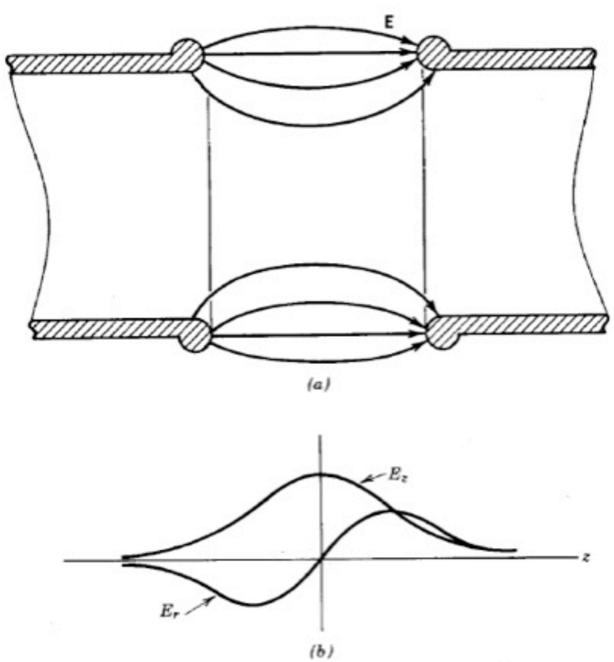
3. Nonlinear effects in beam transport

3.1 Spherical aberrations

So far we analyzed space charge dominated beam transport in linear approximation to space charge forces. In the presence of linear focusing field, the beam emittance remains constant. Realistic focusing elements posses strong aberrations, which result in distortion of phase space area, occupied by the beam. Among others, the spherical aberration cannot be eliminated, and therefore, has the most significant effect on particle dynamics.



Field distribution in electrostatic lens gap.

Potential of axial-symmetric electrostatic lens is defined by Laplace's equation:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial U}{\partial r})+\frac{\partial^2 U}{\partial z^2}=0 \tag{3.1}$$

Solution:

$$U(z,r)=U(z)-\frac{r^2}{4}U''(z)+\frac{r^4}{64}U^{(4)}(z)-\frac{r^6}{2304}U^{(6)}(z)+... \tag{3.2}$$

Field distribution inside each gap is given by near-axis approximation:

$$E_z(r,z)=E_z(z)-\frac{r^2}{4}E_z^{(2)}(z)+\frac{r^4}{64}E_z^{(4)}(z)+ \ldots\ldots+\frac{(-1)^n E_z^{(2n)}}{(n!)}(\frac{r}{2})^{2n}\ , \tag{3.3}$$

$$E_r(r,z)=-\frac{r}{2}E_z^{'}(z)+\frac{r^3}{16}E_z^{(3)}(z)\ldots.+\frac{(-1)^n E_z^{(2n-1)}}{(n!)(n-1)!}(\frac{r}{2})^{2n-1}\ . \tag{3.4}$$

3.

Equation of particle motion $$\frac{d^2x}{dz^2} = \frac{q}{mv_z^2} x\left(-\frac{1}{2}\frac{\partial E_z}{\partial z} + \frac{r^2}{16}\frac{\partial^3 E_z}{\partial z^3} + ...\right)$$

Let us neglect the change of particle position in x - direction while crossing the gap. Change of slope of particle trajectory at the entrance of the first gap is

$$\Delta\left(\frac{dx}{dz}\right)_{in} = \frac{1}{v_{in}^2}\frac{q}{m} x\left(-\frac{1}{2}\int_{-\infty}^{d/2}\frac{dE_z}{dz}dz + \frac{r^2}{16}\int_{-\infty}^{d/2}\frac{d^3E_z}{dz^3}dz\right) = -\frac{q}{m}\frac{E_z}{2v_{in}^2} x\left(1 - \frac{r^2}{8E_z}\frac{d^2E_z}{dz^2}\right)$$

where v_{in} is an effective particle velocity at the entrances of the gap, and the values of the field are taken at the center of the gap. Analogously, the change of the slope of the particle trajectory at the exit of the first gap is

$$\Delta\left(\frac{dx}{dz}\right)_{out} = \frac{q}{m}\frac{E_z}{2v_{out}^2} x\left(1 - \frac{r^2}{8E_z}\frac{d^2E_z}{dz^2}\right)$$

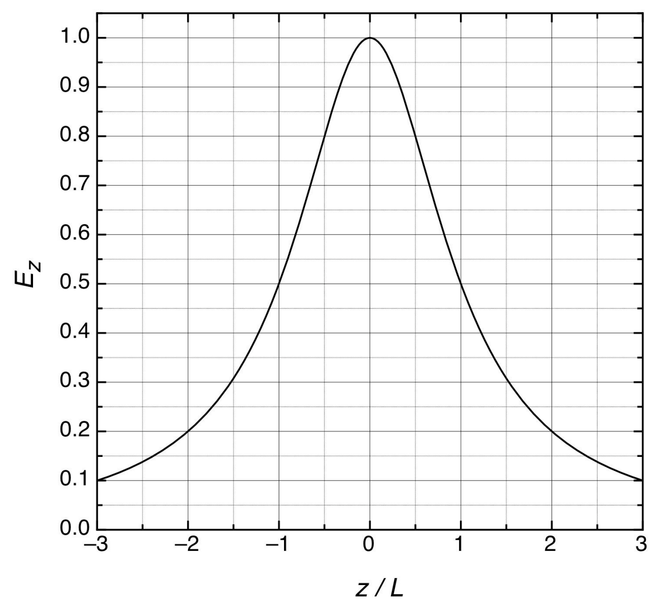
where v_{out} is an effective particle velocity at the exit of the first gap. Total change of slope of the particle at the first gap is

$$\Delta\left(\frac{dx}{dz}\right) = \frac{q}{mc^2}\frac{E_z}{2} x\left(\frac{1}{\beta_{out}^2} - \frac{1}{\beta_{in}^2}\right)\left(1 - \frac{r^2}{8E_z}\frac{d^2E_z}{dz^2}\right)$$

To calculate term in brackets, let us approximate the field in the gap by function $E_z = \dfrac{E_o}{1+(\dfrac{z}{L})^2}$

where L is a half of an effective gap width $L \approx \dfrac{d+a}{2}$

The second derivative $$\frac{d^2E_z}{dz^2} = -\frac{2E_o}{L^2}\frac{[1-3(\frac{z}{L})^2]}{[1+(\frac{z}{L})^2]^3}$$



3. Approximation of the static field in the gap.

The term in bracket taken at the center of the gap: $$1-\frac{r^2}{8E_z}\frac{d^2E_z}{dz^2}=1+\frac{r^2}{4L^2}$$

Finally, the change of slope of particle trajectory at the gap is

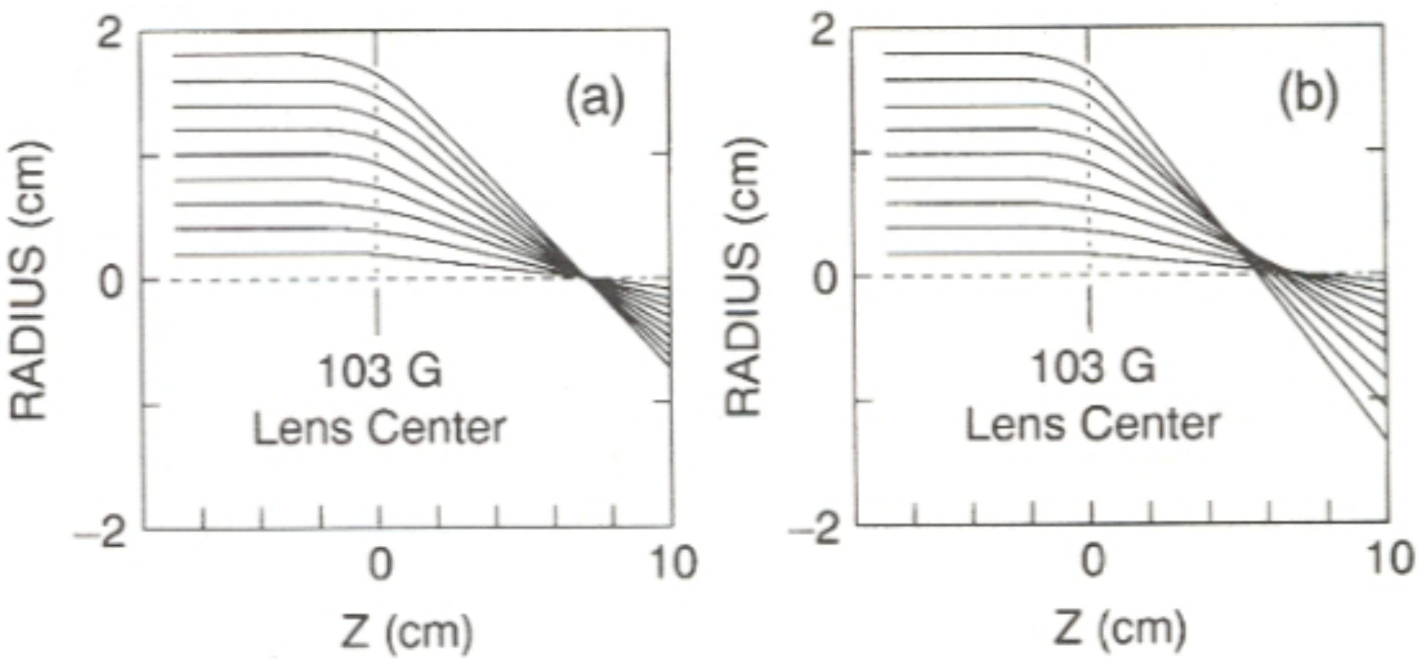
$$\Delta(\frac{dx}{dz})=\frac{q}{mc^2}\frac{E_z}{2}x(\frac{1}{\beta_{out}^2}-\frac{1}{\beta_{in}^2})(1+\frac{r^2}{4L^2})$$

If the field in the gap accelerates particles, $E_z > 0$, then $\beta_{out}>\beta_{in}$, and change of slope of particle trajectory is negative $\Delta(\frac{dx}{dz})<0$

If the field in the gap decelerates particles, $E_z < 0$, then $\beta_{out}<\beta_{in}$, and change of slope of particle trajectory is also negative $\Delta(\frac{dx}{dz})<0$

The *gap with electrostatic field focuses particles*. Change of slope of particle trajectory can be written via focal length f and aberration coefficient C_s:

$$\Delta(\frac{dx}{dz})=-\frac{x}{f}[1+\frac{C_s}{f}(\frac{r}{f})^2]$$



Focusing of a parallel beam by (a) an ideal lens (a) and
(b) by lens with spherical aberrations (from Reiser, 1994, p. 458).

