Unit 4 - Lecture 13

Beam loading & wakefields

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Source: Wake field slides are based on Sannibale lecture 9

Assumptions in our discussion

1. Particle trajectories are parallel to z-axis in the region of interest

2. The particles are highly relativistic

3. (1) + (2) ==> The beam is rigid,
   - ➙ Particle trajectories are not changed in the region of interest

4. Linearity of the particle motion
   - ➙ Particle dynamics are independent of presence of other particles

5. Linearity of the electromagnetic fields in the structure
   - ➙ The beam does not detune the structure

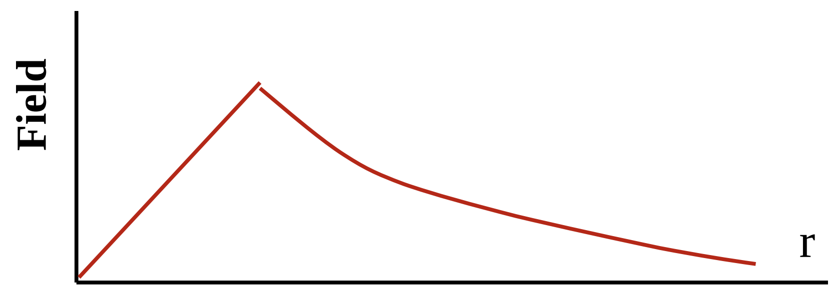
6. The power source is unaffected by the beam

7. The interaction between beam and structure is linear

Recall our discussion of space charge fields

- Coulomb interaction ==> space charge effect
  - A generic particle in the bunch experiences the *collective* Coulomb force due to fields generated by all the other particles in the bunch
- Such self-fields are usually nonlinear
  - Their evaluation usually requires numerical techniques
  - Special cases can be evaluated analytically

We've already written the expressions
for an axisymmetric beam with uniform charge density



Lee Teng's solution for fields inside the beam

- Conditions:
  - Continuous beam with constant linear charge density l
  - Stationary uniform elliptical distribution in the transverse plane
  - *a* and *b* the ellipse half-axes,
  - the beam moves along *z* with velocity βc.

$$E_x = \frac{1}{\pi\varepsilon_0}\frac{\lambda x}{a(a+b)} \qquad E_y = \frac{1}{\pi\varepsilon_0}\frac{\lambda y}{b(a+b)}$$

$$B_x = -\frac{\mu_0}{\pi}\frac{\lambda\beta c y}{b(a+b)} \qquad B_y = \frac{\mu_0}{\pi}\frac{\lambda\beta c x}{a(a+b)}$$

$$B_x = -\frac{\beta}{c}E_y, \qquad B_y = \frac{\beta}{c}E_x,$$

Space charge for Gaussian distribution

- Conditions
  - Charge density is gaussian in the transverse plane
  - $x << \sigma_x$ and $y << \sigma_y$:

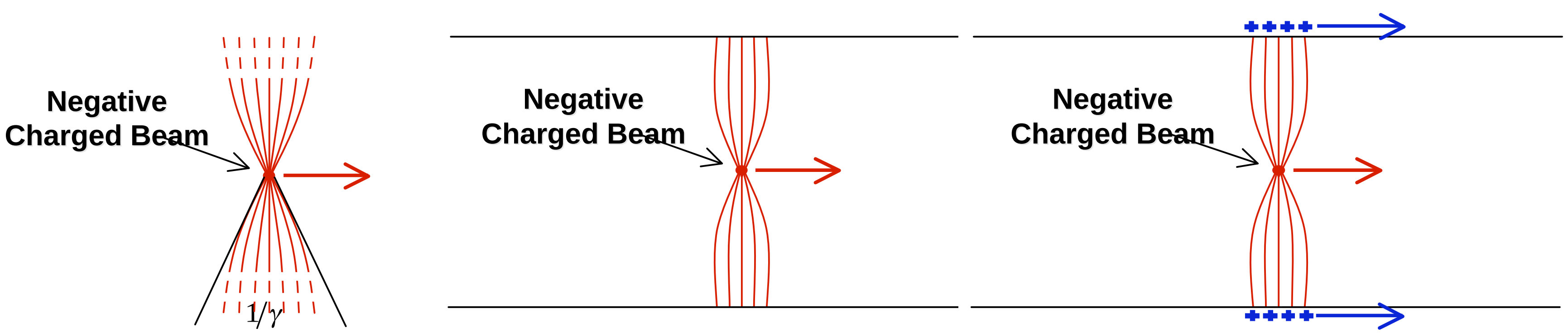
$$E_x = \frac{1}{2\pi\varepsilon_0} \frac{\lambda x}{\sigma_x(\sigma_x + \sigma_y)} \qquad E_y = \frac{1}{2\pi\varepsilon_0} \frac{\lambda y}{\sigma_y(\sigma_x + \sigma_y)}$$

$$B_x = -\frac{\mu_0}{2\pi} \frac{\lambda \beta c y}{\sigma_y(\sigma_x + \sigma_y)} \qquad B_y = \frac{\mu_0}{2\pi} \frac{\lambda \beta c x}{\sigma_x(\sigma_x + \sigma_y)}$$

$$B_x = -\frac{\beta}{c} E_y, \qquad B_y = \frac{\beta}{c} E_x,$$

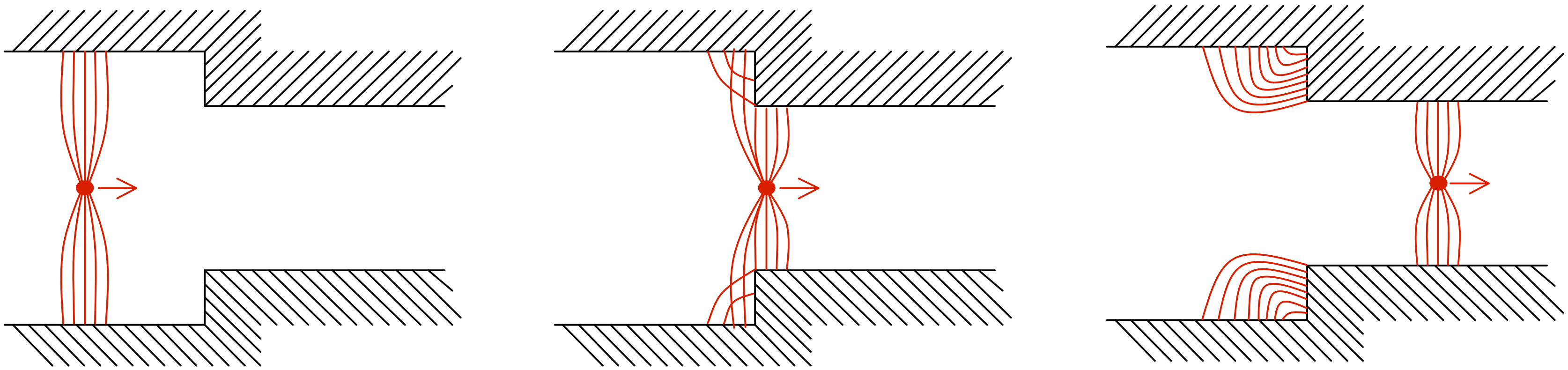
Vacuum Chamber Effects:Image Charge

- In the lab frame, the EM field of a relativistic particle is transversely confined within a cone of aperture of ~ $1/\gamma$
- Particle accelerators operate in an ultra high vacuum environment provided by a metal *vacuum chamber*
- By Maxwell equations, the beam's E field terminates perpendicular to the chamber (conductive) walls
- An equal image charge, but with opposite sign, travels on the vacuum chamber walls following the beam



Vacuum Chamber Wake Fields

- Any variation in chamber profile, chamber material, or material properties perturbs this configuration.
- The beam loses part of its energy to establish EM (wake) fields that remain after the passage of the beam.

