Lecture 1b

Basic Principles: Relativity, Maxwell's Equation's, and Accelerator Coordinate Systems

Basic Units and Relationships

Energy and momentum in accelerators are usually expressed in units of "electron Volts":

$$1 \text{ eV} = 1.602 \times 10^{-19} \text{ Joules}$$

We will use energy units:

keV = 10^3 eV
MeV = 10^6 eV
GeV = 10^9 eV
TeV = 10^{12} eV

Similarly, the units of momentum, p, are eV/c.

And finally, for mass, the units are eV/c^2. For instance

m_p = mass proton = 938 MeV/c^2
m_e = mass electron = 511 keV/c^2

In practice, we will sometimes drop the factor of c.

For all other quantities, we will alternate between CGS and MKS units (see Tables 1.1 and 1.2 in Weidemann for conversion factors).

Relativistic Relationship

In most accelerators, particles move at relativistic speeds, and therefore we need to use relativistic mechanics to describe particle motion and fields.

Einstein's Special Theory of Relativity:

1) The laws of physics apply in all inertial (non-accelerating) reference frames.
2) The speed of light in vacuum is the same for all inertial observers.

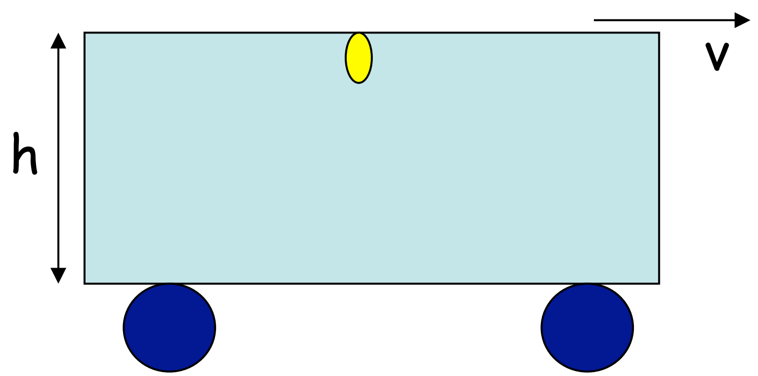
Notice that (1) does not mean that the answer to a physics calculation is the same in all inertial reference frames. It only means that the physics law's governing the calculation are the same.

The Relativistic Factor γ

Example: Consider a light bulb hanging in a boxcar moving at relativistic velocity. How long does it take a light ray, moving directly down in the boxcar frame, from the bulb to reflect off the floor and return to the ceiling:

a) as computed by an observer in the car?

b) as computed by an observer on the ground? (**Calculation**)



a) Inside the boxcar, $t^* = \frac{2h}{c}$

b) On the ground, $t = \frac{2h}{c} \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$

The answers differ by a factor of:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \text{ where } \beta = \frac{v}{c}$$

Therefore time is *dilated* for the observer on the ground, compared with the observer in the boxcar.

Other Relativistic Relationships

These principles give rise to time dilation and length contraction:

$$t = \gamma t^*$$
$$L = L^*/\gamma$$

The LHS quantities are given in the rest frame of the observer who perceives an object in motion. We often call this the "Lab frame". The RHS quantities (*) are in the rest frame of the moving object.

Time dilation is an important concept in particle physics because many particles have limited lifetimes. Time dilation says that the particle lifetimes are longer in the "Lab frame".

For an observer in the lab frame, the mass of an object also appears to increase at high velocity. The object becomes infinitely heavy as it approaches the speed of light.

$$m = \gamma m_o$$

Relativistic Energy Equations

The factors γ and β are commonplace in most relativistic equations:

$$\beta = \frac{v}{c}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \beta^2}}$$

In fact, the total energy of a particle (sum of kinetic and rest energy), is given by:

$$E = mc^2 = \gamma m_o c^2 = T + m_o c^2$$

For accelerators, it is often convenient to find γ using the kinetic energy, T, of a particle:

