A. Nassiri -ANL
Impedance matching is important for the following reasons:

- Maximum power is delivered when the load is matched to the line (assuming the generator is matched), and power loss in is minimized.
- Impedance matching sensitive receiver components (antenna, low-noise amplifier, etc.) improves the signal-to-noise ratio of the system.
• Factors that may be important in the selection of a particular matching network include the following:
  
  – **Complexity:**
    • A simpler matching network is usually cheaper, more reliable, and less lossy than a more complex design.
  
  – **Bandwidth:**
    • Narrow or broadband.
  
  – **Implementation:**
    • According to the technology used the matching network can be decided on.
  
  – **Adjustability:**
    • Required if dealing with variable load.
Matching with lumped elements (L-Networks)

Network for $z_L$ inside the $1 + jx$ circle

Network for $z_L$ outside the $1 + jx$ circle

$jX$ and $jB$ Can be capacitors or inductors

8 possibilities for matching
Matching with lumped elements (L-Networks)

**Analytic Solutions**

\[
Z_L = R_L + jX_L, \quad z_L = \frac{Z_L}{Z_0}
\]

\[
Z_o = Z_L // \left( \frac{1}{jB} \right) + jX
\]

\[
Z_o = jX + \frac{1}{jB + 1/(R_L + jX_L)}
\]

\[
Z_o = jX + \frac{R_L + jX_L}{jB(R_L + jX_L) + 1} = jX + \frac{R_L + jX_L}{(1 - BX_L) + jBR_L}
\]

\[
Z_o = jX \left( (1 - BX_L) + jBR_L \right) + R_L + jX_L
\]

\[
Z_o = \frac{(R_L - XBR_L) + j(X_L + X(1 - BX_L))}{(1 - BX_L) + jBR_L}
\]

\[
R_L > Z_o
\]
Matching with lumped elements (L-Networks)

\[ \Re e_{LH} = \Re e_{RH} \]

\[ Z_o (1 - BX_L) = (R_L - XBR_L) \]

\[ B(XR_L - X_L Z_o) = R_L - Z_o \]

\[ \Im m_{LH} = \Im m_{RH} \]

\[ BR_L Z_o = X_L + X(1 - BX_L) \]

\[ X(1 - BX_L) = BZ_o R_L - X_L \]

Solving the 2\textsuperscript{nd} order equation

\[ B = \frac{X_L \pm \sqrt{R_L/Z_o} \sqrt{R_L^2 + X_L^2 - Z_o R_L}}{R_L^2 + X_L^2} \]

But \( R_L > Z_o \)

\[ B(XR_L - X_L Z_o) = R_L - Z_o \]

\[ X = \frac{1}{B} + \frac{X_L Z_o}{R_L} - \frac{Z_o}{BR_L} \]
Matching with lumped elements (L-Networks)

**Analytic Solutions**

\[ Z_L = R_L + jX_L \]

\[ \frac{1}{Z_o} = jB + \frac{1}{R_L + j(X + X_L)} \]

\[ \frac{1}{Z_o} = jB(R_L + j(X + X_L)) + 1 \]

\[ R_L + j(X + X_L) = Z_o \left( (1 - B(X + X_L)) + jBR_L \right) \]

\[ \Re e_{LH} = \Re e_{RH} \]

\[ \Im m_{LH} = \Im m_{RH} \]

\[ B Z_o (X + X_L) = Z_o - R_L \]

\[ (X + X_L) = B Z_o R_L \]

\[ X = \pm \sqrt{R_L (Z_o - R_L)} - X_L \]

\[ B = \pm \frac{\sqrt{(Z_o - R_L)/R_L}}{Z_o} \]
Smith Chart Solutions

Example

Design an L-section matching network to match a series RC load with an impedance $Z_L = 200 - j100 \, \Omega$, to a $100 \, \Omega$, at a frequency of 500 MHz.

