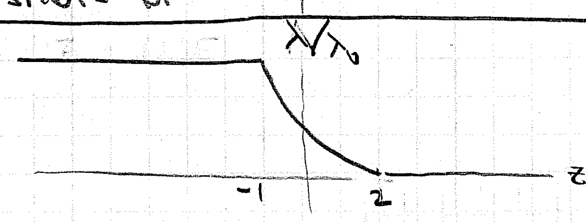
$$\left.\begin{array}{l} V = \frac{2}{3}(x+1) \\ \Lambda = \left[-\frac{1}{3}x + \frac{2}{3}\right]^2 \end{array}\right\} \text{ for } -1 < x < 2$$

RECALL $x = \frac{V_0 z}{C_s S}$ SO $x = 2 \Rightarrow z = 2C_s\left(\frac{S}{V_0}\right)$

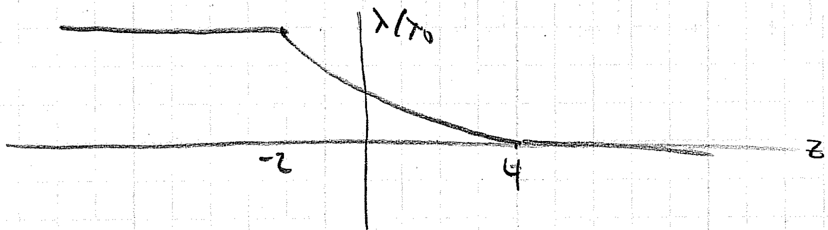
$x = -1 \Rightarrow z = -C_s\left(\frac{S}{V_0}\right)$

SO BEAM END EXPANDS AT TWICE SPACE-CHARGE WAVE SPEED & RAREFACTION WAVE PROPAGATES INWARD AT THE SPACE-CHARGE WAVE SPEED.

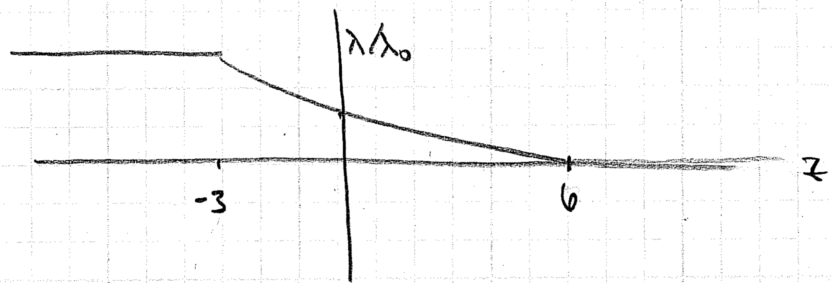
SNAPSHOTS OF λ/λ_0 VS z AT VARIOUS s



$s = v_0/c_s$



$s = 2v_0/c_s$

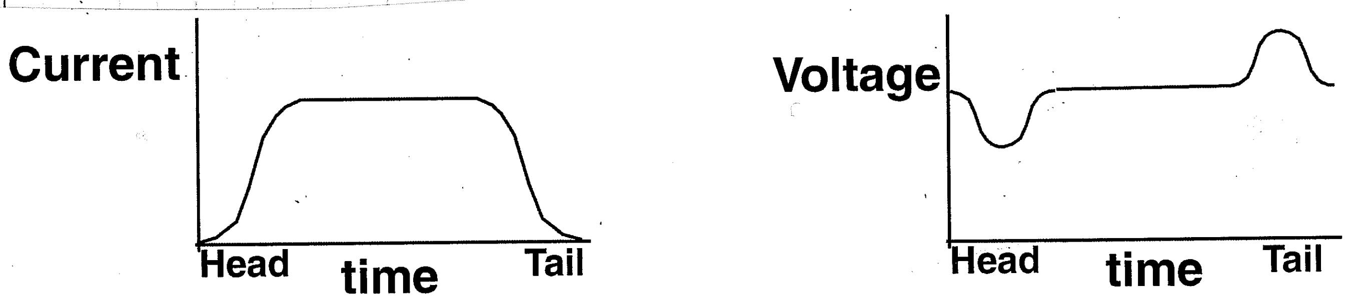


$s = 3v_0/c_s$

HOW DOES ONE PREVENT "END EROSION"?

