LECTURE 8: BEAM OPTICS AND EMITTANCE GROWTH

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Abstract. The objectives of this lecture are to classify and describe the various sources of emittance growth in an electron injector.

1. Introduction

There are four types of emittance in RF guns: thermal (aka cathode, $\epsilon_{thermal}$), rf (ϵ_{rf}), space charge (σ_{sc}) and aberrations due to the optical focusing elements (σ_{optics}). These emittances are usually assumed to be un-correlated such that the total emittance is given by,

$$\epsilon_{total} = \sqrt{\epsilon^2_{thermal} + \epsilon^2_{rf} + \epsilon^2_{sc} + \epsilon^2_{optics}} \tag{1}$$

A previous lecture described the thermal emittance in it's three forms of thermionic, photo-electric and field emission. This lecture discusses the rf emittance, geometric and chromatic abberations of the beam, and space charge emittance growth. Along the way, the concepts of projected emittance and slice emittance are introduced.

2. RF Emittance in Photocathode Guns

The rf emittance refers to a time-dependent focusing of the beam by the rf fields at the entrance and exit of each rf cavity. The rf focusing kick occurs at the ends of the cavities as a simple result of Maxwell's equations as described by the Panofsky-Wenzel theorem. This theorem simply states that the radial electric and azimuth magnetic fields are proportional to the z- and t-derivatives, respectively of the of the longitudinal field,

$$E_r = -\frac{r}{2}\frac{\partial}{\partial z}E_z \tag{2}$$

$$cB_\theta = \frac{r}{2c}\frac{\partial}{\partial t}E_z \tag{3}$$

Where the radial force is,

$$F_r = e(E_r - \beta c B_\theta). \tag{4}$$

where E_z and E_r are the electric field components of a cylindrically symmetric rf cavity. The change in radial momentum is the integral of the force,

$$p_r = \frac{1}{mc}\int F_r dt = \frac{1}{mc^2}\int F_r \frac{dz}{\beta}. \tag{5}$$

To a good approximation, the longitudinal rf field can be represented as,

$$E_z = E(z)\cos kz \sin(\omega t + \phi_0), \tag{6}$$

resulting in the following integrals for the radial momentum,

(7)

$$F_r = er\left[-\frac{1}{2}\frac{dE(z)}{dz}\cos kz\sin(\omega t+\phi_0) - \frac{1}{2c}\frac{d}{dt}\left(E(z)\sin kz\cos(\omega t+\phi_0)\right) + \frac{\beta}{2}\frac{dE(z)}{dz}\sin kz\cos(\omega t+\phi_0)\right]$$

We assume the impulse approximation, such that $E(z)$ is constant inside the gun with a sharp cutoff at the cathode and gun exit, $E(z) = E_0\theta(z_f - z)$. Then the integral for the radial momentum is easily performed by relating time to the z-coordinate,

$$p_r = \int_0^{t_f} F_r dt = \frac{1}{c}\int_0^{z_f} F_r \frac{dz}{\beta} \tag{8}$$

At the exit iris of the gun the beam is relativistic, $\beta = 1$, and the radial momentum kick becomes,

$$\Delta p_r = \frac{eE_0 r}{2c}\sin(kz_f - \omega t_f - \phi_0) = \frac{eE_0 r}{2c}\sin\phi_e \tag{9}$$

Where the electron phase at the exit of the gun is ϕ_e . Writing this in terms of the radial angle change gives the focal strength,

$$\Delta p_r = \gamma mc\Delta r' \Rightarrow \Delta r' = r\frac{eE_0\sin\phi_e}{2\gamma mc^2} = -\frac{r}{f_{rf}} \tag{10}$$

and

$$f_{rf} = -\frac{2\gamma mc^2}{eE_0\sin\phi_e} \tag{11}$$

Therefore electrons at various longitudinal positions along the bunch length arriving at different phases at the gun exit will experience different kicks as illustrated in Figure 2, causing an increase in the projected emittance. Clearly the rf emittance is a minimum when the rf focal length is independent of ϕ_e . This occurs when

$$\frac{df_{rf}}{d\phi_e} = \frac{2\gamma mc^2}{eE_0}\frac{cos\phi_e}{sin^2\phi_e} \Rightarrow 0 \tag{12}$$

which occurs when $\phi_e = \frac{\pi}{2}$.

