Unit 3 - Lecture 5
RF-accelerators:
Synchronism conditions

William A. Barletta

Director, United States Particle Accelerator School

Dept. of Physics, MIT

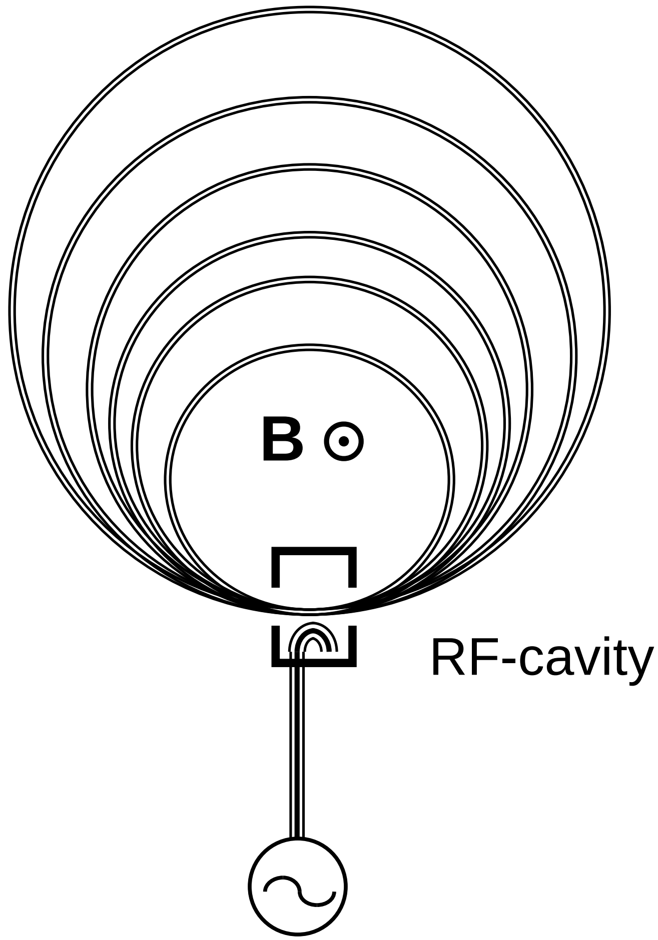
Final Exam schedule:

8.277 Introduction to Particle Accelerators

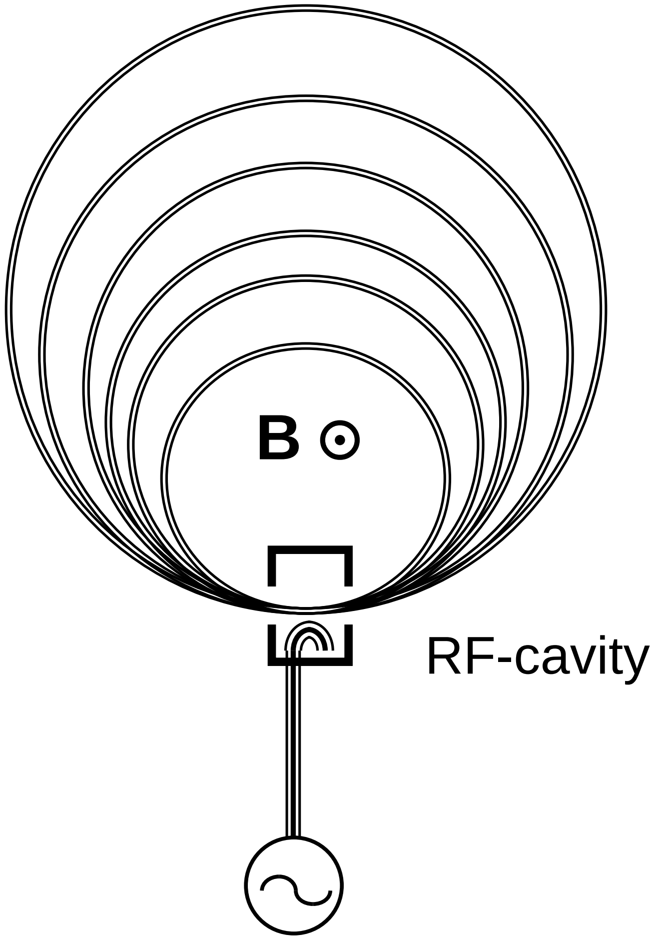
Room 4-145 Thursday, May 22 9:00AM - 12:00NOON

You may use your lecture notes

Will this work?



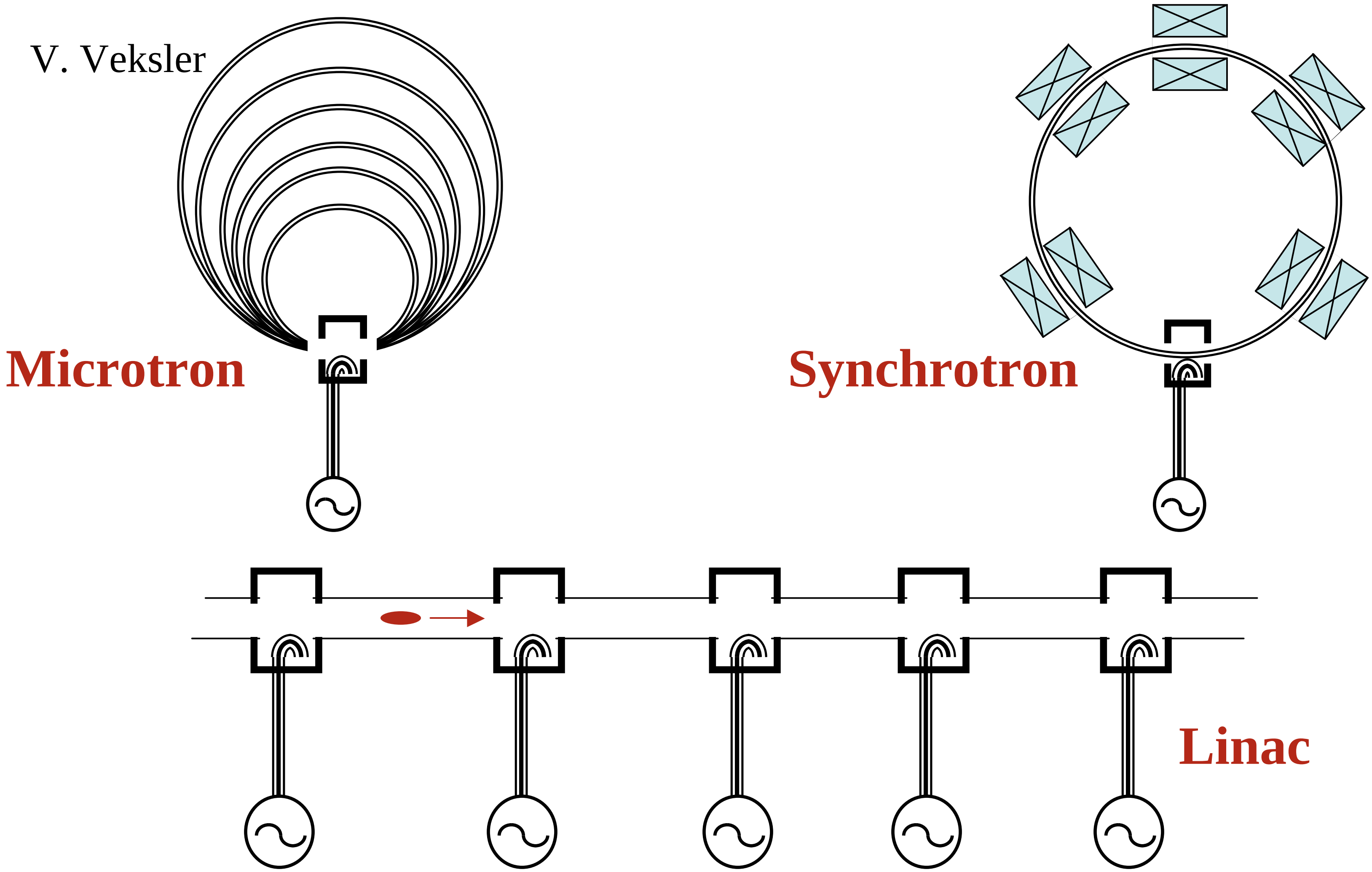
We can vary B in an RF cavity



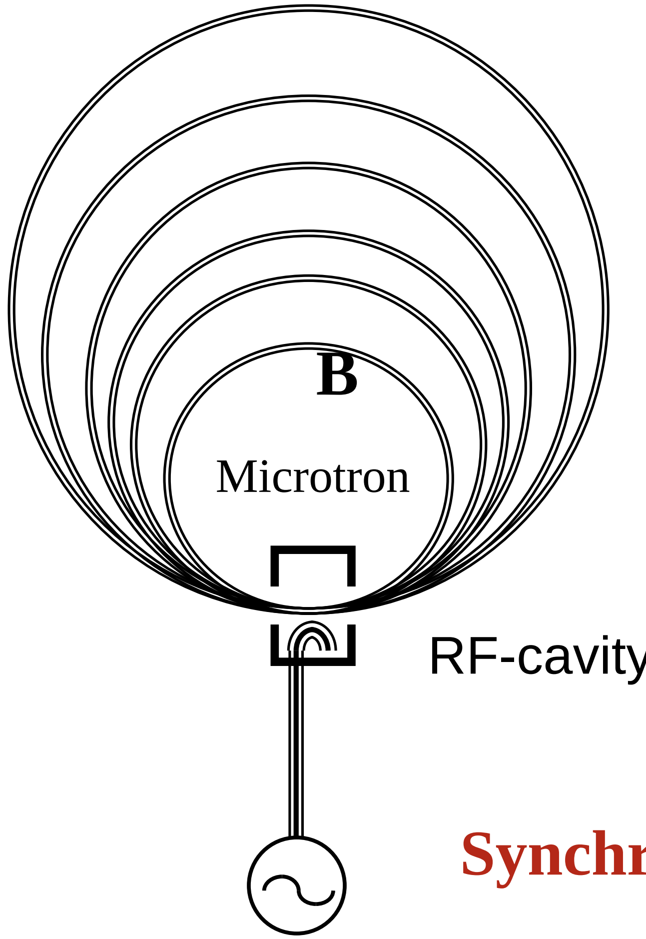
Note that inside the cavity

$dB/dt \neq 0$

RF-cativties for acceleration



Linac size is set by E_{gap}; why not one gap?



Note that in cavity
$dB/dt \neq 0$

Synchronism condition:
$\Delta\tau_{rev} = N/f_{rf}$

RF accelerators

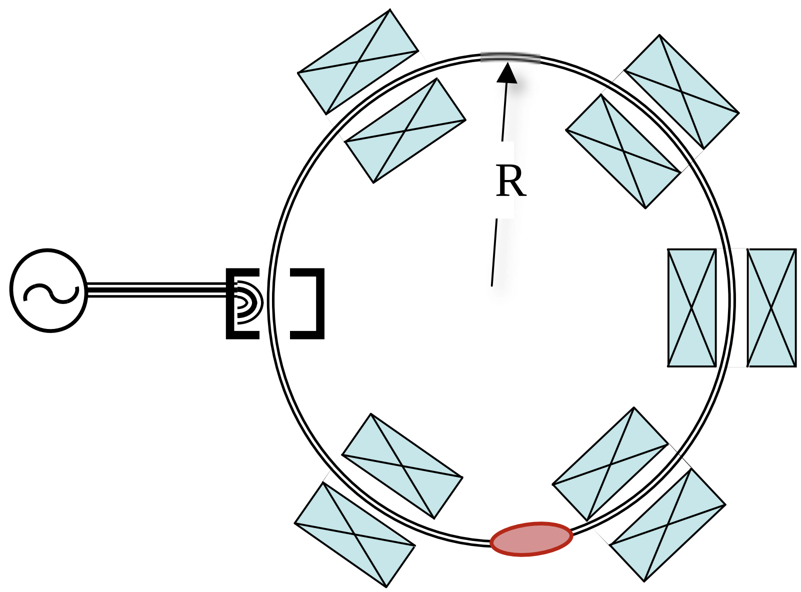
MacMillan's first synchrotron

The synchrotron introduces two new ideas: change B_{dipole} & change ω_{rf}

- For low energy ions, f_{rev} increases as E_{ion} increases
- ==> Increase ω_{rf} to maintain synchronism
- For any E_{ion} circumference must be an integral number of rf wavelengths

$$L = h\,\lambda_{rf}$$

- h is the harmonic number



$$L = 2\pi R$$

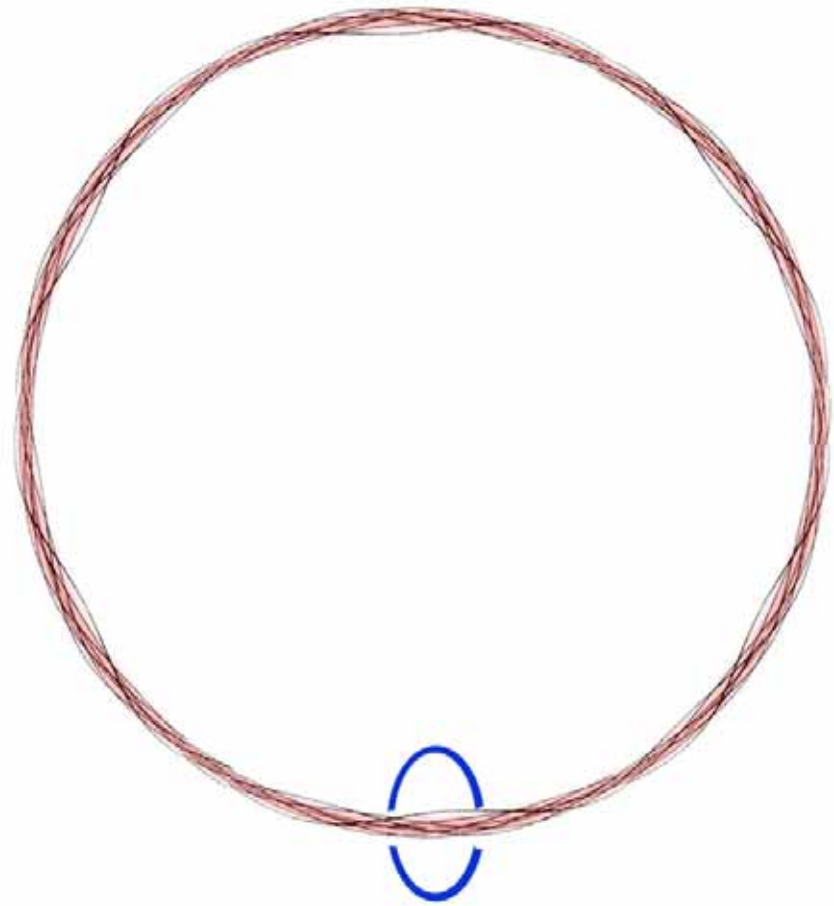
$$f_{rev} = 1/\tau = v/L$$

Ideal closed orbit in the synchrotron

- Beam particles will not have identical orbital positions & velocities
- In practice, they will have transverse oscillatory motion (betatron oscillations) set by radial restoring forces
- An ideal particle has zero amplitude motion on a closed orbit along the axis of the synchrotron

