Unit 9 - Lecture 18

Deviations from the design orbit

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Off- momentum particles
&
Momentum dispersion
Momentum dispersion function of the lattice

- *Off-momentum* particles undergo betatron oscillations about a new class of closed orbits in circular accelerators.

- Orbit displacement arises from dipole fields that establish the ideal trajectory + less effective quadrupole focusing.
We have derived
\[ \frac{d^2x}{ds^2} - \frac{\rho + x}{\rho^2} = -\frac{B_y}{(B\rho)} \left(1 + \frac{x}{\rho}\right)^2 \]

Using \( p = (B\rho) \)
\[ \frac{d^2x}{ds^2} - \frac{\rho + x}{\rho^2} = -\frac{B_y}{(B\rho)_{\text{design}}} \left(1 + \frac{x}{\rho}\right)^2 \frac{p_o}{p} \]

Consider fields that vary linearly with transverse position
\[ B_y = B_o + B'x \]

Then neglecting higher order terms in \( x/\rho \) we have
\[ \frac{d^2x}{ds^2} + \left[ \frac{1}{\rho^2} \frac{2p_o - p}{p} + \frac{B'}{(B\rho)_{\text{design}}} \frac{p_o}{p} \right] x = \frac{1}{\rho} \frac{p - p_o}{p} \equiv \frac{1}{\rho} \Delta p \]
Equation for the dispersion function

Define $D(x,s)$ such that $x = D(x,s) \left( \Delta p/p_o \right)$

Look for a closed periodic solution; $D(x,s+L) = D(x,s)$ of the inhomogeneous Hill’s equation

\[
\frac{d^2 D}{ds^2} + \left[ \frac{1}{\rho^2} \frac{2p_o - p}{p} + \frac{B'}{(B\rho)_{\text{design}}} \frac{p_o}{p} \right] D = \frac{1}{\rho} \frac{p_o}{p}
\]

For a piecewise linear lattice the general solution is

\[
\begin{pmatrix} D \\ D' \end{pmatrix}_{\text{out}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} D \\ D' \end{pmatrix}_{\text{in}} + \begin{pmatrix} e \\ f \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} D \\ D' \end{pmatrix}_{\text{out}} = \begin{pmatrix} a & b & e \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D \\ D' \end{pmatrix}_{\text{in}}
\]
Solution for D

- The solution for the homogeneous portion is the same as that for $x$ and $x'$

- The values of $M_{13}$ and $M_{23}$ for ranges of $K$ are

<table>
<thead>
<tr>
<th>$K$</th>
<th>$e$</th>
<th>$f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt; 0$</td>
<td>$\frac{e}{p</td>
<td>K</td>
</tr>
<tr>
<td>$0$</td>
<td>$\frac{1}{2} \frac{eB_o l}{p}$</td>
<td>$\frac{eB_o l}{p}$</td>
</tr>
<tr>
<td>$&gt; 0$</td>
<td>$\frac{e}{pK}B_o\left[1 - \cos(\sqrt{K} l)\right]$</td>
<td>$\frac{e}{p\sqrt{K}}B_o\left[\sin(\sqrt{K} l)\right]$</td>
</tr>
</tbody>
</table>
What is the shape of D?

- In the drifts $D'' = 0$
  - $\Rightarrow$ D has a constant slope

- For focusing quads, $K > 0$
  - $\Rightarrow$ D is sinusoidal

- For defocusing quads, $K < 0$
  - $\Rightarrow$ D grows (decays) exponentially

- In dipoles, $K_x(s) = G^2$
  - $\Rightarrow$ D is sinusoidal section “attracted to” $D = 1/G = \rho$
SPEAR-I dispersion

From: Sands SLAC - pub 121
The condition for the achromatic cell

We want to start with zero dispersion and end with zero dispersion.