However, even when $\phi_e = \frac{\pi}{2}$ there is still an increase due to the second-order curvature of the rf waveform. For this reason, RF guns typically are operated with bunch lengths no more than 10 degrees of rf phase long. The emittance growth due to rf curvature can be greatly reduced by introducing a third harmonic of the fundamental rf frequency . The above derivation follows that of Kim [] who also gives the rf emittance for a Gaussian beam with an root-mean-square width of exiting the gun at $\phi_e = \frac{\pi}{2}$ due to this curvature effect,

$$\epsilon_{rf} = \frac{eE_0}{2mc^2}\frac{\langle x^2\rangle\sigma_\phi^2}{\sqrt{2}} \tag{13}$$

Additional useful analytic formulae and their accuracy are discussed by Travier.

This radial field is significant in a high gradient rf gun as shown in Figure 1 where the longitudinal and transverse rf fields are shown for a gun with 100 MV/m peak electric field on the cathode. In this case, the peak transverse field is 15 MV/m or 1/6 of the longitudinal field, corresponding to a focal length of only 10 cm for a beam with exit energy of 5 MeV.

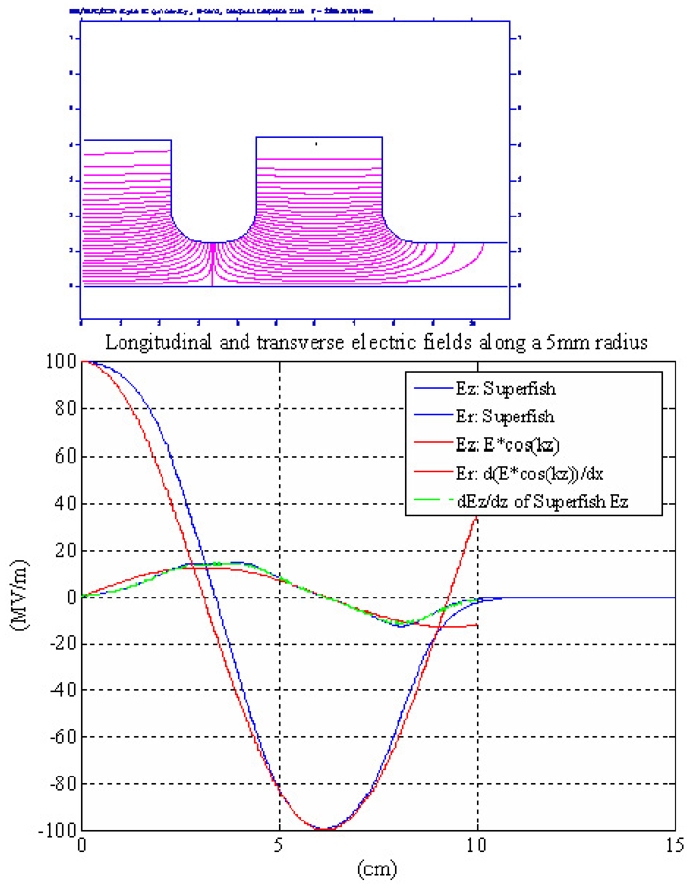

FIGURE 1

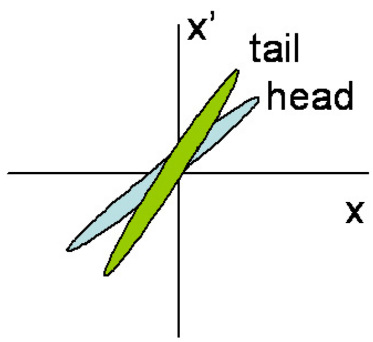

FIGURE 2

The accuracy of this analytic calculation of the strong negative rf focusing can be seen in Figure 3 which shows the results of a numerical ray trace calculation [ref:GPT] for a gun with a 115 MV/m cathode field.

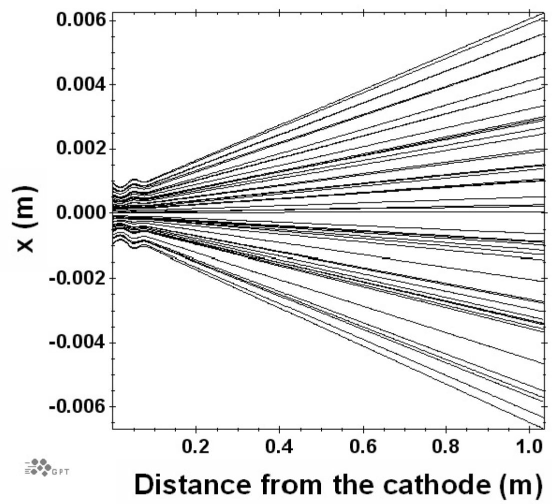

FIGURE 3. Electron trajectories for a rf gun with a cathode field on 115 MV/m as computed by General Particle Tracker Program, GPT.