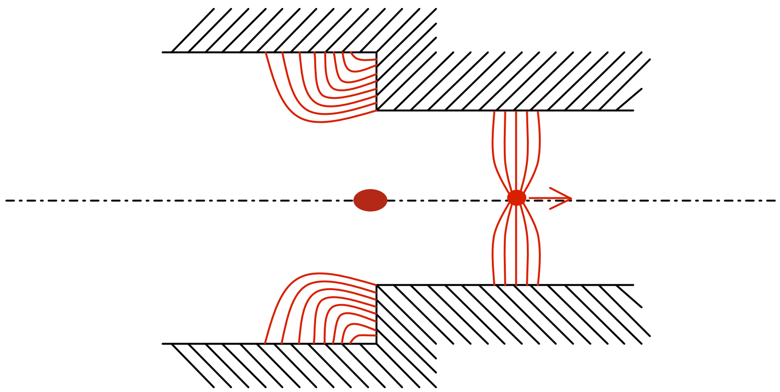


- By causality in the case of ultra-relativistic beams, chamber wakes can <u>only</u> affect trailing particles

The accelerator cavity is, by design, such a variation

Longitudinal wakes & beam loading

- If the structure is axisymmetric & if the beam passes on the axis of symmetry…



- … the force on axis can only be longitudinal

In a cavity the longitudinal wake (HOMs)
is closely related to beam loading via the cavity impedance

Fundamental theorem of beam loading

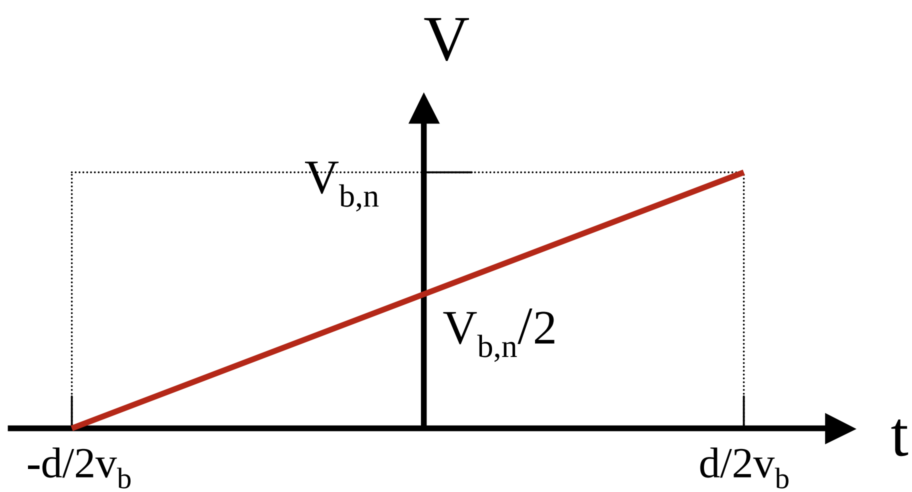
A point charge crosses a cavity initially empty of energy.

After the charge leaves the cavity, a beam-induced voltage $V_{b,n}$ remains in each mode.

By energy conservation the particle must have lost energy equal to the work done by the induced voltage on the charge

What fraction (f) of $V_{b,n}$ does the charge itself see?

The naïve guess is correct for any cavity

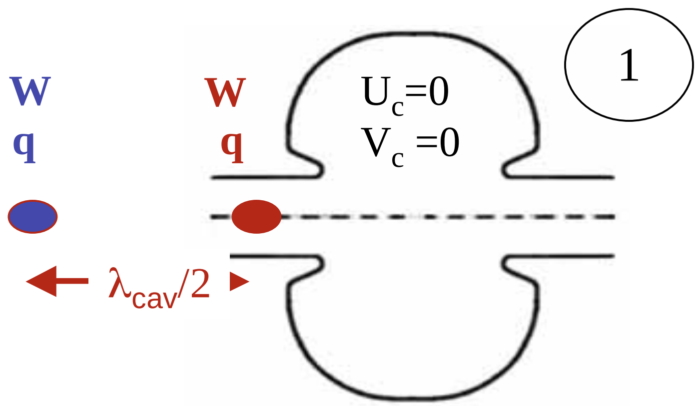


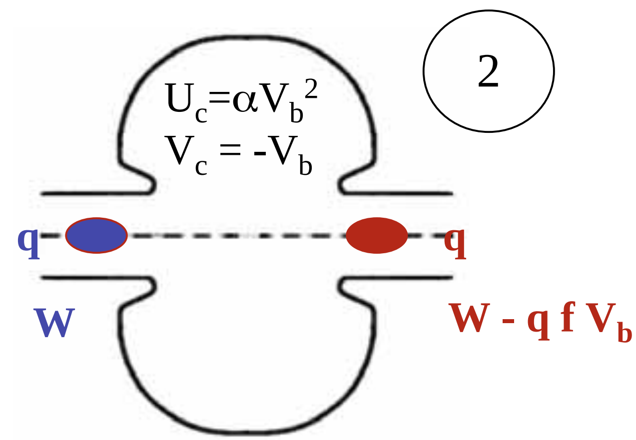
This theorem relates the energy loss by a charge passing through a structure to the electromagnetic properties of modes of that structure.

By superposition,
$V_{b,n}$ in a cavity is the same whether or not a generator voltage is present.