3.2 Beam emittance growth due to spherical aberrations

Let us estimate emittance growth of the beam passing through the lens. We assume that position of particles is not changed while crossing the lens, and only slope of particle trajectory is changed. Transformation from initial particle variables before lens (*x_o, x_o'*) to that after crossing the lens (*x, x'*) is given by:

$$x = x_o \ , \tag{3.9}$$

$$x' = x_o' - \left(1 + \frac{C_s}{f^3} r_o^2\right) \frac{x_o}{f} \ . \tag{3.10}$$

Suppose, initial phase space volume is bounded by the ellipse

$$\frac{x_o^2}{R^2}\text{э} + \frac{x_o'^2}{\text{э}} R^2 = \text{э}, \tag{3.11}$$

To find the deformation of the boundary of the beam phase space after passing through the lens, let us substitute inverse transformation

$$x_o = x, \tag{3.12}$$

$$x_o' = x' + \left(1 + \frac{C_s}{f^3} r^2\right) \frac{x}{f} \ , \tag{3.13}$$

The boundary of the new phase space volume, occupied by the beam after passing through the lens at phase plane (*x, x'*) is given by:

$$\frac{x^2}{R^2}\text{э} + \frac{R^2}{\text{э}}\left(x' + \frac{x}{f} + \frac{C_s x^3}{f^4}\right)^2 = \text{э}. \qquad (3.14)$$

Let us introduce new variables (*J, ψ)* instead of (*x, x')* according to transformation:

$$\frac{x}{R}\sqrt{\text{э}} = \sqrt{2J}\, cos\, \psi, \qquad (3.15)$$

$$\left(x' + \frac{x}{f}\right)\frac{R}{\sqrt{\text{э}}} = \sqrt{2J}\, sin\psi. \qquad (3.16)$$

In new variables, the shape of beam emittance is:

$$2J + 2\frac{C_s R^4 (2J)^2}{f^4\ \text{э}^2} sin\psi\, cos^3\psi + \left(\frac{C_s R^4}{f^4}\right)^2 \frac{(2J)^3}{\text{э}^4} cos^6\varphi = \text{э}. \qquad (3.17)$$

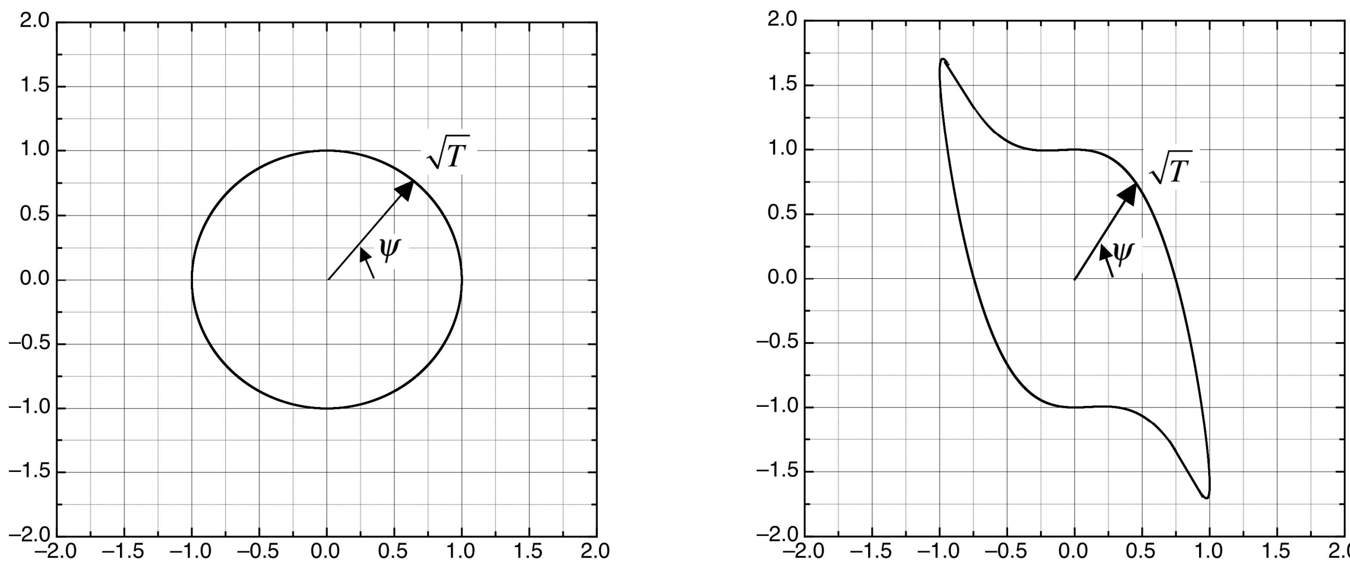
Let us rewrite Eq. (3.17) as

$$T + T^2 2\upsilon \sin\psi \cos^3\psi + T^3 \upsilon^2 \cos^6\psi = 1 , \qquad (3.18)$$

where

$$T = \frac{2J}{\varepsilon} , \qquad \upsilon = \frac{C_s R^4}{\varepsilon f^4} . \qquad (3.19)$$

Without nonlinear perturbation, $\upsilon = 0$, equation (3.18) describes ellipse (circle) in phase space. If $\upsilon \neq 0$, equation (3.18) describes *S* - shape figure of beam emittance.



Distortion of beam emittance due to spherical aberrations, Eq. (3.18): (left) υ = 0, (right) υ = 1.6.

In general case, arbitrary transformation

$$x = x_o \,, \qquad\qquad x' = x_o' + f(x_o, y_o) \,, \tag{3.20}$$

conserves phase space area due to Jacobian of transformation (3.20) always equals unity:

$$\begin{vmatrix} \dfrac{\partial x}{\partial x_o} & \dfrac{\partial x}{\partial x_o'} \\ \dfrac{\partial x'}{\partial x_o} & \dfrac{\partial x'}{\partial x_o'} \end{vmatrix} = 1 \tag{3.21}$$

While phase space areas occupied by the beam before and after lens are the same, the effective area, occupied by the beam, is increased. The value of beam emittance can be estimated as a total area of elements *dx dx'* occupied by the beam. Actual areas of the beam in both cases are the same, but the number of elements covered by the beam at phase pane is different.

Let us denote the increase of effective beam emittance as a square of product of minimum and maximum values of *T*:

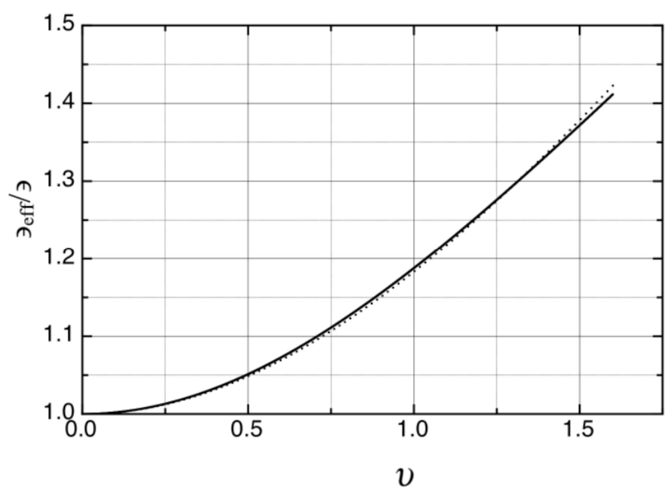
$$\frac{\epsilon_{eff}}{\epsilon} = \sqrt{T_{max}\, T_{min}}\ . \tag{3.22}$$

Values T_{max}, T_{min} are determined numerically from Eq. (3.18). Dependence of emittance growth versus parameter υ is presented at figure below. Dependence can be approximated by the function:

$$\frac{\epsilon_{eff}}{\epsilon} = \sqrt{1 + K\upsilon^2}, \tag{3.23}$$

where parameter $K \approx 0.4$. Substitution of Eq. (3.19) into Eq. (3.23) gives for effective beam emittance growth:

$$\epsilon_{eff} = \sqrt{\epsilon^2 + K\left(\frac{C_s R^4}{f^4}\right)^2} \tag{3.24}$$



Beam emittance growth after beam passing through axial-symmetric lens as a function of parameter υ : (sold line) Eq. (3.22), (dotted line) approximation by Eq.(3.23).

3.3 Beam emittance growth in drift space

Consider now emittance growth of space-charge dominated Gaussian beam in drift space. Space charge density and space charge field of the initial beam are:

$$\rho(r_o) = \frac{2I}{\pi R_o^2 \beta c} \exp(-2\frac{r_o^2}{R_o^2}) , \qquad (3.25)$$

$$E_b = \frac{I}{2\pi \varepsilon_o \beta c} \frac{1}{r_o} [1 - \exp(-2\frac{r_o^2}{R_o^2})] . \qquad (3.26)$$

Nonlinear function in space charge field is expanded as

$$f(r_o) = 1 - \exp(-2\frac{r_o^2}{R_o^2}) \approx 2\frac{r_o^2}{R_o^2} - 2\frac{r_o^4}{R_o^4} + \ldots . \qquad (3.27)$$

At the initial stage of beam emittance growth we can assume, that particle radius is unchanged, while the slope of the trajectory is changed. It gives us the nonlinear transformation:

$$r = r_o , \qquad (3.28)$$

$$r' = r_o' + \frac{2zP^2}{R_o^2} r_o - \frac{2zP^2}{R_o^4} r_o^3 . \qquad (3.29)$$

where P^2 is the generalized perveance.

Transformation, Eqs. (3.28), (3.29), is similar to that of lens aberration, Eqs. (3.12), (3.13) with formal substitution:

$$\frac{C_s}{f^4} = \frac{2\,z\,P^2}{R_o^4} \tag{3.30}$$

Substitution of Eq. (3.30) into Eq. (3.23), gives for initial beam emittance growth in free space:

$$\text{Э}_{eff} = \sqrt{\text{Э}^2 + K\left[\frac{I\,z}{I_c\,(\beta\gamma)^3}\right]^2}\,. \tag{3.31}$$

As follows from Eq. (3.31), initial emittance growth does not depend on initial beam radius R_o. Coefficient K depends on nonuniformity of the beam. For KV beam coefficient K = 0, while for Gaussian beam it reaches the value of K = 0.6.

3.4 Beam uniforming in drift space

Space charge forces of Gaussian beam are nonlinear function of radius which result in nonlinear redistribution of space charge density. At the beam drift, there is a certain distance where different layers of the beam do not cross each other: the radial motion of the particles is nonlinear, but the beam flow is still laminar. The assumption of laminar flow holds as long as $p_x(x)$ remains single-valued function. For laminar flow, the number of particles contained in an arbitrary cylinder with initial radius r_o remains constant. Use of Gauss theorem yields the result:

$$E_r r = E_{r_o} r_o = \frac{1}{\varepsilon_o} \int_0^{r_0} \rho(r^{'}) r^{'} dr^{'} = const \ . \qquad (3.32)$$

Here, r is the radius of our hypothetical cylinder, which expands as the beam drift, thus $r = r(z)$, $r(z=0)=r_o$. Use of Eq. (3.32) yields the radial space charge force at any location:

$$E_r = \frac{I}{2\pi \varepsilon_o \beta c} \frac{f(r_o)}{r} \ , \qquad (3.33)$$

$$f(r_o) = [1 - exp(-2 \frac{r_o^2}{R_o^2})] \ . \qquad (3.34)$$

Taking into account expression for space charge field, Eq. (3.33), the equation of motion of an arbitrary particle in drift region $dp_r / dt = qE_r / \gamma^2$ under the self space charge forces has the form:

$$\frac{d^2 r}{d z^2} = \frac{2 I}{I_c \beta^3 \gamma^3} \frac{f(r_o)}{r} . \qquad (3.35)$$

Let us introduce the new variables: $\bar{R} = \frac{r}{r_o}$, $\qquad Z = \frac{z}{r_o} \sqrt{\frac{4 I f(r_o)}{I_c \beta^3 \gamma^3}}$, (3.36)

Then the Eq. (3.35) becomes: $\frac{d^2 \bar{R}}{dZ^2} = \frac{1}{2\bar{R}}$ (3.37)

Eq. (3.37) is an equation for a single particle within the beam. It coinsides with envelope equation for the beam with negligible emittance. The first integral of the equation (3.37) is:

$$\left(\frac{d\bar{R}}{d Z}\right)^2 - \left(\frac{d\bar{R}}{d Z}\right)_o^2 = ln \bar{R} . \qquad (3.38)$$

The approximate solution of this equation is $\bar{R}(Z) = 1 + 0.25 Z^2 - 0.017 Z^3$. It gives for evolution of particle radius in drift space:

$$r = r_o \left[1 + \frac{1}{4} \eta \left(\frac{R_o}{r_o}\right)^2 f(r_o) - 0.017 \eta^{3/2} \left(\frac{R_o}{r_o}\right)^3 f^{3/2}(r_o)\right] , \qquad (3.39)$$

$$\eta = \frac{4 I}{I_o \beta^3 \gamma^3} \frac{z^2}{R_o^2} , \qquad (3.40)$$

Let us take into account that the number of particles *dN* inside a thin ring (r, $r + dr$) is constant during the drift of the beam at certain distance, hence the particle density $\rho(r)=dN/(2\pi r dr)$ at any z is connected with the initial density $\rho(r_o)$ by the equation:

$$\rho(r) = \rho(r_o)\,\frac{r_o\, d\, r_o}{r\, d\, r}\ . \tag{3.41}$$

Calculation of derivatives (3.41) gives the redistribution of the Gaussian beam under self space charge forces :

$$\rho(r) = \frac{\rho_o\, exp\,(-2\xi_o^2)}{a_o + a_1 F + a_2 F^2 + a_3 F^3 + a_4 F^4 + a_5 F^5 + a_6 F^6}\ , \tag{3.42}$$