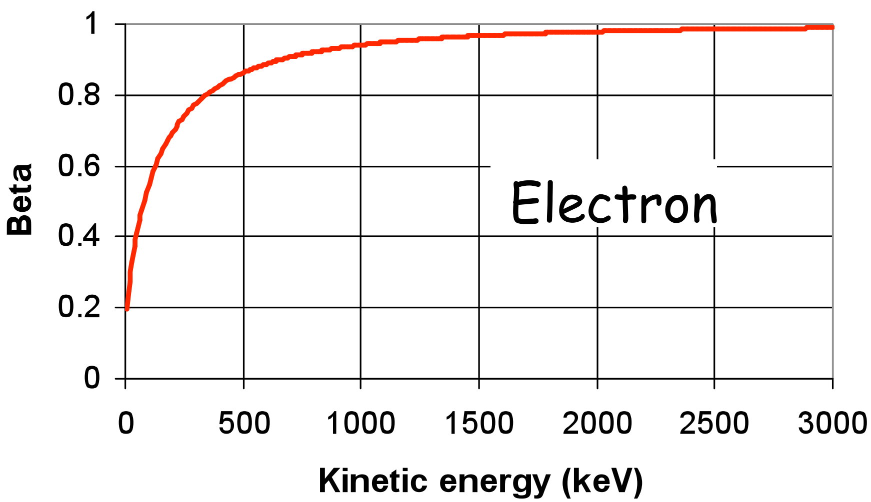
$$T = m_o c^2 (\gamma - 1) \Rightarrow \gamma = 1 + \frac{T}{m_o c^2}$$

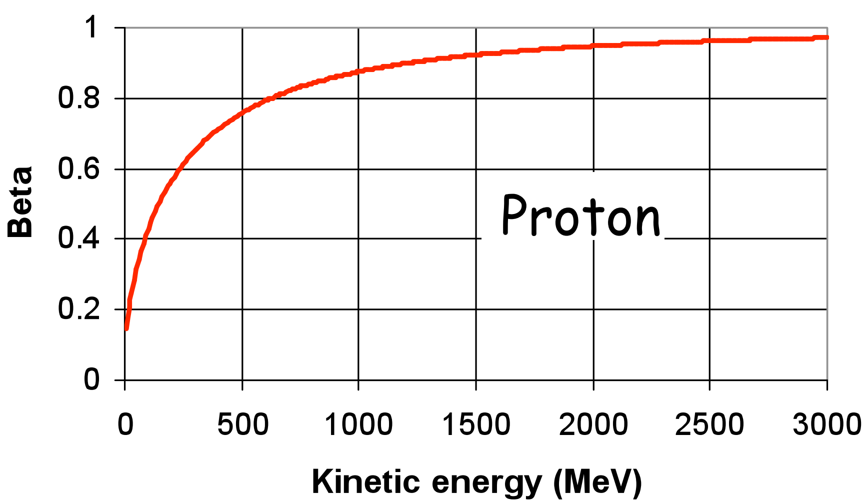
And finally, for the relationship between momentum and energy, we have:

$$\left.\begin{aligned} E &= \gamma m_o c^2 \\ p &= \gamma m_o v = \gamma m_o \beta c \end{aligned}\right\} \Rightarrow cp = \beta E$$

Relativistic Beta Function

The β function is the speed of a particle divided by the speed of light. As a massive particle is accelerated, β increases asymptotically towards 1 (speed of light), but never gets there:





- Heavier particles become relativistic at higher energies.
- No particle with finite mass can travel at the speed of light in vacuum (β=1). Massless particles always satisfy β=1.

Maxwell's Equations

In accelerators, we use electric fields to accelerate particles and magnetic fields to guide and focus particles. The standard equations used to describe the fields are Maxwell's equations:

(For vacuum or "well-behaved" materials)

CGS	MKS
$\nabla \bullet E = \frac{4\pi}{\varepsilon_r}\rho$	$\nabla \bullet E = \frac{\rho}{\varepsilon}$
$\nabla \times E = -\frac{1}{c}\frac{\partial}{\partial t}B$	$\nabla \times E = -\frac{\partial}{\partial t}B$
$\nabla \bullet B = 0$	$\nabla \bullet B = 0$
$\nabla \times \frac{B}{\mu_r} = \frac{4\pi}{c}J + \frac{1}{c}\frac{\partial}{\partial t}E$	$\nabla \times B = \mu J + \mu\varepsilon\frac{\partial}{\partial t}E$

$$\varepsilon = \varepsilon_r \varepsilon_o, \qquad \varepsilon_o = \text{permittivity of free space}$$

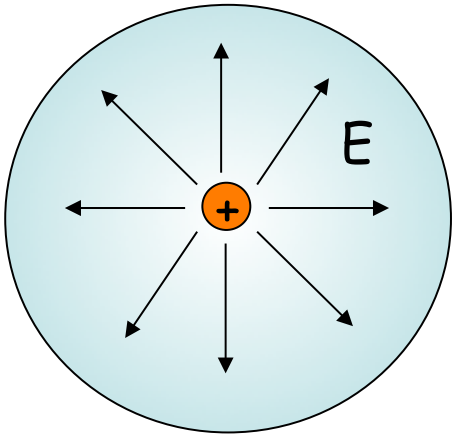
$$\mu = \mu_r \mu_o, \qquad \mu_o = \text{permeability of free space}$$

A Closer Look

Divergence theorem: Divergence integrated over the volume of a region is equal to the flux through the surface area of the region.