A simple proof

Half an rf period later, the voltage has changed in phase by π





For simplicity:

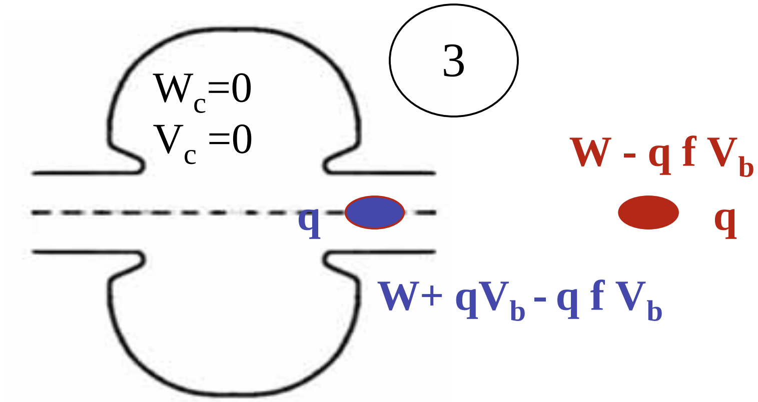
Assume that the change in energy of the particles does not appreciably change their velocity

Notice:

$$\boldsymbol{\alpha V_b^2 = q\,f\,V_b \implies V_b = q\,f / \alpha}$$

V_b is proportional to q

The simplest wakefield accelerator



Note that **the second charge** has gained energy

$$\Delta W = 1/2\ qV_b$$

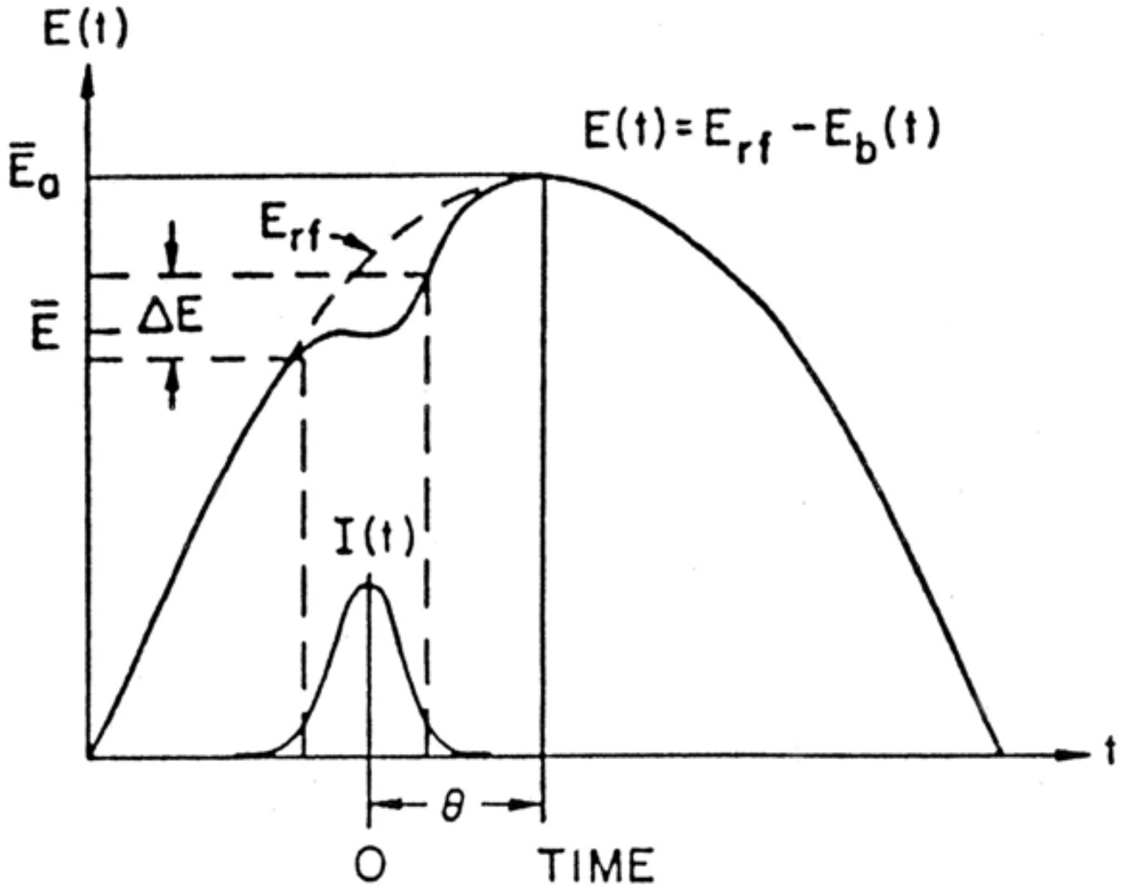
from longitudinal wake field of **the first charge**

By energy conservation:

$$W + qV_b - q\,fV_b + W - q\,fV_b = W + W$$

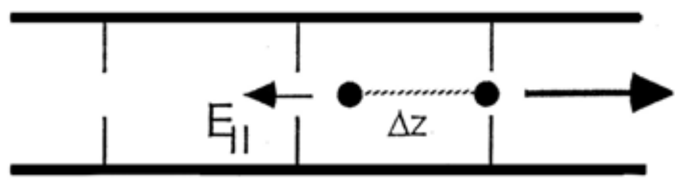
$$\Longrightarrow f = 1/2$$

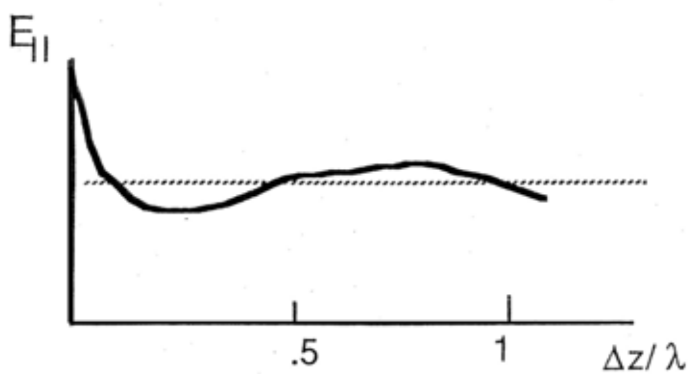
Beam loading lowers accelerating gradient