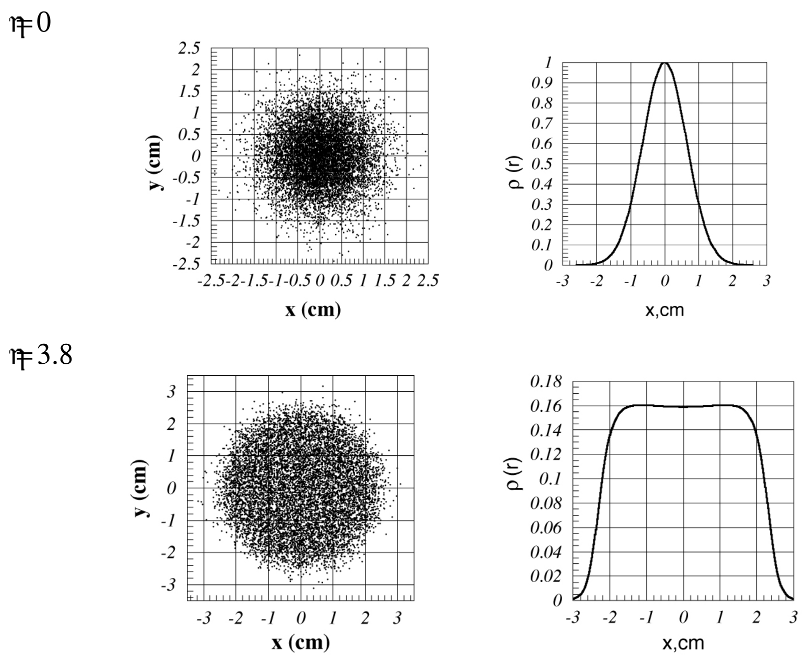


Fig. 3.6. Redistribution of Gaussian beam in drift space.

In Eq. (3.42) the following notations are used:

$$\xi_o = \frac{r_o}{R_o} \, , \tag{3.43}$$

$$F = \sqrt{\frac{1 - exp(-2\xi_o^2)}{\xi_o^2}} \, , \tag{3.44}$$

$$a_o = 1 + \eta \; exp(-2\xi_o^2), \tag{3.45}$$

$$a_1 = - 0.102 \; \eta^{3/2} \; exp(-2\xi_o^2), \tag{3.46}$$

$$a_2 = \frac{1}{4} \eta^2 \; exp(-2\xi_o^2) \, , \tag{3.47}$$

$$a_3 = 0.017 \; \eta^{3/2} - 0.0425 \; \eta^{5/2} \; exp(-2\xi_o^2) \tag{3.48}$$

$$a_4 = 1.734 \cdot 10^{-3} \; \eta^3 exp\,(-2\xi_o^2) - \frac{1}{16} \eta^2 \, , \tag{3.49}$$

$$a_5 = 0.01275 \; \eta^{5/2} \, , \tag{3.50}$$

$$a_6 = - 5.78 \cdot 10^{-4} \; \eta^3 \, . \tag{3.51}$$

According to Eq. (3.42), the beam with initial Gaussian distribution becomes more uniform when the parameter η is close to 4.

3.5 Beam emittance growth in a focusing channel

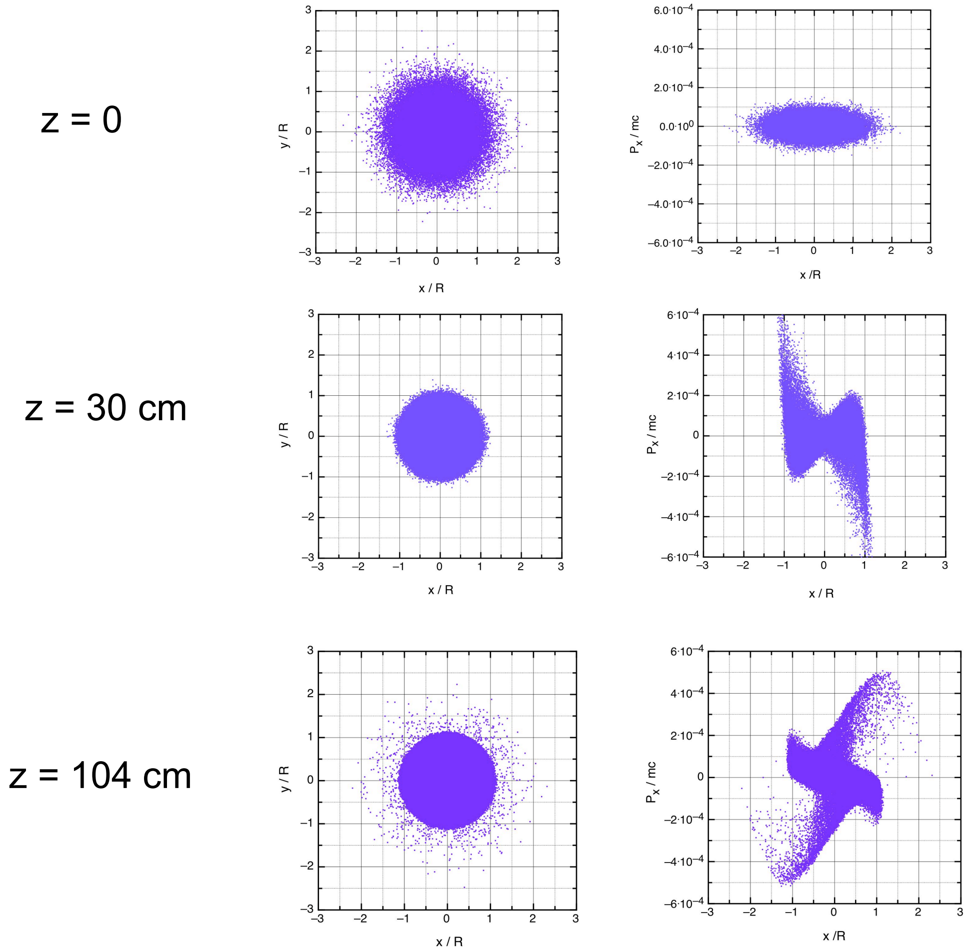


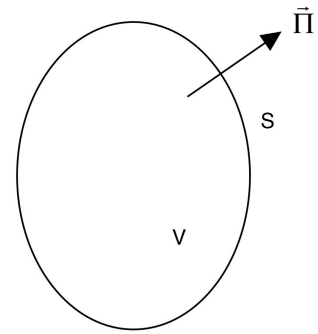
Fig. 3.7. Injection of 135 keV, 100 mA, 0.07 π cm mr ad proton beam with Gaussian distribution in a focusing channel with linear field. It results in (a) beam uniforming

(b) beam emittance growth

(c) halo formation.

Conservation of energy for electromagnetic field (Umov-Poynting's theorem)

$$\oint_S [\vec{E},\vec{H}]\, d\vec{S} = -\frac{d}{dt}\int_V \left(\frac{\mu_o H^2}{2} + \frac{\varepsilon_o E^2}{2}\right) dV - \int_V \vec{j}\vec{E}\, dV \qquad (3.52)$$



Expression on the left side is an integral of Poynting's vector

$$\vec{\Pi} = [\vec{E},\vec{H}] \qquad (3.53)$$

over surface *S* surrounding volume *V* and is equal to the power of electromagnetic irradiation, or energy of electromagnetic field coming through the surface *S* per second. The first integral in right side of Eq. (3.52) is a change of energy of electromagnetic field per second:

$$\frac{dW}{dt} = \frac{d}{dt}\int_V \left(\frac{\mu_o H^2}{2} + \frac{\varepsilon_o E^2}{2}\right) dV \qquad (3.54)$$

where electromagnetic energy in volume *V* is

$$W = \frac{1}{2}\int_V (\mu_o H^2 + \varepsilon_o E^2) dV \qquad (3.55)$$

Second term in right side of Eq. (3.52) can be expressed as a sum over all charges in the beam

$$\int_V \vec{j}\vec{E}\, dV = \int_V \rho \vec{v}\vec{E}\, dV = \sum q\vec{v}\vec{E} \tag{3.56}$$

Change of kinetic energy $W_{kin} = mc^2(\gamma - 1)$ of particle in time is

$$\frac{dW_{kin}}{dt} = mc^2 \frac{d\gamma}{dt} \tag{3.57}$$

where derivative of reduced particle energy $\gamma = \sqrt{1 + (p / mc)^2}$ over time is

$$\frac{d\gamma}{dt} = \frac{1}{\gamma (mc)^2} \vec{p}\frac{d\vec{p}}{dt} = \frac{1}{mc^2} \vec{v}\frac{d\vec{p}}{dt} = \frac{1}{mc^2} q\vec{v}\vec{E} \tag{3.58}$$

Therefore,

$$q\vec{v}\vec{E} = \frac{dW_{kin}}{dt} \tag{3.59}$$

and second term, Eq. (3.52), is the change of kinetic energy of the beam in time:

$$\sum q\vec{v}\vec{E} = \sum \frac{dW_{kin}}{dt} \tag{3.60}$$

Consider non-relativistic case (no magnetic field):

$$\frac{d}{dt}\left(\frac{\varepsilon_o}{2}\int E^2\,dV+\sum_{i=1}^{N}W_{kin}\right)=0\ ,\tag{3.61}$$

where E is the total electrostatic field in the structure, and W_{kin} is the kinetic energy of particles:

$$W_{kin}=mc^2\sqrt{1+\frac{p_x^2+p_y^2+p_z^2}{(mc)^2}}\approx mc^2\gamma+\frac{p_x^2+p_y^2}{2m\gamma}\tag{3.62}$$

where summation is performed over all particles of the beam. Assume that energy is the same for all particles, and is not changed during beam transport. Below consider only transverse particle motion and kinetic energy, associated with this motion. According to definition of rms beam values, kinetic energy of particles is:

$$\sum_{i=1}^{N}W_{kin}=\frac{N}{2m\gamma}\left[<p_x^2>+<p_y^2>\right]\ .\tag{3.63}$$

where rms value of transverse momentum is $$<p_x^2>=\left(\frac{mc\varepsilon}{2R}\right)^2\ .\tag{3.64}$$

In a round beam rms values in both transverse directions are the same, $<p_x^2>=<p_y^2>$, therefore

$$\sum_{i=1}^{N}W_{kin}=N\frac{mc^2}{\gamma}\left(\frac{\varepsilon}{2R}\right)^2\ .\tag{3.65}$$

We consider continuous beam, therefore Eq. (3.61) can be rewritten as

$$L_b \frac{\varepsilon_o}{2}\int_o^{\infty} E^2\, dS + N\frac{mc^2}{\gamma}\left(\frac{\varepsilon}{2R}\right)^2 = const \quad , \tag{3.66}$$

where L_b is an arbitrary length along the beam, containing N particles. Using beam current $I = q\beta cN/L_b$, Eq. (3.66) becomes:

$$\frac{4q\gamma\,\beta c}{mc^2 I}\left(\frac{\varepsilon_o}{2}\int_0^{\infty} E^2 dS\right) + \left(\frac{\varepsilon}{R}\right)^2 = const \tag{3.67}$$

Applying the last equation to the initial and final beam, one has,

$$\frac{\varepsilon_f^2}{\varepsilon_i^2} = \frac{R_f^2}{R_o^2} + \frac{4q\gamma\,\beta c R_f^2}{mc^2 I\varepsilon_i^2}\left(\frac{\varepsilon_o}{2}\int_o^{\infty} E_i^2\, dS - \frac{\varepsilon_o}{2}\int_o^{\infty} E_f^2\, dS\right) . \tag{3.68}$$