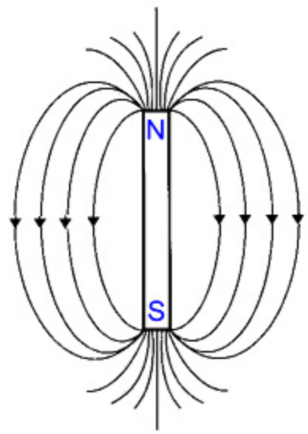
$$\nabla \bullet E = \frac{4\pi}{\varepsilon_r}\rho \Rightarrow \int (\nabla \bullet E)dV = \frac{4\pi}{\varepsilon_r} Q_{\text{enclosed}}$$

$$\rightarrow \oint_{\text{surface}} \mathrm{E} \bullet \mathrm{dA} = \frac{4\pi}{\varepsilon_r} Q_{\text{enclosed}}$$



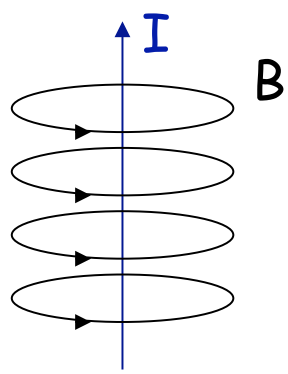
<u>Gauss Law for Electric Fields:</u> The total electric field flux through a surface is equal to the charge enclosed by the surface (to within a multiplicative factor)

A Closer Look



Magnetic fields do not diverge. Net magnetic flux through a closed surface is <u>zero.</u>

$$\nabla \bullet B = 0$$



There are no magnetic monopoles!

Magnetic fields lines for a dipole run from North to South. For a field generated by a current, I, point your right thumb in the direction of current – your fingers will curl in the direction of B.

A Closer Look

$$\nabla \times E = -\frac{1}{c}\frac{\partial}{\partial t}B$$

$$\nabla \times \frac{B}{\mu_r} = \frac{4\pi}{c}J + \frac{1}{c}\frac{\partial}{\partial t}E$$

A changing magnetic field induces an electric field...

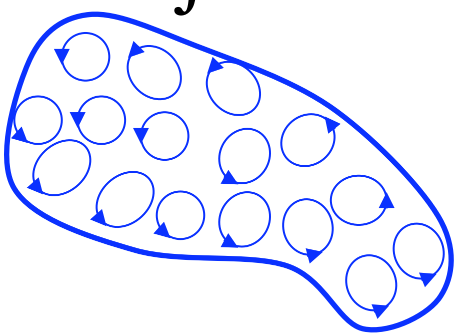
A changing electric field induces a magnetic field...

This concept is important in RF acceleration of particles.

A Closer Look

<u>Stokes Theorem:</u> Curl integrated over an area inside a closed curve equals the line integral around the curve.

$$\int (\nabla \times V) \bullet dA$$



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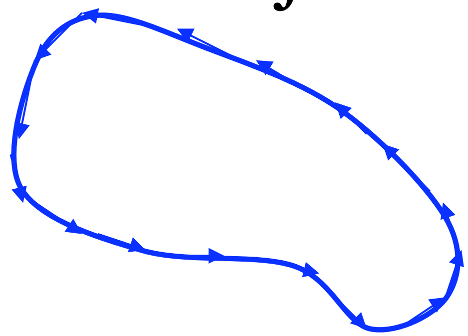
$$\int V \bullet dl$$



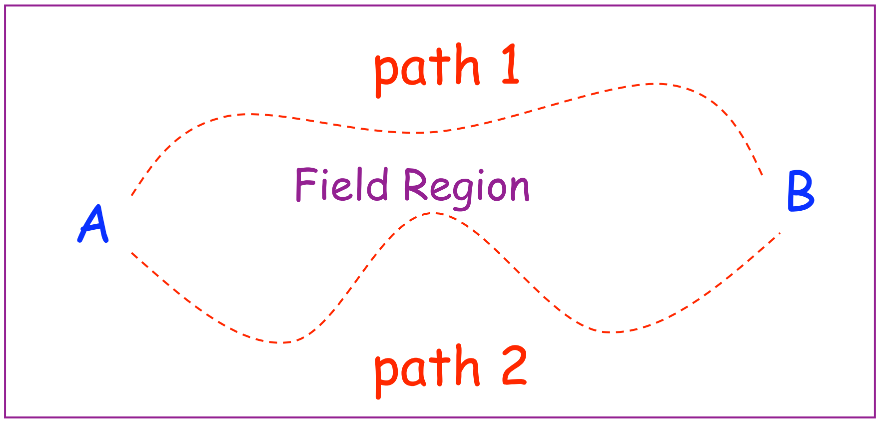
The "curl" of a vector function is a measure of its "swirl" or "twist". For the total "swirl", all contributions cancel except those at the boundary.

