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Space Charge Dominated Beam Transport and Acceleration

Yuri Batygin

Los Alamos National Laboratory

United States Particle Accelerator School

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Abstract

The course is intended to give a broad overview of self-consistent beam dynamics with strong space charge forces in beamlines and in Radio Frequency accelerators. Special emphasis is on the physics of high brightness beams in phase space. The topics include: Hamiltonian self-consistent dynamics of particles, equations of motion, emittance and brightness of the beam, beam transport in quadrupole focusing channel and in longitudinal magnetic field, averaging method in particle dynamics, Kapchinsky-Vladimirsky beam envelope equations, beam current limit in beamlines, nonlinear effects in beam transport, beam emittance growth due to space charge forces, halo formation in particle beams, beam equilibrium in focusing channels, space charge dominated beam in RF linear accelerators. The course consist of 23 hours of lectures, focusing on the theoretical understanding of the course content, as well as sessions on how to solve practical problems.

Contents

1. Preliminaries of beam dynamics
2. Beam focusing in transport channels
3. Nonlinear effects in beam transport
4. Space charge effects in RF linear accelerators
5. Computer classes
6. Problems

1. Preliminaries of beam dynamics

1.1. Self-consistent particle dynamics

1.2. Hamiltonian equations

1.3. Applicability of Vlasov's equation to particle dynamics

1.4. Canonical transformations

1.5. Dynamics in axial-symmetric structures. Busch' theorem

1.6. Beam emittance

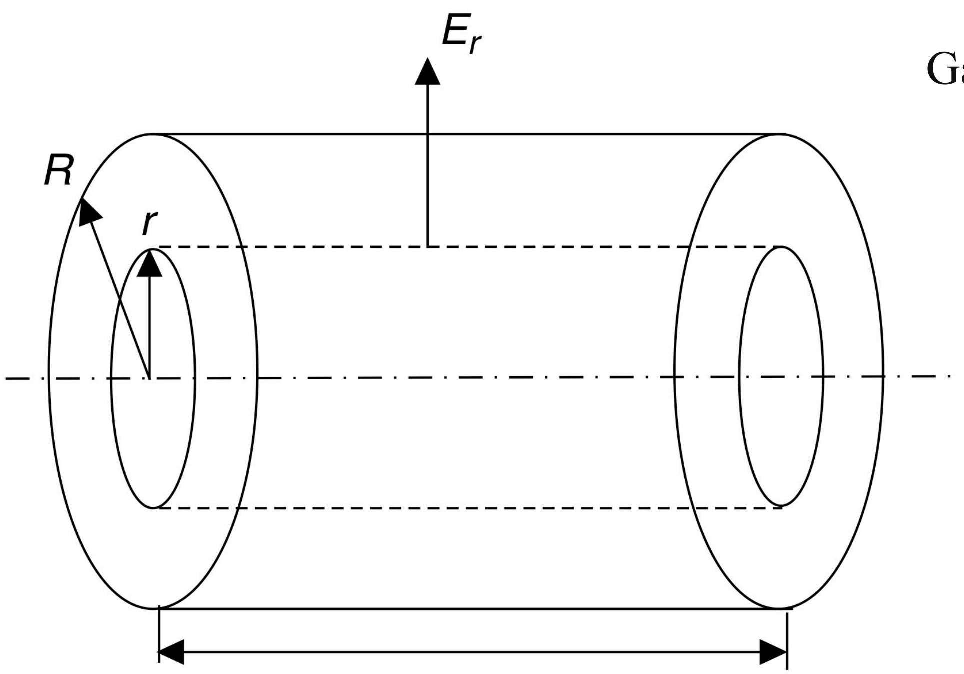
1.7. Root-mean-square emittance

1.8. Particle distributions in phase space

1.9. Emittance of the beam in particles sources

1.10. Space charge effects in the extraction region of particle sources

Example: beam drift in free space



Gauss theorem: $$\oint \vec{E}\, d\vec{S} = \frac{1}{\varepsilon_o} \int \rho\, dV$$

$$E_r 2\pi r L = \frac{\rho \pi r^2 L}{\varepsilon_o}$$

Space charge field: $$E_r = \frac{\rho r}{2\varepsilon_o}$$

Space charge density: $$\rho = \frac{Q}{V} = \frac{I\,T}{\pi R^2 \beta c T} = \frac{I}{\pi R^2 \beta c}$$

Space charge field: $$E_r = \frac{I}{2\varepsilon_o \pi R^2 \beta c} r$$

From Maxwell equations for magneto-static field: $\oint \vec{H}\, d\vec{l} = \int \vec{j}\, dS$

$$\frac{B_\theta}{\mu_o} 2\pi r = j\pi r^2$$

Magnetic field generated by current flow: $B_\theta = \mu_o \frac{Ir}{2\pi R^2} = \frac{\beta}{c} E_r$

Equation of single particle within the beam: $\frac{dp_r}{dt} = q(E_r - \beta c B_\theta) = \frac{qE_r}{\gamma^2}$

Equation for boundary particle $\frac{d^2 r}{dz^2} = \frac{qE_r}{m\gamma^3 (\beta c)^2} = \frac{2I}{I_c (\beta\gamma)^3 r}$

Characteristic current: $I_c = 4\pi\varepsilon_o \frac{mc^3}{q}$

Equation for dimensionless beam radius $$\frac{d^2\bar{R}}{dz^2} = \frac{2I}{I_c(\beta\gamma)^3 r_o^2 \bar{R}} \qquad \bar{R} = \frac{r}{r_o}$$

Let us multiply by $\frac{d\bar{R}}{dz}$ and integrate $$\frac{d\bar{R}}{dz}\frac{d^2\bar{R}}{dz^2} = \frac{2I}{I_c(\beta\gamma)^3 r_o^2 \bar{R}}\frac{d\bar{R}}{dz}$$

$$\frac{d\bar{R}}{dz}\frac{d^2\bar{R}}{dz^2} = \frac{1}{2}\frac{d}{dz}[(\frac{d\bar{R}}{dz})^2] \qquad \frac{1}{\bar{R}}\frac{d\bar{R}}{dz} = \frac{d}{dz}(\ln\bar{R})$$

Equations for dimensionless variables $$(\frac{d\bar{R}}{dZ})^2 = \ln\bar{R} \qquad Z = 2\frac{z}{r_o}\sqrt{\frac{I}{I_c(\beta\gamma)^3}}$$

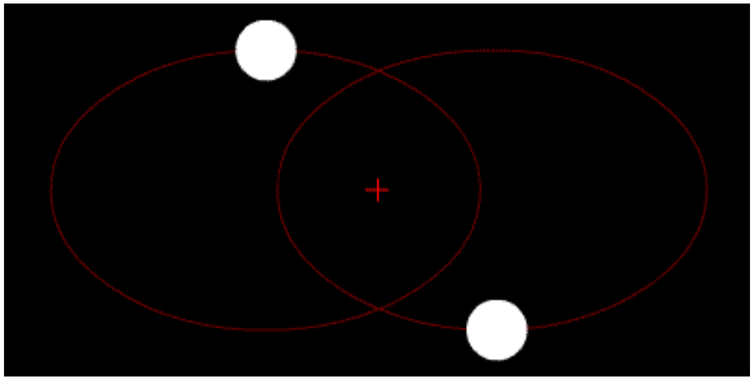
Approximate solution $$\bar{R} \approx 1 + 0.25Z^2 - 0.017Z^3$$

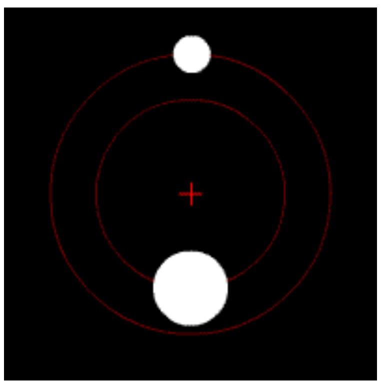
Let us determine distance z where the beam radius is doubled: $\bar{R} = 2 \quad Z \approx 2 \quad z = r_o \dfrac{1}{\sqrt{\dfrac{I}{I_c(\beta\gamma)^3}}}$

When $I \approx I_c(\beta\gamma)^3$, such beam cannot exists because it is diverged at the distance of $z \approx r_o$ equal to beam radius

1.1. Self-consistent particle dynamics

Example: Two - body problem*





Every point mass attracts every single other point mass by a force pointing along the line intersecting both points. The force is directly proportional to the product of the two masses and inversely proportional to the square of the distance between the point masses:

$$F = G\frac{m_1 m_2}{r^2},$$

where:

- F is the magnitude of the gravitational force between the two point masses,
- G is the gravitational constant,
- m_1 is the mass of the first point mass,
- m_2 is the mass of the second point mass, and
- r is the distance between the two point masses.