Eq. (3.68) can be rewritten as

$$\frac{\varepsilon_f^2}{\varepsilon_i^2}=\frac{R_f^2}{R_o^2}\left(1+b\frac{W_i - W_f}{W_o}\right), \qquad (3.69)$$

where initial, W_i, and final, W_f, energy stored in electrostatic field are

$$W_i=\frac{\varepsilon_o}{2}\int_o^{\infty} E_i^2\, dS \quad W_f=\frac{\varepsilon_o}{2}\int_o^{\infty} E_f^2\, dS, \qquad (3.70)$$

and normalization constant is

$$W_o = 2\pi\varepsilon_o\left(\frac{I}{I_c}\frac{mc^2}{q\beta\gamma}\right)^2 \qquad (3.71)$$

If the beam is initially rms-matched, then the rms beam radius is changing insignificantly, so we can put $R_f \approx R_o$. Additionally, taking into account expression

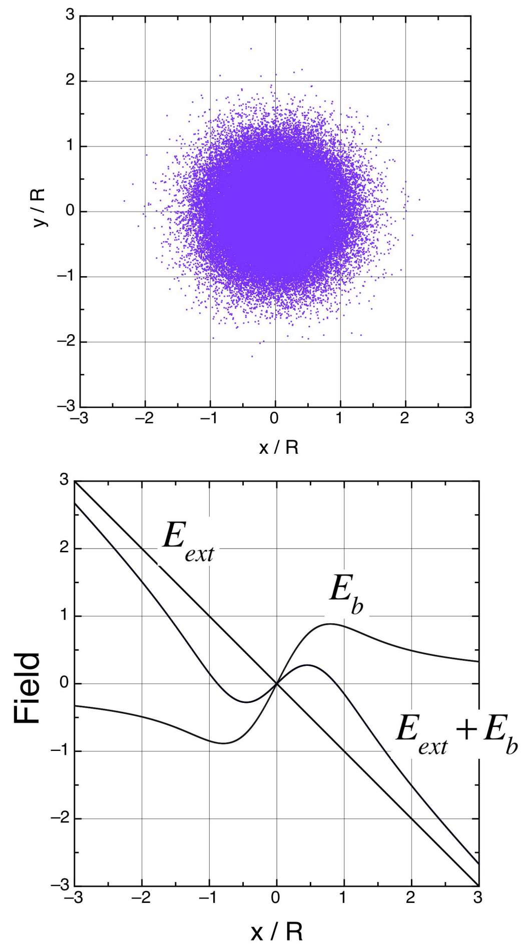
$$b=\frac{\mu_o^2}{\mu^2}-1$$

one can write:

$$\frac{\varepsilon_f}{\varepsilon_i}=\sqrt{1+\left(\frac{\mu_o^2}{\mu^2}-1\right)\left(\frac{W_i - W_f}{W_o}\right)}\ . \qquad (3.72)$$

In emittance-dominated regime $\mu \approx \mu_o$, and Eq. (3.72) gives us conservation of beam emittance. Consider space charge dominated regime. Initial total field E_i is given by:

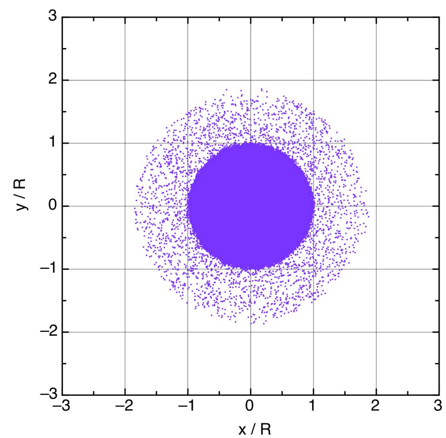
$$E_i = \frac{mc^2}{qR\gamma}\frac{2I}{\beta\gamma I_c}\{-\frac{r}{R}+\frac{R}{r}[1-exp(-\frac{2r^2}{R^2})\}. \qquad (3.73)$$

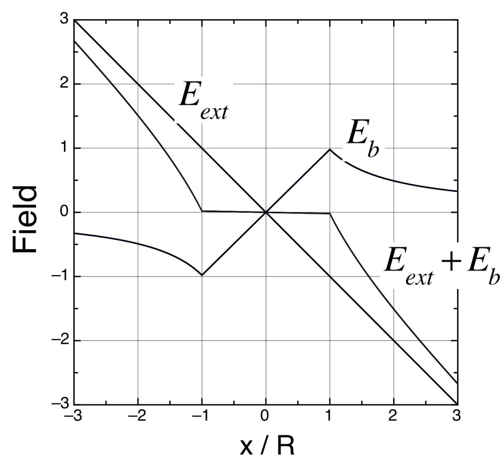


3. External focusing field E_{ext}, space charge field of Gaussian beam E_b, and total field E_{ext} + E_b at initial moment of time.

Final beam distribution is close to uniform with the same value of beam radius R. It is a general property of space-charge dominated regime, that self-field of the beam almost compensates for external field within the beam. We can put $E_f \approx 0$ within the beam and $E_f = E_{ext} + E_b$ outside the beam

$$E_f = \begin{cases} 0, & r \le R \\ \dfrac{mc^2}{qR}\dfrac{2I}{\beta\gamma^2 I_c}\left(-\dfrac{r}{R}+\dfrac{R}{r}\right), & r > R \end{cases} \quad . \tag{3.74}$$





3. External focusing field E_{ext}, space charge field E_b, and total field E_{ext} + E_b after beam uniforming.

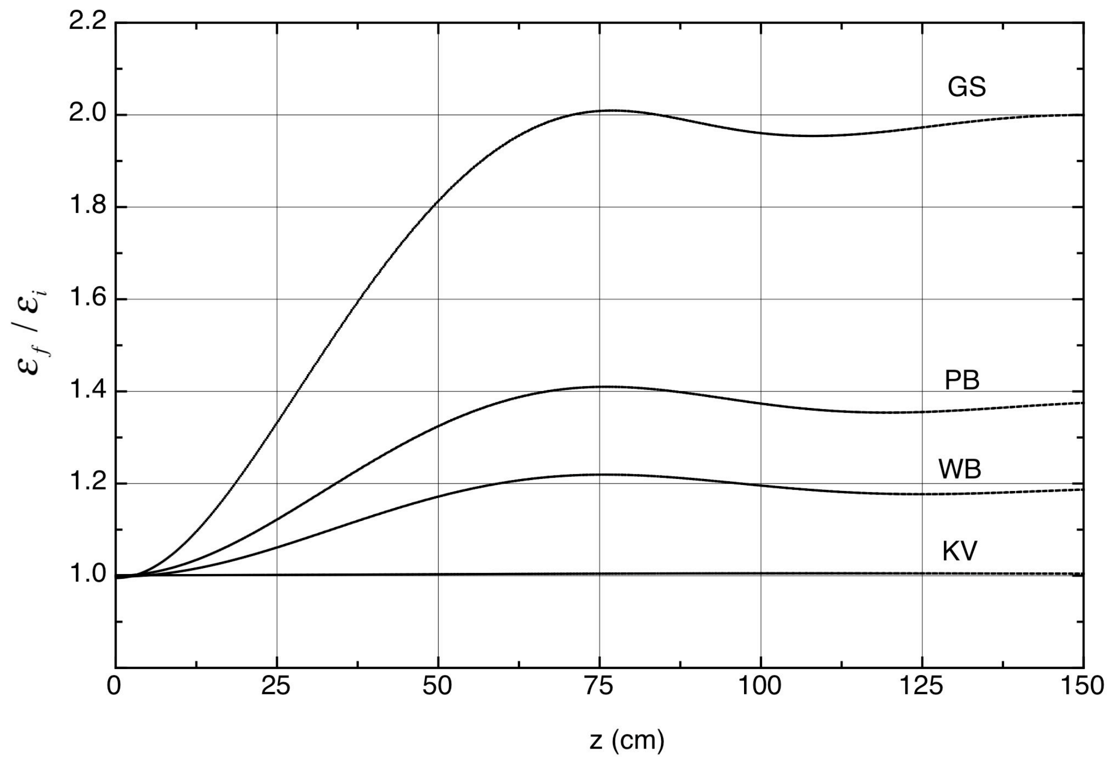
Substitution of E_f and E_i into Eq.(3.70) gives for

$$\frac{W_f - W_i}{W_o} = \int_o^{\xi_{max}} [-\xi + \frac{1}{\xi}(1 - e^{-2\xi^2})]^2 \xi d\xi - \int_1^{\xi_{max}} (-\xi + \frac{1}{\xi})^2 \xi d\xi \approx 0.077 \qquad (3.75)$$

where $\xi = r / R$. In Eq. (3.75) the upper limit of integration is arbitrary and usually is determined by the aperture of the channel, $\xi_{max} = a / R$.

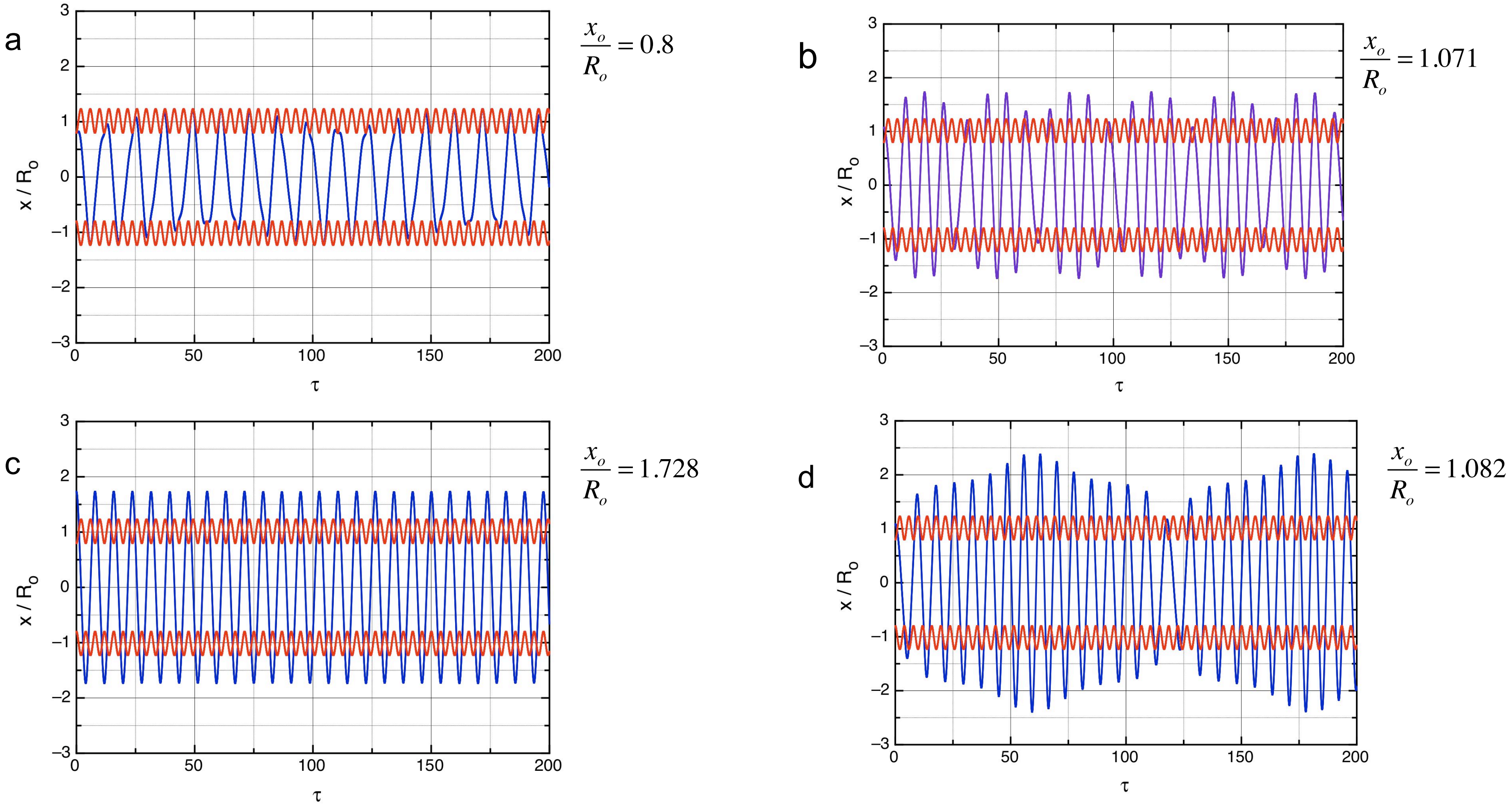
Free energy parameter for different beam distributions