Ideal closed orbit & synchronous particle

- The ideal synchronous particle always passes through the rf-cavity when the field is at the same phase

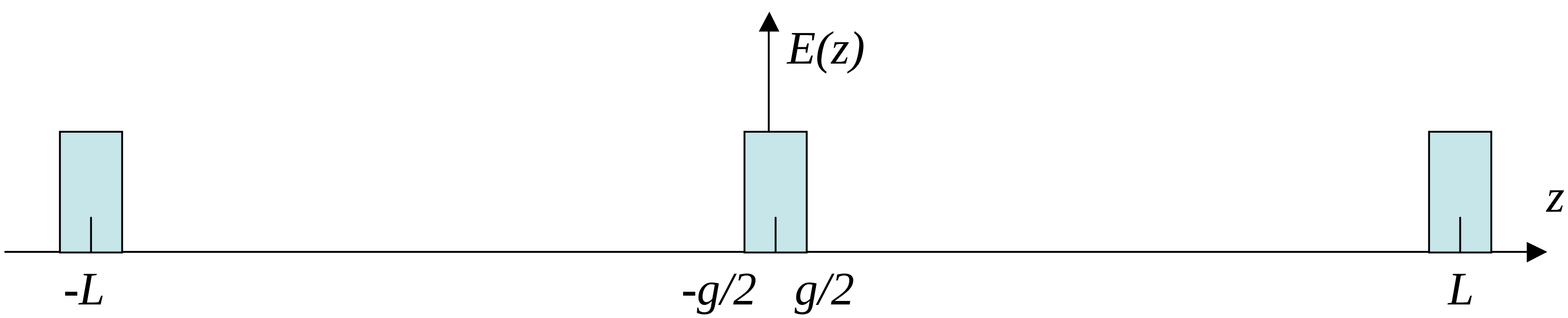


Synchrotron acceleration

- The rf cavity maintains an electric field at $\omega_{rf} = h\, \omega_{rev} = h\, 2\pi v / L$
- Around the ring, describe the field as $E(z,t) = E_1(z) E_2(t)$
- $E_1(z)$ is periodic with a period of L

$$E_2(t) = E_o \sin\left(\int_{t_o}^{t} \omega_{rf}\, dt + \varphi_o \right)$$

- The particle position is $z(t) = z_o + \int_{t_o}^{t} v\, dt$



Energy gain

- The energy gain for a particle that moves from 0 to L is given by:

$$W = q\int_0^L E(z,t)\cdot dz = q\int_{-g/2}^{+g/2} E_1(z)E_2(t)dz =$$

$$= qgE_2(t) = qE_o \sin\left(\int_{t_o}^{t} \omega_{rf}\, dt + \varphi_o\right) = qV$$

- V is the voltage gain for the particle.
  - ➙ depends only on the particle trajectory
  - ➙ includes contributions from all electric fields present
    - (RF, space charge, interaction with the vacuum chamber, …)

- Particles can experience energy variations $U(E)$ that depend on energy
  - ➙ synchrotron radiation emitted by a particle under acceleration

$$\Delta E_{Total} = qV + U(E)$$

Energy gain -II

- The synchronism conditions for the synchronous particle
  - condition on rf- frequency,
  - relation between rf voltage & field ramp rate

- The rate of energy gain for the synchronous particle is

$$\frac{dE_s}{dt} = \frac{\beta_s c}{L} eV \sin\varphi_s = \frac{c}{h\lambda_{rf}} eV \sin\varphi_s$$

- Its rate of change of momentum is

$$\frac{dp_s}{dt} = eE_o \sin\varphi_s = \frac{eV}{L} \sin\varphi_s$$

Beam rigidity links B, p and ρ

- Recall that $p_s = e\rho B_o$
- Therefore,

$$\frac{dB_o}{dt} = \frac{V \sin\varphi_s}{\rho L}$$

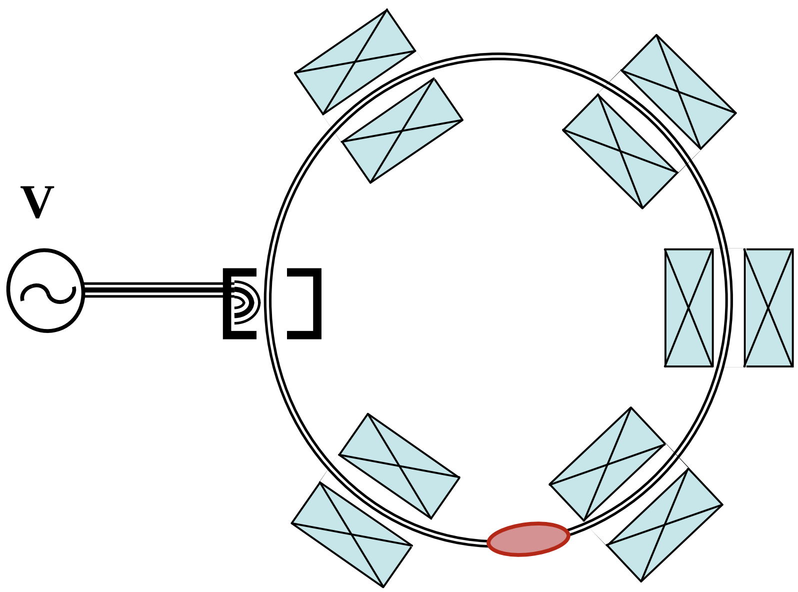
- If the ramp rate is uniform then $V\sin\phi_s = constant$
- In rapid cycling machines like the Tevatron booster

$$B_o(t) = B_{\min} + \frac{B_{\max} - B_{\min}}{2}\left(1 - \cos 2\pi f_{cycle} t\right)$$

- Therefore $V\sin\phi_s$ varies sinusoidally

Phase stability
&
Longitudinal phase space

Phase stability: Will bunch of finite length stay together & be accelerated?

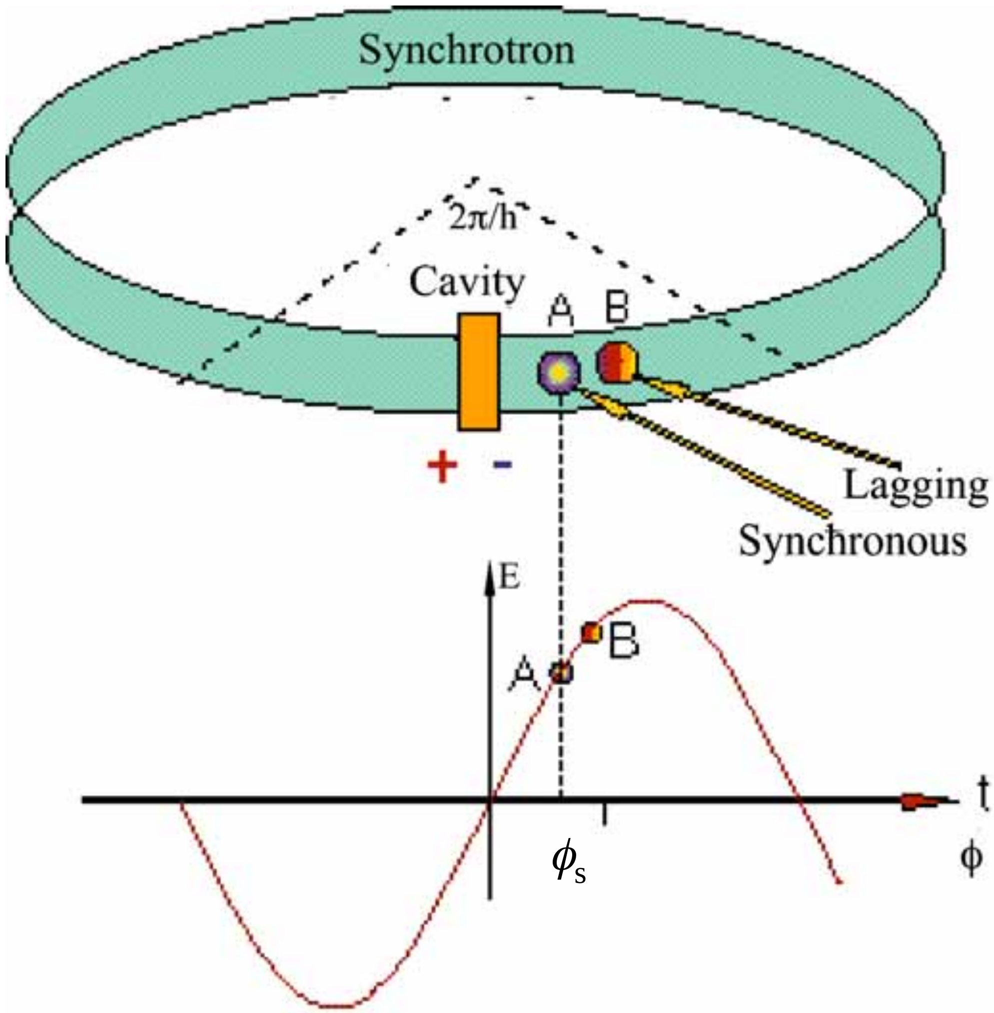


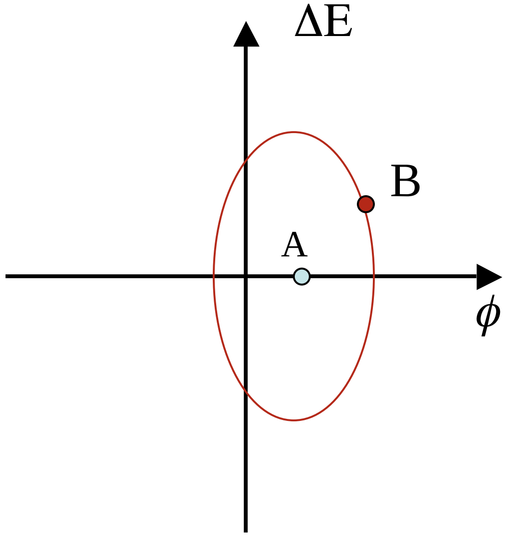
Let's say that the synchronous particle makes the ith revolution in time: T_i

Will particles close to the synchronous particle in phase stay close in phase?

Discovered by MacMillan & by Veksler

What do we mean by phase?
Let's consider non-relativistic ions

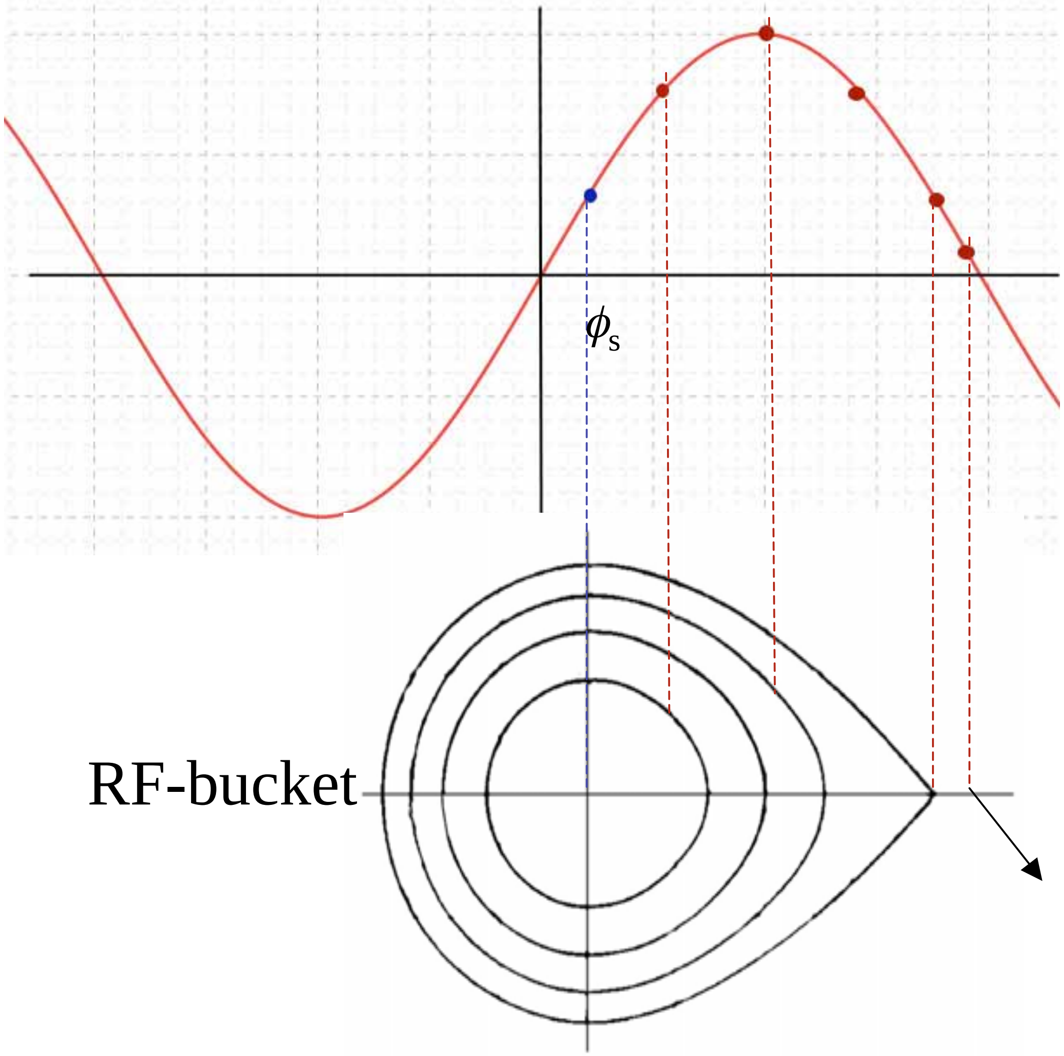




How does the ellipse change as B lags further behind A?