This requires

\[ I_a = \int_0^s a(s) \frac{ds}{\rho(s)} = 0 \]

and

\[ I_b = \int_0^s b(s) \frac{ds}{\rho(s)} = 0 \]

In the DBA this requires adjusting the center quad so that the phase advance through the dipoles is \( \pi \).
Momentum compaction

Consider bending by sector magnets

The change in the circumference is

$$\Delta C = \oint \left( \rho + D \frac{\Delta p}{p_0} \right) d\theta - \oint \rho d\theta$$

Therefore

$$\frac{\Delta C}{C} = \oint \frac{D}{\rho} \frac{ds}{p_0} \frac{\Delta p}{p_0} = \left\langle \frac{D}{\rho} \right\rangle \frac{\Delta p}{p_0} \quad \text{or} \quad \alpha = \left\langle \frac{D}{\rho} \right\rangle = \frac{1}{\gamma_t}$$

For simple lattices $\gamma_t \sim Q \sim$ number of cells of an AG lattice
Total beam size due to betatron oscillations plus momentum spread.

- Displacement from the ideal trajectory of a particle
  - First term = increment to closed orbit from off-momentum particles
  - Second term = free oscillation about the closed orbit

\[ x_{total} = D \frac{\Delta p}{p_o} + x_\beta \]

- Average the square of \( x_{total} \) to obtain the rms displacement

\[ \sigma_x^2(s) = \frac{\varepsilon \beta(s)}{\pi} + D^2(s) \left\langle \left( \frac{\Delta p}{p_o} \right)^2 \right\rangle \]

- \( \therefore \) in a collider, design for \( D = 0 \) in the interaction region
The focusing strength of a quadrupole depends on the momentum of the particle

\[
\frac{1}{f} \propto \frac{1}{p}
\]

=> Off-momentum particles oscillate around a chromatic closed orbit NOT the design orbit

Deviation from the design orbit varies linearly as

\[
x_D = D(s) \frac{\Delta p}{p}
\]

The tune depends on the momentum deviation

\[Q'_x = \frac{\Delta Q}{\Delta p / p_o}\] or \[\xi_x = \frac{\Delta Q_x / Q_x}{\Delta p / p_o}\]

\[Q'_y = \frac{\Delta Q}{\Delta p / p_o}\] or \[\xi_x = \frac{\Delta Q_y / Q_y}{\Delta p / p_o}\]
Example of chromatic aberation
Chromatic aberration in muon collider ring

From: Alex Bogacz and Hisham Sayed presentation
The uncorrected, "natural" chromaticity is negative & can lead to a large tune spread and consequent instabilities

→ Correction with sextupole magnets

\[ \xi_{\text{natural}} = -\frac{1}{4\pi} \oint \beta(s)K(s)ds \approx -1.3Q \]
Measurement of chromaticity

* Steer the beam to a different mean radius & different momentum by changing rf frequency, \( f_a \), & measure \( Q \)

\[
\Delta f_a = f_a \eta \frac{\Delta p}{p} \quad \text{and} \quad \Delta r = D_{av} \frac{\Delta p}{p}
\]

* Since \( \Delta Q = \xi \frac{\Delta p}{p} \)

\[
\therefore \quad \xi = f_a \eta \frac{dQ}{df_a}
\]
Chromaticity correction with sextupoles

From: Wiedemann, Ch. 7, v.1
Sextupole correctors

- Placing sextupoles where the betatron function is large, allows weak sextupoles to have a large effect

- Sextupoles near F quadrupoles where $\beta_x$ is large affect mainly horizontal chromaticity

- Sextupoles near D quadrupoles where $\beta_y$ is large affect mainly horizontal chromaticity
Rotated quadrupoles & misalignments can couple the motion in the horizontal & vertical planes

A small rotation can be regarded a normal quadrupole followed by a weaker quad rotated by 45°

\[ B_{s,x} = \frac{\partial B_x}{\partial y} x \quad \text{and} \quad B_{s,y} = \frac{\partial B_y}{\partial x} y \]