4D Distribution	2D Projection	$\frac{W_i - W_f}{W_o}$
KV	ρ_o	0
Water Bag	$\rho_o(1 - \frac{r^2}{R^2})$	0.01126
Parabolic	$\rho_o(1 - \frac{r^2}{R^2})^2$	0.02366
Gaussian	$\rho_o \exp(-\frac{r^2}{R^2})$	0.077



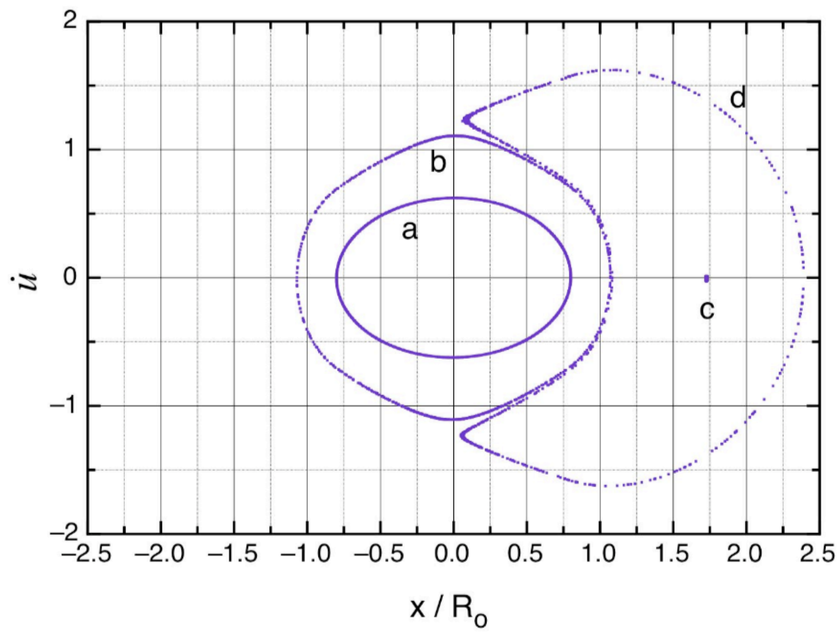
Beam emittance growth in a uniform focusing channel for different particle distributions.

3.6 Halo formation



Envelope oscillations of the beam with space charge parameter b=3, amplitude Δ= 0.2 and single particle trajectories with initial conditions (a) x_o/R_o=0.8, (b) x_o/R_o =1.071, (c) x_o/R_o =1.728, (d) x_o/R_o =1.082.

Stroboscopic particle motion



Stroboscopic particle trajectories at phase plane $(u, du/d\tau)$ taken after each two envelope oscillation periods: (a) x_o/R_o=0.8, (b) x_o/R_o =1.071, (c) x_o/R_o=1.728, (d) x_o/R_o =1.082.

Dimensionless envelope equation for round beam

Envelope equation for round beam
$$\frac{d^2R}{dz^2} - \frac{\varepsilon^2}{R^3} + \left(\frac{\mu_o}{L}\right)^2 R - \frac{2\,I}{I_c\,\beta^3\gamma^3 R} = 0$$

Let us multiply everything by
$$\frac{1}{R_e}\left(\frac{L}{\mu_o}\right)^2$$

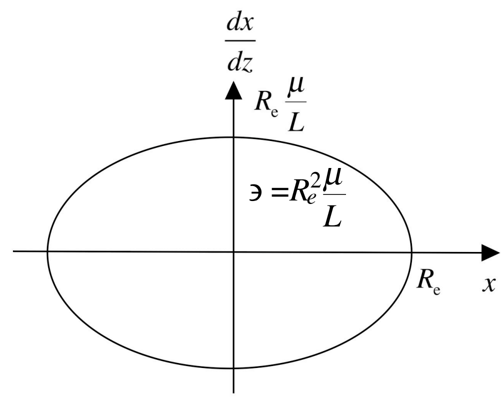
Dimensionless time $\tau = \mu_o \dfrac{z}{L}$

First term
$$\frac{d^2\left(\frac{R}{R_e}\right)}{d\left(\mu_o \frac{z}{L}\right)^2} = \frac{d^2\left(\frac{R}{R_e}\right)}{d\tau^2}$$

Second term
$$\frac{\varepsilon^2}{R^3 R_e}\left(\frac{L}{\mu_o}\right)^2 = \frac{1}{\left(\frac{R}{R_e}\right)^3}\left(\frac{\varepsilon L}{R_e^2\mu_o}\right)^2 = \frac{1}{\left(\frac{R}{R_e}\right)^3}\left(\frac{\mu}{\mu_o}\right)^2 = \frac{1}{\left(\frac{R}{R_e}\right)^3}\frac{1}{(1+b)}$$

$$\left(\frac{\mu}{\mu_o}\right)^2 = \frac{1}{1+b}$$

Recall:
$$\frac{d^2x}{dz^2} + \left(\frac{\mu}{L}\right)^2 x = 0$$



Third term:

$$(\frac{\mu_o}{L})^2 \frac{R}{R_e}(\frac{L}{\mu_o})^2 = \frac{R}{R_e}$$

Fourth term:

$$\frac{2\,I}{I_c\,(\beta\gamma)^3 R R_e}(\frac{L}{\mu_o})^2 = \frac{2\,I}{I_c(\beta\gamma)^3}\frac{1}{(\frac{R}{R_e})}(\frac{L R_e}{\mu_o R_e^2})^2 = \frac{2\,I R_e^2}{I_c(\beta\gamma)^3 \text{э}^2}\frac{1}{(\frac{R}{R_e})}(\frac{\mu}{\mu_o})^2 = \frac{1}{(\frac{R}{R_e})}\frac{b}{(1+b)}$$

Finally:

$$\frac{d^2(\frac{R}{R_e})}{d\tau^2} + \frac{R}{R_e} - \frac{1}{(1+b)(\frac{R}{R_e})^3} - \frac{b}{(1+b)(\frac{R}{R_e})} = 0$$

Dimensionless single-particle equation of motion

Hamiltonian of particle motion in uniform focusing channel

$$H = \frac{p_x^2 + p_y^2}{2\, m\gamma} + m\gamma\Omega_r^2 \frac{(x^2+y^2)}{2} + q\frac{U_b}{\gamma^2}$$

Equation of particle motion

$$\frac{d^2x}{dt^2} + \Omega_r^2 x - \frac{q}{m\gamma^3} E_b = 0$$

Space charge field

$$E_b = \frac{I}{2\pi\varepsilon_o \beta c} \begin{cases} \frac{x}{R^2}, & r \le R \\ \frac{1}{x}, & r > R \end{cases}$$

Frequency of transverse oscillations

$$\Omega_r = \beta c \frac{\mu_o}{L}$$

Let us multiply equation of motion by

$$\frac{1}{R_e}\left(\frac{L}{\mu_o \beta c}\right)^2$$

Beam envelope (core) equation:

$$\frac{d^2r}{d\tau^2} + r - \frac{1}{(1+b)r^3} - \frac{b}{(1+b)r} = 0$$

Single particle equation of motion:

$$\frac{d^2u}{d\tau^2} + u = \begin{cases} \frac{b}{(1+b)}\frac{u}{r^2}, & |u| \le r \\ \frac{b}{(1+b)u}, & |u| > r \end{cases}$$

Dimensionless envelope $r = \frac{R}{R_e}$

Dimensionless coordinate $u = \frac{x}{R_e}$

Space charge parameter

$$b = \frac{2}{\beta\gamma}\frac{I}{I_c}\frac{R_e^2}{\varepsilon^2}$$

I beam current

$I_c = 4\pi\varepsilon_o mc^3 / q$ characteristic beam current

ε normalized beam emittance

β particles velocity,

γ particle energy

R_o radius of the equilibrium envelope

Small intensity beam $b \approx 0$

High intensity beam b >> 1

Envelope oscillations

Envelope equation $$\frac{d^2r}{d\tau^2}+r-\frac{1}{(1+b)r^3}-\frac{b}{(1+b)r}=0$$

Expansions $r=1+\vartheta$ $\quad \frac{1}{r}\approx 1-\vartheta$ $\quad \frac{1}{r^3}\approx 1-3\vartheta$ $\quad \frac{d^2\vartheta}{d\tau^2}+2(\frac{2+b}{1+b})\vartheta=0$

Equation for small deviation from equilibrium

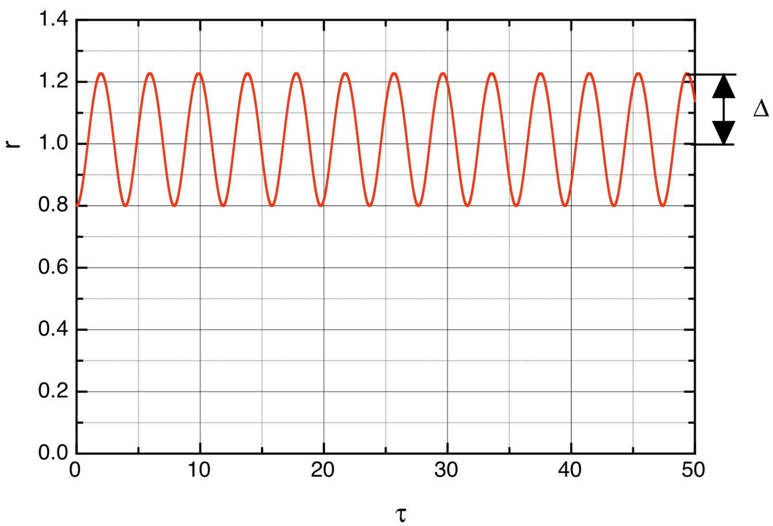
$$r=1+\Delta\cos(2\Omega\tau)$$

Envelope oscillation frequency

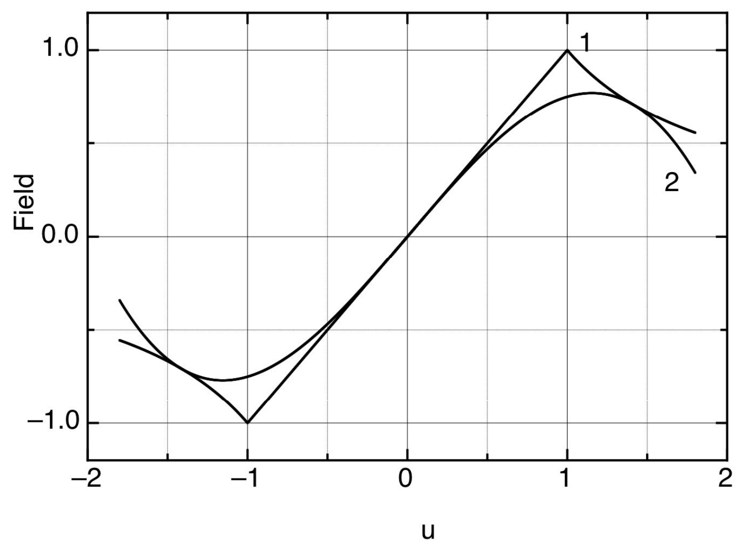
$$2\Omega=\sqrt{2(\frac{2+b}{1+b})}$$

For small intensity beam $b\approx 0$ $\quad r=1+\Delta\cos 2\tau$

For high intensity beam b >>1 $\quad r=1+\Delta\cos\sqrt{2}\tau$



Approximation of space charge field (R.Gluckstern, 1994)



(1) Field of uniformly charged beam

$$F = \frac{b}{(1+b)}\left\{ \begin{array}{ll} \frac{u}{r^2}, & |u| \le r \\ \frac{1}{u}, & |u| > r \end{array} \right.$$

(2) Field approximation:

$$F = \frac{b}{(1+b)}\left(-\frac{u}{r^2} + \frac{u^3}{4}\right)$$

3.

