Unit 4 - Lecture 9
RF-accelerators: RF-cavities

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RF-cativties for acceleration

Microtron

Synchrotron

Linac
S-band (~3 GHz) RF linac
RF cavities: Basic concepts

- Fields and voltages are complex quantities.
  - For standing wave structures use phasor representation
    \[ \tilde{V} = Ve^{i\omega t} \]
    where \( V = |\tilde{V}| \)
  - At \( t = 0 \) particle receives maximum voltage gain

\[ \text{zo is the reference plane} \]

- For cavity driven externally, phase of the voltage is
  \[ \theta = \omega t + \theta_o \]
- For electrons \( v \approx c \); therefore \( z = z_o + ct \)
Basic principles and concepts

- Superposition
- Energy conservation
- Orthogonality (of cavity modes)
- Causality
If you can kick the beam, the beam can kick you

\[ \text{Total cavity voltage} = V_{\text{generator}} + V_{\text{beam-induced}} \]

\[ \text{Fields in cavity} = E_{\text{generator}} + E_{\text{beam-induced}} \]
**Basic principles: Energy conservation**

- Total energy in the particles and the cavity is conserved
  - Beam loading

\[ W_c = U_i - U_f \]
Basics: Orthogonality of normal modes

- Each mode in the cavity can be treated independently in computing fields induced by a charge crossing the cavity.

- The total stored energy is equals the sum of the energies in the separate modes.

- The total field is the phasor sum of all the individual mode fields at any instant.
There can be no disturbance ahead of a charge moving at the velocity of light.

In a mode analysis of the growth of the beam-induced field, the field must vanish ahead of the moving charge for each mode.
Example: Differential superposition

A point charge \( q \) induces a voltage \( V_0 \) passing through a cavity, what voltage is induced by a Gaussian bunch of charge \( q \)?

A differential charge induces the differential voltage

\[
d\tilde{V} = \tilde{V}_o \frac{dq}{q} = V_0 e^{j\omega_0 t} \frac{dq}{q}
\]

Say \( dq \) passes \( z = 0 \) at \( t_o \); at time \( t \) the induced voltage will be

\[
d\tilde{V} = \frac{V_o}{q} e^{j\omega_0 (t-t_o)} dq(t_o)
\]

The bunch has a Gaussian distribution in time

\[
dq(t_o) = \frac{q}{\sqrt{2\pi\sigma}} e^{-t_o^2/(2\sigma^2)} dt_o
\]

\[
V = V_0 e^{j\omega_0 t} e^{-\omega_o^2\sigma^2/2} dt_o
\]
Basic components of an RF cavity

Outer region: Large, single turn Inductor

Central region: Large plate Capacitor

Power feed from rf - generator

Beam (Load) current

Wall current

Displacement current
Lumped circuit analogy of resonant cavity

\[ Z(\omega) = \left[ j\omega C + (j\omega L + R)^{-1} \right]^{-1} \]

\[
Z(\omega) = \frac{1}{j\omega C + (j\omega L + R)^{-1}} = \frac{(j\omega L + R)}{(j\omega L + R) j\omega C + 1} = \frac{(j\omega L + R)}{(1 - \omega^2 LC) + j\omega RC}
\]

The resonant frequency is \( \omega_o = \frac{1}{\sqrt{LC}} \)
Converting the denominator of $Z$ to a real number we see that

$$|Z(\omega)| \sim \left[\left(1 - \frac{\omega^2}{\omega_o^2}\right)^2 + (\omega RC)^2\right]^{-1}$$

The width is

$$\frac{\Delta \omega}{\omega_0} = \frac{R}{\sqrt{L/C}}$$
More basics from circuits - $Q$

$Q = \frac{\omega_0 \circ \text{Energy stored}}{\text{Time average power loss}} = \frac{2\pi \circ \text{Energy stored}}{\text{Energy per cycle}}$

$E = \frac{1}{2} LI_o I_o^*$ and $\langle P \rangle = \langle i^2(t) \rangle R = \frac{1}{2} I_o I_o^* R_{\text{surface}}$

$\therefore Q = \frac{\sqrt{L/C}}{R} = \left( \frac{\Delta \omega}{\omega_o} \right)^{-1}$
Translate circuit model to a cavity model:
Directly driven, re-entrant RF cavity

Outer region: Large, single turn Inductor

\[ L = \frac{\mu_0 \pi a^2}{2\pi(R + a)} \]