Solution

\[
Z_L = 200 - j100 \, \Omega \\

z_L = 2 - j1 \, \Omega
\]
$z_L = 2 - j1 \, \Omega$

1st Step
2\textsuperscript{nd} Step

\[ y_I = 0.4 + j0.2 \]

\[ OP = OQ \]
3rd Step

\[ y_1 = 0.4 + j0.2 \]
$y_1 = 0.4 + j0.2$

$jb_1 = j0.3$

$jb_2 = -j0.7$

$y_1 = 0.4 + j0.5$

$y_2 = 0.4 - j0.5$
4th Step

Sol. 1

\[ y_1 = 0.4 + j0.2 \]
\[ jb_1 = j0.3 \]
\[ jx_1 = +j1.2 \]

\[ z_1 = 1 - j1.2 \]
$y_1 = 0.4 + j0.2$

$jb_2 = -j0.7$

$jx_2 = -j1.2$

$z_1 = 1 + j1.2$
Example

\[ jb_1 = j0.3 \quad jx_1 = +j1.2 \]
\[ jb_2 = -j0.7 \quad jx_2 = -j1.2 \]

**Solution 1**

\[ jB = jb \times \frac{1}{Z_0} = j\omega C \]
\[ C = \frac{b}{\omega Z_o} = \frac{0.3}{2\pi(500 \times 10^6)100} = 0.955 \text{ pF} \]
\[ jX = jx \times Z_o = j\omega L \]
\[ L = \frac{xZ_o}{\omega} = \frac{1.2 \times 100}{2\pi(500 \times 10^6)} = 38.2 \text{ nH} \]

**Solution 2**

\[ jB = jb \times \frac{1}{Z_0} = -j \frac{1}{\omega L} \]
\[ L = \frac{-Z_o}{\omega b} = \frac{-100}{2\pi(500 \times 10^6)(-0.7)} = 45.5 \text{ nH} \]
\[ jX = jx \times Z_o = -j \frac{1}{\omega C} \]
\[ C = \frac{-1}{\omega xZ_o} = \frac{-1}{2\pi(500 \times 10^6) \times (-1.2) \times 100} = 2.65 \text{ pF} \]
Example
The Quarter Wave Transformer

\[ Z_{in} = Z_o \frac{Z_L + jZ_o \tan \beta l}{Z_o + jZ_L \tan \beta l} \]

\[ Z_{in} = Z_1 \frac{Z_L}{\tan \beta l} + jZ_1 \]

\[ Z_1 = \sqrt{Z_o Z_L} \]

Need to drive the mismatch versus frequency?

Let \( t = \tan \beta l = \tan \theta \) at \( f = f_o \) \( \theta = \frac{\pi}{2} \)

\[ Z_{in} = Z_1 \frac{Z_L + jZ_1 t}{Z_1 + jZ_L t} \]

\[ \Gamma = \frac{Z_{in} - Z_o}{Z_{in} + Z_o} = \frac{Z_1 \left( \frac{Z_L + jZ_1 t}{Z_1 + jZ_L t} \right) - Z_o}{Z_1 \left( \frac{Z_L + jZ_1 t}{Z_1 + jZ_L t} \right) + Z_o} = \frac{Z_1 (Z_L + jZ_1 t) - Z_o (Z_1 + jZ_L t)}{Z_1 (Z_L + jZ_1 t) + Z_o (Z_1 + jZ_L t)} \]
The Quarter Wave Transformer

\[ \Gamma = \frac{Z_1(Z_L - Z_o) + j t(Z_1^2 - Z_o Z_L)}{Z_1(Z_L + Z_o) + j t(Z_1^2 + Z_o Z_L)} \]

\[ \Gamma = \frac{Z_L - Z_o}{Z_L + Z_o + j 2 \sqrt{Z_o Z_L}} \]

\[ |\Gamma| = \frac{|Z_L - Z_o|}{\left( (Z_L + Z_o)^2 + 4 t^2 Z_o Z_L \right)^{1/2}} = \frac{1}{\left( (Z_L + Z_o)^2 / (Z_L - Z_o)^2 + \left( 4 t^2 Z_o Z_L / (Z_L - Z_o)^2 \right) \right)^{1/2}} \]

\[ \frac{(Z_L + Z_o)^2}{(Z_L - Z_o)^2} = 1 + \frac{4 Z_o Z_L}{(Z_L - Z_o)^2} \]