Anharmonic oscillator with parametric excitation for single particle motion

With field approximation, equation of particle motion is

$$\frac{d^2u}{d\tau^2}+u-\left(\frac{b}{1+b}\right)\left[\frac{u}{(1+\Delta\cos 2\Omega\tau)^2}-\frac{u^3}{4}\right]=0$$

Using expansion

$$\frac{1}{(1+\Delta\cos 2\Omega\tau)^2}\approx 1-2\Delta\cos 2\Omega\tau$$

Equation of particle motion

$$\frac{d^2u}{d\tau^2}+u\left(\frac{1}{1+b}\right)(1+2b\Delta\cos 2\Omega\tau)+\left(\frac{b}{1+b}\right)\frac{u^3}{4}=0$$

Equation corresponds to Hamiltonian

$$H=\frac{\dot{u}^2}{2}+\varpi^2\frac{u^2}{2}(1-h\cos 2\Omega\tau)+\alpha\frac{u^4}{4}$$

with the following notations

$$\varpi^2=\frac{1}{1+b} \qquad h=-2b\Delta \qquad \alpha=\frac{b}{4\,(1+b)}$$

Canonical transformation of Hamiltonian

Change the variables ($\dot{u}$, u) to new variables (Q, P) using a generating function

$$F_2(u,P,\tau) = \frac{uP}{\cos\Omega\tau} - \left(\frac{P^2}{2\varpi} + \varpi\frac{u^2}{2}\right)tg\Omega\tau$$

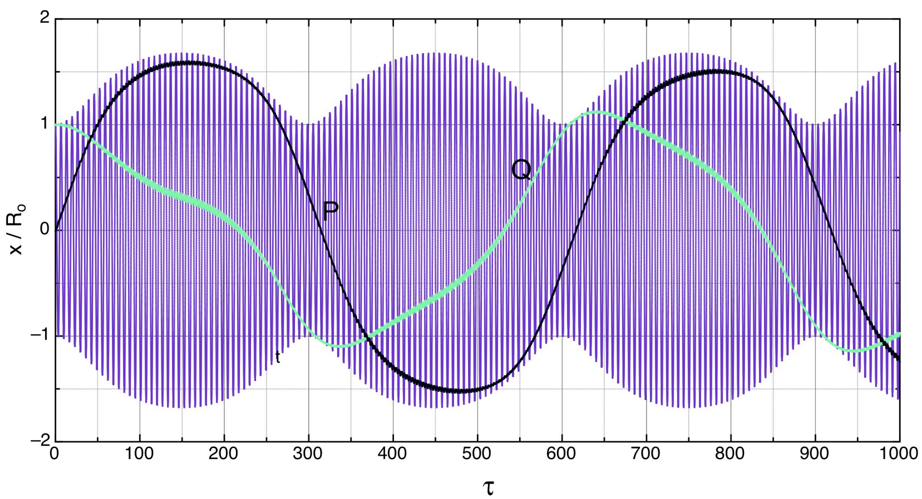
Relationships between variables are given by:

$$\begin{cases} Q = \dfrac{\partial F_2}{\partial P} = \dfrac{u}{cos\ \Omega\ \tau} + \dfrac{P}{\varpi}\ tg\Omega\ \tau \\ \dot{u} = \dfrac{\partial F_2}{\partial u} = \dfrac{P}{cos\ \Omega\ \tau} - \varpi\ u\ tg\Omega\ \tau \end{cases}$$

or

$$\begin{cases} u = Q\ cos\,\Omega\ \tau + \dfrac{P}{\varpi}\ sin\,\Omega\ \tau \\ \dot{u} = -\ \varpi\ Q\ sin\,\Omega\ \tau + P\,cos\ \Omega\ \tau \end{cases}$$

3.



Averaged Hamiltonian

New Hamiltonian $K = H + \frac{\partial F_2}{\partial \tau}$

$$K = \frac{P^2}{2} + \varpi^2 \frac{Q^2}{2} - \frac{\varpi^2 h}{2}(Q\cos\Omega\tau + \frac{P}{\varpi}\sin\Omega\tau)^2 \cos 2\Omega\tau + \frac{\alpha}{4}(Q\cos\Omega\tau + \frac{P}{\varpi}\sin\Omega\tau)^4 - \frac{P^2\Omega}{2\varpi} - \frac{\Omega\varpi}{2}Q^2$$

After averaging all time-dependent terms over period of $2\pi/\Omega$

$$\bar{K} = \frac{\varpi^2 \bar{Q}^2}{2}(1 - \frac{\Omega}{\varpi} - \frac{h}{4}) + \frac{\bar{P}^2}{2}(1 - \frac{\Omega}{\varpi} + \frac{h}{4}) + \frac{3}{32}\alpha(\bar{Q}^2 + \frac{\bar{P}^2}{\varpi^2})^2$$

Second canonical transformation

Change variables $(\bar{Q},\bar{P})$ to action-angle variables $(J,\ \psi)$ using generating function

$$F_1(\bar{Q},\psi)=\frac{\varpi\,\bar{Q}^2}{2tg\psi}$$

Transformation is given by

$$\left\{\begin{array}{l}\bar{Q}=\sqrt{\dfrac{2J}{\varpi}}\sin\psi\\ \bar{P}=\sqrt{2J\varpi}\cos\psi\end{array}\right.$$

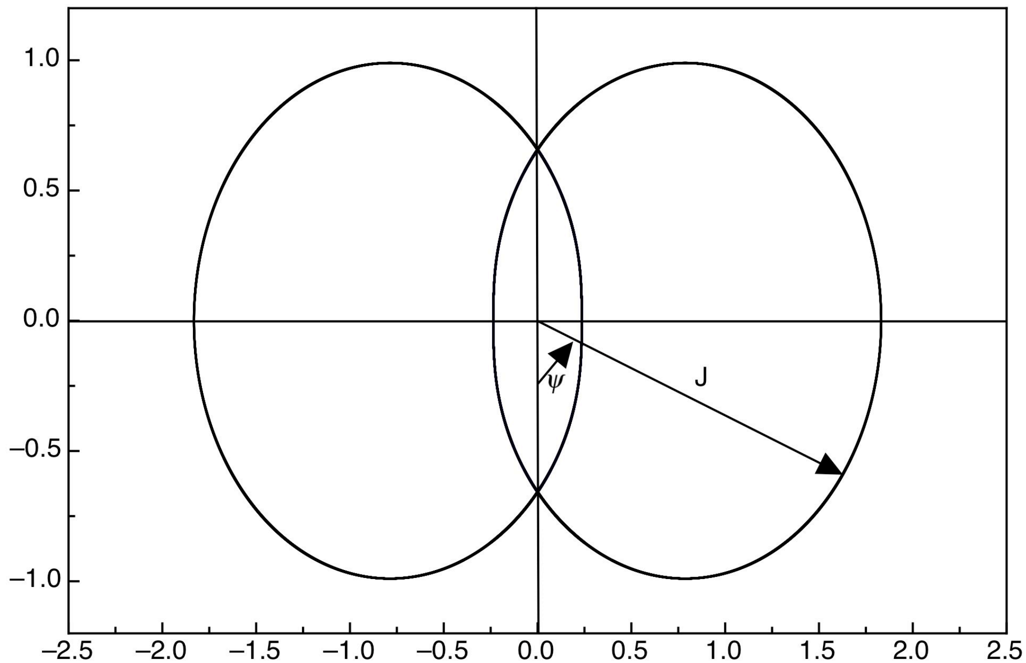
New Hamiltonian

$$\boxed{\bar{K}=\upsilon J+\kappa J^2+2\chi J\cos 2\psi}$$

with the following notations

$$\upsilon=\varpi-\Omega=\frac{\sqrt{2}-\sqrt{2+b}}{\sqrt{2(1+b)}}\qquad \kappa=\frac{3}{32}b\qquad \chi=-\frac{1}{4}\frac{b\Delta}{\sqrt{1+b}}$$

Nonlinear parametric resonance

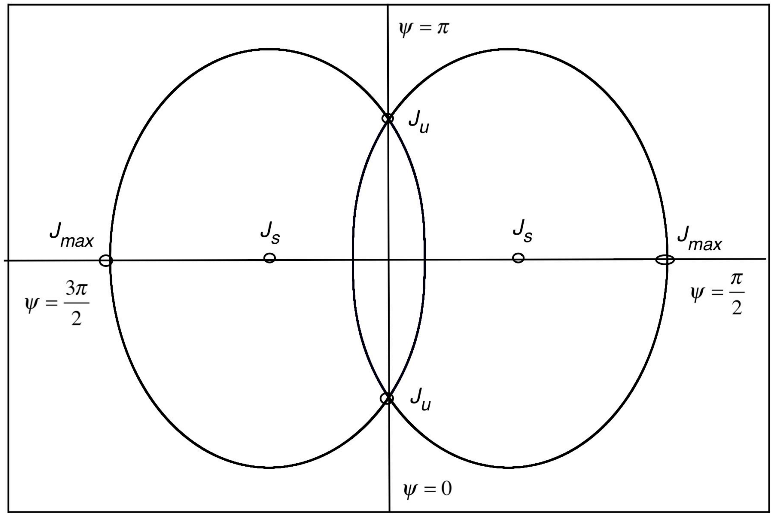


Lines of equal values of Hamiltonian $\bar{K} = \upsilon J + \kappa J^2 + 2\chi J \cos 2\psi = -4.079 \cdot 10^{-2}$

corresponding to the beam with space charge parameter b = 1, core amplitude $\Delta = -0.1$

and parameters $\upsilon = -0.159,\ \kappa = 0.09445, \chi = 0.01777$

Fixed points at phase plane



Fixed points:

$$\frac{dJ}{d\tau} = -\frac{\partial \bar{K}}{\partial \psi} = 4\chi J \sin 2\psi = 0 \longrightarrow \sin 2\psi = 0$$

$$\frac{d\psi}{d\tau} = \frac{\partial \bar{K}}{\partial J} = \upsilon + 2\kappa J + 2\chi \cos 2\psi = 0$$

Unstable points $\quad cos 2\psi = 1 \quad \psi = 0, \pi \quad J_u = -\frac{\upsilon + 2\chi}{2\kappa}$

Stable points $\quad cos 2\psi = -1 \quad \psi = \frac{\pi}{2}, \frac{3\pi}{2} \quad J_s = \frac{-\upsilon + 2\chi}{2\kappa}$

Value of Hamiltonian at unstable point $\bar{K}_u = \bar{K}(J_u) \quad \bar{K}_u = -\frac{(\upsilon + 2\chi)^2}{4\kappa}$

Maximum value of variable *J*

Particle with the value of Hamiltonian $\bar{K}_u = \bar{K}(J_u)$ can reach the point J_{max} having $\psi = \pi/2$

Equation for J_{max}

$$\kappa J_{\max}^2 + J_{\max}(\upsilon - 2\chi) - \bar{K}_u = 0$$

$$J_{\max} = \frac{(-\upsilon + 2\chi) + \sqrt{8|\upsilon\chi|}}{2\kappa}$$

The value of J is connected with variables (u, $du/d\tau$) as

$$J = \frac{1}{2}(u^2\varpi + \frac{\dot{u}^2}{\varpi})$$

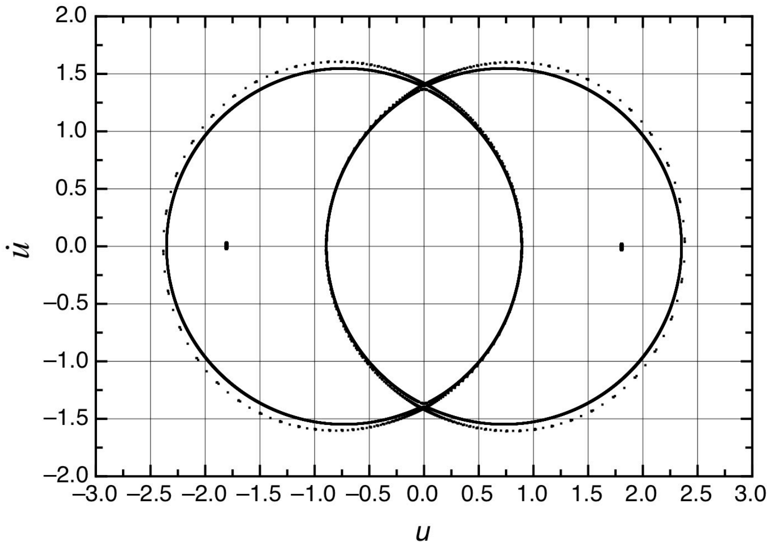
Maximum deviation of particle from the axis