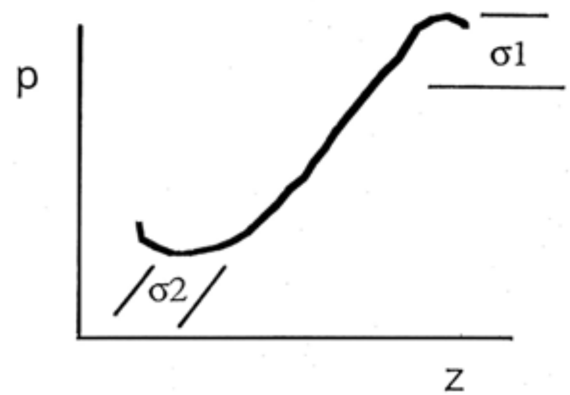


Locating the bunch at the best rf-phase minimizes energy spread

Longitudinal wake field determines the (minimum) energy spread







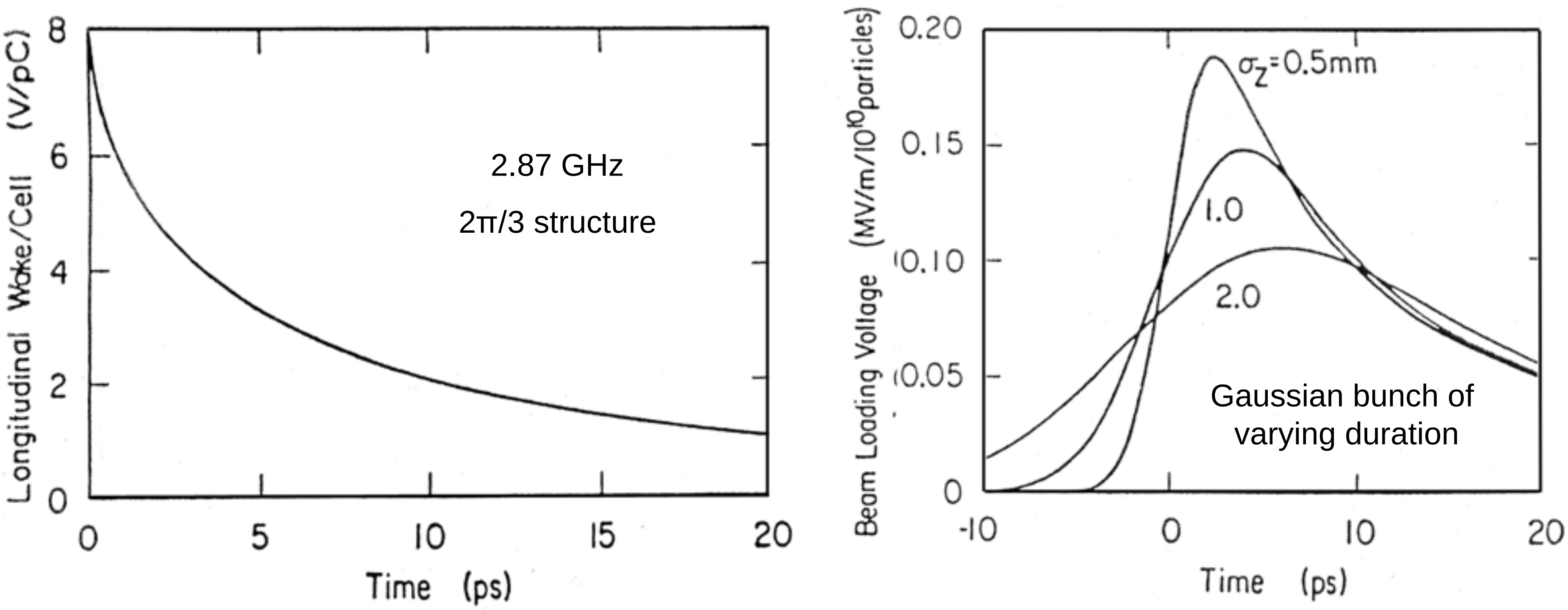
The wake potential, $W_{||}$ varies roughly linearly with distance, s, back from the head

$$W_{ll}(s) \approx W'_{ll} s$$

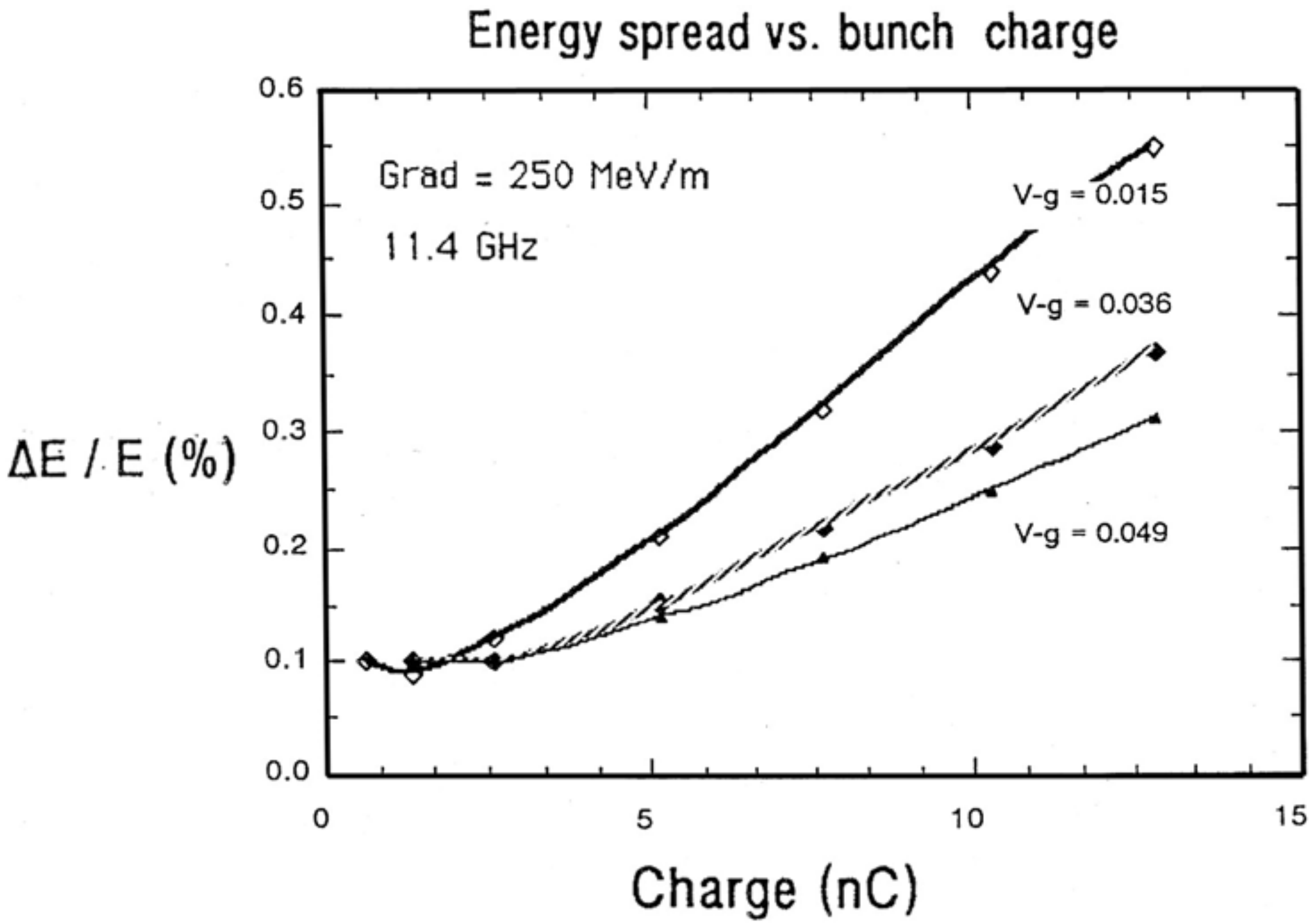
The energy spread per cell of length d for an electron bunch with charge q is