\[ |\Gamma| = \frac{1}{\left( 1 + 4 Z_o Z_L / (Z_L - Z_o)^2 + \left( 4 t^2 Z_o Z_L / (Z_L - Z_o)^2 \right) \right)^{1/2}} = \frac{1}{\left( 1 + \left( 4 Z_o Z_L / (Z_L - Z_o)^2 \right) \sec^2 \theta \right)^{1/2}} \]

\[ 1 + t^2 = 1 + \tan^2 \theta = \sec^2 \theta \]

\[ \left( 4 Z_o Z_L / (Z_L - Z_o)^2 \right) \sec^2 \theta > 1 \]
The Quarter Wave Transformer

If we are operating close to \( f_o \)

\[
|\Gamma| \approx \left| \frac{Z_L - Z_o}{2\sqrt{Z_o Z_L}} \right| \cos \theta
\]

\[
\Delta \theta = 2 \left( \frac{\pi}{2} - \theta_m \right)
\]

\[
\frac{1}{\Gamma_m^2} = 1 + \left( \frac{2\sqrt{Z_o Z_L}}{Z_L - Z_o} \sec \theta_m \right)^2
\]

\[
\cos \theta_m = \frac{\Gamma_m}{\sqrt{1 - \Gamma_m^2}} \frac{2\sqrt{Z_o Z_L}}{|Z_L - Z_o|}
\]

\[
\theta = \beta l = \frac{2\pi f}{v_p} \frac{v_p}{4f_o} = \frac{\pi f}{2f_o}
\]

\[
f_m = \frac{2\theta_m f_o}{\pi}
\]

\[
\Gamma_m = \text{Maximum reflection coefficient magnitude that can be tolerated}
\]
The Quarter Wave Transformer

\[
\frac{\Delta f}{f_o} = 2\left(\frac{f_o - f_m}{f_o}\right) = 2 - \frac{2f_m}{f_o} = 2 - \frac{4\theta_m}{\pi} = 2 - \frac{4}{\pi} \cos^{-1}\left(\frac{\Gamma_m}{\sqrt{1 - \Gamma_m^2}}\right) \frac{2\sqrt{Z_oZ_L}}{|Z_L - Z_o|}
\]
Example

Design a single-section quarter-wave matching transformer to match a 10 Ω load to a 50 Ω line, at \( f_0 = 3 \) GHz. Determine the percent bandwidth for which the SWR \( \leq 1.5 \).

Solution

\[
Z_1 = \sqrt{Z_o Z_L} = \sqrt{(50)(10)} = 22.36 \Omega
\]

Length \( \lambda / 4 \) @ 3 GHz \( \Gamma_m = \frac{SWR - 1}{SWR + 1} = \frac{1.5 - 1}{1.5 + 1} = 0.2 \)

\[
\Delta f = 2 - \frac{4}{\pi} \cos^{-1} \left[ \frac{\Gamma_m}{\sqrt{1 - \Gamma_m^2}} \frac{2\sqrt{Z_o Z_L}}{|Z_L - Z_o|} \right]
\]

\[
\Delta f = 2 - \frac{4}{\pi} \cos^{-1} \left[ \frac{0.2}{\sqrt{1 - (0.2)^2}} \frac{2\sqrt{(50)(10)}}{|10 - 50|} \right] = 0.29, \text{ or } 29\%
\]
• Single-Section Transformer

\[ \Gamma \]

\[ \Gamma_1 = \frac{Z_2 - Z_1}{Z_2 + Z_1} \]
\[ \Gamma_2 = -\Gamma_1 \]
\[ T_{21} = 1 + \Gamma_1 = \frac{2Z_2}{Z_1 + Z_2} \]
\[ T_{12} = 1 + \Gamma_2 = \frac{2Z_1}{Z_1 + Z_2} \]

\[ \beta l = \theta \]
The Theory of Small Reflections

\[ \Gamma = \Gamma_1 + T_{12} T_{21} \Gamma_3 e^{-2j\theta} + T_{12} T_{21} \Gamma_3^2 \Gamma_2 e^{-4j\theta} + \ldots = \Gamma_1 + T_{12} T_{21} \Gamma_3 e^{-2j\theta} \sum_{n=0}^{\infty} \Gamma_2^n \Gamma_3^n e^{-2jn\theta} \]