From E. J. N. Wilson CAS lecture

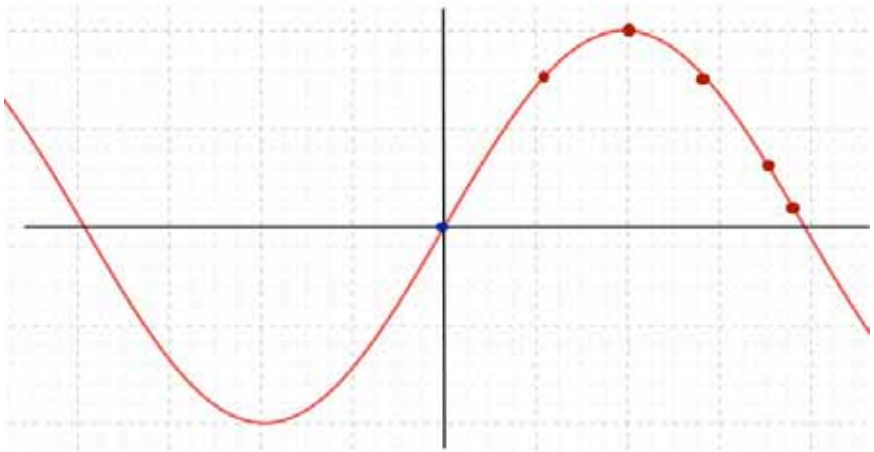
How does the ellipse change as B lags further behind A?



How does the size of the bucket change with ϕ_s ?

This behavior can be though of as phase or longitudinal focusing

- Stationary bucket: A special case obtains when $\phi_s = 0$
  - ➙ The synchronous particle does not change energy
  - ➙ All phases are trapped

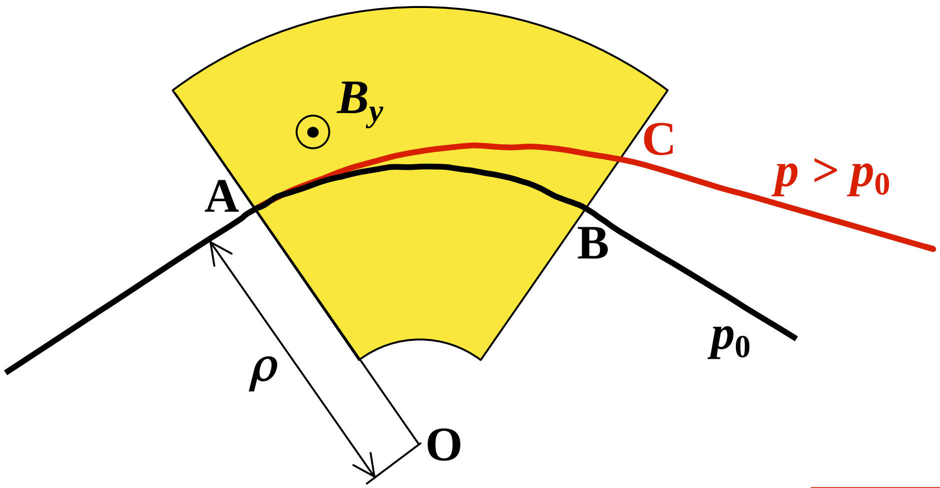


- We can expect an equation of motion in ϕ of the form

$$\frac{d^2\varphi}{ds^2} + \Omega^2 \sin\varphi = 0$$

Pendulum equation

Length of orbits in a bending magnet



$$\rho = \frac{p}{qB_z} = \frac{\beta \gamma m_0 c}{q B_z}$$

L_0 = Trajectory length between A and B

L = Trajectory length between A and C

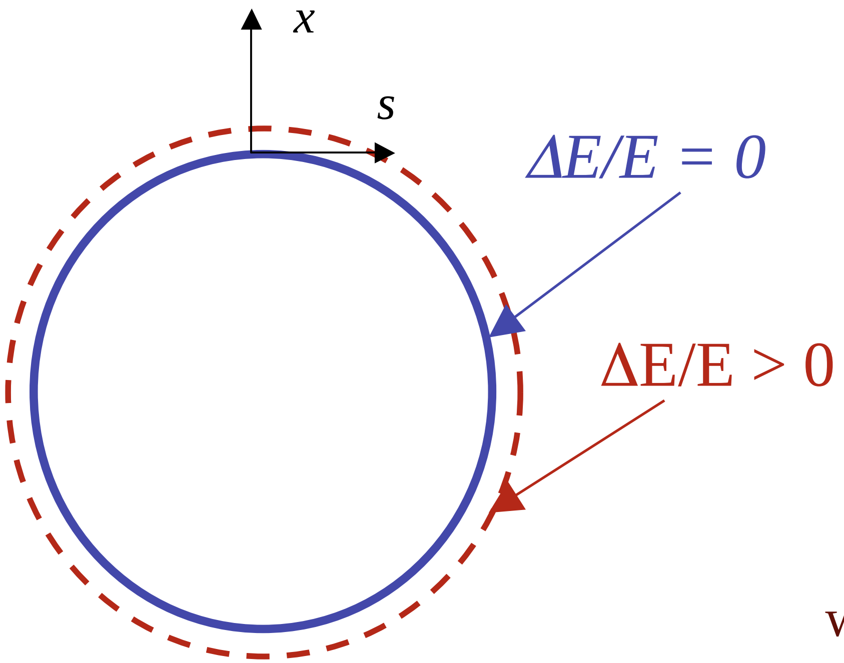
$$\frac{L - L_0}{L_0} \propto \frac{p - p_0}{p_0} \quad \Rightarrow \quad \boxed{\frac{\Delta L}{L_0} = \alpha \frac{\Delta p}{p_0}}$$

where α is constant

$$\text{For } \gamma >> 1 \quad \Rightarrow \frac{\Delta L}{L_0} = \alpha \frac{\Delta p}{p_0} \cong \alpha \frac{\Delta E}{E_0}$$

In the sector bending magnet $L > L_0$ so that a > 0
Higher energy particles will leave the magnet later.

Definition: Momentum compaction



$$\frac{\Delta L}{L} = \alpha \frac{\Delta p}{p}$$

$$\alpha = \int_0^{L_o} \frac{D_x}{\rho} ds$$

where dispersion, D_x, is the change in the closed orbit as a function of energy

Momentum compaction, α, is the change in the closed orbit length as a function of momentum.

Phase stability: Basics

- Distance along the particle orbit between rf-stations is *L*
- Time between stations for a particle with velocity *v* is

$$\tau = L/v$$

- Then

$$\frac{\Delta\tau}{\tau} = \frac{\Delta L}{L} - \frac{\Delta v}{v}$$

- Note that

$$\frac{\Delta v}{v} = \frac{1}{\gamma^2}\frac{\Delta p}{p}$$ (Exercise)

- For circular machines, L can vary with p
- For linacs L is independent of p

Phase stability: Slip factor & transition

- Introduce γ_t such that

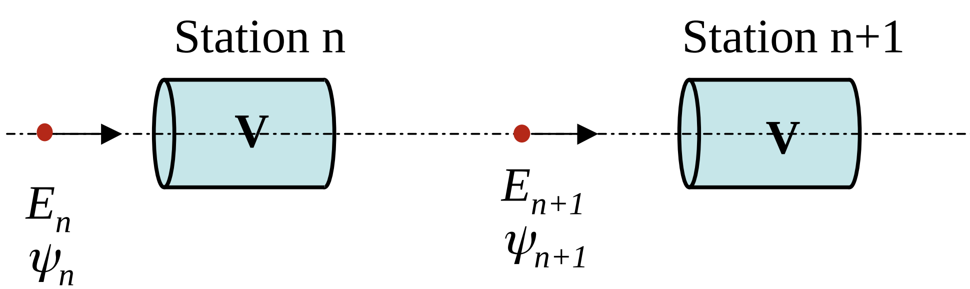
$$\frac{\Delta L}{L} = \frac{1}{\gamma_t^{\,2}} \frac{\Delta p}{p}$$

- Define a slip factor

$$\eta \equiv \frac{1}{\gamma_t^{\,2}} - \frac{1}{\gamma^2}$$

- At some *transition energy* η changes sign

- Now consider a particle with energy E_n and phase ψ_n w.r.t. the rf that enters station n at time T_n



Equation of motion for particle phase

- The phase at station *n+1* is

$$\psi_{n+1} = \psi_n + \omega_{rf}(\tau + \Delta\tau)_{n+1}$$

$$= \psi_n + \omega_{rf}\tau_{n+1} + \omega_{rf}\tau_{n+1}\left(\frac{\Delta\tau}{\tau}\right)_{n+1}$$

- By definition the synchronous particle stays in phase (mod 2π)
- Refine the phase mod 2π

$$\phi_n = \psi_n - \omega_{rf} T_n$$

$$\boxed{\phi_{n+1} = \phi_n + \omega_{rf}\tau_{n+1}\left(\frac{\Delta\tau}{\tau}\right)_{n+1} = \phi_n + \eta\underbrace{\omega_{rf}\tau_{n+1}}_{\text{harmonic number} = 2\pi N}\left(\frac{\Delta p}{p}\right)_{n+1}}$$

Equation of motion in energy

$$(E_s)_{n+1} = (E_s)_n + eV\sin\phi_s$$ and in general $$E_{n+1} = E_n + eV\sin\phi_n$$

Define ΔE= E - E_s

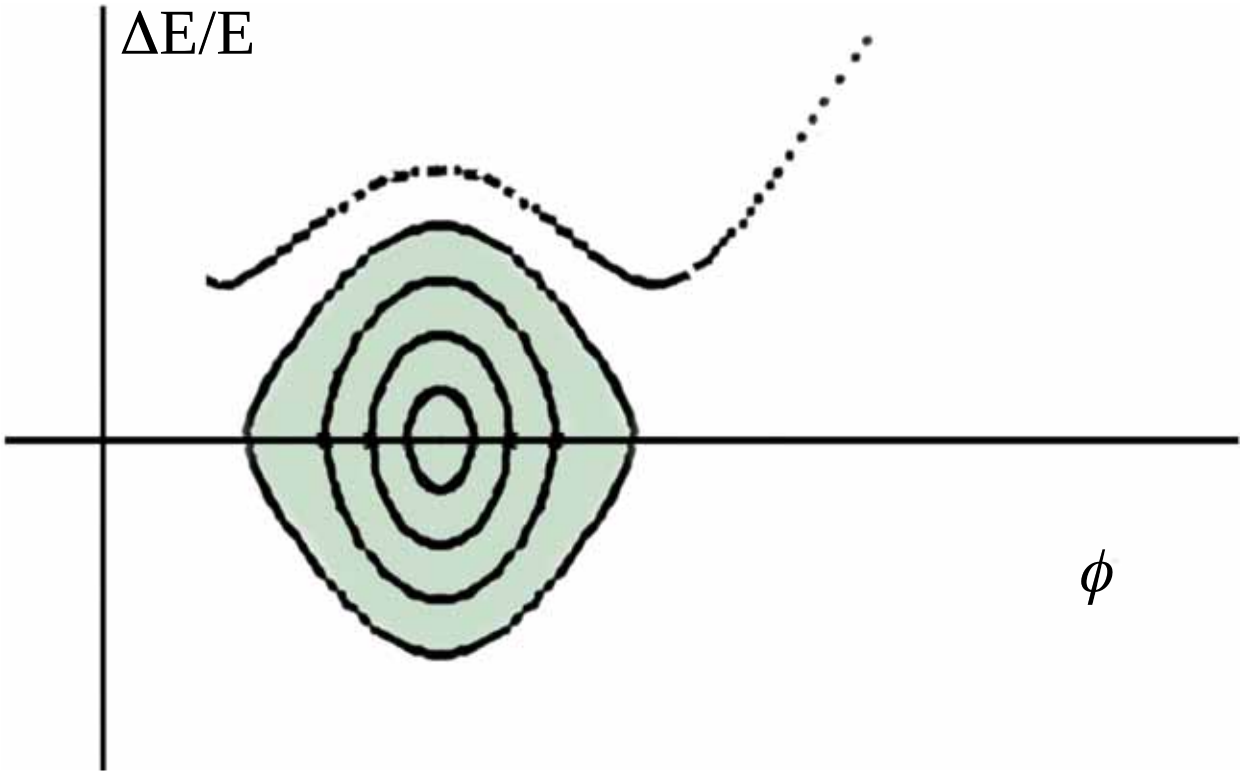
$$\boxed{\Delta E_{n+1} = \Delta E_n + eV(\sin\phi_n - \sin\phi_s)}$$

Exercise: Show that $$\frac{\Delta p}{p} = \frac{c^2}{v^2}\frac{\Delta E}{E}$$

Then

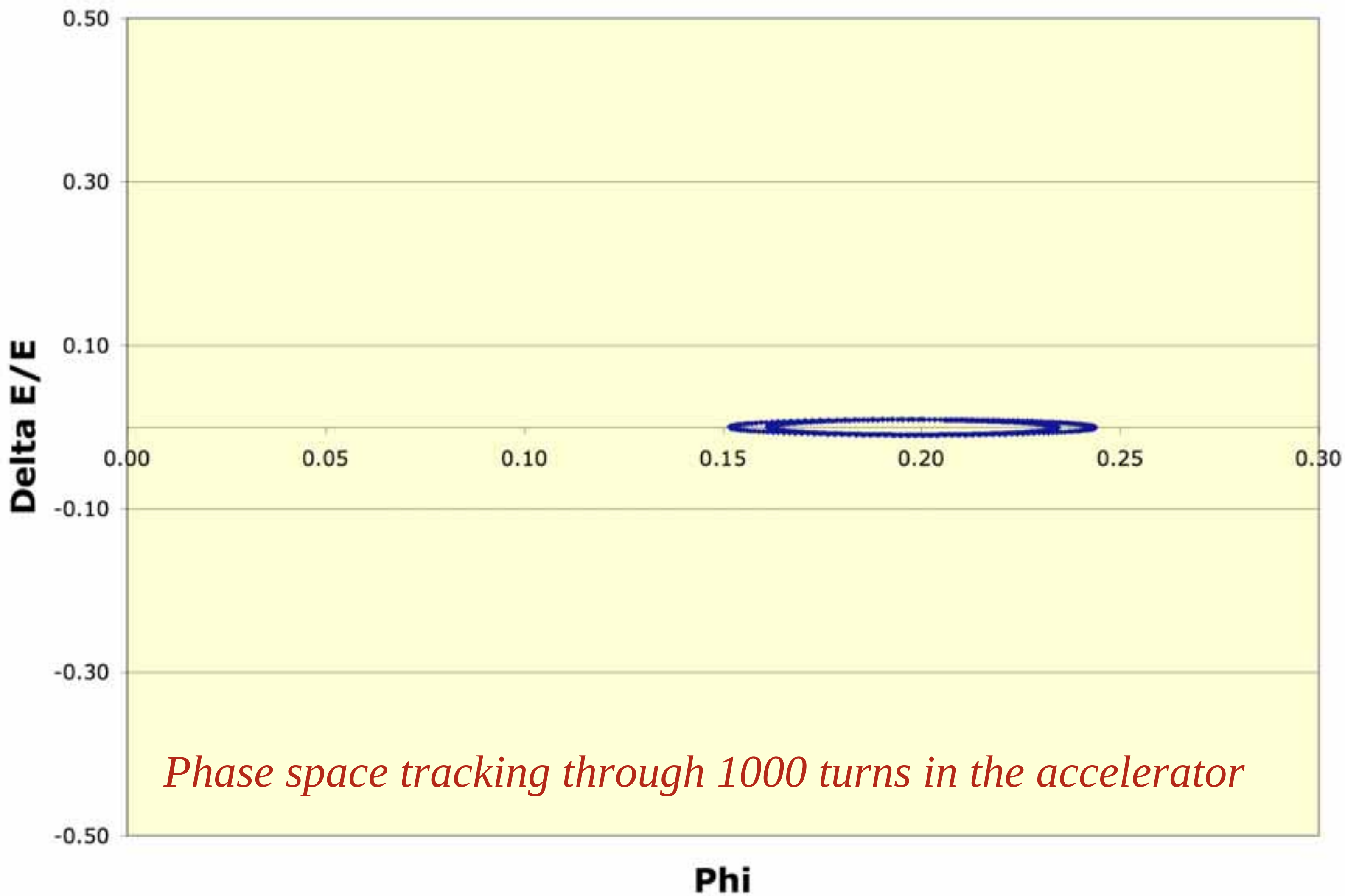
$$\boxed{\phi_{n+1} = \phi_n + \frac{\omega_{rf}\tau\eta c^2}{E_s v^2}\Delta E_{n+1}}$$