In classical mechanics, the two-body problem is to determine the motion of two point particles that interact only with each other.

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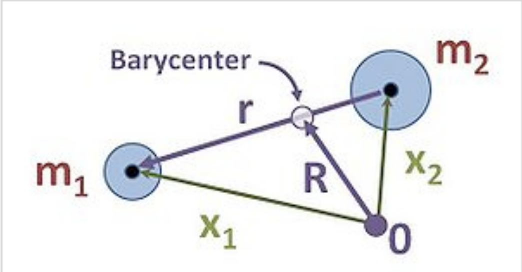
Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the positions of the two bodies, and m_1 and m_2 be their masses. The goal is to determine the trajectories $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ for all times t, given the initial positions $\mathbf{x}_1(t=0)$ and $\mathbf{x}_2(t=0)$ and the initial velocities $\mathbf{v}_1(t=0)$ and $\mathbf{v}_2(t=0)$.



Jacobi coordinates for two-body problem; Jacobi coordinates are $\boldsymbol{R} = \frac{m_1}{M}\boldsymbol{x}_1 + \frac{m_2}{M}\boldsymbol{x}_2$ and $\boldsymbol{r} = \boldsymbol{x}_1 - \boldsymbol{x}_2$ with $M = m_1 + m_2$.[1]

When applied to the two masses, Newton's second law states that

$$\mathbf{F}_{12}(\mathbf{x}_1, \mathbf{x}_2) = m_1\ddot{\mathbf{x}}_1$$
$$\mathbf{F}_{21}(\mathbf{x}_1, \mathbf{x}_2) = m_2\ddot{\mathbf{x}}_2$$

where $\mathbf{F}_{12}$ is the force on mass 1 due to its interactions with mass 2, and $\mathbf{F}_{21}$ is the force on mass 2 due to its interactions with mass 1.

Adding and subtracting these two equations decouples them into two one-body problems, which can be solved independently. *Adding* equations (1) and (2) results in an equation describing the center of mass (barycenter) motion. By contrast, *subtracting* equation (2) from equation (1) results in an equation that describes how the vector $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$ between the masses changes with time. The solutions of these independent one-body problems can be combined to obtain the solutions for the trajectories $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$.

1.2. Hamiltonian dynamics

Hamiltonian of charged particle with charge q and mass m

$$H = c\sqrt{m^2c^2 + (P_x - qA_x)^2 + (P_y - qA_y)^2 + (P_z - qA_z)^2} + q\,U$$

$x,\ y,\ z$ position in real space

P_x, P_y, P_z components of canonical momentum

A_x, A_y, A_z components of the vector – potential

$U(x,y,z)$ scalar potential of the electromagnetic field

Equations of motion:

$$\frac{d\vec{x}}{dt} = \frac{\partial H}{\partial \vec{P}} \qquad \frac{d\vec{P}}{dt} = -\frac{\partial H}{\partial \vec{x}}$$

Canonical momentum $\vec{P} = (P_x, P_y, P_z)$ and mechanical momentum $\vec{p} = (p_x, p_y, p_z)$ are related:

$$\vec{p} = \vec{P} - q\,\vec{A}$$

Element of phase space: $dV = dx\,dy\,dz\,dP_x\,dP_y\,dP_z$

Phase space density (beam distribution function):

$$f(x, y, z, P_x, P_y, P_z) = \frac{dN}{dx\,dy\,dz\,dP_x\,dP_y\,dP_z}$$

Liouville’s theorem: if the motion of a system of mechanical particles obeys Hamilton’s equations, then phase space density remains constant along phase space trajectories and phase space volume occupied by the particles is invariant (Liouville's Equation):

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \vec{x}} \frac{d\vec{x}}{dt} + \frac{\partial f}{\partial \vec{P}} \frac{d\vec{P}}{dt} = 0$$

Being applied to ensemble of particles in electromagnetic field it is called the *Vlasov equation*.

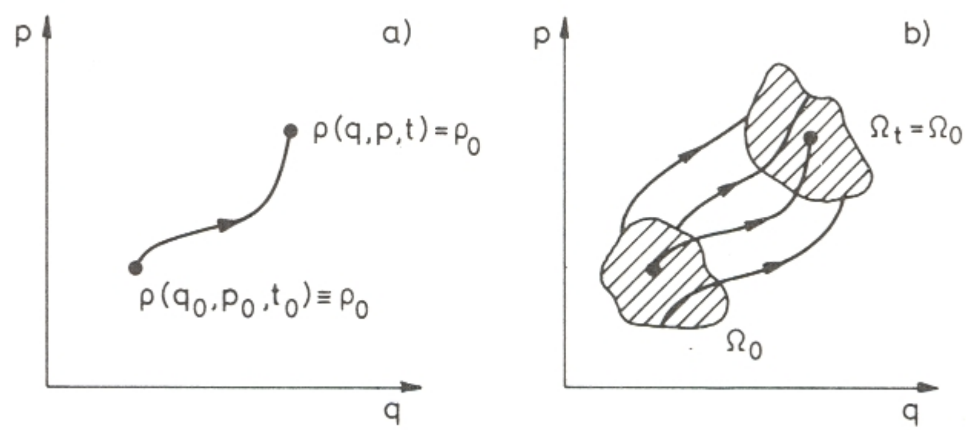


Illustration of conservation of phase space volume (A.Sorensen, 1987, CERN 87-10).

Self-consistent approach to N-particle dynamics

Solution to the equations of motion of the particles, together with the equations for the electromagnetic field which they create.

Solution of self-consistent problem: the phase space density, as a constant of motion can be expressed as a function of other constants of motion $I_1, I_2, \ldots.$

$$f = f(I_1, I_2, \ldots.)$$

This equation automatically obeys Liouville's equation

$$\frac{df}{dt} = \frac{\partial f}{\partial I_1}\frac{dI_1}{dt} + \frac{\partial f}{\partial I_2}\frac{dI_2}{dt} + \ldots. = 0$$

because of vanishing derivatives, $dI_i/dt = 0$.