\[ \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for} \quad |x| < 1 \]

\[ \Gamma = \Gamma_1 + \frac{T_{12} T_{21} \Gamma_3 e^{-2j\theta}}{1 - \Gamma_2 \Gamma_3 e^{-2j\theta}} \]

\[ \Gamma_2 = -\Gamma_1 \]

\[ T_{21} = 1 + \Gamma_1 = \frac{2Z_2}{Z_1 + Z_2} \]

\[ T_{12} = 1 + \Gamma_2 = \frac{2Z_1}{Z_1 + Z_2} \]
The Theory of Small Reflections

\[ \Gamma = \frac{\Gamma_1 + \Gamma_3 e^{-2j\theta}}{1 + \Gamma_1 \Gamma_3 e^{-2j\theta}} \]

If the discontinuities between the impedances \( Z_1, Z_2, \) and \( Z_2, Z_L \) are small, then \( |\Gamma_1 \Gamma_3| << 1. \)

\[ \Gamma \approx \Gamma_1 + \Gamma_3 e^{-2j\theta} \]
Multi-section Transformer

\[ \Gamma_{0} = \frac{Z_{1} - Z_{o}}{Z_{1} + Z_{o}} \]
\[ \Gamma_{n} = \frac{Z_{n+1} - Z_{n}}{Z_{n+1} + Z_{n}} \]
\[ \Gamma_{N} = \frac{Z_{L} - Z_{N}}{Z_{L} + Z_{N}} \]

\[ \Gamma \cong \Gamma_{1} + \Gamma_{3}e^{-2j\theta} \]

\[ \Gamma = \Gamma(\theta) \cong \Gamma_{0} + \Gamma_{1}e^{-2j\theta} + \Gamma_{2}e^{-4j\theta} + \cdots + \Gamma_{N-2}e^{-2j(N-2)\theta} + \Gamma_{N-1}e^{-2j(N-1)\theta} + \Gamma_{N}e^{-2jN\theta} \]

Assume that the transformer is symmetrical

\[ \Gamma_{0} = \Gamma_{N} \quad \Gamma_{1} = \Gamma_{N-1} \quad \Gamma_{2} = \Gamma_{N-2} \quad \text{etc.} \]

\[ \Gamma(\theta) = \Gamma_{0} + \Gamma_{1}e^{-2j\theta} + \Gamma_{2}e^{-4j\theta} + \cdots + \Gamma_{2}e^{-2j(N-2)\theta} + \Gamma_{1}e^{-2j(N-1)\theta} + \Gamma_{0}e^{-2jN\theta} \]

\[ \Gamma(\theta) = \Gamma_{0}\left(1 + e^{-2jN\theta}\right) + \Gamma_{1}\left(e^{-2j\theta} + e^{-2j(N-1)\theta}\right) + \Gamma_{2}\left(e^{-4j\theta} + e^{-2j(N-2)\theta}\right) + \cdots \]
\[ \Gamma(\theta) = e^{-jN\theta} \left[ \Gamma_0 (e^{jN\theta} + e^{-jN\theta}) + \Gamma_1 (e^{j(N-2)\theta} + e^{-j(N-2)\theta}) + \Gamma_2 (e^{j(N-4)\theta} + e^{-j(N-4)\theta}) + \cdots \right] \]

\[ \Gamma(\theta) = 2e^{-jN\theta} \left[ \Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \cdots + \Gamma_n \cos(N-2n)\theta + \frac{1}{2} \Gamma_{N/2} \right] \]

For \( N \) even

\[ \Gamma(\theta) = 2e^{-jN\theta} \left[ \Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \cdots + \Gamma_n \cos(N-2n)\theta + \frac{1}{2} \Gamma_{(N-1)/2} \cos\theta \right] \]

For \( N \) odd

**Finite Fourier Cosine Series**

By choosing the \( \Gamma_{ns} \) and enough sections \((N)\) we can achieve the required response.