$$\Delta W_{ll}(s) \approx -qeW'_{ll} s_{tail}$$

Beam loading effects for the SLAC linac



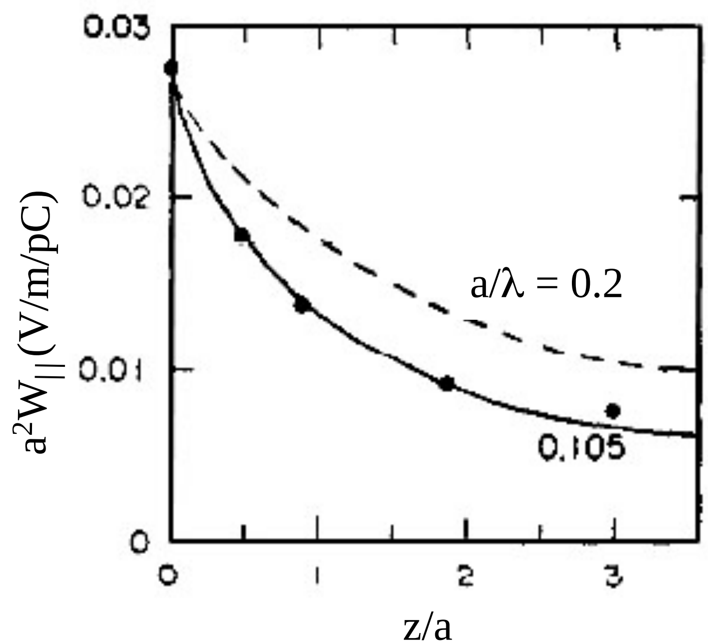
My calculation for a CLIC-like structure

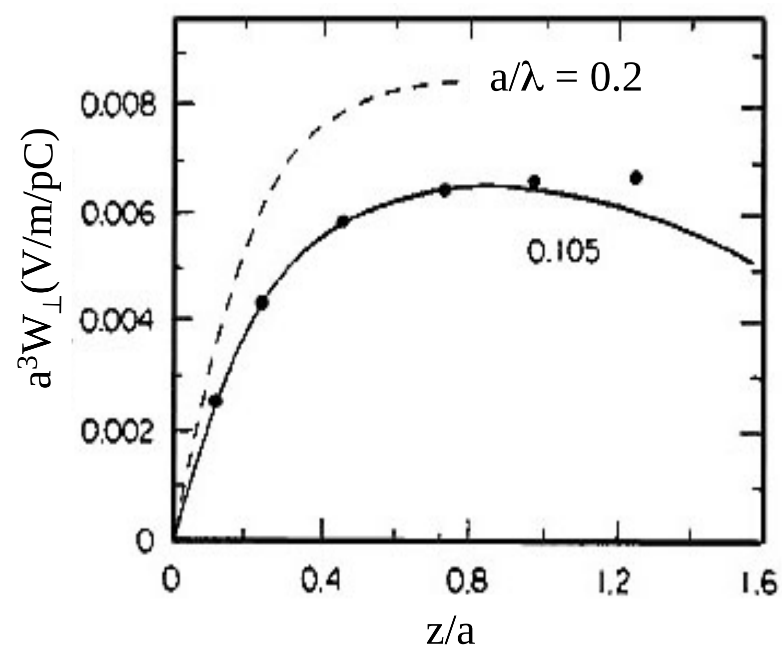


Scaling of wakefields with geometry & frequency in axisymmetric structures

For the disk-loaded waveguide structure (and typically)

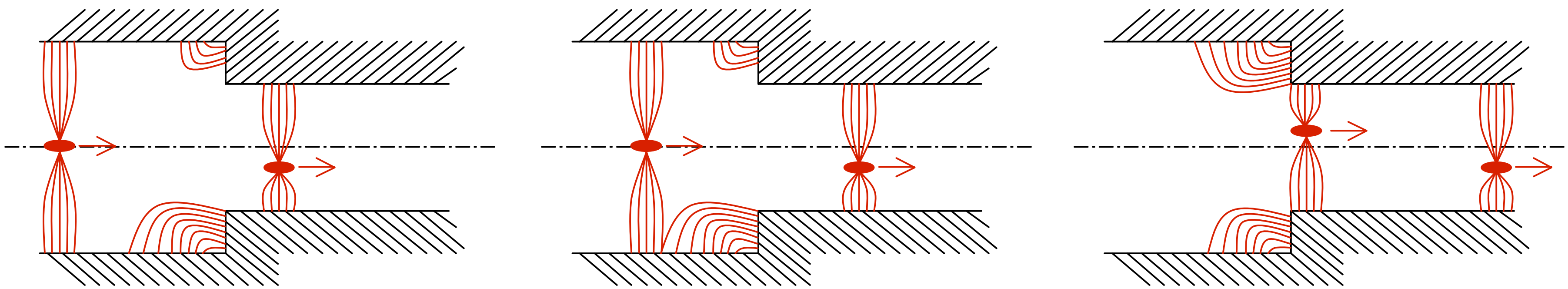
- Longitudinal wake field scales as $\boxed{a^{-2} \sim \lambda_{rf}^{-2}}$
- Transverse wakes scale as $\boxed{a^{-3} \sim \lambda_{rf}^{-3}}$





Wakes are transient fields generated during the beam passage

- Duration depends on the geometry & material of the structure
- Case 1: Wake persists for the duration of a bunch passage
  - ➙ Particles in the tail can interact with wakes due to particles in the head.
  - ➙ *Single bunch instabilities* can be triggered
    - (distortion of the longitudinal distribution, bunch lengthening, transverse instabilities)

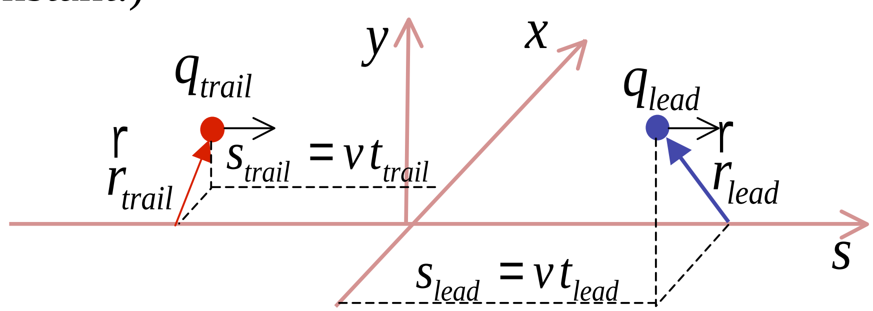


- Case 2: The wake field lasts longer than the time between bunches
  - ➙ Trailing bunches can interact with wakes from leading bunches to generate *multi-bunch bunch instabilities*