Longitudinal phase space of beam

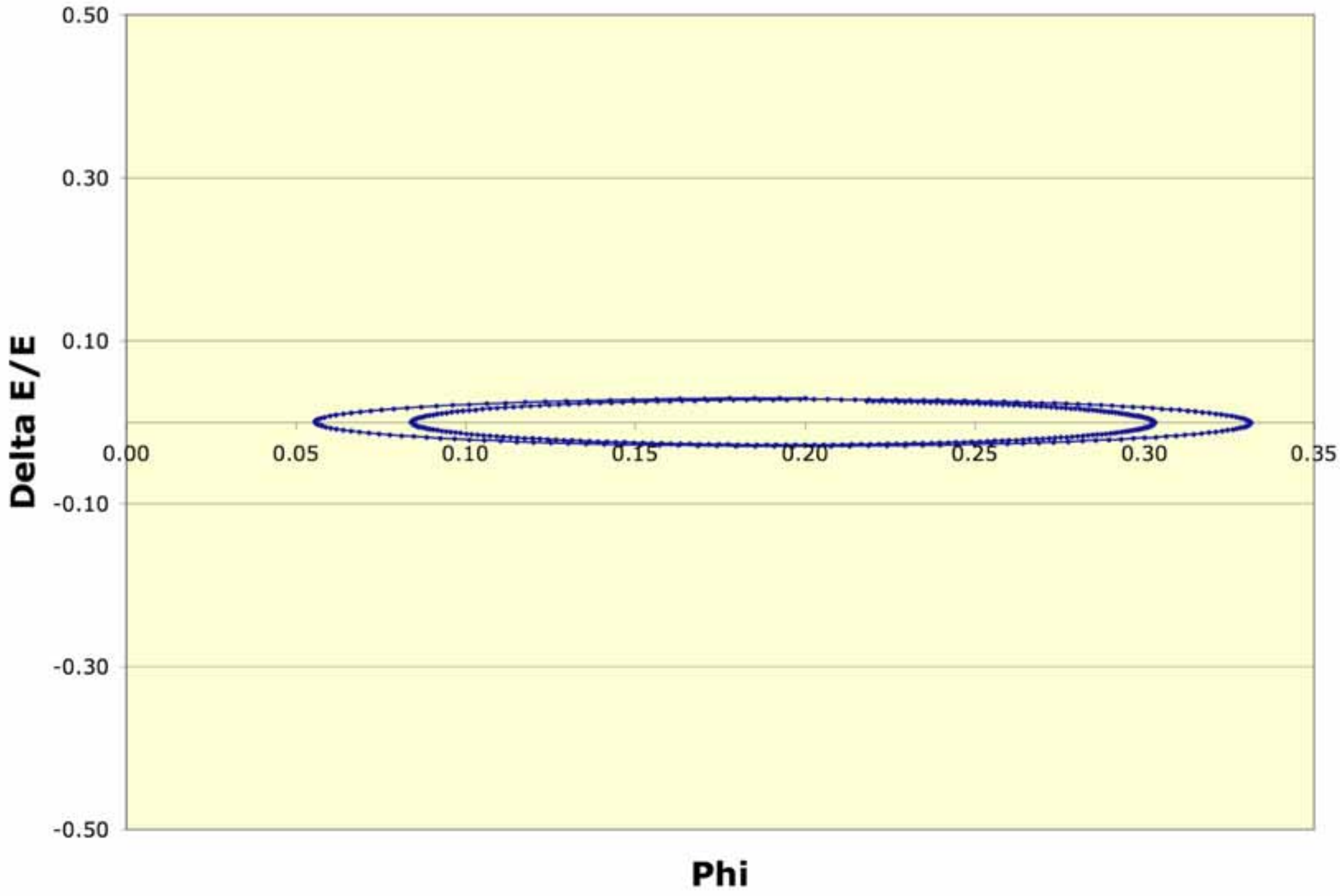


Solving the difference equations will show if there are areas of stability in the ($\Delta E/E$, ϕ) longitudinal phase space of the beam

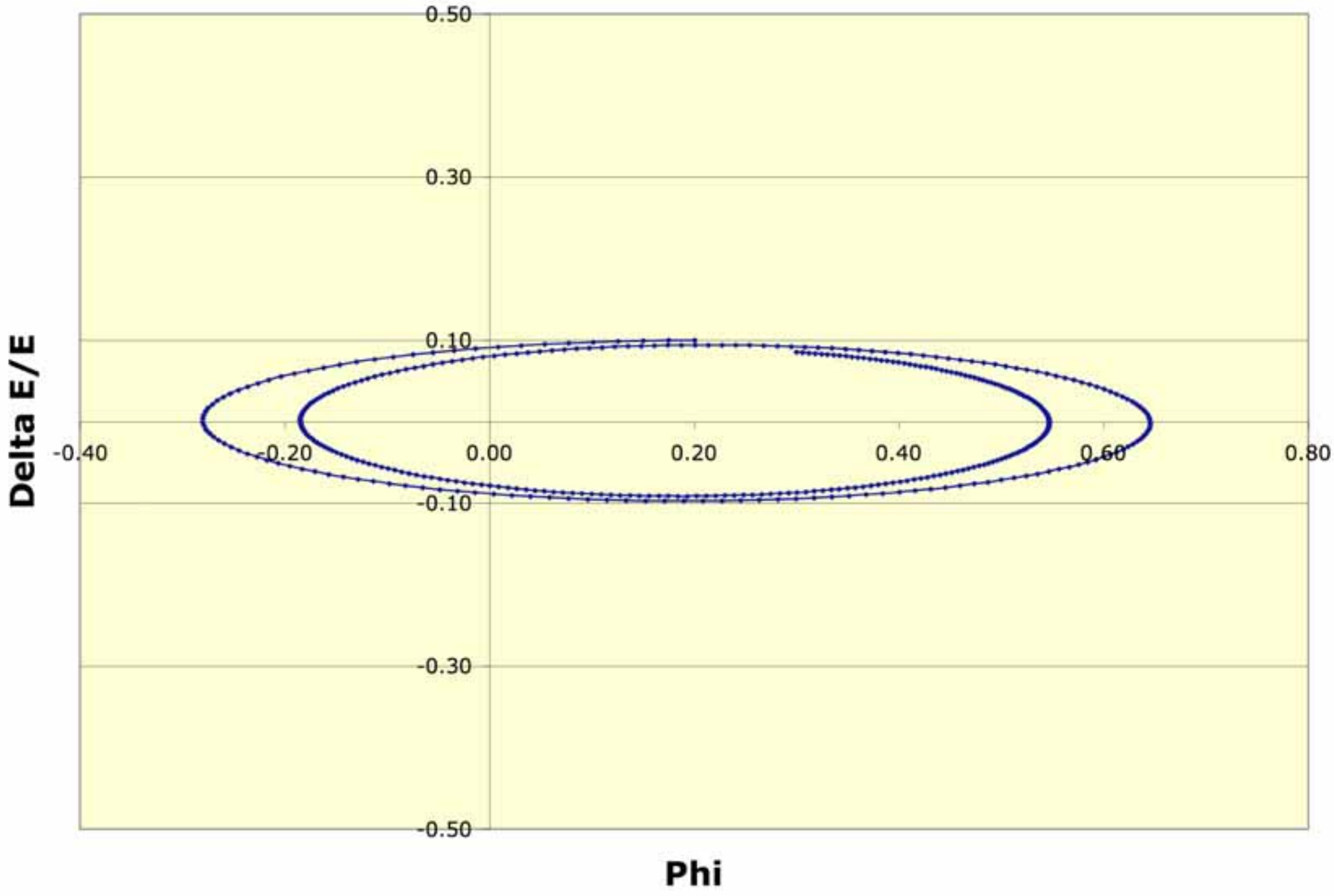
Phase stability, ΔE/E = 0.03, $\phi_n=\phi_s$



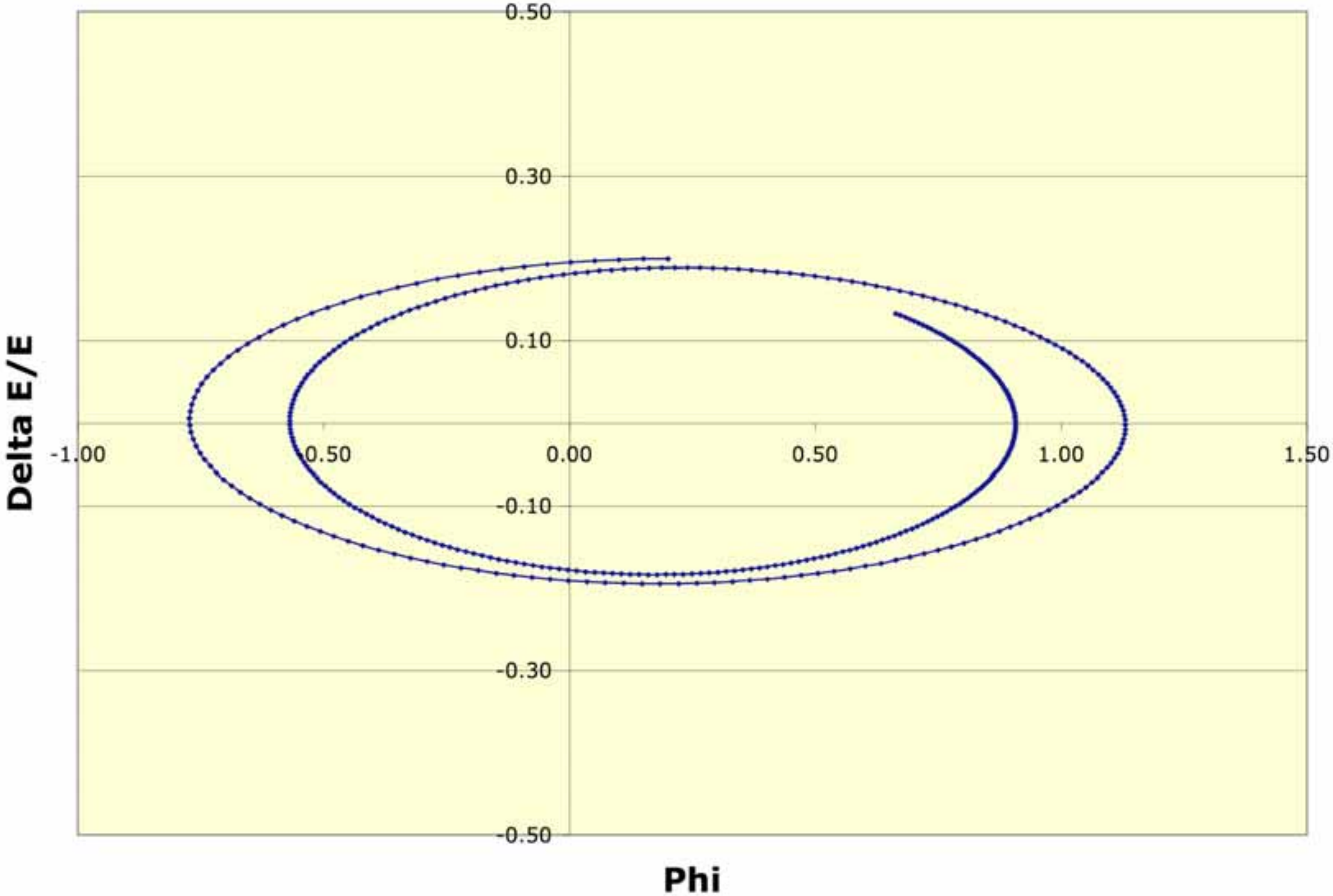
Phase stability, ΔE/E = 0.05, $\phi_n=\phi_s$



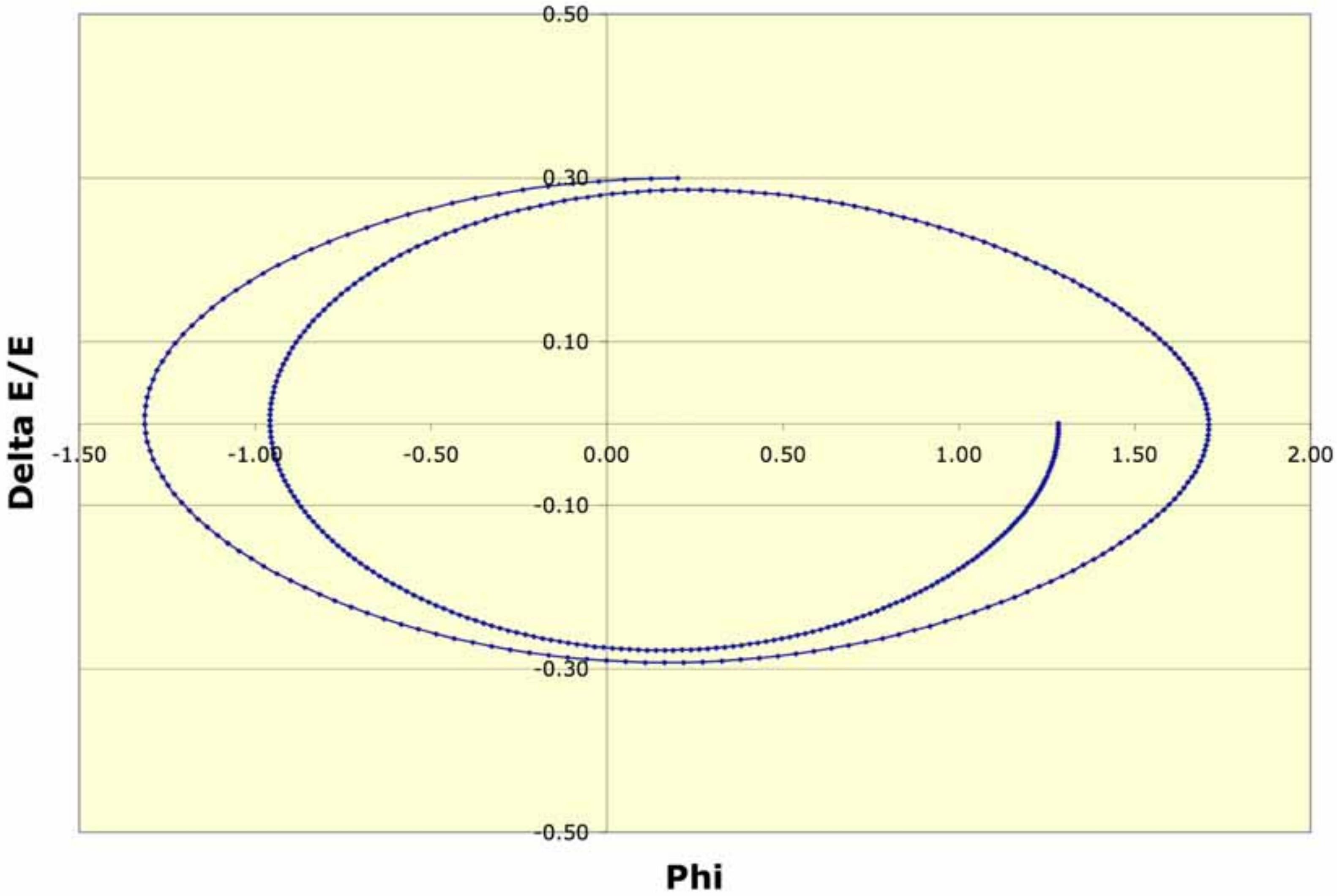
Phase stability, ΔE/E = 0.1, $\phi_n=\phi_s$