$$\nabla \times \frac{B}{\mu_r} = \frac{4\pi}{c} J \qquad \Rightarrow \oint_{loop} \frac{B}{\mu_r} \bullet dl = I_{enclosed} \quad (\text{if } \partial E / \partial t = 0)$$

<u>"Stokes' Law for Magnetic Fields"</u>: For a constant E field, the component of the B field along any closed path is equal to the total current enclosed.

Scalar Potential

For any material-free field region, if the integral from point A to point B is independent of the path, then the field can be expressed as the gradient of a scalar potential.



$$\int_{\text{Path A}} (\text{Field})\, \text{ds} = \int_{\text{Path B}} (\text{Field})\, \text{ds}$$

$$\Rightarrow Field = -\nabla V$$

$$\nabla V = \frac{dV}{dx}\hat{x} + \frac{dV}{dy}\hat{y} + \frac{dV}{dz}\hat{z}$$

So, for electric and magnetic fields in a material-free region, we can write:

$$E = -\nabla V_E$$

$$B = -\Delta V_B$$

We will find these expressions useful!

The Lorentz Force Equation

$$\vec{F} = m\vec{a} = \frac{d\vec{p}}{dt}$$

A force is the change in momentum with respect to time.

For a charged particle passing through an E or B field the force is governed by the ***Lorentz Force Equation***:

$$\vec{F} = q(\vec{E} + \frac{\vec{v}}{c} \times \vec{B})$$

Force from the electric field is in the direction of E

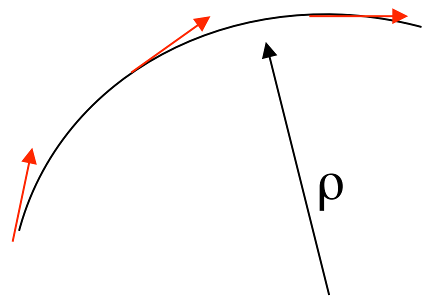
Force from the magnetic field is perpendicular to the direction of v and B, as given by the "Right Hand Rule"

<u>Right Hand Rule for **a=b x c**</u> : Point your fingers in the direction of **b**, then curl your fingers toward the direction of **c**, and then your thumb will point in the direction of **a**.

Accelerator Coordinate Systems

In general, any accelerator will be designed (shaped) to give a "reference trajectory" for particle travel. This reference trajectory usually includes a number of bends:

Longitudinal axis points in the direction of the reference trajectory at any point (tangent to the reference path).



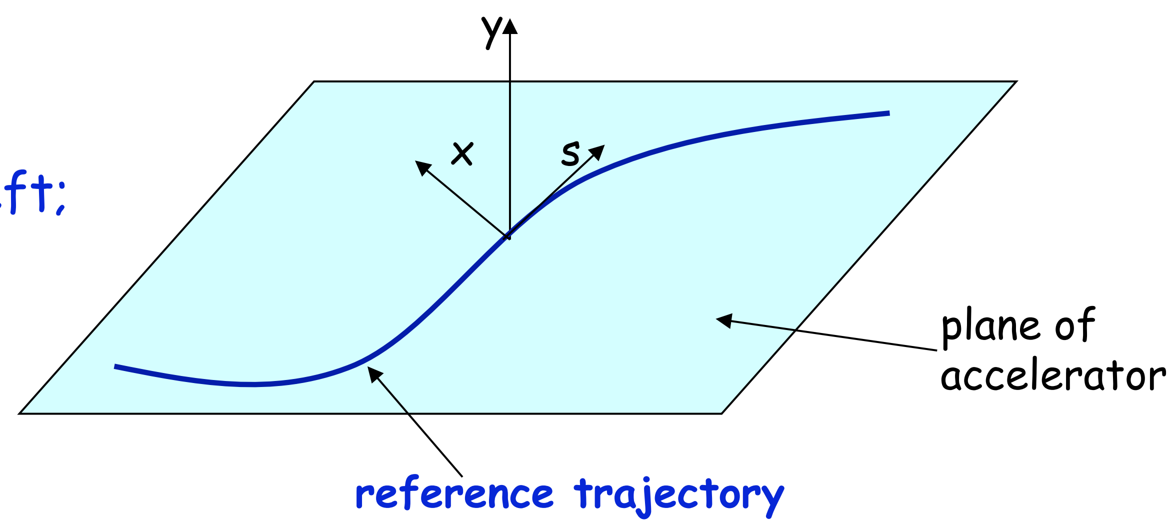
In beam physics, we are generally interested in deviations from the reference trajectory. Therefore it is most convenient to place the coordinate system origin on the reference trajectory, and align one (the longitudinal) coordinate axis with the reference trajectory. Remaining (transverse) axes are chosen perpendicular to the longitudinal axis.