Field created by the beam is described by Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

space charge density

$$\rho = q \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f \, dP_x \, dP_y \, dP_z$$

beam current density

$$\vec{j} = q \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \vec{v} f \, dP_x \, dP_y \, dP_z$$

$\varepsilon_o = 8.85 \times 10^{-12}$ F/m is the electric permittivity
$\mu_o = 4\pi\ 10^{-7}$ H/m is the magnetic permeability of free space

Instead of electric field $\vec{E}$ and magnetic field $\vec{B}$, it is common to use vector potential $\vec{A}$ and scalar potential U:

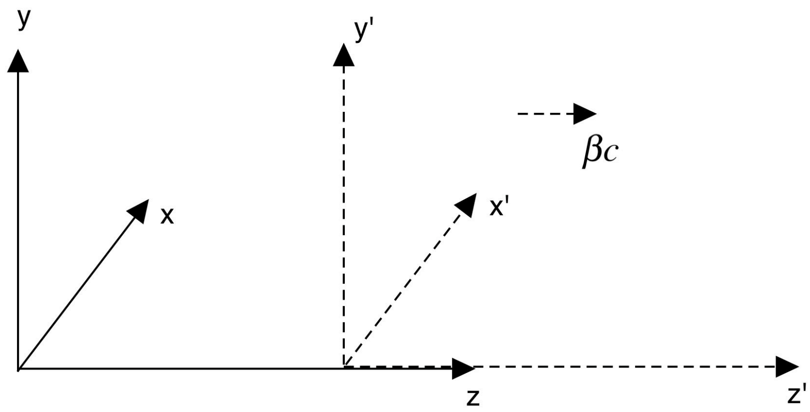
$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - gradU$$

$$\vec{B} = rot\vec{A}$$

The field of the beam is described by the equations

$$\Delta U_b - \frac{1}{c^2}\frac{\partial^2 U_b}{\partial t^2} = -\frac{\rho}{\varepsilon_o}$$

$$\Delta \vec{A_b} - \frac{1}{c^2}\frac{\partial^2 \vec{A_b}}{\partial t^2} = -\mu_o \vec{j}$$



Consider system of coordinates, which moves with the average beam velocity β. We will denote all values in this frame by prime symbol. Potentials $U', \vec{A}'$ are connected with that in laboratory system, U, $\vec{A}$, by Lorentz transformation

$$A_z = \gamma(A_z' + \frac{\beta}{c}U')$$

$$U = \gamma\,(U' + \beta c A_z')$$

$$A_x = A_x', \quad A_y = A_y'$$

1.

In the moving system of coordinates, particles are static, therefore, vector potential of the beam equals to zero, $\vec{A}_b' = 0$. According to Lorentz transformations, components of vector potential of the beam are converted into laboratory system of coordinates as follow

$$A_{xb} = 0\ , \qquad A_{yb} = 0\ , \qquad A_{zb} = \beta \frac{U_b}{c}$$

In a particle beam, the vector potential and the scalar potential are related via the expression $\vec{A}_b = \vec{v}_z\ / c^2 U_b$, therefore, it is sufficient to only solve the equation for the scalar potential. The unknown distribution function of the beam is then found by substituting equation for distribution function into the field equation and solving it. For example, for beam transport, equation for unknown space charge potential is

$$\Delta U_b = - \frac{q}{\varepsilon_o} \int_{-\infty}^{\infty} f(I_1, I_2, \ldots)\, d\vec{P}$$

Equation for unknown potential of the beam together with Vlasov's equation for beam distribution function constitute *self-consistent system of equations* describing beam evolution in the field created by the beam itself.

1.3. Applicability of Vlasov's equation to particle dynamics

Vlasov's equation describes behavior of non-interactive particles in given external field. Charged particles within the beam interact between themselves:
(i) interaction of large number of particles resulted in smoothed collective charge density and current density distribution
(ii) individual particle - particle collisions, when particles approach to each other at the distance, much smaller than the average distance between particles.

First type of interaction results in generation of smoothed electromagnetic field, which, being added to the field of external sources, act at the beam as an external field. The second type of interaction has a meaning of particle collisions resulting in appearance of additional fluctuating electromagnetic fields.

Using Vlasov's eqauiton, we *formally* expand it to dynamics of interacting charged particles, assuming that the total electromagnetic filed of the structure ($U, \vec{A}$)

$$U = U_{ext} + U_b$$

$$\vec{A} = \vec{A}_{ext} + \vec{A}_b$$

U_{ext}, $\vec{A}_{ext}$, external field

U_b, $\vec{A}_b$ field created by the beam

and *neglecting* individual particle-particle interactions.

Vlasov's equation treats collisionless plasma, where individual particle-particle interactions are negligible in comparison with the collective space charge field

Quantative treatment of validity of collisionless approximation dynamics to particle dynamics:

n - particle density within the beam

$\bar{r}$ - the average distance between particles.

$$n\bar{r}^3 = 1 \qquad \text{, or} \qquad \bar{r} = n^{-1/3}$$

Individual particle-particle collisions are neglected, when kinetic energy of thermal particle motion within the beam is much larger than potential energy of Coulomb particle-particle interaction:

$$\frac{mv_t^2}{2} >> \frac{q^2}{4\pi\varepsilon_o \bar{r}}$$

v_t is the root-mean square velocity of chaotic particle motion within the beam:

$$\frac{mv_t^2}{2} = \frac{kT}{2}$$

T is the "temperature" of chaotic particle motion

$k = 8.617342 \times 10^{-5}$ eV K^{-1} = $1.3806504 \times 10^{-23}$ J K^{-1} is the Boltsman's constant.

Radius of Debye shielding in plasma: $$\lambda_D = \sqrt{\frac{\varepsilon_o kT}{q^2 n}}$$

Combining all equation one gets:

$$\bar{r} << \sqrt{2\pi}\,\lambda_D \quad \text{or} \quad N_D >> 1, \quad \text{or} \quad N_D = (2\pi)^{3/2} n\lambda_D$$

where N_D is the number of particles within Debye sphere.

Individual particle-particle collisions can be neglected if number of particles w ithin Debye sphere is much larger than unity (or average distance between particles is much smaller than λ_D).

Particle density within uniformly charged cylindrical beam of radius R, with current I, propagating with longitudinal velocity βc, is

$$n = \frac{I}{\pi\, q\, \beta c\, R^2}$$

Hamiltonian equations of motion

Motion of a charged classical particle in an electromagnetic field is described by Hamiltonian dynamics. The three corresponding canonical conjugate variable pairs are (x, P_x), (y, P_y), (z, P_z). The equations of motion then follow from Hamilton's equations:

$$\frac{dx}{dt} = \frac{\partial H}{\partial P_x}, \qquad \frac{dy}{dt} = \frac{\partial H}{\partial P_y}, \qquad \frac{dz}{dt} = \frac{\partial H}{\partial P_z}, \tag{1.27}$$

$$\frac{dP_x}{dt} = -\frac{\partial H}{\partial x}, \qquad \frac{dP_y}{dt} = -\frac{\partial H}{\partial y}, \qquad \frac{dP_z}{dt} = -\frac{\partial H}{\partial z}. \tag{1.28}$$

As an example, taking a partial derivative of the Hamiltonian with respect to P_x yields the equation for the rate of change of the particle's x-position

$$\frac{dx}{dt} = \frac{c\,(P_x - q\,A_x)}{\sqrt{m^2c^2 + (P_x - q\,A_x)^2 + (P_y - q\,A_y)^2 + (P_z - q\,A_z)^2}}. \tag{1.29}$$

Canonical momentum $\vec{P} = (P_x, P_y, P_z)$ is related to mechanical momentum $\vec{p} = (p_x, p_y, p_z)$ via the expression:

$$\vec{p} = \vec{P} - q\,\vec{A} \tag{1.30}$$

Note that the denominator in Eq.(1.29) is actually $mc\gamma$, where the relativistic factor γ is:

$$\gamma = \sqrt{1 + \frac{(P_x - qA_x)^2 + (P_y - qA_y)^2 + (P_z - qA_z)^2}{m^2c^2}} \;. \tag{1.31}$$

Analogously, the equations for the rates of change of the y- and z - positions of the particle can be derived. So, the set of equations for the rate of change of the particle's position is

$$\frac{dx}{dt} = \frac{(P_x - qA_x)}{m\gamma}, \qquad \frac{dy}{dt} = \frac{(P_y - q\,A_y)}{m\gamma}, \qquad \frac{dz}{dt} = \frac{(P_z - q\,A_z)}{m\gamma}. \tag{1.32}$$