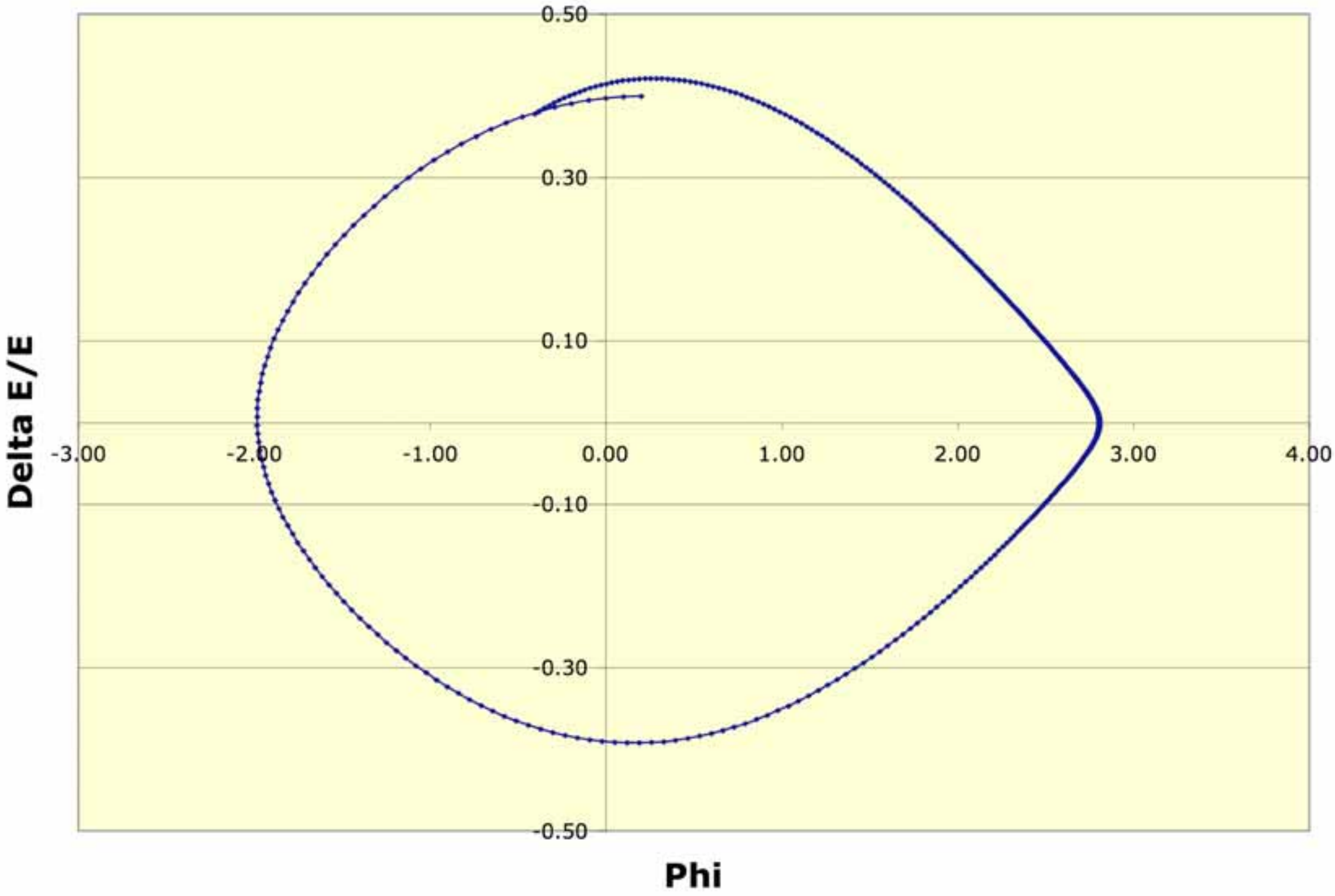
Phase stability, $\Delta E/E = 0.2$, $\phi_n=\phi_s$



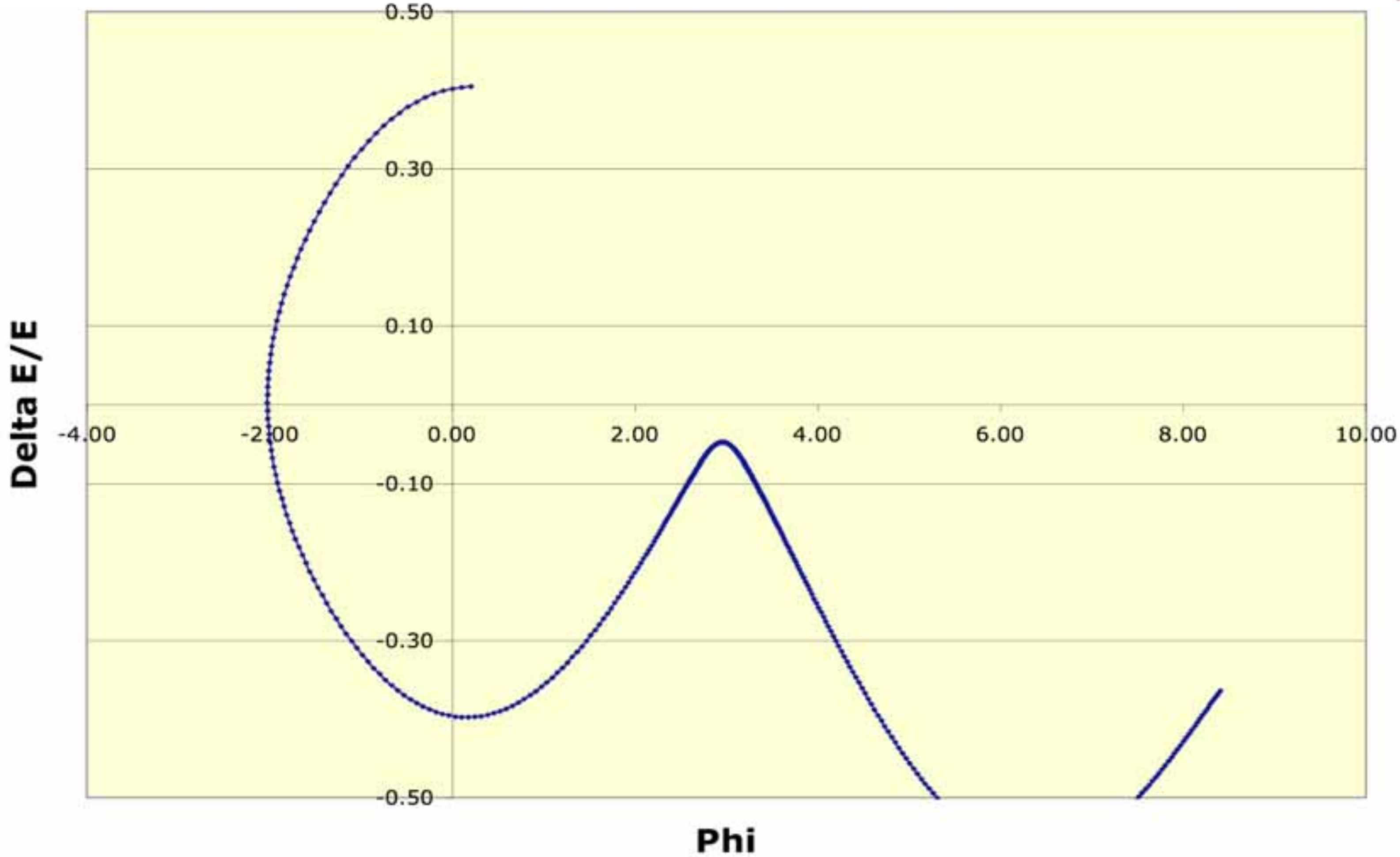
Phase stability, $\Delta E/E = 0.3$, $\phi_n = \phi_s$