Of course this strong de-focussing at the gun exit requires compensation by an equally strong focusing, which is usually provided by a solenoid with a longitudinal magnetic field. It is interesting to note that the dual role of this solenoid. For it not only cancels the strong negative rf lens, but it also plays the crucial function of aligning the beam's transverse properties longitudinally to reduce the bunch's projected emittance. This concept of emittance compensation is discussed next.

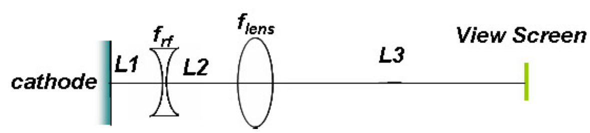

FIGURE 4. Diagram of a simple linear model of and rf gun and solenoid to be used in class exercise.

Class Exercise: Using the ABCD-matrix formulism of linear optics, derive the 2x2 transformation from the cathode, through the gun exit and a thin focussing lens to a view screen located after the gun. The relevant variables of the geometry are shown in Figure 4. What are the conditions for creating an image of the cathode surface at the screen location? Describe how this simple model can be used to determine the uniformity of the cathode's quantum efficiency.

3. Chromatic Aberration of the Solenoid

The section discusses the geometric and chromatic aberrations of the solenoid's long axial magnetic field. The typical location of the solenoid after the gun is shown in Figure xy. This lens cancels the strong defocus at the gun's exit and compensates for the space charge emittance.

The transverse, trace-space emittance is given by

$$\epsilon_x = \sqrt{\langle p_x^2 \rangle \langle x^2 \rangle - \langle p_x x \rangle^2} \tag{14}$$

where $p_x = \beta\gamma x' = x\alpha k \sin\phi$, for the rf lens.

Consider a solenoid producing a uniform axial magnetic field. For electrons moving along the magnetic flux lines there is a focusing strength K,

$$K = \frac{B(0)}{2B\rho_0} \tag{15}$$

where B(0) is the solenoid field and $B\rho_0$ is the beam's magnetic rigidity. Which in useful units is given by,

$$B\rho_0 = 33.356p(GeV/c)kG - m. \tag{16}$$

Due to the beam's rotation in the axial field, the x- and y-transverse trajectories become coupled and a 4x4 matrix is necessary for an optical calculation. However, if one rotates the x-y coordinates with the beam then the two planes decouple. In this rotating frame the transformation becomesTransport Manual, slac-r-091, pp.104-106,

$$R(-KL)R(solenoid) = \begin{pmatrix} C & S/K & 0 & 0 \\ -KS & C & 0 & 0 \\ 0 & 0 & C & S/K \\ 0 & 0 & -KS & C \end{pmatrix} \tag{17}$$

with $C = \cos(KL)$, $S = \sin(KL)$, L the magnetic field effective length, and the rotation matrix,

$$R = \begin{pmatrix} C & 0 & S & 0 \\ 0 & C & 0 & S \\ -S & 0 & C & 0 \\ 0 & -S & 0 & C \end{pmatrix} \tag{18}$$

In the beam frame the focal strength, element R_{21}, is

$$\frac{1}{f_{sol}} = K \sin KL \tag{19}$$

For small values of KL,

$$\frac{1}{f_{sol}} = K^2 L = \left(\frac{B(0)}{2B\rho_0}\right)^2 L \tag{20}$$

Therefore the focal strength of a solenoid is proportional to the axial field squared, this is unlike a quadrupole's strength which scales linearly with the field.

The emittance growth in the solenoid due to the beam's energy spread can be computed using the symmetric beam matrix, σ, and the transformation for a simple lens. The beam matrix is defined as

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \tag{21}$$

The transformation of the beam matrix through a simple lens is given by

$$\sigma(1) = R_{lens}^T \sigma(0) R_{lens} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \begin{pmatrix} 1 & -1/f \\ 0 & 1 \end{pmatrix} \tag{22}$$