Taking partial derivatives of the Hamiltonian with respect to the particle's positions, the equations for the rate of change of the canonical momentum vector are:

$$\frac{dP_x}{dt} = \frac{q}{m\gamma}\,[(P_x - qA_x)\frac{\partial A_x}{\partial x} + (P_y - qA_y)\frac{\partial A_y}{\partial x} + (P_z - qA_z)\frac{\partial A_z}{\partial x}] - q\frac{\partial U}{\partial x}\,, \qquad (1.33)$$

$$\frac{dP_y}{dt} = \frac{q}{m\gamma}\,[(P_x - qA_x)\frac{\partial A_x}{\partial y} + (P_y - qA_y)\frac{\partial A_y}{\partial y} + (P_z - qA_z)\frac{\partial A_z}{\partial y}] - q\frac{\partial U}{\partial y} \qquad (1.34)$$

$$\frac{dP_z}{dt} = \frac{q}{m\gamma}\,[(P_x - qA_x)\frac{\partial A_x}{\partial z} + (P_y - qA_y)\frac{\partial A_y}{\partial z} + (P_z - qA_z)\frac{\partial A_z}{\partial z}] - q\frac{\partial U}{\partial z}\,. \qquad (1.35)$$

It is more common to integrate the equations of motion for mechanical momentum, and use electric, $\vec{E}$, and magnetic, $\vec{B}$, fields instead of vector potential $\vec{A}$ and scalar potential U:

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - gradU \ , \quad \vec{B} = rot\ \vec{A}\ . \tag{1.36}$$

The left-hand side of the equation for the rate of change of the x-component of the canonical momentum, $P_x = p_x + qA_x$, can be represented as follows:

$$\frac{dP_x}{dt} = \frac{dp_x}{dt} + q\ (\frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\frac{dx}{dt} + \frac{\partial A_x}{\partial y}\frac{dy}{dt} + \frac{\partial A_x}{\partial z}\frac{dz}{dt}). \tag{1.37}$$

A combination of this equation with Eq. (1.33), gives:

$$\frac{dp_x}{dt} = q\ (-\frac{\partial A_x}{\partial t} - \frac{\partial U}{\partial x}) + q\ [v_y(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}) + v_z(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z})], \tag{1.38}$$

Applying the same derivations for p_y and p_z, the final set of equations in Cartesian coordinates is:

$$\frac{dx}{dt} = \frac{p_x}{m\gamma}, \tag{1.39}$$

$$\frac{dy}{dt} = \frac{p_y}{m\gamma}, \tag{1.40}$$

$$\frac{dz}{dt} = \frac{p_z}{m\gamma}, \tag{1.41}$$

$$\frac{dp_x}{dt} = q\left(E_x + \frac{p_y}{m\gamma} B_z - \frac{p_z}{m\gamma} B_y\right), \tag{1.42}$$

$$\frac{dp_y}{dt} = q\left(E_y - \frac{p_x}{m\gamma} B_z + \frac{p_z}{m\gamma} B_x\right), \tag{1.43}$$

$$\frac{dp_z}{dt} = q\left(E_z + \frac{p_x}{m\gamma} B_y - \frac{p_y}{m\gamma} B_x\right), \tag{1.44}$$

or

$$\frac{d\vec{x}}{dt} = \frac{\vec{p}}{m\gamma} \qquad \frac{d\vec{p}}{dt} = q\{\vec{E} + [\vec{v}\vec{B}]\}$$

1.4. Canonical Transformations

In Hamiltonian mechanics, a canonical transformation is a change of canonical coordinates (q,p,*t*) → (Q,P,*t*) that preserves the form of Hamilton's equations. Hamiltonian equations of motions are

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

New variables also obey canonical equations of motion

$$\frac{dQ_i}{dt} = \frac{\partial H'}{\partial P_i}, \qquad \frac{dP_i}{dt} = -\frac{\partial H'}{\partial Q_i} \qquad (5.1)$$

where H' is a new Hamiltonian. New variables can be considered as functions of old variables and time $Q_i = Q_i(p_i, q_i, t)$, $P_i = P_i(p_i, q_i, t)$. Transformations from old variables to new variables, which keep canonical structure of the equation of motion (5.1) are called canonical transformations.

From classical mechanics it follows, that both new and old variables obey principle of least action :

$$\delta \int (\sum p_i \, dq_i - Hdt) = 0 \qquad (5.2)$$

$$\delta \int (\sum P_i \, dQ_i - H' \, dt) = 0 \qquad (5.3)$$

That means, that integrands in eqs. (5.2), (5.3) are different as total differential of arbitrary function F of coordinates, momentum and time:

$$\sum p_i \, dq_i - H \, dt = \sum P_i \, dQ_i - H' \, dt + dF, \text{ or} \qquad (5.4)$$

$$dF = \sum p_i \, dq_i - \sum P_i \, dQ_i + (H' - H) \, dt \qquad (5.5)$$

Function F is called generating function of transformation.

Type 1 generating function

To be a total differential, equation (5.5) has to have the following form:

$$dF = \sum \frac{\partial F}{\partial q_i} dq_i + \sum \frac{\partial F}{\partial Q_i} dQ_i + \frac{\partial F}{\partial t} dt \qquad (5.6)$$

From comparison of equations (5.5) and (5.6) it is clear, that the variables and the new Hamiltonian have to obey the following equatons:

$$p_i = \frac{\partial F}{\partial q_i}, \qquad P_i = -\frac{\partial F}{\partial Q_i} \qquad (H' - H)\, dt = \frac{\partial F}{\partial t} dt \qquad (5.7)$$

Therefore new Hamiltonian is connected with the old one via relationship

$$\boxed{H' = H + \frac{\partial F}{\partial t}} \qquad (5.8)$$

Equations (5.7) provide canonical transformation from old variables to new variables, if generating function depends on old and new coordinates:

$$\boxed{p_i = \frac{\partial F_1}{\partial q_i}} \qquad \boxed{P_i = -\frac{\partial F_1}{\partial Q_i}} \qquad \boxed{F_1 = F_1(q, Q, t)} \qquad (5.9)$$

Type 2 generating function

Let us rewrite eq. (5.5) as follow:

$$dF = \sum p_i\, dq_i - \sum P_i\, dQ_i + \sum Q_i\, dP_i - \sum Q_i\, dP_i + (H' - H)\, dt \qquad (5.11)$$

Let us introduce new generating function F_2

$$F_2 = F + \sum P_i Q_i, \qquad dF_2 = dF + \sum P_i\, dQ_i + \sum Q_i\, dP_i \qquad (5.12)$$

For new generting function the following equation is valid:

$$dF_2 = \sum p_i\, dq_i + \sum Q_i\, dP_i + (H' - H)\, dt \qquad (5.13)$$

Equation (5.13) indicates, that generating function of the second type is a function of old coordinates and new momentum $F_2 = F_2(q, P, t)$. Relationship between new Hamiltonian and the old one is given by equation (5.8). Again, to be a total differential, the following eqautions have to be valid, which form the second canonical transformation:

$$\boxed{p_i = \frac{\partial F_2}{\partial q_i}} \qquad \boxed{Q_i = \frac{\partial F_2}{\partial P_i}} \qquad \boxed{F_2 = F_2(q, P, t)} \qquad (5.14)$$

Type 3 generating function

To find third canonical transformation, let us add and subtract $\sum q_i \, dp_i$ from eq. (5.5):

$$dF = \sum p_i \, dq_i - \sum P_i \, dQ_i + \sum q_i \, dp_i - \sum q_i \, dp_i + (H' - H) \, dt \qquad (5.16)$$

Introducing generating function of the 3rd type

$$F_3 = F - \sum p_i \, q_i, \qquad dF_3 = dF - \sum p_i \, dq_i - \sum q_i \, dp_i \qquad (5.17)$$

the eqution for total differential of the generating function is as follow:

$$dF_3 = -\sum P_i \, dQ_i - \sum q_i \, dp_i + (H' - H) \, dt \qquad (5.18)$$

Last equation forms the canonical transformation of the 3rd type:

$$\boxed{P_i = -\frac{\partial F_3}{\partial Q_i}} \qquad \boxed{q_i = -\frac{\partial F_3}{\partial p_i}} \qquad \boxed{F_3 = F_3\,(Q, p, t)} \qquad (5.19)$$