Phase stability, $\Delta E/E = 0.4$, $\phi_n = \phi_s$

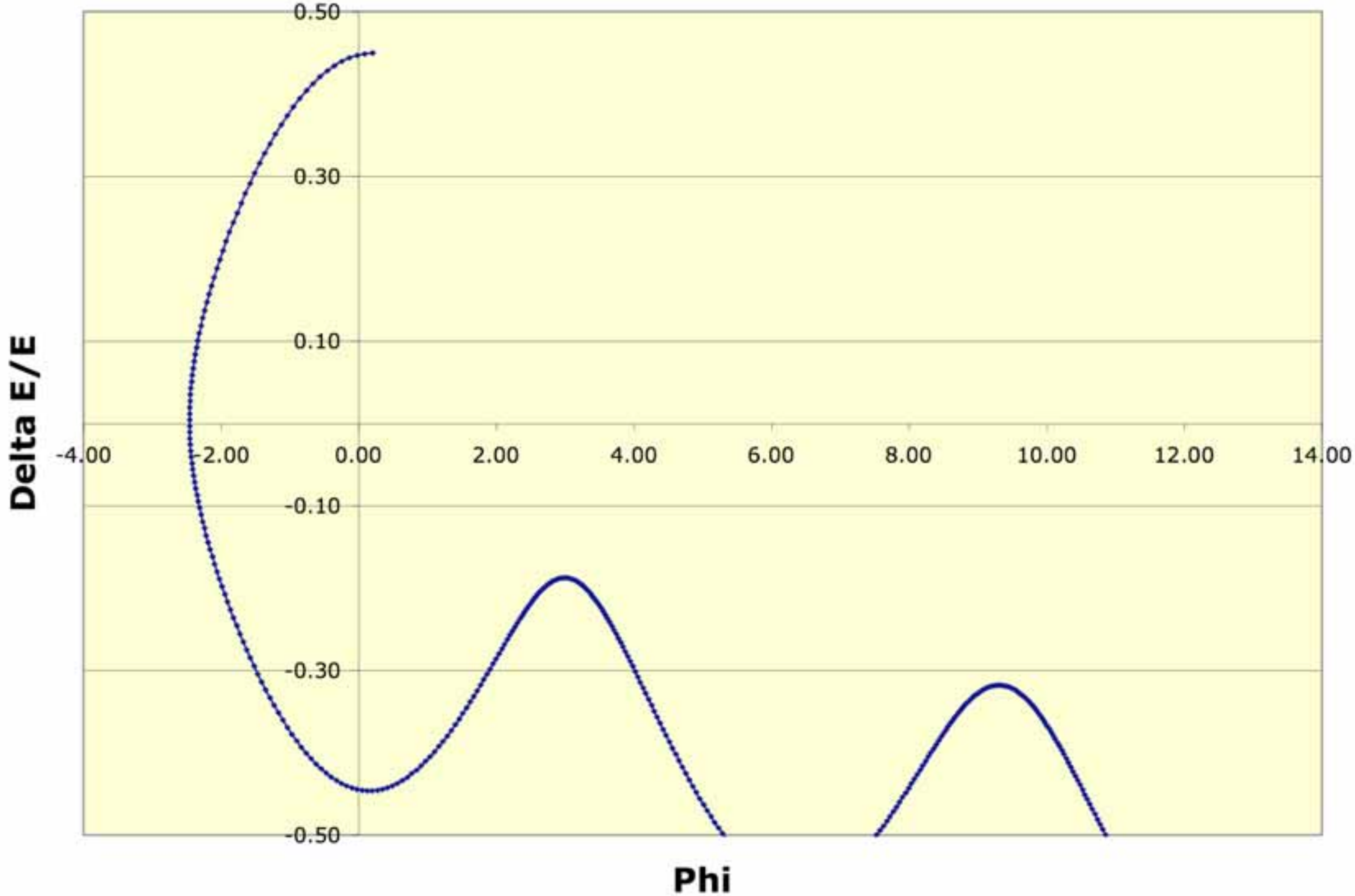


Phase stability, $\Delta E/E$ = 0.405, $\phi_n=\phi_s$

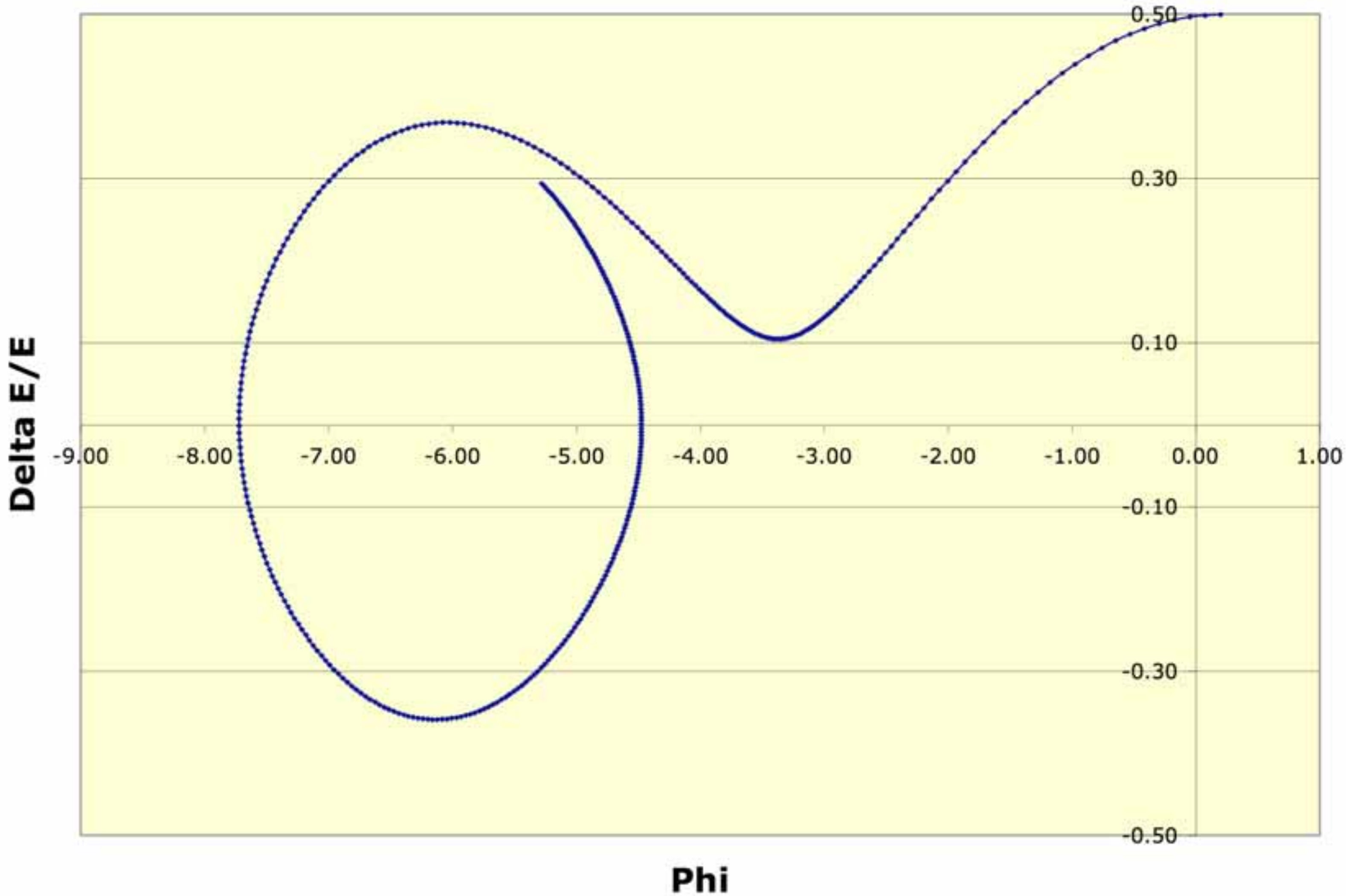


Regions of stability and instability are sharply divided

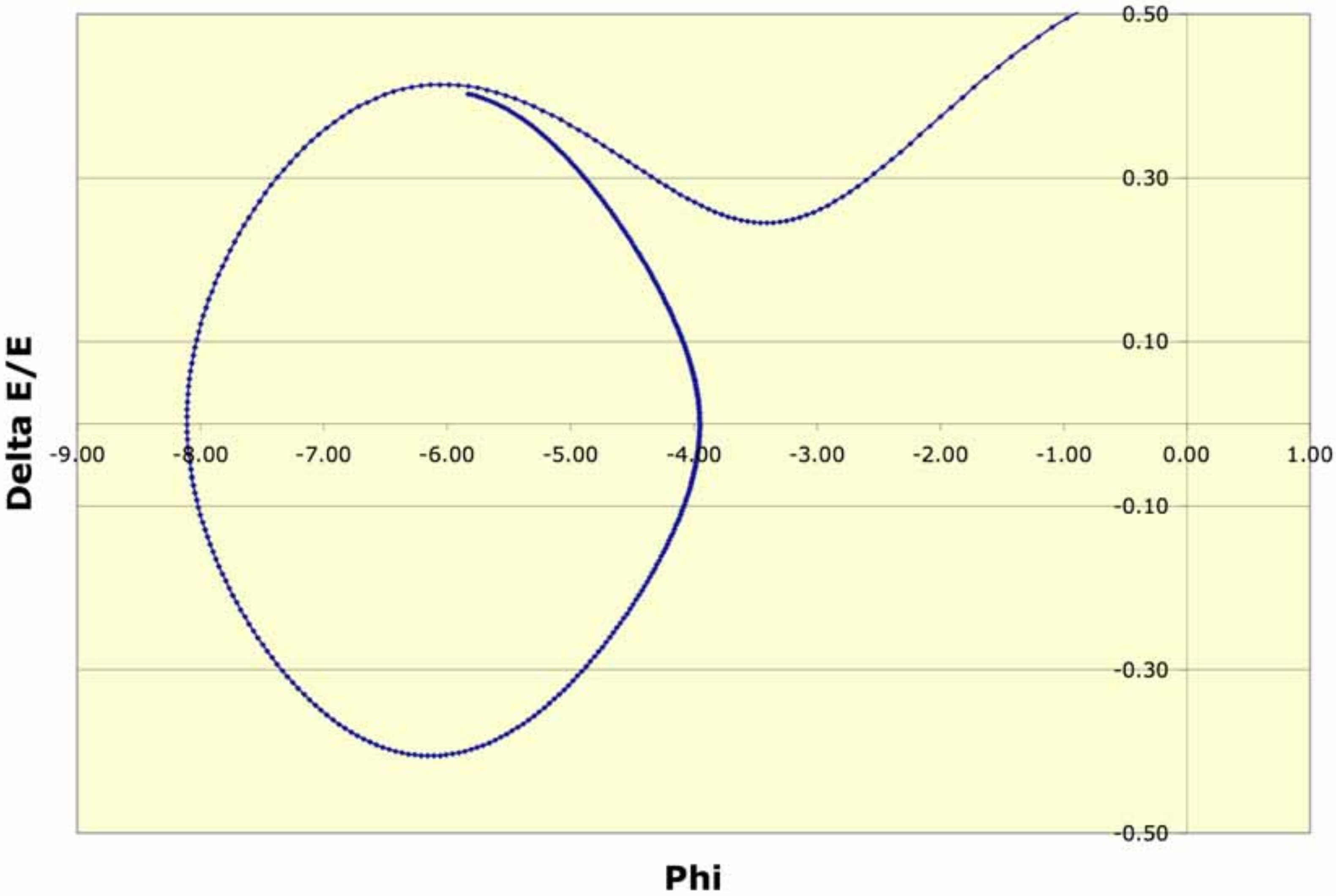
Phase stability, $\Delta E/E = 0.45$, $\phi_n = \phi_s$



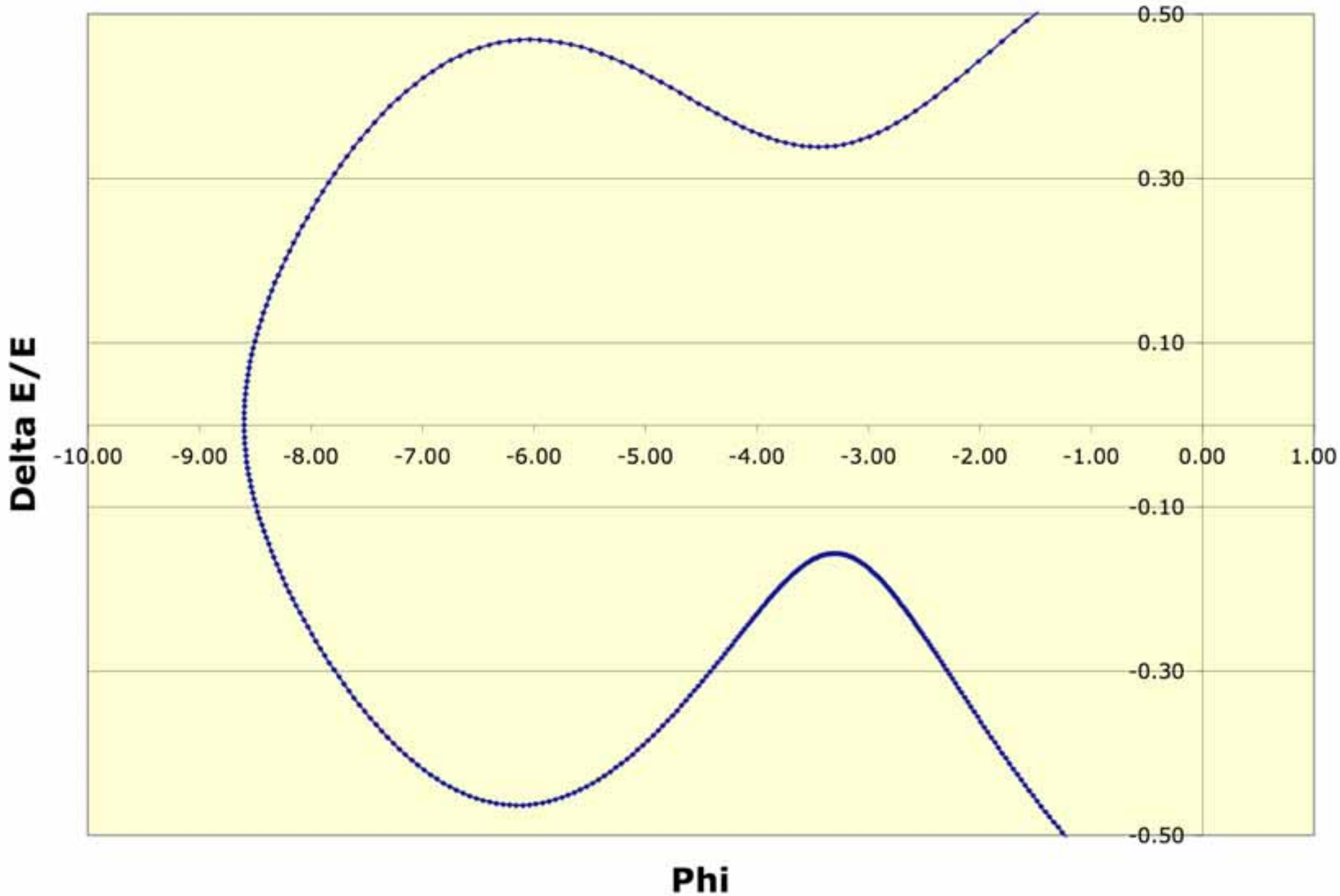
Phase stability, $\Delta E/E$ = 0.5, $\phi_n=\phi_s$



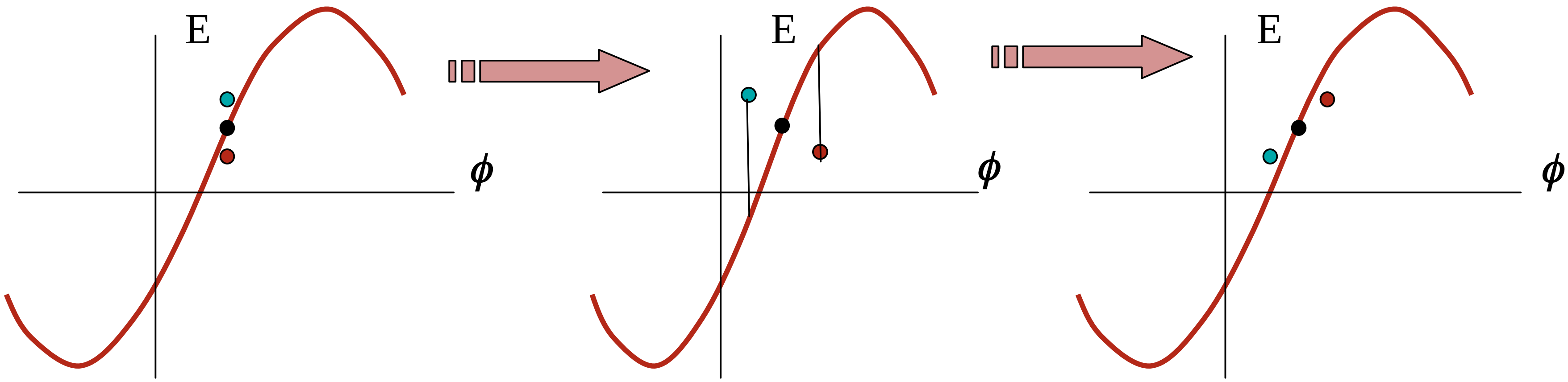
Phase stability, $\Delta E/E$ = 0.55, $\phi_n = \phi_s$