Maximum value of particle deviation from the axis is $u_{\max} = \sqrt{\frac{2J_{\max}}{\varpi}}$ or

$$\frac{x_{\max}}{R_e} = \sqrt{\frac{32}{3}\frac{\sqrt{1+\frac{b}{2}} - 1 + \frac{b|\Delta|}{2} + \sqrt{2b|\Delta|(\sqrt{1+\frac{b}{2}} - 1)}}{b}}$$

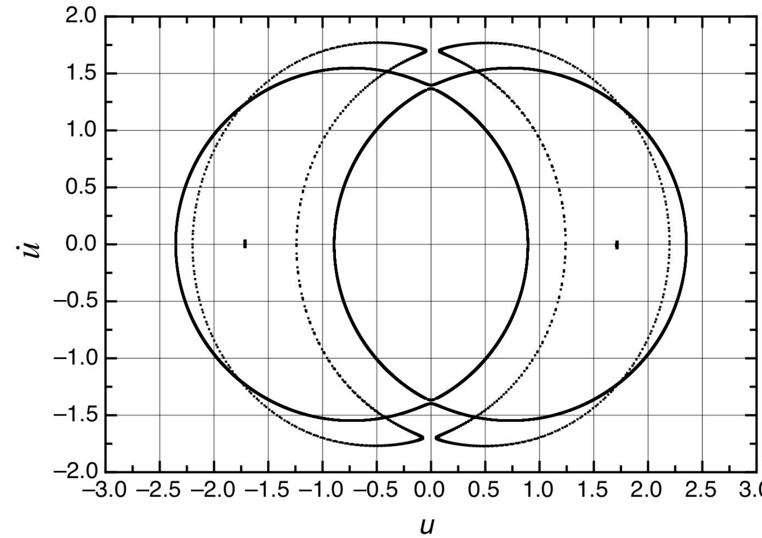
3.

Comparison of analytical and numerical results: low intensity beam



Numerical integration in approximate space charge field

$$F = \frac{b}{(1+b)}\left(-\frac{u}{r^2} + \frac{u^3}{4}\right)$$

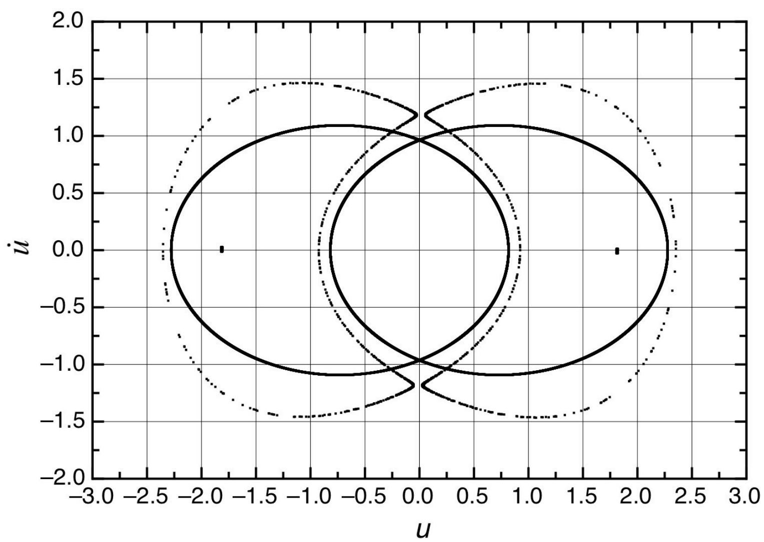


Numerical integration in space charge field of uniform core

$$F = \frac{b}{(1+b)}\begin{cases} \frac{u}{r^2}, & |u| \le r \\ \frac{1}{u}, & |u| > r \end{cases}$$

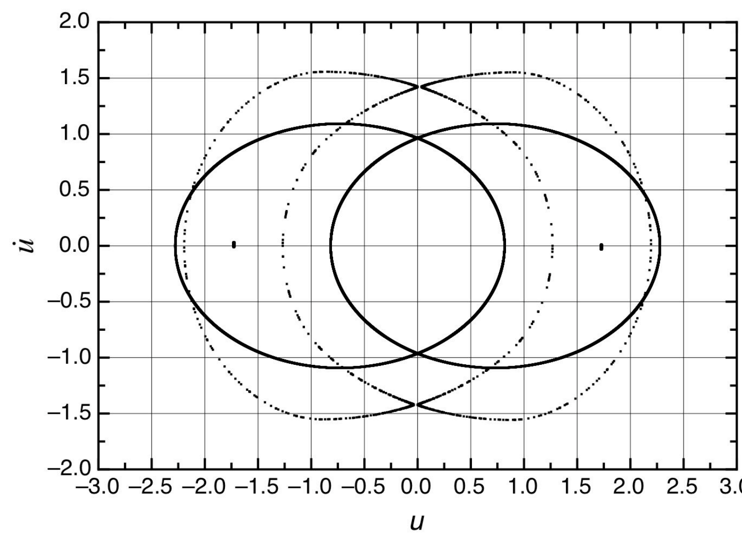
(Solid) analytical and (dotted) numerical results of averaged phase space trajectories, b=0.1, Δ= - 0.1.

Comparison of analytical and numerical results: moderate intensity beam



Numerical integration in approximate space charge field

$$F = \frac{b}{(1+b)}\left(-\frac{u}{r^2} + \frac{u^3}{4}\right)$$

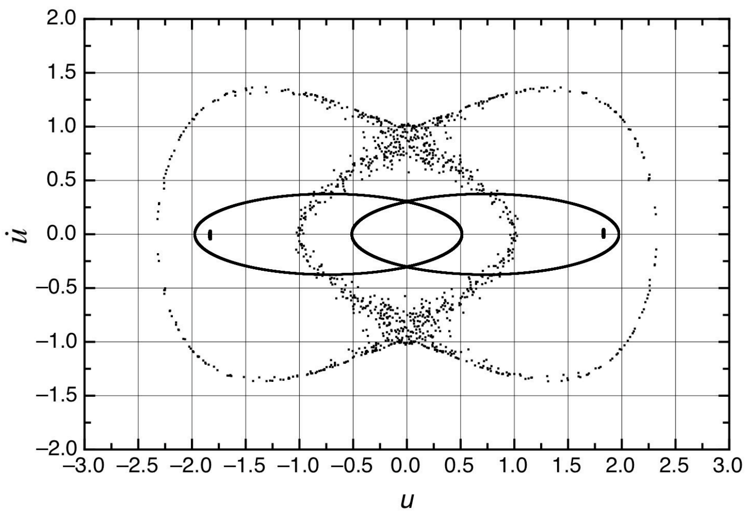


Numerical integration in space charge field of uniform core

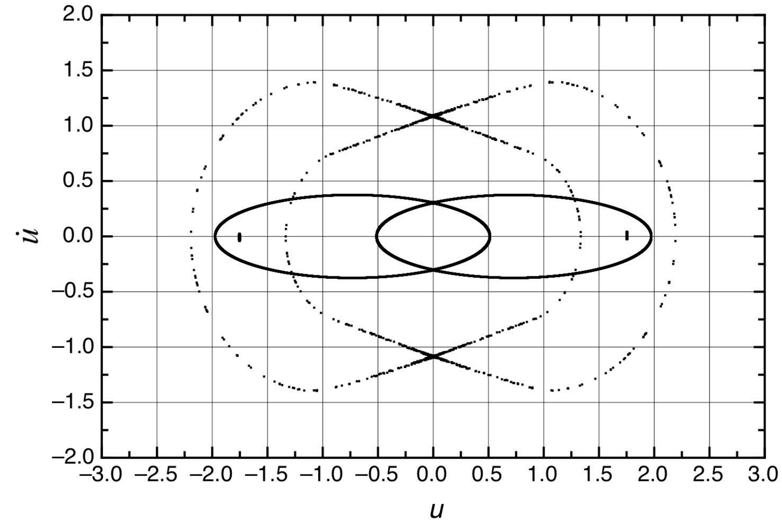
$$F = \frac{b}{(1+b)}\begin{cases} \frac{u}{r^2}, & |u| \le r \\ \frac{1}{u}, & |u| > r \end{cases}$$

(Solid) analytical and (dotted) numerical results of averaged phase space trajectories, b=1, $\Delta = 0.1$

Comparison of analytical and numerical results: high intensity beam



Numerical integration in approximate space charge field



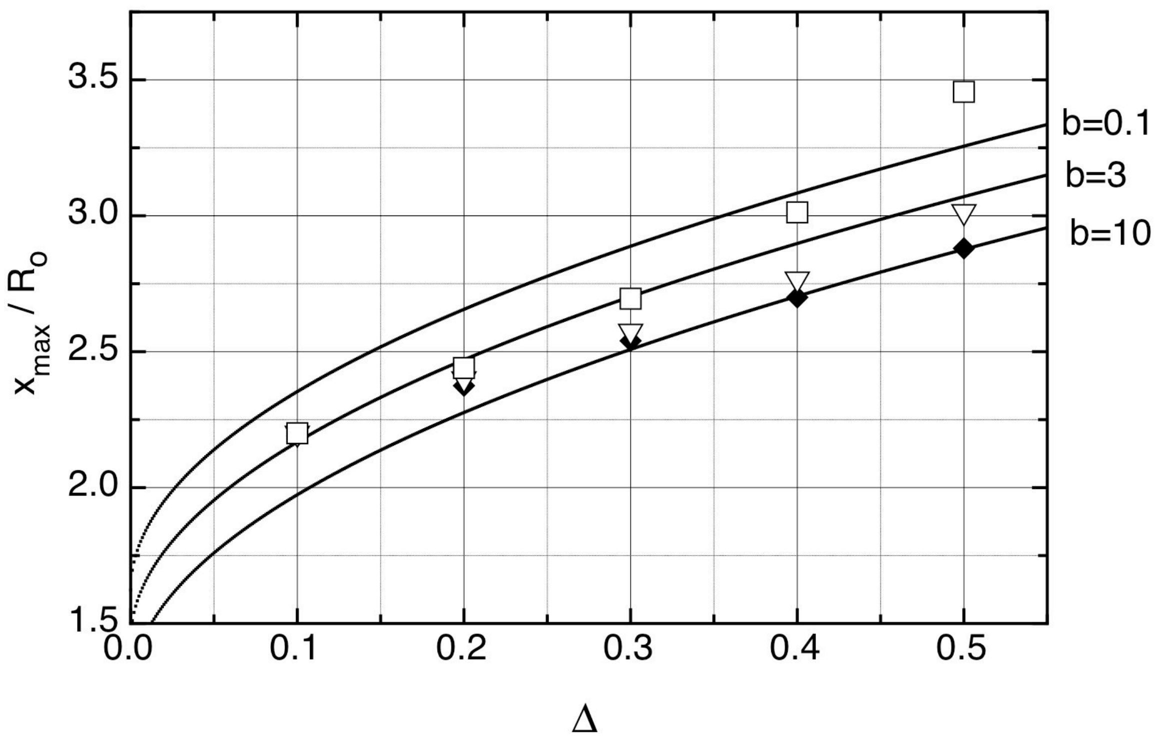
Numerical integration in space charge field of uniform core

$$F = \frac{b}{(1+b)}\left(-\frac{u}{r^2} + \frac{u^3}{4}\right)$$

$$F = \frac{b}{(1+b)}\begin{cases} \dfrac{u}{r^2}, & |u| \le r \\ \dfrac{1}{u}, & |u| > r \end{cases}$$