Type 4 generating function

Forth canonical transformation is attained via adding and subtracting of the $\sum Q_i \, dP_i$ from Eq. (5.5):

$$dF = \sum p_i \, dq_i - \sum P_i \, dQ_i + \sum q_i \, dp_i - \sum q_i \, dp_i + \sum Q_i \, dP_i - \sum Q_i \, dP_i + (H' - H) \, dt$$

Generating function of the 4th ype is defined as follow:

$$F_4 = F - \sum p_i q_i + \sum P_i Q_i \qquad (5.22)$$

It results in the eqution for total differential of the generating function:

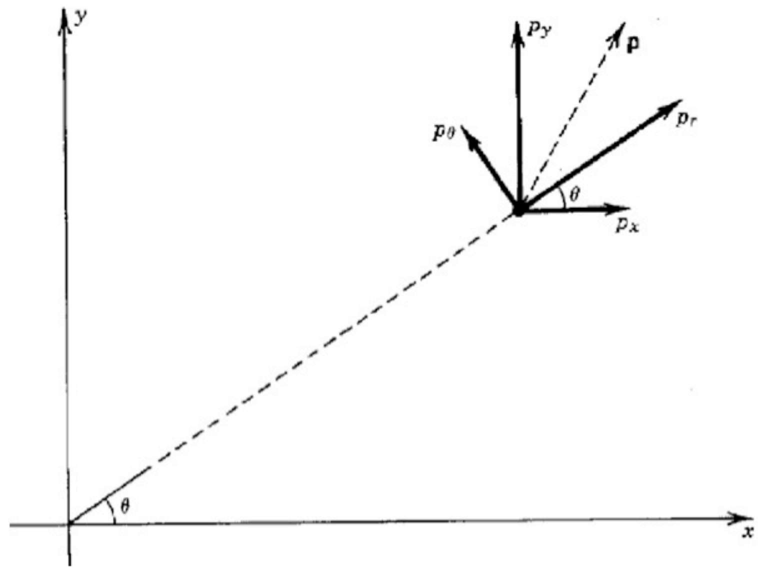
$$dF_4 = - \sum q_i \, dp_i + \sum Q_i \, dP_i + (H' - H) \, dt \qquad (5.23)$$

Canonical transformation of the 4th type are descibed by equations:

$$\boxed{q_i = - \frac{\partial F_4}{\partial p_i}} \qquad \boxed{Q_i = \frac{\partial F_4}{\partial P_i}} \qquad \boxed{F_4 = F_4 (p, P, t)} \qquad (5.24)$$

Example: Canonical transformation from Cartesian to cylindrical coordinates

Very often, particle dynamics in accelerators is described in a cylindrical system of coordinates (r, θ, z), because of axial symmetry inherent to accelerating structures.



Relationship between cylindrical and Cartesian coordinates.

A canonical transformation of the Hamiltonian from Cartesian to cylindrical system of coordinates is accomplished by selecting a generating function of the transformation, as a function of new position variables and old momentum:

$$F_3\,(r, \theta, z, P_x, P_y, P_z) = - r\,P_x \cos\theta \;- r\,P_y \sin\theta - z\,P_z\,. \qquad (1.45)$$

The relationships between new and old variables in a canonical transformation are obtained using the equations

$$x = -\frac{\partial F_3}{\partial P_x}, \qquad y = -\frac{\partial F_3}{\partial P_y}, \qquad z = -\frac{\partial F_3}{\partial P_z}, \qquad (1.46)$$

$$P_r = -\frac{\partial F_3}{\partial r}, \qquad P_\theta = -\frac{\partial F_3}{\partial \theta}, \qquad P_z = -\frac{\partial F_3}{\partial z}. \qquad (1.47)$$

Calculation of the partial derivatives, Eqs. (1.46), (1.47), gives the relationship between Cartesian and cylindrical coordinates:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z, \qquad (1.48)$$

$$P_r = P_x\cos\theta + P_y\sin\theta, \qquad (1.49)$$

$$P_\theta = r\,(-P_x\sin\theta + P_y\cos\theta), \qquad (1.50)$$

$$P_z = P_z\,. \qquad (1.51)$$

Inverse transformation of Eqs. (1.49) (1.50), (1.52), (1.53) gives

$$P_x = P_r\, cos\theta - \frac{P_\theta}{r}\, sin\theta\,, \qquad (1.56)$$

$$P_y = P_r\, sin\theta + \frac{P_\theta}{r}\, cos\theta\,, \qquad (1.57)$$

$$P_z = P_z\,. \qquad (1.51)$$

$$A_x = A_r\, cos\theta - A_\theta\, sin\theta\,, \qquad (1.58)$$

$$A_y = A_r\, sin\theta + A_\theta\, cos\theta\,. \qquad (1.59)$$

$$A_z = A_z \qquad (1.54)$$

After a canonical transformation, the new Hamiltonian is expressed in terms of the old one as

$$K = H + \frac{\partial F_3}{\partial t} \,. \qquad (1.55)$$

Since the generating function, Eq. (1.45), does not depend on time explicitly, the new Hamiltonian equals the old one, $K = H$:

$$H = c\sqrt{(mc)^2 + (\frac{P_\theta}{r} - qA_\theta)^2 + (P_r - qA_r)^2 + (P_z - qA_z)^2} + qU \,. \qquad (1.60)$$

Hamilton's equations in cylindrical coordinates read

$$\frac{dr}{dt} = \frac{\partial H}{\partial P_r}, \qquad \frac{d\theta}{dt} = \frac{\partial H}{\partial P_\theta}, \qquad \frac{dz}{dt} = \frac{\partial H}{\partial P_z}, \qquad (1.61)$$

$$\frac{dP_r}{dt} = -\frac{\partial H}{\partial r}, \qquad \frac{dP_\theta}{dt} = -\frac{\partial H}{\partial \theta}, \qquad \frac{dP_z}{dt} = -\frac{\partial H}{\partial z}. \qquad (1.62)$$

Calculating the partial derivatives, Eqs. (1.61), the equations for particle position are

$$\frac{dr}{dt} = \frac{P_r - qA_r}{m\gamma}, \qquad (1.63)$$

$$\frac{d\theta}{dt} = \frac{1}{m\gamma r}\left(\frac{P_\theta}{r} - qA_\theta\right), \qquad (1.64)$$

$$\frac{dz}{dt} = \frac{P_z - qA_z}{m\gamma}, \qquad (1.65)$$

Again, instead of canonical momentum, it is more common to use mechanical momentum, components of which are obtained from Eqs. (1.63) – (1.65) by

$$p_r = m\gamma \frac{dr}{dt} = P_r - qA_r, \qquad (1.69)$$

$$p_\theta = m\gamma\, r \frac{d\theta}{dt} = \frac{P_\theta}{r} - qA_\theta, \qquad (1.70)$$

$$p_z = m\gamma \frac{dz}{dt} = P_z - qA_z. \qquad (1.71)$$

Equations of motion in cylindrical coordinates are

$$\frac{dr}{dt} = \frac{p_r}{m\gamma}, \qquad \frac{d\theta}{dt} = \frac{p_\theta}{m\gamma\, r} \qquad \frac{dz}{dt} = \frac{p_z}{m\gamma} \tag{1.81}$$

$$\frac{dp_r}{dt} = \frac{p_\theta^2}{m\gamma\, r} + q\,(E_r + \frac{p_\theta}{m\gamma} B_z - \frac{p_z}{m\gamma} B_\theta), \tag{1.84}$$

$$\frac{1}{r}\frac{d(rp_\theta)}{dt} = q\,(E_\theta + \frac{p_z}{m\gamma} B_r - \frac{p_r}{m\gamma} B_z), \tag{1.85}$$

$$\frac{dp_z}{dt} = q\,(E_z + \frac{p_r}{m\gamma} B_\theta - \frac{p_\theta}{m\gamma} B_r). \tag{1.86}$$