Wake Potentials

- Wake fields effects can be longitudinal or transverse.
  - ➙ Longitudinal wakes change the energy of beam particles
    - For longitudinal wakes it suffices to consider *only its electric field*
  - ➙ Transverse wakes affect beam particles' transverse momentum
- The wake potential is the energy variation induced by the wake field of the lead particle on a *unit charge* trailing particle

(Assume *v* constant.)



$$\boxed{V_W(\vec{r}_{lead}, \vec{r}_{trail}, t_{trail} - t_{lead}) = \int_{-\infty}^{\infty} \vec{E}_W(s, \vec{r}_{lead}, \vec{r}_{trail}, t_{trail} - t_{lead}) \cdot d\vec{s}}$$

Wake function is the wake potential induced per unit charge

$$W\left(\vec{r}_{lead}, \vec{r}_{trail}, t_{trail} - t_{lead}\right) = \frac{V_W\left(\vec{r}_{lead}, \vec{r}_{trail}, t_{trail} - t_{lead}\right)}{q_{lead}}$$

- For a bunch with charge distribution $i(\mathbf{r},t)$

$$\int i\left(\vec{r}, t\right)\, d\vec{r}\, dt = Nq$$

the total energy variation that the trailing particle experiences due to the whole bunch is

$$V\left(\vec{r}_{trail}, t_{trail}\right) = \int W\left(\vec{r}, \vec{r}_{trail}, t_{trail} - t\right) i\left(\vec{r}, t\right) d\vec{r}\, dt$$

- In real accelerators, the transverse beam size << chamber aperture.
  - ➙ It suffices to use the on-axis expression for the wakes (*monopole wake* approximation), using r and $r_{trail} = 0$ in the previous expressions.

Coupling Impedance

- The wake function describes the interaction of the beam with its external environment in the *time domain*

- The frequency domain "alter ego" of W is the coupling impedance (in Ohms) and defined as the *Fourier transform of the wake function*

$$\boxed{Z(\vec{r}, \vec{r}_{trail}, \omega) = \int_{-\infty}^{\infty} W(\vec{r}, \vec{r}_{trail}, \tau) e^{-j\omega\tau} d\tau \qquad with \quad \tau = t_{trail} - t}$$

- If I is the Fourier transform of the charge distribution, the Fourier transform of the total induced voltage is simply given by:

$$\tilde{V}(\vec{r}, \vec{r}_{trail}, \omega) = Z(\vec{r}, \vec{r}_{trail}, \omega)\; I(\vec{r}, \omega)$$

- Then

$$V(\vec{r}, \vec{r}_{trail}, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{V}(\vec{r}, \vec{r}_{trail}, \omega)\, e^{j\omega\tau}\, d\omega$$

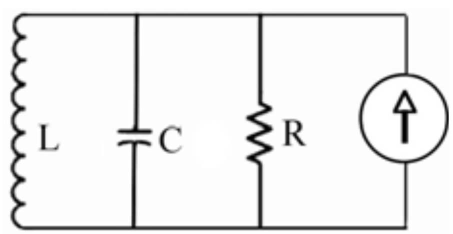
Interpretation of the coupling impedance

- The impedance is a complex quantity

$$Z(\vec{r}, \vec{r}_{trail}, \omega) = Z_R(\vec{r}, \vec{r}_{trail}, \omega) + j\, Z_j(\vec{r}, \vec{r}_{trail}, \omega)$$

  - → Z_R is responsible for the energy losses
  - → Z_j defines the phase between the beam response & exciting wake potential

- The impedance can be modeled by a parallel RLC model of the structure



$$Z(\omega) = \frac{R}{1 + jQ\left(\dfrac{\omega}{\omega_R} - \dfrac{\omega_R}{\omega}\right)}, \qquad \omega_R = \frac{1}{\sqrt{LC}}, \quad Q = R\sqrt{\frac{C}{L}}$$

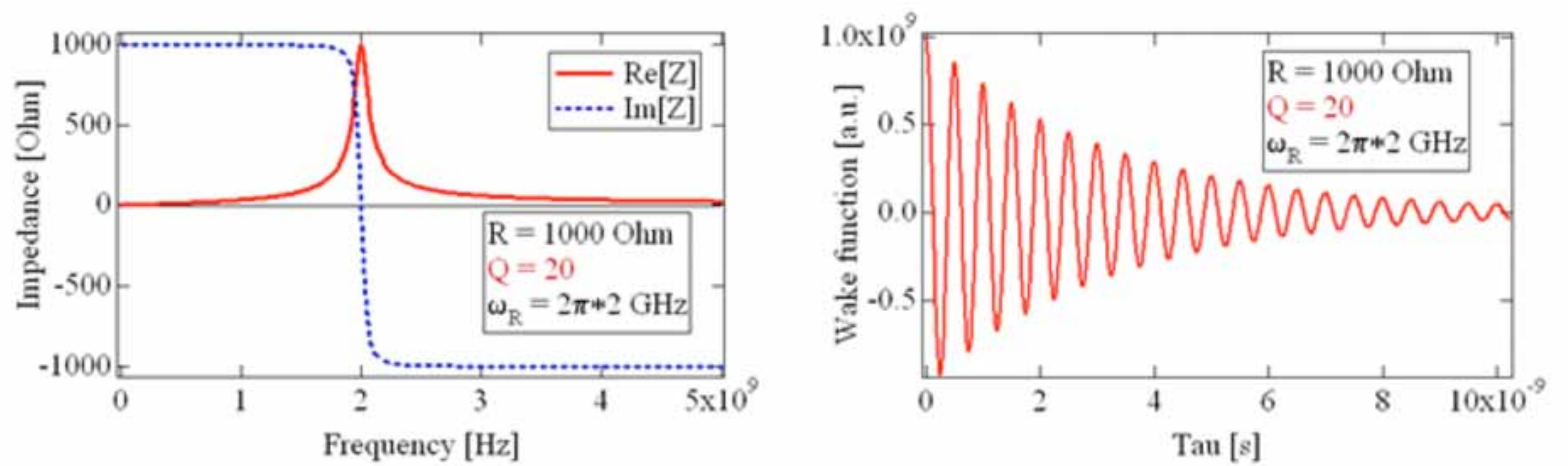
$$W(\tau) = \begin{cases} 0 & \tau < 0 \\ \dfrac{e^{-\omega_R \tau / 2Q}}{C}\left[\cos\left(\omega_R\, \tau\, \sqrt{1 - 1/4Q^2}\right) - \dfrac{\sin\left(\omega_R\, \tau\, \sqrt{1 - 1/4Q^2}\right)}{\sqrt{4Q^2 - 1}}\right] & \tau > 0 \end{cases}$$