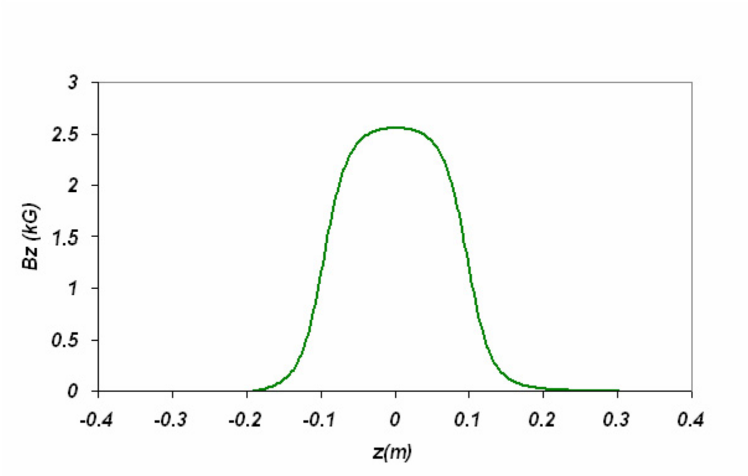

FIGURE 5. The longitudinal field, B_z, vs. z for the LCLS gun solenoid. The effective length of the magnetic field is 19.35 cm.

Performing the matrix multiplications gives,

$$\sigma(1) = \begin{pmatrix} \sigma_{11} & -\frac{\sigma_{11}}{f} + \sigma_{12} \\ -\frac{\sigma_{11}}{f} + \sigma_{12} & \frac{\sigma_{11}}{f^2} - \frac{2\sigma_{12}}{f} + \sigma_{22} \end{pmatrix} \tag{23}$$

The change in the beam matrix due to a variation in the beam momentum is

$$\Delta\sigma(1) = \frac{d\sigma(1)}{dp}\Delta p = \begin{pmatrix} 0 & -\sigma_{11}\frac{d}{dp}(\frac{1}{f})\Delta p \\ -\sigma_{11}\frac{d}{dp}(\frac{1}{f})\Delta p & [...] \end{pmatrix} \tag{24}$$

The emittance due to the momentum spread is then the given by the differential form of the emittance formula,

$$\epsilon_{n,chromatic} = \beta\gamma\sqrt{\det\Delta\sigma(1)} = \beta\gamma\sigma_x^2 \left| \frac{d}{dp}\left(\frac{1}{f}\right) \right| \Delta p \tag{25}$$

Where we have used $\sigma_{11} = \sigma_x^2$. As described earlier, in the rotating frame of the beam the focal length is given by

$$\frac{1}{f} = KS = K\sin KL \tag{26}$$

And taking the derivative gives,

$$\frac{d}{dp}\left(\frac{1}{f}\right) = (\sin KL + KL\cos KL)\frac{dK}{dp} = -(\sin KL + KL\cos KL)\frac{K}{p} \tag{27}$$

Inserting this into the expression for the emittance results in the final result,

$$\epsilon_{n,chromatic} = \beta\gamma K\sigma_x^2 (\sin KL + KL\cos KL)\frac{\sigma_p}{p} \tag{28}$$

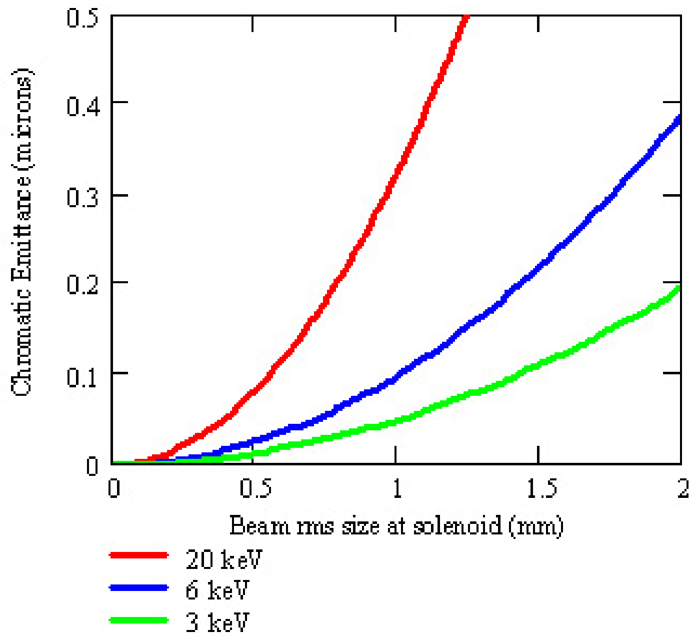

FIGURE 6. The chromatic emittance of the LCLS gun solenoid vs. the rms beam size at the solenoid for 3, 6 and 20 KeV rms energy spread.