APPLY EAR PULSES AT END OF BEAM:

$$V \sim E_z = \frac{-g}{4\pi\epsilon_0}\frac{\partial\lambda}{\partial z}$$



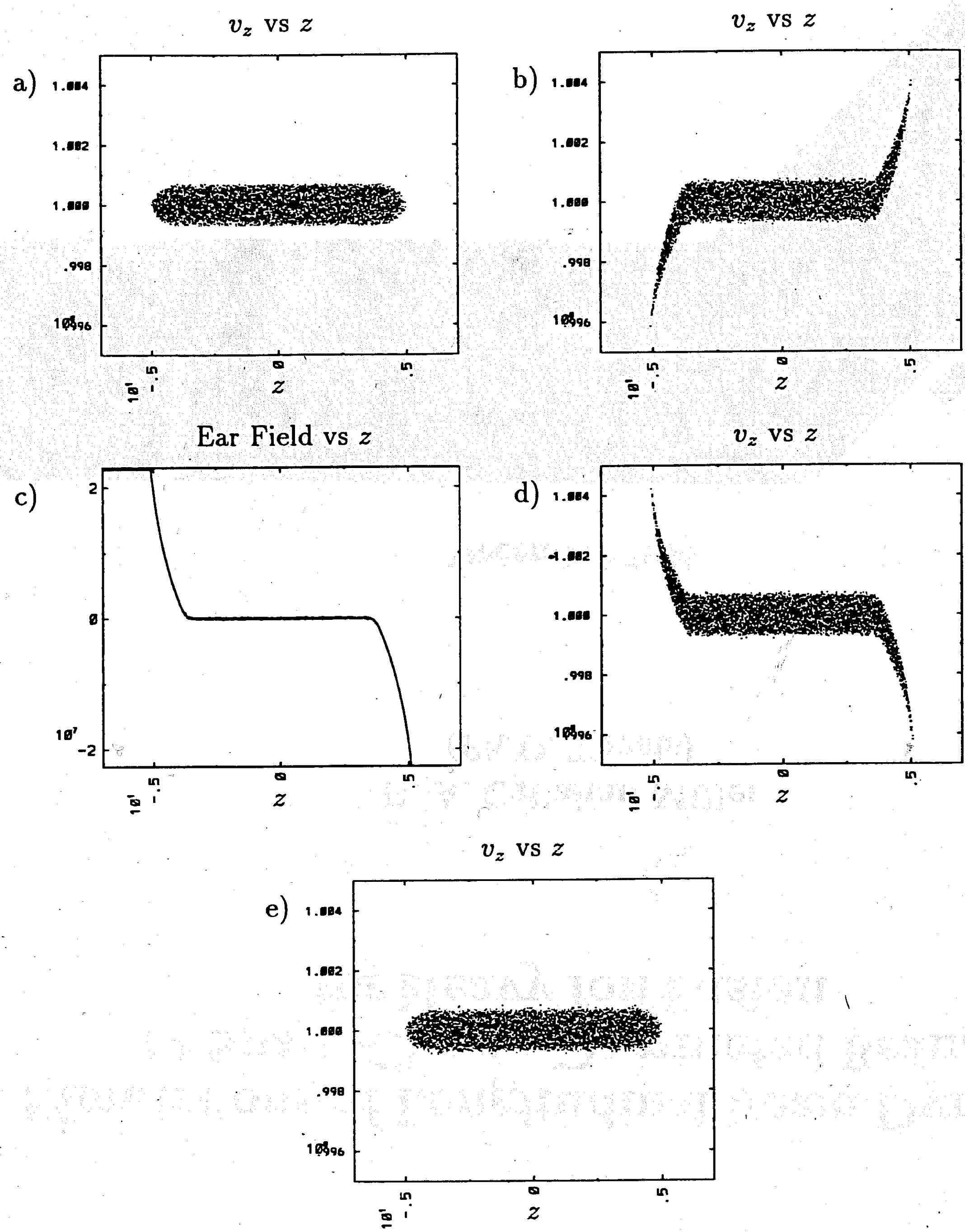


Figure 6.4: One cycle of intermittently-applied ears. (a) Initial phase space (b) Beam expands (c) Ear Field is applied (d) Beam is compressed (e) Beam expands back to its initial state

From D. Callahan Miller
PhD thesis, U.C. Davis, 1994.