Accelerator Coordinate Systems

In this *curvilinear coordinate system*, the direction of all axes change along the reference path.

Of the two remaining axis, we choose the horizontal axis to go in the plane of the reference trajectory, and the vertical axis to be perpendicular to this plane. This is convenient since most accelerators are laid out entirely in a horizontal plane.

As we look in the direction of s,
positive x is to the left;
and positive y is up.



Transverse Equations of Motion

The choice of this coordinate system, together with the Lorentz Force Equation, gives rise to the following equations of motion.

(**Derivation**)

$$x'' = \frac{d^2 x}{ds^2} = -\frac{e}{pc} B_y$$

$$y'' = \frac{d^2 y}{ds^2} = \frac{e}{pc} B_x$$

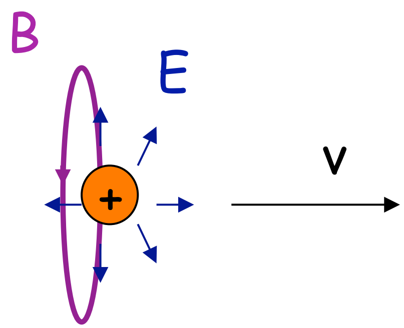
For the ideal particle, these equations determine the reference trajectory.

In practice, we will use these equations only to define the reference trajectory. For the motion of other particles, we will subtract out the reference trajectory, thus making it the origin of the coordinate system.

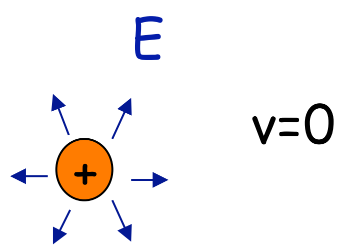
Lorentz Transformation of Fields

Do the fields E and B look the same in all inertial reference frames?

Example: A particle is passing by an observer at velocity v.



In the “lab frame”, the moving charged particle produces a current, and thus it has both an E field and a B field.



But, in the frame of reference moving with the particle, the particle is at rest and has only an E field.

The Special Theory of Relativity states that the laws of physics, i.e., Maxwell’s equations in this case, are the same in all inertial reference frames. But the results of the laws can appear different in different reference frames.

Lorentz Transformation of the Fields

The transverse fields, E and B, transform according to the following equations.

$$E_x^* = \gamma(E_x + \beta_s B_y) \qquad B_x^* = \gamma(B_x - \beta_s E_y)$$

$$E_y^* = \gamma(E_y - \beta_s B_x) \qquad B_y^* = \gamma(B_y + \beta_s E_x)$$

$$E_s^* = E_s \qquad B_s^* = B_s$$

Here, the (*) quantities on the left hand side are taken in the reference frame moving with velocity β_s, relative to the non-(*) quantities, which are in the lab frame.

Energy In Particle Collisions

Many high energy accelerators are particle colliders. How much energy is yielded in particle collisions?

We choose to work in the "center of mass" frame, where the overall momentum is zero. The center of mass energy is conserved. The "center of mass energy" is written as:

$$E_{CM} = \gamma_{CM}^{1} \times m_1 + \gamma_{CM}^{2} \times m_2$$

Two-Particle Collision

The net energy left over for the experiment is the center of mass energy, less the rest mass energy of the particles which result from the collision.

Suppose for instance that we collide two identical particles.

a) What is the total energy in the CM frame?

$$E_{CM} = 2\gamma_{CM} m$$

b) What is the equivalent energy of the moving particle if one particle is at rest?

$$E_{LAB} = (\gamma_{LAB} + 1)m = (\gamma_{CM}^2 (1 + \beta_{CM}^2) + 1)m$$

For LCH, $\gamma_{CM} \approx 7 \times 10^3 \rightarrow \gamma_{LAB} \approx 10^8$.