1.5. Dynamics in axial-symmetric field. Busch's theorem

An area of special interest in beam dynamics is an axially-symmetric static field, $E_\theta = 0$, $B_\theta = 0$, which is common in beam transport. In this case, all partial derivatives over the azimuth angle are equal to zero, $\partial/\partial\theta = 0$, and the canonical angular momentum is a constant of motion:

$$P_\theta = m\gamma\, r^2 \frac{d\theta}{dt} + r\, qA_\theta = const\,. \tag{1.87}$$

The angular component of the vector – potential is given by

$$A_\theta = \frac{\Psi}{2\pi r}\,, \tag{1.88}$$

where Ψ is the magnetic flux

$$\Psi = \int_o^r B_z\, 2\pi\, r'\, dr'\,. \tag{1.89}$$

Substitution of Eq. (1.88) into Eq. (1.87) gives:

$$r^2 \frac{d\theta}{dt} + q\frac{\Psi}{2\pi m\gamma} = const. \qquad (1.90)$$

If we denote the initial conditions as $\dot{\theta}_o$, r_o, Ψ_o, Eq. (1.90) can be rewritten as

$$r^2\,\dot{\theta} - r_o^2\,\dot{\theta}_o = \frac{q}{2\pi m\gamma}(\Psi - \Psi_o), \qquad (1.91)$$

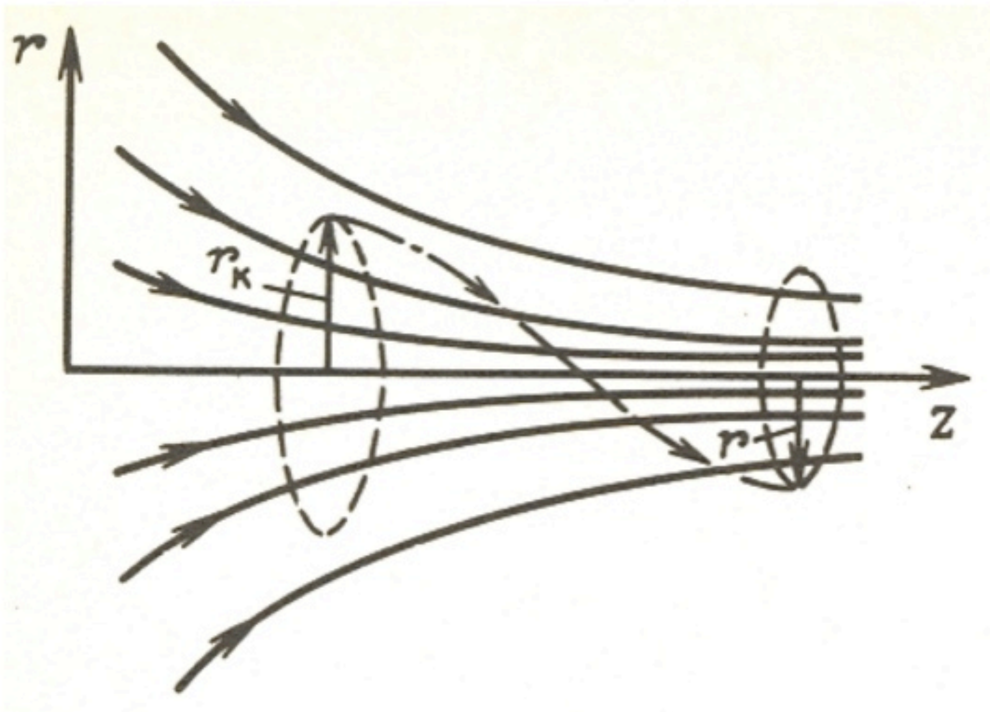
which is known as Busch's theorem. It states that change in angular momentum of a particle in a static magnetic field is defined by the change in magnetic flux comprised by the particle trajectory.

Busch's theorem can be represented as

$$\dot{\theta} = \frac{P_\theta}{m\gamma\, r^2} - \omega_L\,, \qquad (1.93)$$

where ω_L is the Larmor frequency of particle oscillations in a longitudinal magnetic field

$$\omega_L = \frac{q\,B}{2m\gamma}\,. \qquad (1.94)$$

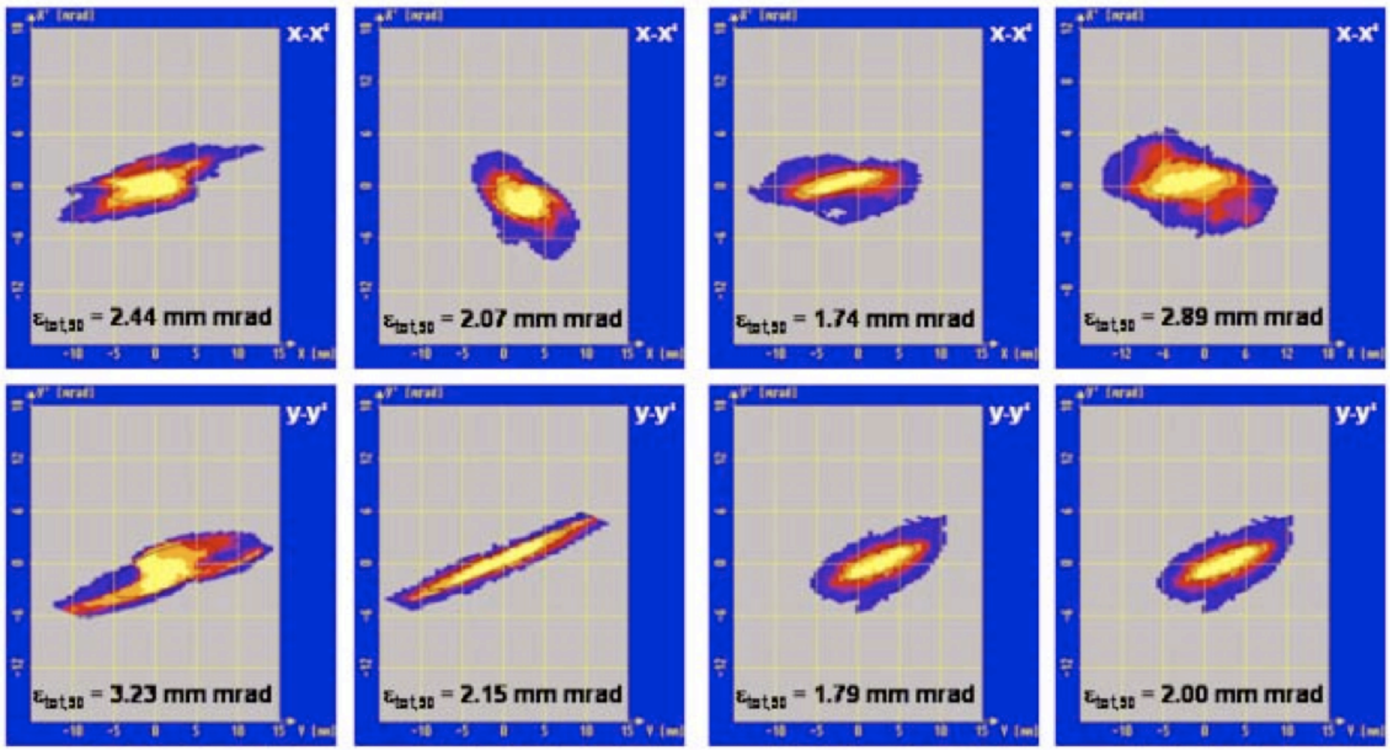


On Busch's theorem for particle in axial-symmetric magnetic field.