Phase stability, ΔE/E = 0.6, $\phi_n=\phi_s$



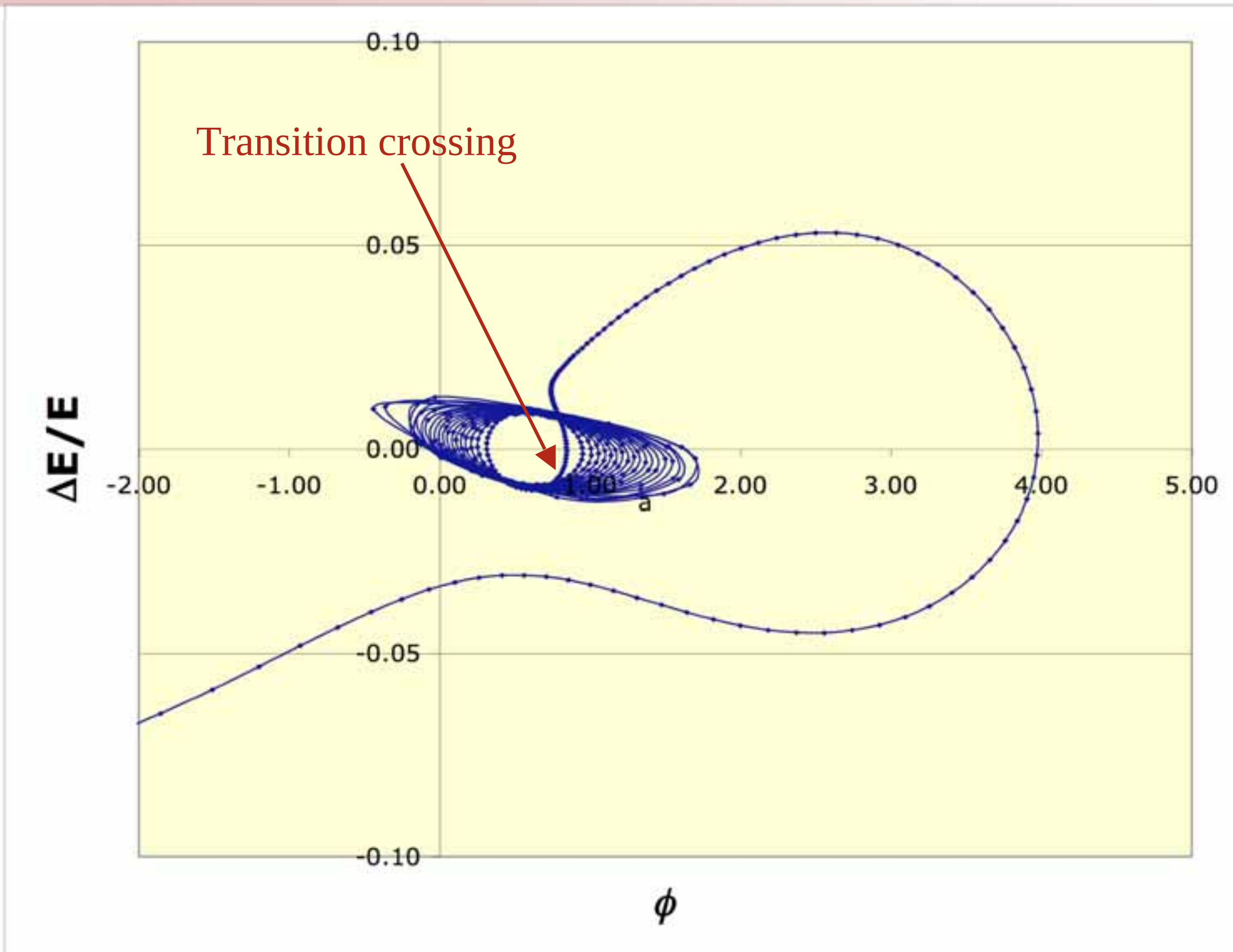
Physical picture of phase stability



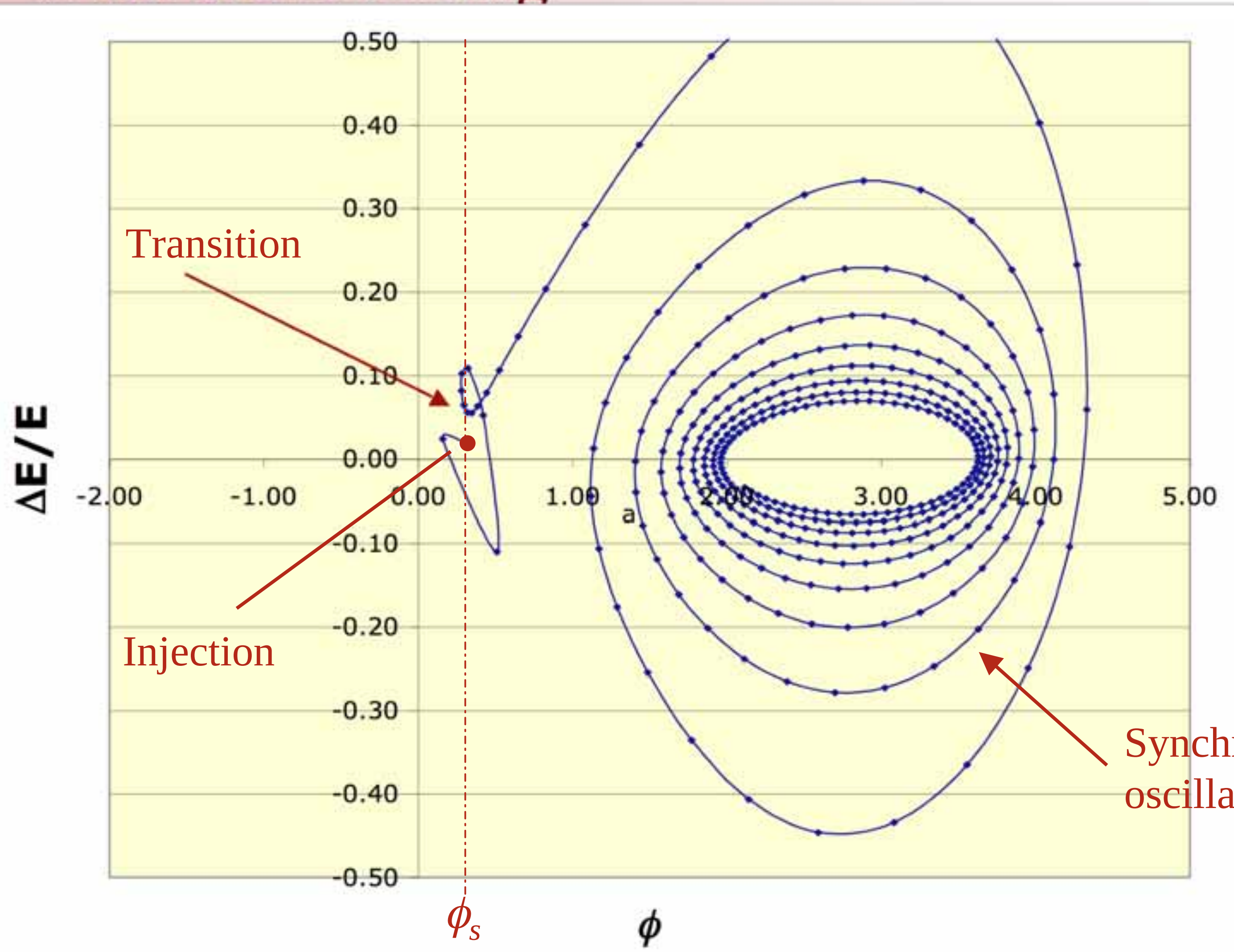
Here we've picked the case in which we are above the transition energy

(typically the case for electrons)

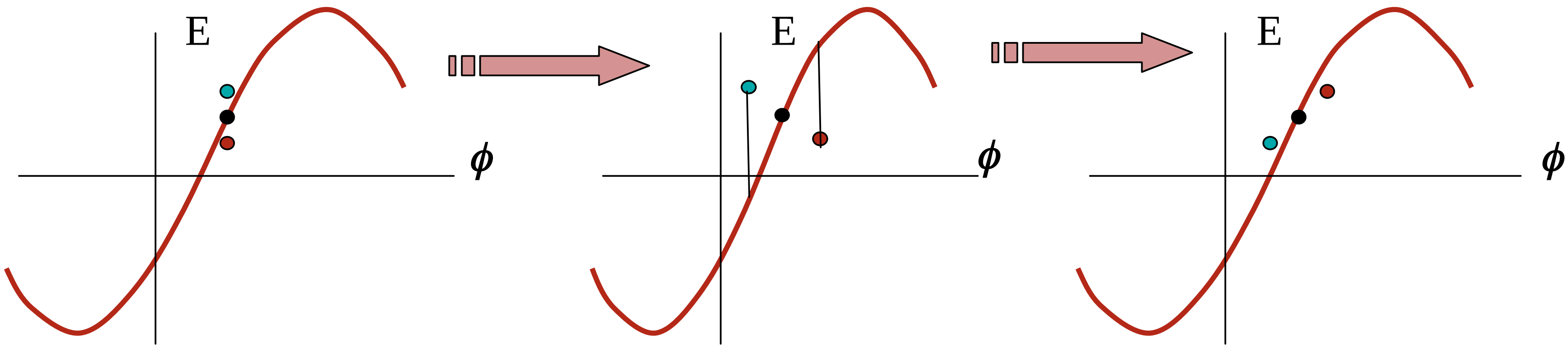
Consider this case for a proton accelerator



Case of favorable transition crossing in an electron ring



Frequency of synchrotron oscillations



- Phase-energy oscillations mix particles longitudinally within the beam
- What is the time scale over which this mixing takes place?
- If ΔE and ϕ change slowly, approximate difference equations by differential equations with n as independent variable

Two first order equations ==> one second order equation

$$\frac{d\varphi}{dn} = \frac{\eta\omega_{rf}\tau}{\beta^2 E_s}\,\Delta E$$

and

$$\frac{\mathrm{d}\Delta \mathrm{E}}{dn} = eV(\sin\varphi - \sin\varphi_s)$$

yield

$$\boxed{\frac{d^2\varphi}{dn^2} = \frac{\eta\omega_{rf}\tau}{\beta^2 E_s}\,eV(\sin\varphi - \sin\varphi_s)}$$

(Pendulum equation)

if

$$V = \text{constant and } \frac{\mathrm{dE_s}}{\mathrm{dn}} \text{ is sufficiently small}$$

Multiply by dϕ/dn & integrate

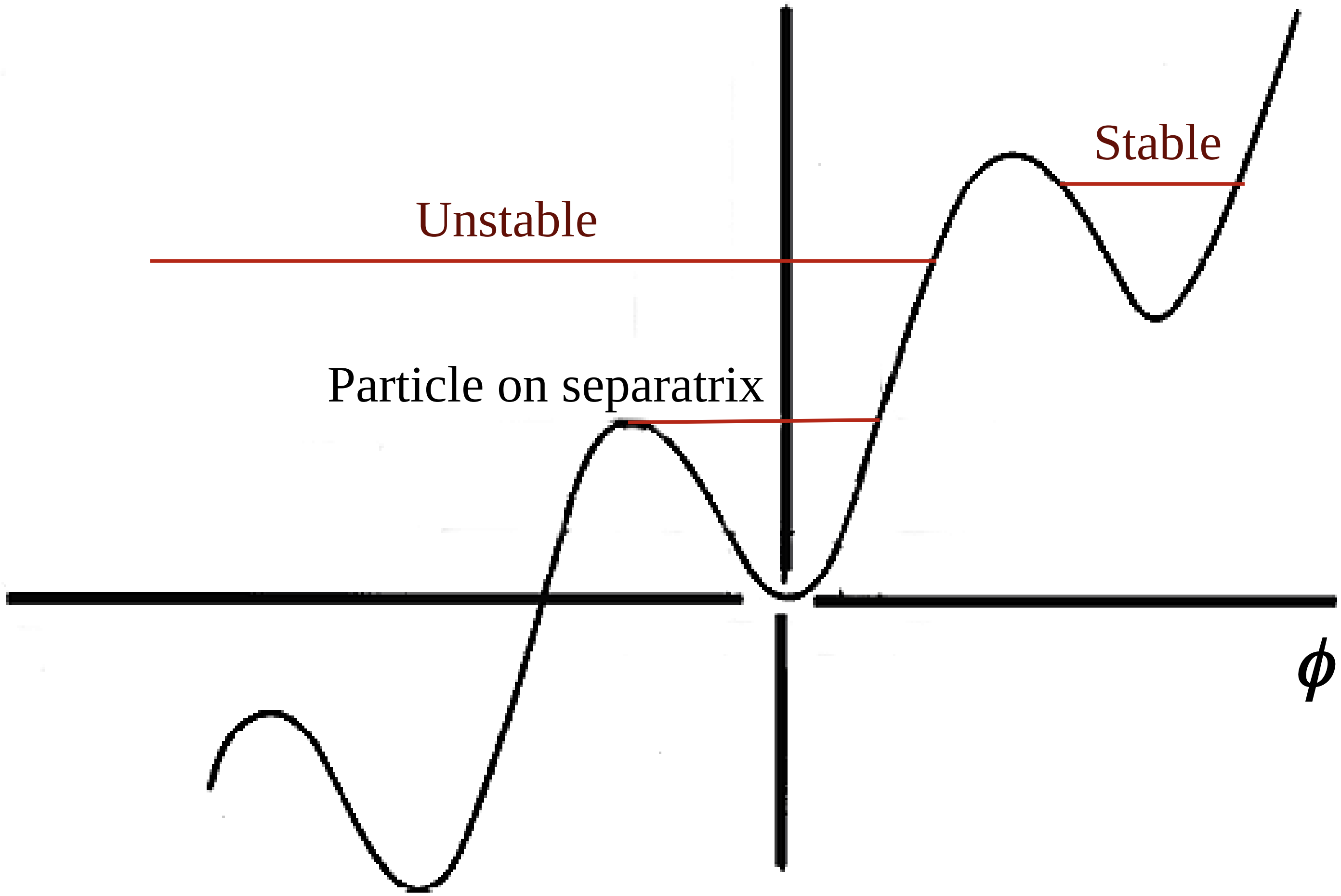
$$\int \frac{d^2\varphi}{dn^2}\frac{d\varphi}{dn}dn = \frac{\eta\omega_{rf}\tau}{\beta^2 E_s} eV \int \frac{d\varphi}{dn}(\sin\varphi - \sin\varphi_s)dn$$

==> $$\frac{1}{2}\left(\frac{d\varphi}{dn}\right)^2 = -\frac{\eta\omega_{rf}\tau}{\beta^2 E_s} eV(\cos\varphi - \sin\varphi_s) + const$$

Rearranging

$$\underbrace{\frac{1}{2}\left(\frac{d\varphi}{dn}\right)^2}_{\text{“K.E.”}} + \underbrace{\frac{\eta\omega_{rf}\tau}{\beta^2 E_s} eV(\cos\varphi - \sin\varphi_s)}_{\text{“P.E”}} = \underbrace{const}_{\text{Total}}$$

"Energy" diagram for $\cos\phi + \phi\sin\phi_s$



Stable contours in phase space

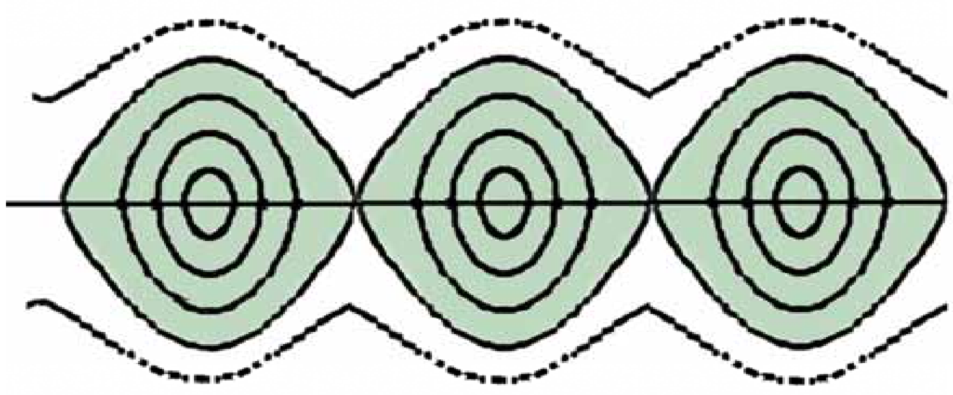
Insert $$\frac{d\varphi}{dn} = \frac{\eta\omega_{rf}\tau}{\beta^2 E_s}\,\Delta E$$

into $$\frac{1}{2}\left(\frac{d\varphi}{dn}\right)^2 + \frac{\eta\omega_{rf}\tau}{\beta^2 E_s} eV(\cos\varphi - \sin\varphi_s) = const$$