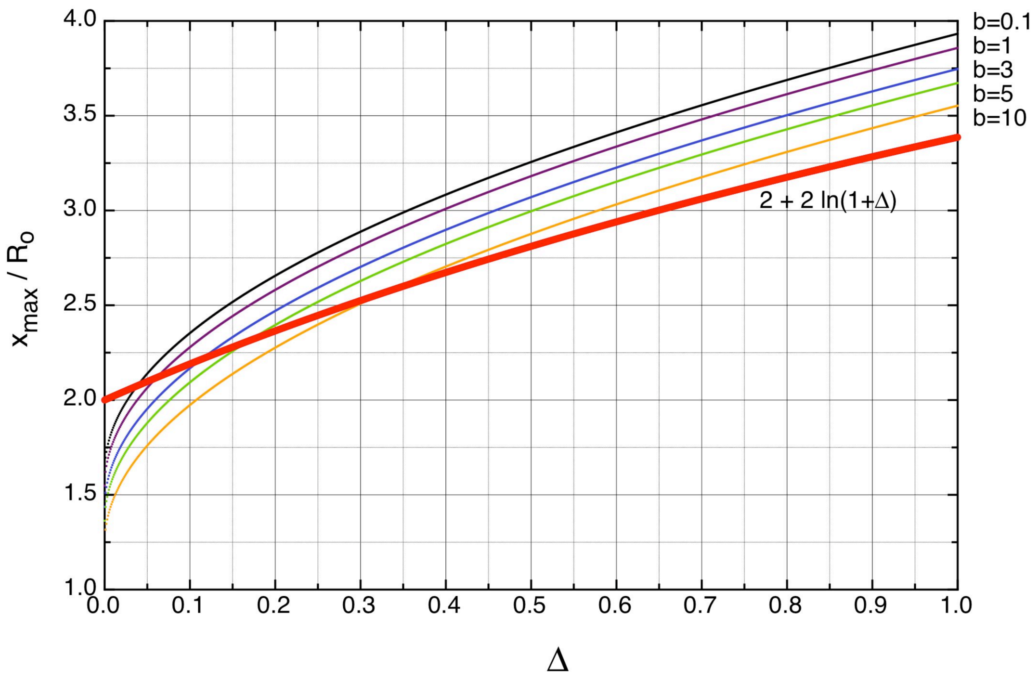
(Solid) analytical and (dotted) numerical results of averaged phase space trajectories, b=10, $\Delta = 0.1$

Comparison of analytical and numerical results (cont.)



Comparison of analytical (solid lines) and numerical simulation of maximum particle deviation from the axis: (black) $b = 10$, (triangle) $b = 3$, (square) $b = 0.1$.

Comparison of analytical and numerical results (cont.)



Maximum values of particle deviation from the axis as a function of amplitude of core oscillations. (Red) model of Tom Wangler (*RF Linear Accelerators*, Wiley, 1998)

$$\frac{x_{\max}}{R_o/2} = A + B\ln(\mu)$$

where $A = B = 4$, $\mu = 1+\Delta$.

3.7. Non-uniform beam equilibrium

Non-uniform beam in general case is intrinsically mismatched with linear focusing channel. Meanwhile, it is possible to find matched solution for non-uniform beam without emittance growth, if we refuse from linearity of focusing field.

Assume that the beam is propagating in continuously focusing channel, and is matched with the channel. Hence, the Hamiltonian is a constant of motion:

$$H = \frac{p_x^2 + p_y^2}{2m\gamma} + q\, U(x,y) = const\,. \qquad (4.3)$$

The total potential of the structure is a combination of the external focusing potential, U_{ext}, and the space charge potential U_b of the beam, $U = U_{ext} + U_b \gamma^{-2}$. The time-independent distribution function of a matched beam obeys Vlasov's equation, where the partial derivative of the distribution function over time is equal to zero due to assumption of a matched beam:

$$\frac{1}{m\gamma}\left(\frac{\partial f}{\partial x} p_x + \frac{\partial f}{\partial y} p_y\right) - q\left(\frac{\partial f}{\partial p_x}\frac{\partial U}{\partial x} + \frac{\partial f}{\partial p_y}\frac{\partial U}{\partial y}\right) = 0\,. \qquad (4.4)$$

Eq. (4.4) can be solved to find the total potential of the structure, U, as a function of beam distribution function $f\,(x, p_x, y, p_y)$. The distribution function of the beam is supposed to be given. Therefore, the self - potential of the beam U_b is also a known function derived from Poisson's equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U_b}{\partial r}\right) = -\frac{\rho(r)}{\varepsilon_o}, \tag{4.5}$$

Combining solutions of Vlasov's equation for total potential of the structure, U, and space charge potential of the beam, obtained from Poisson's equation, U_b, the external potential of the focusing structure can be found

$$U_{ext} = U - \frac{U_b}{\gamma^2}. \tag{4.6}$$

The solution of this problem is unique for every specific particle distribution.

Consider a z - uniform beam with Gaussian distribution function in four - dimensional phase space:

$$f = f_o \, exp(-2\frac{x^2 + y^2}{R^2} - 2\frac{p_x^2 + p_y^2}{p_o^2}). \tag{4.7}$$

This distribution has an elliptical phase space projection at every phase plane with normalized root-mean-square beam emittance:

$$\varepsilon = \frac{4}{m\ c}\sqrt{<x^2><p_x{}^2> - <xp_x>^2} = R\ \frac{p_o}{m\ c}\ . \tag{4.8}$$

Substituting the distribution function, Eq. (4.7), into Vlasov's equation yields an expression for the total unknown potential of the structure:

$$\frac{mc^2}{q}\frac{1}{\gamma}(x\,p_x + y p_y) = \frac{R^4}{\varepsilon^2}\ (p_x\frac{\partial U}{\partial x} + p_y\frac{\partial U}{\partial y}). \tag{4.9}$$

Vlasov's equation can be separated into two independent parts for x- and y- coordinates respectively:

$$\frac{\partial U}{\partial x} = \frac{mc^2\,\varepsilon^2}{\gamma\, q R^4}\ x\ , \qquad \frac{\partial U}{\partial y} = \frac{mc^2\,\varepsilon^2}{\gamma\, q R^4}\ y\ . \tag{4.10}$$

Combining solutions of Eq. (4.10), the total potential of the structure is a quadratic function of coordinates which creates linear focusing field E_{tot}:

$$U(x,y) = \frac{mc^2}{q}\frac{1}{\gamma}\frac{\varepsilon^2}{R^4}\left(\frac{x^2+y^2}{2}\right), \qquad (4.11)$$

$$E_{tot} = -\frac{mc^2}{q}\frac{1}{\gamma}\frac{\varepsilon^2}{R^4}\, r\,. \qquad (4.12)$$

The appearance of quadratic terms in the total potential of the structure is quite clear because phase space projections of the beam have elliptical shape which is conserved in linear field. The space charge field of the beam, E_b, is calculated from Poisson's equation using a known space charge density function of the beam ρ_b:

$$\rho_b = \rho_o\, exp\left(-2\frac{r^2}{R^2}\right), \qquad (4.13)$$

$$E_b = -\frac{\partial U_b}{\partial r} = \frac{I}{2\pi\,\varepsilon_o\,\beta\, c}\frac{1}{r}\left[1 - exp\left(-2\frac{r^2}{R^2}\right)\right], \qquad (4.14)$$

where $\rho_o = 2I/(\pi c\beta R^2)$ is the space charge density at the axis.

Subtraction of the space charge field from the total field of the structure gives the expression for the external focusing field of the structure which is required for conservation of the beam:

$$E_{ext} = -\frac{mc^2}{qR\gamma}\left[\frac{\varepsilon^2 r}{R^3} + 2\frac{I}{I_c\beta\gamma}\frac{R}{r}\left(1 - exp\left(-2\frac{r^2}{R^2}\right)\right)\right], \quad (4.15)$$

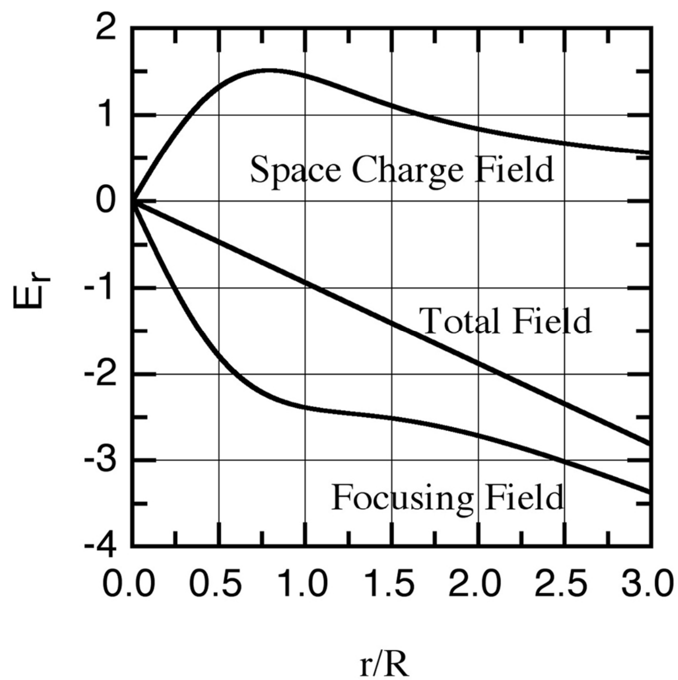
The relevant potential of the focusing field is given by the expression:

$$U_{ext}(r) = \frac{mc^2}{q\gamma}\left[\left(\frac{\varepsilon^2}{2R^4} + \frac{2I}{I_c\beta\gamma R^2}\right)r^2 + \frac{2I}{I_c\beta\gamma}\left(-\frac{r^4}{2R^4} + \frac{2}{9}\frac{r^6}{R^6} + \ldots + \frac{(-1)^{k+1}2^k\ r^{2k}}{2k\ k!\,R^{2k}}\right)\right]. \quad (4.16)$$

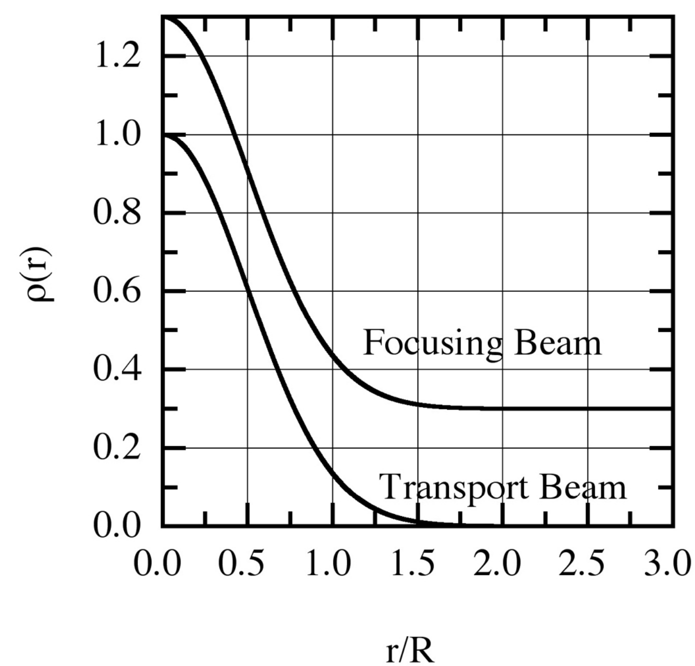
External potential of the structure, E q. (4.16), consists of two parts: quadratic (which produces linear focusing) and the part with higher order terms which describe nonlinear focusing. The linear part depends on the values of beam emittance and on the beam current, while the nonlinear part depends on beam current only. This means that the external field has t o compensate the nonlinearity of self-field of th e beam and produ ce required linear focusing of th e beam to kee p the elliptical beam phase space distribution.

Required potential distributio n can be crea ted by introducing inside the transport channel an opposite charged cloud of particles (plasma lens) with the space charge density:

$$\rho_{ext} = \rho_o \, exp(-2\frac{r^2}{R^2}) + \frac{I_c \, \varepsilon^2}{2\pi c \, R^4}. \qquad (4.17)$$



Total field of the structure E_{tot}, required external focusing field E_{ext}, and space-charge field of the Gaussian beam E_b.



Charged particle density of the transported beam with Gaussian distribution, and of the external focusing beam