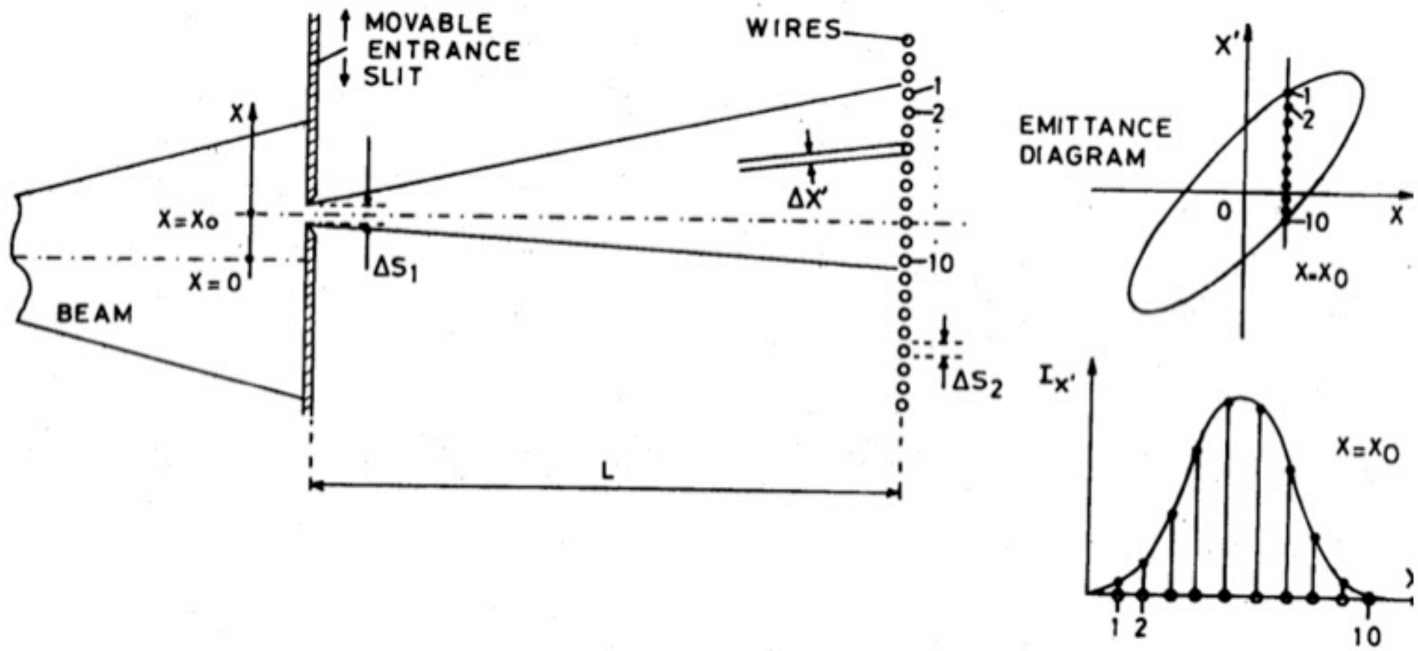
1.6. Beam emittance

Beam emittance is the area, occupied by the particles in the phase plane (x, dx/dz)

$$Э_x = \frac{1}{\pi}\int\int dx\, dx'$$



Results of beam emitance measurements in GSI UNILAC accelerator (W. Bayer et al., Proceedings of PAC07, Albuquerque, New Mexico, p. 1413 (2007)).



Emittance measuring device.

The phase-space area occupied by the particles on a plane of canonical-conjugate variables (x, P_x), is called the normalized emittance, and is given by

$$\varepsilon_x = \frac{1}{\pi m c}\int\int dx\, dP_x$$

Taking into account that $dx/dz = p_x/p_z$, natural and normalized beam emittances are connect via the relationship

$$\varepsilon_x = \beta_z \gamma\, э_x$$

With an increase of beam energy, longitudinal momentum p_z also increases. Consequently, the value of dx/dz , which is inversely proportional to p_z, decreases, resulting in a decrease of beam emittance, э. However, normalized beam emittance remains energy-independent. Because of this feature, normalized beam emittance is convenient for comparison of beams with different energies.

Representation of beam emittance as an ellipse

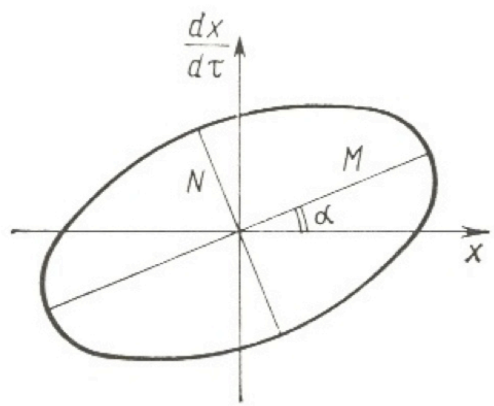
In the phase plane, the beam is usually approximated by an ellipse. The area of ellipse with semi-axes M and N is simply

$$S = \pi M N$$

The general ellipse equation can be written as

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = э$$

parameters α, β, γ are called Twiss parameters



Ellipse of the beam at phase plane of transverse oscillations.

Let us express the ellipse parameters in terms of the semi-axes *M, N* and the angle ψ.
In the $(\bar{x}, \bar{x}')$ system of coordinates, the ellipse is upright, and is described by the equation

$$\left(\frac{\bar{x}}{M}\right)^2 + \left(\frac{\bar{x}'}{N}\right)^2 = 1$$

The transformation to this system of coordinates is given by

$$\bar{x} = x\,cos\psi + x'\ sin\,\psi$$

$$\bar{x}' = -\ x\ sin\psi + x'\ cos\ \psi$$

Comparison with previous ellipse equation yields the relationships between Twiss parameters and ellipse parameters:

$$\alpha = \left(\frac{N}{M} - \frac{M}{N}\right)\ sin\psi\ cos\psi$$

$$\beta = \frac{N}{M}\ sin^2\psi + \frac{M}{N}\ cos^2\psi$$

$$\gamma = \frac{N}{M}\ cos^2\psi + \frac{M}{N}\ sin^2\psi$$

From last equations it follows that $\beta\gamma - \alpha^2 = 1$.

Among the three Twiss parameters $\alpha, \beta, \gamma,$ only two are independent, while the third one is connected via the identity $\beta\gamma - \alpha^2 = 1$. We can take advantage of this fact to reduce the number of variables. Let us introduce two new parameters

$$\sigma = \sqrt{\beta}$$

$$\sigma' = -\frac{\alpha}{\sqrt{\beta}}$$

In terms of these parameters, the ellipse equation reads

$$\frac{x^2}{\sigma^2} + (x\,\sigma' - x'\,\sigma)^2 = э$$

Combining all we get a relationship between the new parameters and the ellipse parameters:

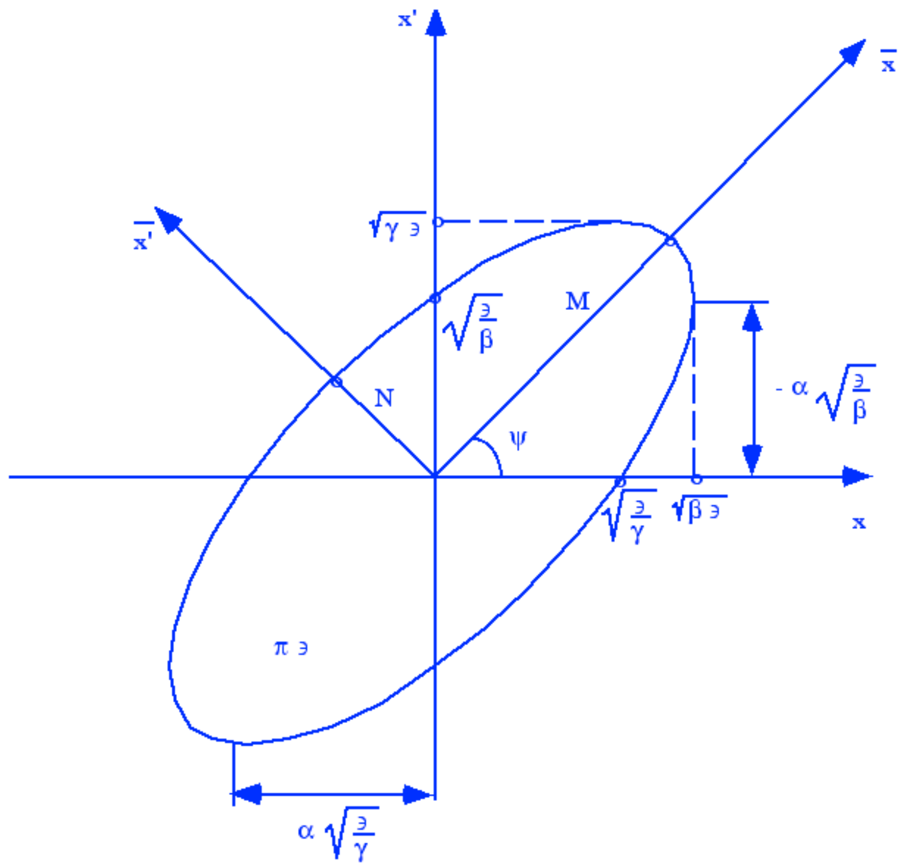
$$\sigma = \sqrt{\sin^2\psi\,\frac{N}{M} + \cos^2\psi\,\frac{M}{N}}$$