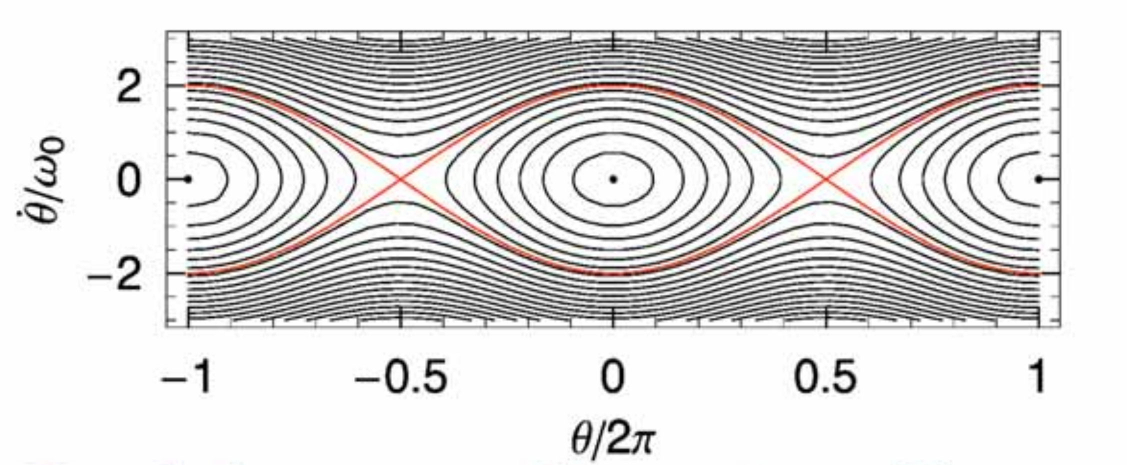
$$\boxed{(\Delta E)^2 + 2eV\,\frac{\beta^2 E_s}{\eta\omega_{rf}\tau}(\cos\varphi - \sin\varphi_s) = const}$$

for all parameters held constant

For $\phi_\sigma = 0$ we have



We've seen this behavior for the pendulum



Now let's return to the question of frequency

For small phase differences, $\Delta\phi = \phi - \phi_s$, we can linearize our equations

$$\frac{d^2\varphi}{dn^2} = \frac{d^2\Delta\varphi}{dn^2} = \frac{\eta\omega_{rf}\tau}{\beta^2 E_s} eV(\sin\varphi - \sin\varphi_s)$$

$$= \frac{\eta\omega_{rf}\tau}{\beta^2 E_s} eV\left(\sin(\varphi_s + \Delta\varphi) - \sin\varphi_s\right)$$

$$\approx 4\pi^2 \underbrace{\left(\frac{\eta\omega_{rf}\tau}{4\pi^2\beta^2 E_s} eV\cos\varphi_s\right)}_{-\nu_s^2}\Delta\varphi$$

(harmonic oscillator in $\Delta\phi$)

$-\nu_s^2$ *Synchrotron tune*

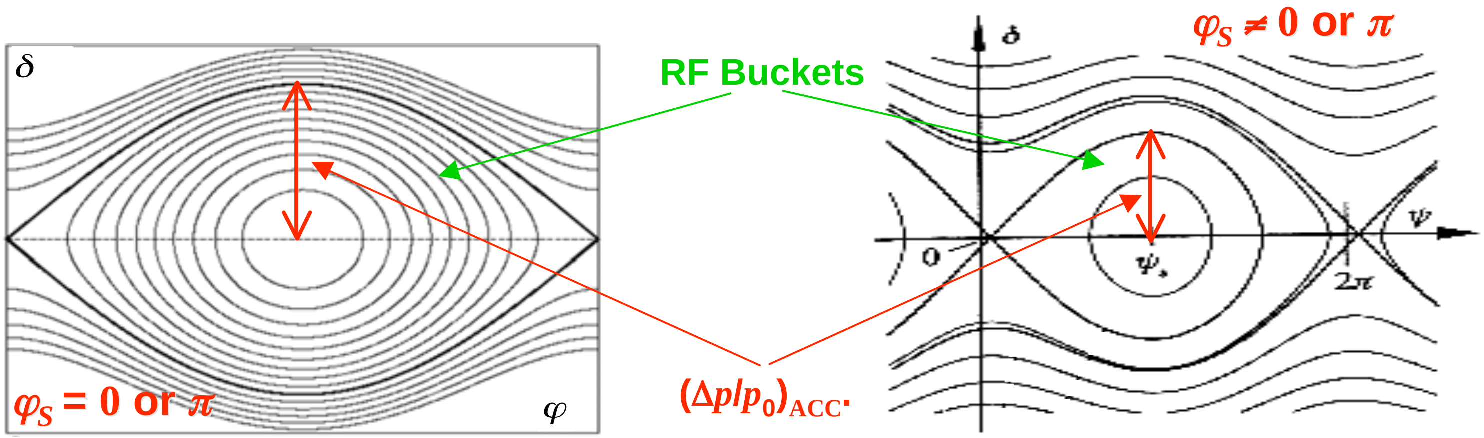
$$\Omega_s = \frac{2\pi\nu_s}{\tau} = \sqrt{-\frac{\eta\omega_{rf}}{\tau\beta^2 E_s} eV\cos\varphi_s} \;=\; \text{synchrotron angular frequency}$$

Choice of stable phase depends on η

$$\Omega_s = \sqrt{-\frac{\eta\omega_{rf}}{\tau\beta^2 E_s} eV\cos\varphi_s}$$

- Below transition ($\gamma < \gamma_t$),
  - ➙ $\eta < 0$, therefore $\cos\phi_s$ must be > 0
- Above transition ($\gamma > \gamma_t$),
  - ➙ $\eta > 0$, therefore $\cos\phi_s$ must be < 0
- At transition $\Omega_s = 0$; there is no phase stability
- Circular accelerators that must cross transition shift the synchronous phase at $\gamma > \gamma_t$
- Linacs have no path length difference, $\eta = 1/\gamma^2$; particles stay locked in phase and $\Omega_s = 0$

Momentum acceptance: maximum momentum of any particle on a stable orbit



$$\boxed{\left(\frac{\Delta p}{p_0}\right)^2_{ACC} = \frac{2|q|\hat{V}}{\pi\, h|\eta_C|\beta\, c\, p_0}}$$

$$\boxed{\left(\frac{\Delta p}{p_0}\right)^2_{ACC} = \frac{F(Q)}{2Q}\frac{2|q|\hat{V}}{\pi\, h|\eta_C|\beta\, c\, p_0}}$$

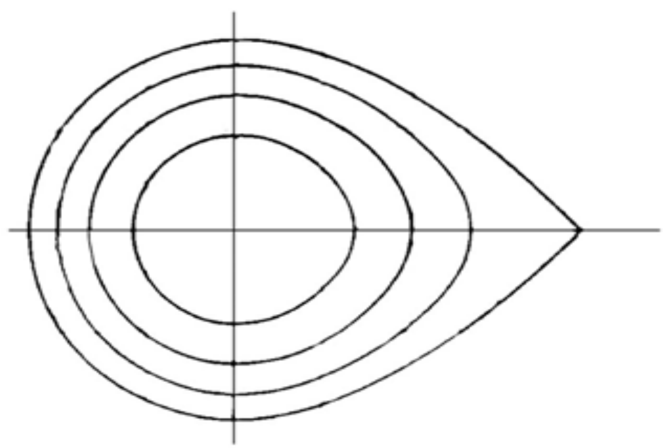
$$F(Q) = 2\left(\sqrt{Q^2 - 1} - \arccos\frac{1}{Q}\right)$$

$$\boxed{\begin{array}{c} Q = \dfrac{1}{\sin\varphi_s} = \dfrac{q\hat{V}}{U_0} \\ \textit{Over voltage factor} \end{array}}$$

How can particles be lost

- Scattering out of the rf-bucket
  - Particles scatter off the collective field of the beam
  - Large angle particle-particle scattering
- RF-voltage too low for radiation losses

$$\Delta E_{Total} = qV + U(E)$$

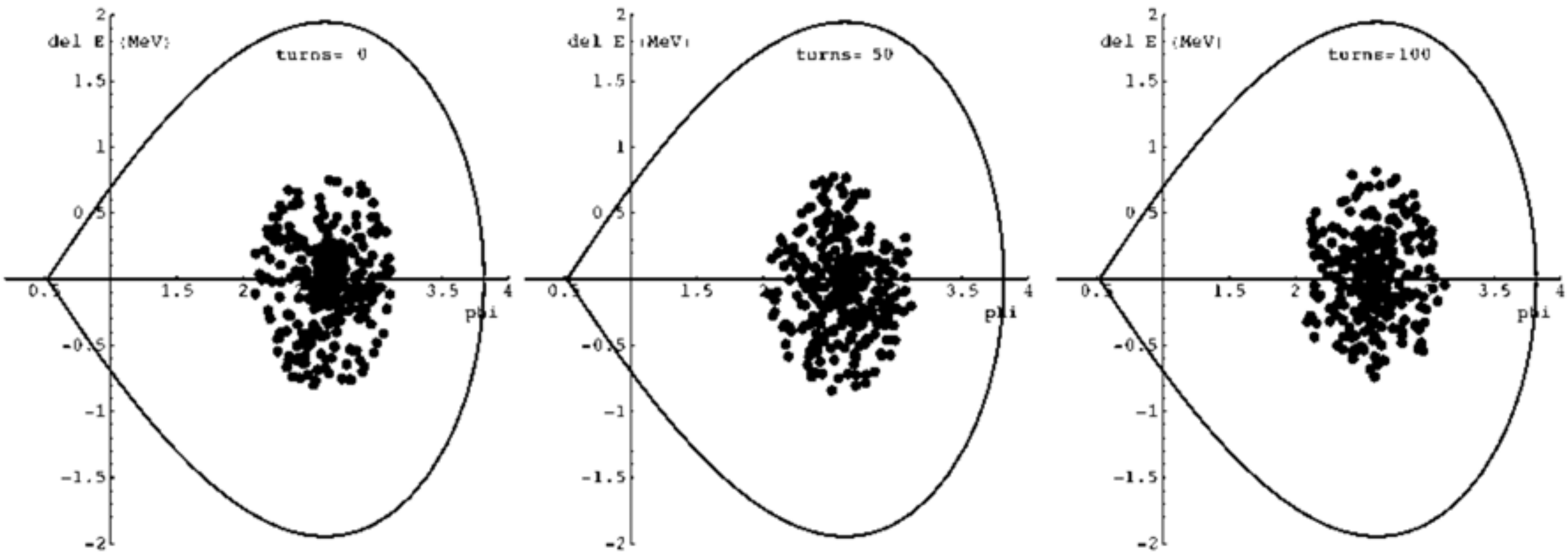


Matching the beam on injection

- Beam injection from another rf-accelerator is typically "bucket-to-bucket"
  - rf systems of machines are phase-locked
  - bunches are transferred directly from the buckets of one machine into the buckets of the other
- This process is efficient for matched beams
  - Injected beam hits the middle of the receiving rf-bucket
  - Two machines are longitudinally matched.
    - They have the same aspect ratio of the longitudinal phase ellipse

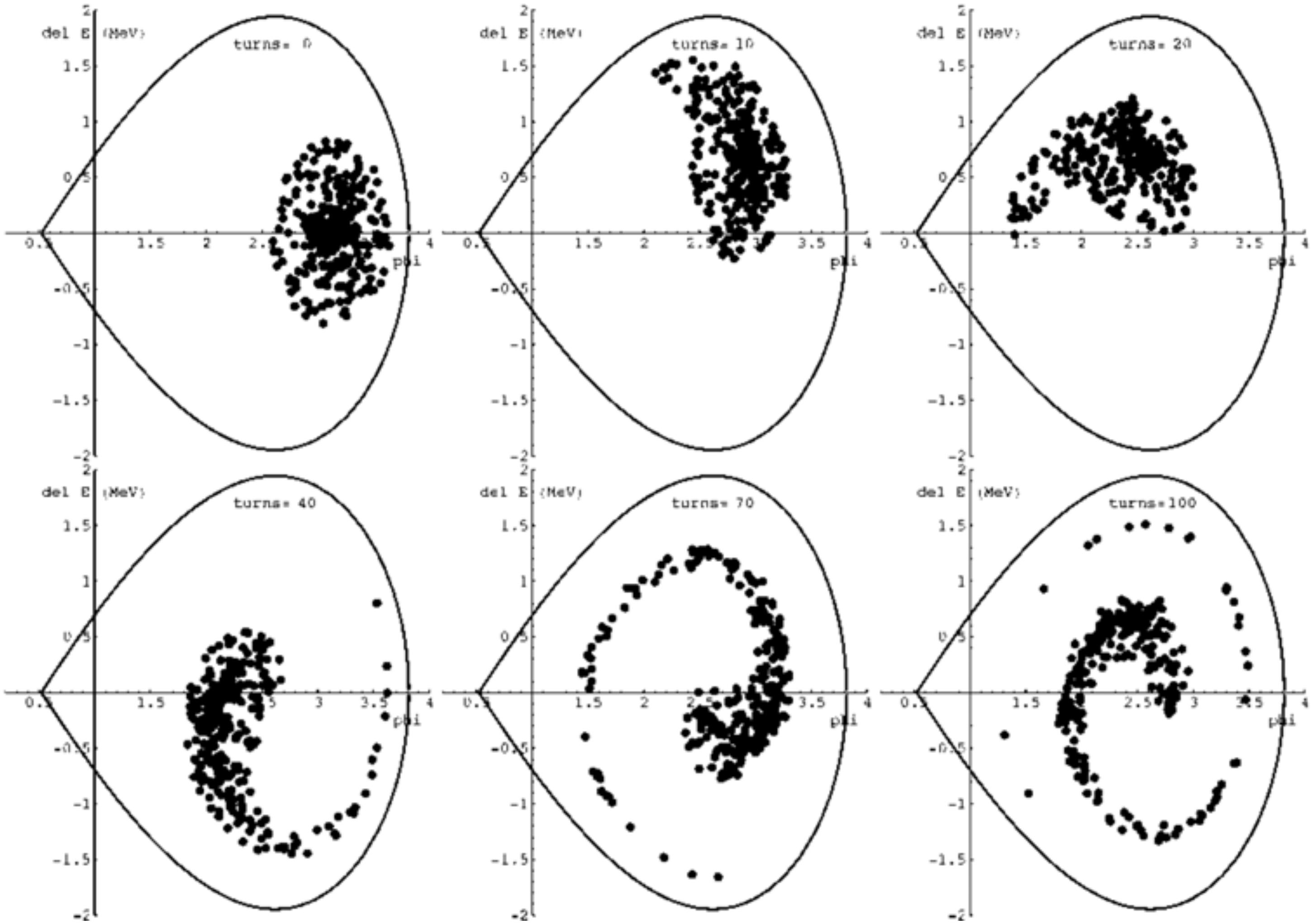
Dugan simulations of CESR injection

Matched transfer - first hundred turns



From Dugan: USPAS lectures - Lecture 11

Example of mismatched CESR transfer: phase error 60°



From Dugan: USPAS lectures - Lecture 11