Narrow-band coupling impedances

- Narrow-band modes are characterized by moderate Q & narrow spectrum

  ==> Associated wake lasts for a relatively long time

  ==> Capable of exciting multi-bunch instabilities



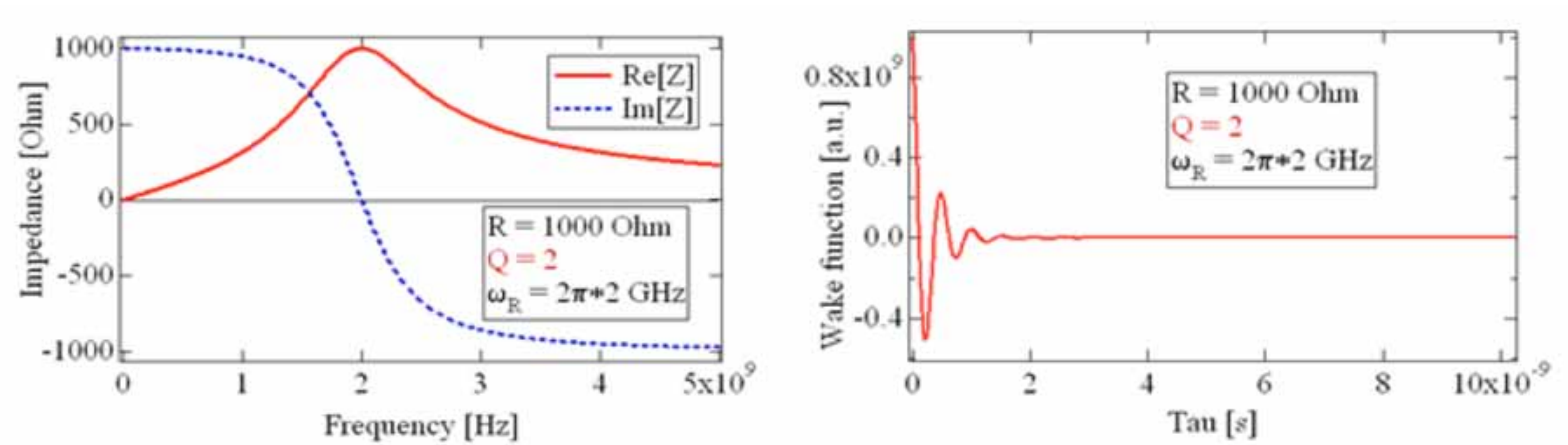
- Narrow band impedances are usually higher order modes of high Q accelerating structures

Broad band coupling impedances

- Broad-band impedance modes have a low Q and a broader spectrum.

  ==> The associated wake last for a relatively short time

  ==> Important only for single bunch instabilities



- Broad band impedances raise from irregularities or variations in the environment of the beam

Same approach applies to transverse wakes

- Transverse wake function is the transverse momentum kick per unit leading charge and unit trailing charge due to the wake fields

- Transverse wake fields are excited when the beam passes off center
  - For small displacements only the *dipole* term proportional to the displacement is important.
  - The *transverse dipole wake function* is the transverse wake function per unit displacement

- The transverse coupling impedance is defined as the Fourier transform of the transverse wake function times *j*

- Longitudinal and transverse wakes represent the same 3D wake field
  - Linked by Maxwell's equations.
  - The Panofsky-Wenzel relations allow one to calculate one wake component when the other is known.

Wakefields in real accelerators

- Accelerator vacuum chambers have complex shapes that include many components that can potentially host wake fields
- Not all wakes excited by the beam can be trapped in the chamber
- Given a chamber geometry, ∃ a cutoff frequency, f_{cutoff}
  - → Modes with frequency > f_{cutoff} propagate along the chamber

$$f_{Cutoff} \approx \frac{c}{b} \qquad where\ \ b \equiv transverse\ chamber\ size$$

Categories of beam-induced wakefields

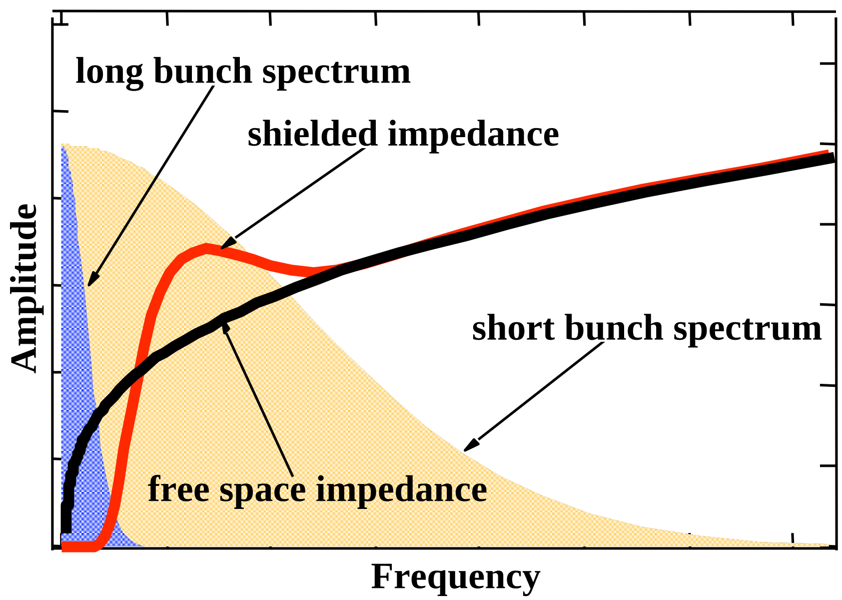
1. Wake fields that travels with the beam (e.g., the space charge)

2. Wake fields that are localized in some parts of the vacuum chamber (narrow and broad band

3. High frequency wakes > f_{cutoff} propagate inside the vacuum chamber.
   - ➙ Do not generate net interaction with the beam as long as they are not synchronous with the beam
   - ➙ A special case is synchrotron radiation which will be discussed later

When are Wakefields Dangerous?

- A wake is potentially dangerous only if it can be excited by the beam. Then, $V_{wake} \sim I_{bunch}$
  - → If V_{wake} exceeds a threshold, it will trigger an instability
    - single bunch instability for broadband impedances
    - coupled bunch instability for narrowband impedances



- Impedance & beam power spectrum must overlap to allow energy transfer from beam to wake & conversely
- The larger the overlap the more dangerous is the wake
- Short bunches have a broader power spectrum than longer ones
  - → bigger overlap with a wake impedance

Examples in linear accelerators

Even smooth structures can have wakes that can destabilize beams

Consider a long pulse of e^- moving through a smooth pipe of infinite σ.

The focusing magnets give a beam a periodic motion transversely with wave number, k_β.