→ This leads to a vertical deflection due to a horizontal displacement

Without such effects \( D_y = 0 \)

In electron rings vertical emittance is caused mainly by coupling or vertical dispersion
Field errors & Resonances
Integer Resonances

- Imperfections in dipole guide fields perturb the particle orbits
  - Can be caused by off-axis quadrupoles
- => Unbounded displacement if the perturbation is periodic
- The motion is periodic when
  \[ mQ_x + nQ_y = r \]
  \( M, n, & r \) are small integers
**Effect of steering errors**

- The design orbit \((x = 0)\) is no longer a possible trajectory.
- Small errors \(\Rightarrow\) a new closed orbit for particles of the nominal energy.
- Say that a single magnet at \(s = 0\) causes an orbit error \(\theta\).
  \[
  \theta = \frac{\Delta B l}{(B \rho)}
  \]
- Determine the new closed orbit.

![Diagram showing the effect of steering errors with orbit deviations and an integral path.](image)
After the steering impulse, the particle oscillates about the design orbit

- At \( s = 0^+ \), the orbit is specified by \((x_o, x'_o)\)

- Propagate this around the ring to \( s = 0^- \) using the transport matrix & close the orbit using \((0, \theta)\)

\[
\begin{pmatrix}
M 
\end{pmatrix}
\begin{pmatrix}
x_o \\
x'_o \\
\end{pmatrix}
+
\begin{pmatrix}
0 \\
\theta \\
\end{pmatrix}
=
\begin{pmatrix}
x_o \\
x'_o \\
\end{pmatrix}
\]

specifies the new closed orbit

\[
\begin{pmatrix}
x_o \\
x'_o \\
\end{pmatrix}
=
(I - M)^{-1}
\begin{pmatrix}
0 \\
\theta \\
\end{pmatrix}
\]
As \((\Delta \phi)_{\text{ring}} = Q\), \(M\) can be written as

\[
M_{\text{ring}} = \begin{pmatrix}
\cos(2\pi Q) + \alpha \sin(2\pi Q), & \beta \sin(2\pi Q) \\
-\gamma \sin(2\pi Q), & \cos(2\pi Q) - \alpha \sin(2\pi Q)
\end{pmatrix}
\]

After some manipulation (see Syphers or Sands)

\[
x(s) = \frac{\theta \beta^{1/2}(s) \beta^{1/2}(0)}{2 \sin \pi Q} \cos(\phi(s) - \pi Q)
\]

As \(Q\) approaches an integer value, the orbit will grow without bound.
The operating point of the lattice in the horizontal and vertical planes is displayed on the tune diagram.

The lines satisfy:

\[ mQ_x + nQ_y = r \]

\( M, n, \text{ and } r \) are small integers.

Operating on such a line leads to resonant perturbation of the beam.

Smaller \( m, n, \text{ and } r \) => stronger resonances.
Example: Quadrupole displacement in the Tevatron

❖ Say a quad is horizontally displaced by an amount $\delta$
  → Steering error, $\Delta x' = \delta/F$ where $F$ is the focal length of the quad

❖ For Tevatron quads $F \approx 25$ m & $Q = 19.4$. Say we can align the quads to the center line by an rms value 0.5 mm
  → For $\delta = 0.5$ mm $\implies \theta = 20$ $\mu$rad
  → If $\beta = 100$ m at the quad, the maximum closed orbit distortion is

\[
\Delta \hat{x}_{\text{quad}} = \frac{20 \ \mu\text{rad} \cdot 100 \ \text{m}}{2 \sin (19.4 \ \pi)} = 1 \ \text{mm}
\]

❖ The Tevatron has ~ 100 quadrupoles. By superposition

\[
\langle \Delta \hat{x} \rangle = N_{\text{quad}}^{1/2} \Delta \hat{x}_{\text{quad}} = 10 \ \text{mm} \text{ for our example}
\]