Central region: Large plate Capacitor

\[ C = \varepsilon_0 \frac{\pi R^2}{d} \]

\[ \omega_o = \frac{1}{\sqrt{LC}} = \frac{c}{\sqrt{c}} \left[ \frac{2((R + a)d)}{\pi R^2 a^2} \right]^{1/2} \]

Q – set by resistance in outer region

\[ Q = \frac{\sqrt{L/C}}{R} \]

Expanding outer region raises Q

Narrowing gap raises shunt impedance

Source: Humphries, Charged Particle Accelerators
Properties of the RF pillbox cavity

- We want lowest mode: with only $E_z$ & $B_\theta$
- Maxwell’s equations are:

$$\frac{1}{r} \frac{\partial}{\partial r} (r B_\theta) = \frac{1}{c^2} \frac{\partial}{\partial t} E_z \quad \text{and} \quad \frac{\partial}{\partial r} E_z = \frac{\partial}{\partial t} B_\theta$$

- Take derivatives

$$\frac{\partial}{\partial t} \left[ \frac{1}{r} \frac{\partial}{\partial r} (r B_\theta) \right] = \frac{\partial}{\partial t} \left[ \frac{\partial B_\theta}{\partial r} + \frac{B_\theta}{r} \right] = \frac{1}{c^2} \frac{\partial^2 E_z}{\partial t^2}$$

$$\frac{\partial}{\partial r} \frac{\partial E_z}{\partial r} = \frac{\partial}{\partial r} \frac{\partial B_\theta}{\partial t}$$

$$\implies \quad \frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r} \frac{\partial E_z}{\partial r} = \frac{1}{c^2} \frac{\partial^2 E_z}{\partial t^2}$$
For a mode with frequency $\omega$

\begin{align*}
&
E_z(r, t) = E_z(r) \ e^{i\omega t} \\
\Rightarrow \quad &
E''_z + \frac{E'_z}{r} + \left( \frac{\omega}{c} \right)^2 E_z = 0 \\
\Rightarrow \quad & \text{(Bessel’s equation, 0 order)} \\
&
\text{Hence,} \quad E_z(r) = E_o \ J_0 \left( \frac{\omega}{c} r \right) \\
\Rightarrow \quad & \text{For conducting walls, } E_z(R) = 0, \text{ therefore} \\
&
\frac{2\pi f}{c} b = 2.405
\end{align*}
E-fields & equivalent circuit: $T_{on1o}$ mode

![Graph showing relative intensity vs. $r/R$ with symbols $E_z$, $B_\theta$, $T_{010}$ and a circuit diagram with a capacitor $C$ and an inductor $L$.]
E-fields & equivalent circuits for $T_{020}$ modes
\( T_{0n0} \) has 
\( n \) coupled, resonant 
circuits; each L & C 
reduced by \( 1/n \)
Simple consequences of pillbox model

- Increasing $R$ lowers frequency
  \[ \implies \text{Stored Energy, } \mathcal{E} \sim \omega^{-2} \]

- \[ \mathcal{E} \sim E_z^2 \]

- Beam loading lowers $E_z$ for the next bunch

- Lowering $\omega$ lowers the fractional beam loading

- Raising $\omega$ lowers $Q \sim \omega^{-1/2}$

- If time between beam pulses,
  \[ T_s \sim \frac{Q}{\omega} \]
  almost all $\mathcal{E}$ is lost in the walls
The beam tube makes the field modes (& cell design) more complicated

- Peak E no longer on axis
  - \( E_{pk} \sim 2 - 3 \times E_{acc} \)
  - FOM = \( E_{pk}/E_{acc} \)
- \( \omega_o \) more sensitive to cavity dimensions
  - Mechanical tuning & detuning
- Beam tubes add length & \( \epsilon \)'s w/o acceleration
- Beam induced voltages \( \sim a^{-3} \)
  - Instabilities
Cavity figures of merit
Make the linac with a series of pillbox cavities

Power the cavities so that $E_z(z,t) = E_z(z)e^{i\omega t}$
The voltage varies during time that bunch takes to cross gap

→ reduction of the peak voltage by \( \Gamma \) (transit time factor)

\[
\Gamma = \frac{\sin\left(\frac{\varphi}{2}\right)}{\frac{\varphi}{2}} \quad \text{where} \quad \varphi = \frac{\omega d}{\beta c}
\]