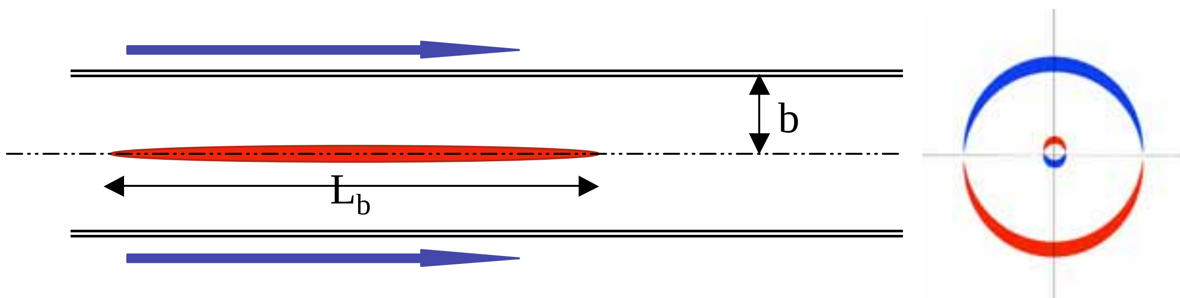


Image charges act to center the beam

Image currents attract the beam to the wall

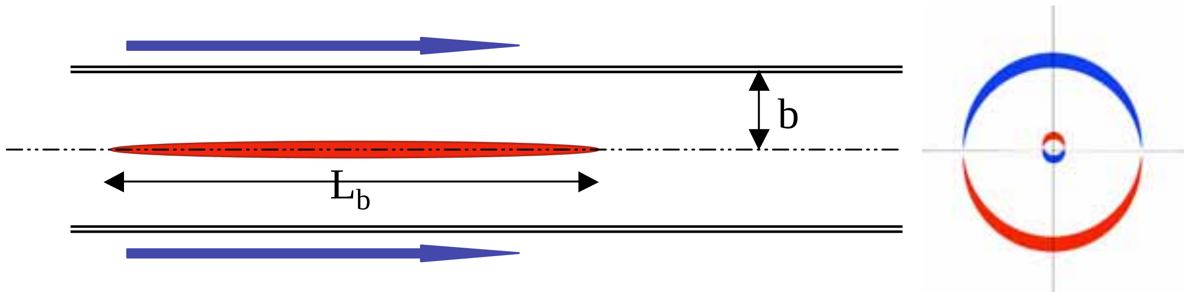
The forces cancel to a factor γ^{-2}

If the beam is off-center, focusing keeps its transverse motion bounded.

Transverse resistive wall instability

Now let the smooth pipe have finite conductivity, σ

As the pulse travels the image current diffuses into the pipe ~ $\sigma^{1/2}$



At a distance z along the pipe the initial displacement will grow as

$$\sim \exp\left[(z/L_{tr})^{2/3}\right]$$

$$L_{tr} = \frac{2\gamma\beta I_A}{I}\sqrt{\frac{\pi\sigma_{pipe}}{\tau_{pulse}}}\frac{k_\beta b^3}{c}$$

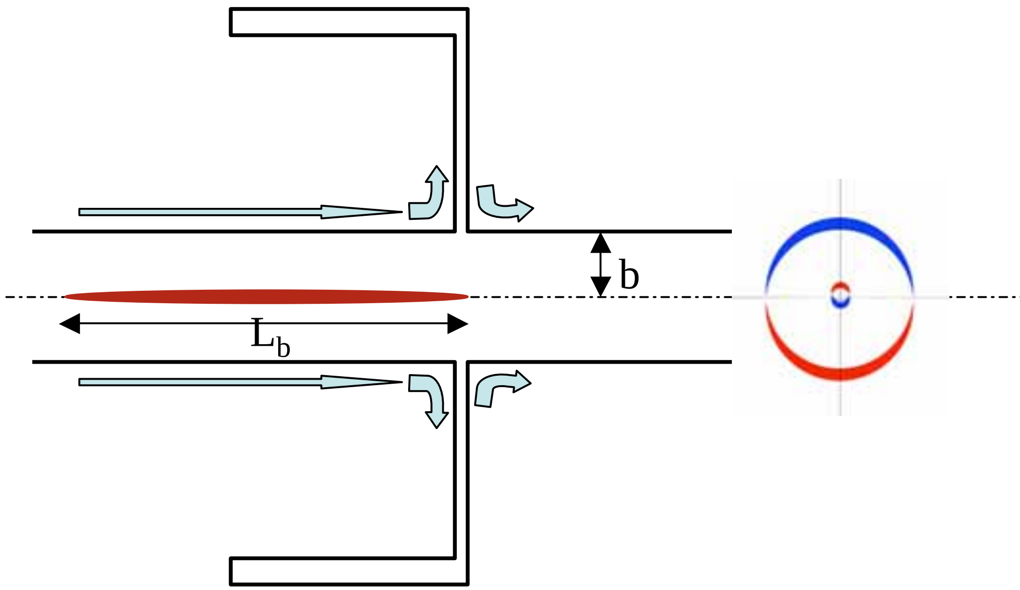
G. Caporaso, W. A. Barletta, V.K. Neil, Part. Accel., **11**, 71 (1980)

Simple example from induction linacs: Image Displacement Instability

Now add a accelerating gaps

At the gap, E_{image} is only slightly perturbed; the image current moves far away.

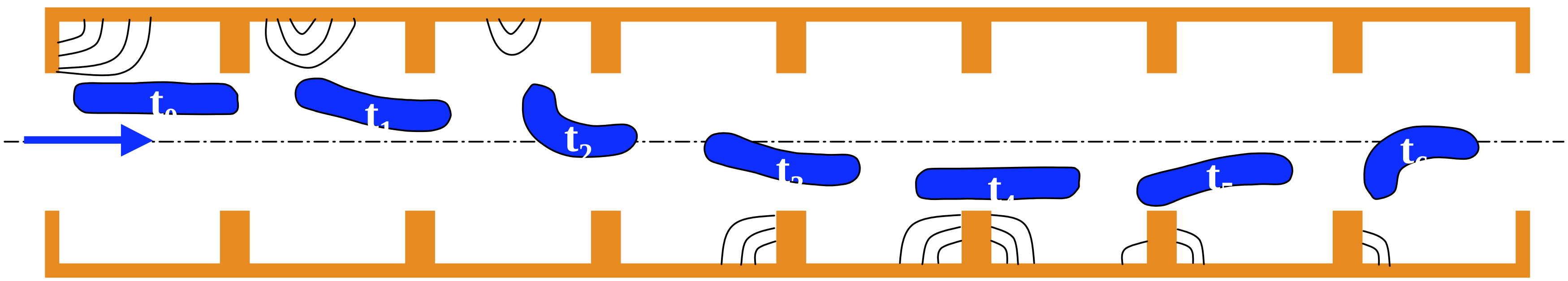
Therefore, the restoring magnetic force is absent at the gap



The displacement will grow exponentially even if σ is infinite

Beam Breakup Instability: High frequency version in rf-accelerators

- Bunch enters off-axis in a linac structure ==> transverse wakes
- Transverse wakes from the bunch head deflect the tail of the bunch
- In long linacs with high I_{bunch}, the effect amplifies distorting the bunch into a “banana” like shape. *(Single-bunch beam break up)*



Snapshots of a single bunch traversing a SLAC structure

- First observed in the 2-mile long SLAC linac

The rest is Pathology