Steering correctors are essential!
Effect of field gradient errors

Let
\[ K_{\text{actual}}(s) = K_{\text{design}}(s) + k(s) \]

where \( k(s) \) is a small imperfection

\[ k(s) \Rightarrow \text{change in } \beta(s) \Rightarrow \Delta Q \]

Consider \( k \) to be non-zero in a small region \( \Delta \) at \( s = 0 \)

\[ \Rightarrow \text{angular kick } \Delta y' \sim y \]

\[ \frac{\Delta y'}{\Delta s} = ky \quad (1) \]
Before $s = 0^-$

$$y = b \cos \frac{s}{\beta_n} \quad (2)$$

At $s = 0^+$ the new (perturbed) trajectory will be

$$y = (b + \Delta b) \cos \left( \frac{s}{\beta_n} + \Delta \phi \right)$$

where

$$\frac{b + \Delta b}{\beta_n} \sin \phi = \Delta y'$$
If $\Delta y'$ is small, then $\Delta b$ and $\Delta \phi$ will also be small

$$
\Rightarrow \quad \Delta \phi \approx \frac{\beta_n \Delta y'}{b}
$$

- Total phase shift is $2\pi Q$; the *tune shift* is

$$(1) \& (2) \quad \Rightarrow \quad \Delta \phi \approx \beta_n k \Delta s \propto \text{phase shift}$$

- Principle effect of the gradient error is to shift the phase by $\Delta \phi$

$$
\Delta Q \approx -\frac{\Delta \phi}{2\pi} = -\beta_n \frac{k \Delta s}{2\pi}
$$

*The total phase advanced has been reduced*
This result overestimates the shift

- The calculation assumes a special case: $\phi_o = 0$
  - The particle arrives at $s = 0$ at the maximum of its oscillation

- More generally for $\phi_o \neq 0$
  - The shift is reduced by a factor $\cos^2 \phi_o$
  - The shift depends on the local value of $\beta$

- On successive turns the value of $\phi$ will change

- $\therefore$ the cumulative tune shift is reduced by $< \cos^2 \phi_o > = 1/2$

$$\Delta Q = -\frac{1}{4\pi} \beta(s)(k\Delta s)$$
Gradient errors lead to half-integer resonances

- For distributed errors

\[ \Delta Q = -\frac{1}{4\pi} \oint \beta(s)k(s)ds \]

- Note that \( \beta \sim K^{-1/2} \implies Q \propto 1/\beta \propto K^{1/2} \)

- \( \therefore \Delta Q \propto k\beta \implies \Delta Q/Q \propto k\beta^2 \propto k/K \) (relative gradient error)

- Or \( \Delta Q \sim Q (\Delta B'/ B') \)

- Machines will large \( Q \) are more susceptible to resonant beam loss

\textit{Therefore, prefer lower tune}
**Tune shifts & spreads**

★ **Causes of tune shifts**
  → Field errors
  → Intensity dependent forces
      • Space charge
      • Beam-beam effects

★ **Causes of tune spread**
  → Dispersion
  → Non-linear fields
      • Sextupoles
  → Intensity dependent forces
      • Space charge
      • Beam-beam effects
Example for the RHIC collider
Think of the resonance lines as having a width that depends on the strength of the effective field error.

Also the operation point has a finite extent.

Resonances drive the beam into the machine aperture.
In real rings, aperture may not be limited by the vacuum chamber size

- Resonances can capture particles with large amplitude orbits & bring them in collision with the vacuum chamber

  ==> “virtual” or *dynamic* aperture for the machine

- Strongly non-linearity ==> numerical evaluation

- *Momentum acceptance* is limited by the size of the RF bucket or by the dynamic aperture for the off-momentum particles.

  ➞ In dispersive regions off-energy particles can hit the dynamic aperture of the ring even if $\Delta p$ is still within the limits of the RF acceptance