For maximum acceleration

\[
T_{\text{cav}} = \frac{d}{c} = \frac{T_{\text{rf}}}{2} \quad \Rightarrow \quad \Gamma = \frac{2}{\pi}
\]
Figure of merit from circuits - $Q$

\[ Q = \frac{\omega_o \circ \text{Energy stored}}{\text{Time average power loss}} = \frac{2\pi \circ \text{Energy stored}}{\text{Energy lost per cycle}} \]

\[ E = \frac{\mu_o}{2} \int |H|^2 dv = \frac{1}{2} L I_o I_o^* \]

\[ \langle \mathcal{P} \rangle = \frac{R_{surf}}{2} \int |H|^2 ds = \frac{1}{2} I_o I_o^* R_{surf} \]

\[ R_{surf} = \frac{1}{\text{Conductivity} \circ \text{Skin depth}} \sim \omega^{1/2} \]

\[ \therefore \quad Q = \frac{\sqrt{L/C}}{R_{surf}} = \left( \frac{\Delta \omega}{\omega_o} \right)^{-1} \]
Measuring the energy stored in the cavity allows us to measure

- We have computed the field in the fundamental mode
  \[ U = \int_0^d dz \int_0^b dr 2\pi r \left( \frac{\varepsilon E_o^2}{2} \right) J_1^2(2.405 r/b) \]
  \[ = b^2 d \left( \varepsilon E_o^2 / 2 \right) J_1^2(2.405) \]

- To measure Q we excite the cavity and measure the E field as a function of time

- Energy lost per half cycle = \( U\pi Q \)

- Note: energy can be stored in the higher order modes that deflect the beam
Keeping energy out of higher order modes

Choose cavity dimensions to stay far from crossovers
Resistive input (shunt) impedance at $\omega_o$ relates power dissipated in walls to accelerating voltage

$$R_{in} = \frac{\langle V^2(t) \rangle}{\mathcal{P}} = \frac{V_o^2}{2\mathcal{P}} = Q\sqrt{\frac{L}{C}}$$

Linac literature commonly defines “shunt impedance” without the “2”

$$\mathcal{R}_{in} = \frac{V_o^2}{\mathcal{P}} \sim \frac{1}{R_{surf}}$$

Typical values 25 - 50 MΩ
Computing shunt impedance

\[ R_{in} = \frac{V_o^2}{\mathcal{P}} \]

\[ \langle \mathcal{P} \rangle = \frac{R_{surf}}{2} \int_s |H|^2 ds \]

\[ R_{surf} = \frac{\mu \omega}{2\sigma_{dc}} = \pi Z_o \frac{\delta_{skin}}{\lambda_{rf}} \quad \text{where} \quad Z_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} = 377 \Omega \]

The on-axis field \( E \) and surface H are generally computed with a computer code such as SUPERFISH for a complicated cavity shape.
The voltage gain seen by the beam can be computed in the co-moving frame, or we can use the transit-time factor, $\Gamma$ & compute $V$ at fixed time

$$V_o^2 = \Gamma \int_{z_1}^{z_2} E(z)dz$$
Exercise: Pillbox array

Derive the Q and $R_{sh}$ for the pillbox cavity as a function of the dimensions of the cavity and the frequency of the fundamental mode.
Note on previous slide

\[ Z(\omega) = \frac{1}{j\omega C + (j\omega L + R)^{-1}} = \frac{(j\omega L + R)}{(j\omega L + R)j\omega C + 1} = \frac{(j\omega L + R)}{1 - \omega^2 LC + j\omega RC} \]

\[ = \frac{(j\omega L + R)}{(1 - \frac{\omega^2}{\omega_o^2})^2 + j\omega RC} = \frac{(j\omega L + R)[1 - \frac{\omega^2}{\omega_o^2} - j\omega RC]}{(1 - \frac{\omega^2}{\omega_o^2})^2 + (\omega RC)^2} = \frac{j\omega \left[ L \left( 1 - \frac{\omega^2}{\omega_o^2} \right) - R^2 C \right] + R \left( 1 - \frac{\omega^2}{\omega_o^2} \right) + \frac{\omega^2}{\omega_o^2} R}{(1 - \frac{\omega^2}{\omega_o^2})^2 + (\omega RC)^2} \]

\[ = \frac{1}{(1 - \frac{\omega^2}{\omega_o^2})^2 + (\omega RC)^2} \left[ R + j\omega \left( L \left( 1 - \frac{\omega^2}{\omega_o^2} \right) - R^2 C \right) \right] \]

\[ \Rightarrow |Z| = \frac{1}{\left(1 - \frac{\omega^2}{\omega_o^2}\right)^2 + (\omega RC)^2} \left[ R^2 + \omega^2 \left( L \left( 1 - \frac{\omega^2}{\omega_o^2}\right) - R^2 C \right)^2 \right]^{1/2} \]