$$\sigma' = \frac{1}{2\sigma}\left(\frac{M}{N} - \frac{N}{M}\right)\sin 2\psi$$

Let us now define *beam spot size* and *beam divergence* via the parameters of a representative ellipse. Zeros of an ellipse are obtained by substitution of the values $x = 0$ or $x' = 0$ into the ellipse equation:

$$x\,(x' = 0) = \pm\sqrt{\frac{\backepsilon}{\gamma}}$$

$$x'\,(x = 0) = \pm\sqrt{\frac{\backepsilon}{\beta}}$$



Ellipse at phase plane $(x,\ x')$.

To find the extrema of an ellipse, let us rewrite the ellipse equation as $F(x, x') = 0$, where

$$F(x, x') = \gamma x^2 + 2\alpha x x' + \beta x'^2 - э$$

Ultimately, we need to find a solution to the equations $\frac{dx}{dx'} = 0$, $\frac{dx'}{dx} = 0$. According to the differentiation rule of an implicit function,

$$\frac{dx}{dx'} = -\frac{\frac{dF}{dx'}}{\frac{dF}{dx}} = -\frac{2\alpha x + 2\beta x'}{2\gamma x + 2\alpha x'} = 0$$

which has a solution $x' = -x\,\alpha/\beta$. Substitution of the obtained value of x' into the ellipse equation gives $x_{max} = \pm\sqrt{\beta э}$. The value of $R = x_{max}$ is associated with the envelope size of the beam

$$R = \sqrt{\beta э}$$

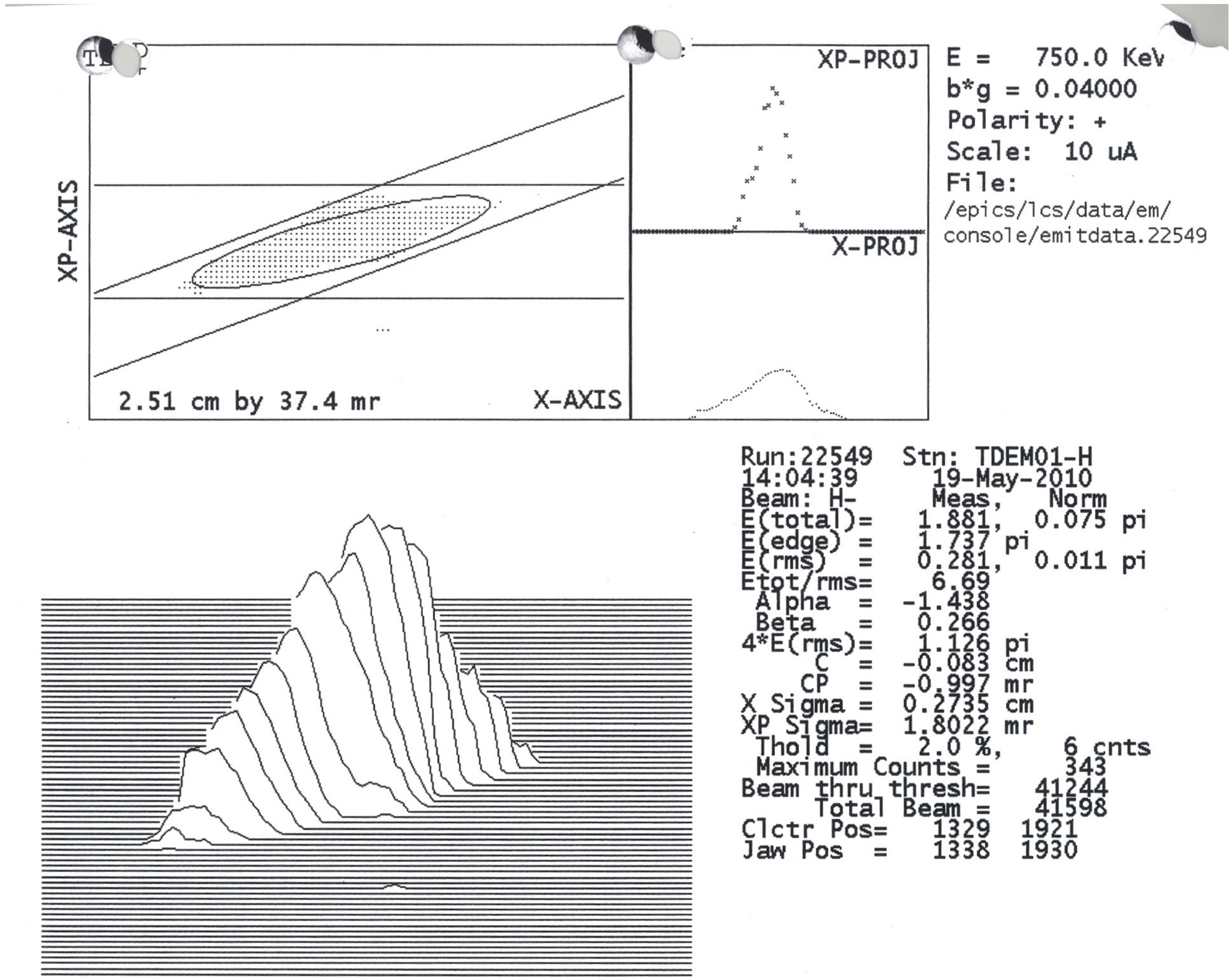
A corresponding point at the ellipse $x'(x_{max})$ is:

$$x'(x_{max}) = \pm\alpha\sqrt{\frac{э}{\beta}}$$

Analogously, for another extreme point:

$$x'_{max} = \pm\sqrt{\gamma э} \qquad x(x'_{max}) = \pm\alpha\sqrt{\frac{э}{\gamma}}$$

1.7. Root-mean-square (rms) beam emittance



1. Realistic beam distribution in phase space.

Consider a beam with a distribution function $f(\vec{x}, \vec{P}, t)$ and let $g(\vec{x}, \vec{P}, t)$ be an arbitrary function of position, momentum, and time. The average value of the function $g(\vec{x}, \vec{P}, t)$ is defined as:

$$<g> = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\vec{x}, \vec{P}, t)\, f(\vec{x}, \vec{P}, t)\, d\vec{x}\, d\vec{P}}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\vec{x}, \vec{P}, t)\, d\vec{x}\, d\vec{P}}$$

The integral in the denominator is just the total number of particles. Now, let us consider some examples of physically significant average values. For $g(\vec{x}, \vec{P}, t) = x$, the average value

$$<x> = \frac{1}{N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, f(\vec{x}, \vec{P}, t)\, d\vec{x}\, d\vec{P}$$

gives the center of gravity of the beam in the x-direction.

Analogously, for $g(\vec{x}, \vec{P}, t) = x^2$, the average value of x^2 is defined as

$$<x^2> = \frac{1}{N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f(\vec{x}, \vec{P}, t)\, d\vec{x}\, d\vec{P}$$

and is called the mean-square value of x. The correlation between variables x and P_x is given by the following expression: taking $g(\vec{x}, \vec{P}, t) = x\, P_x$

$$< xP_x > = \frac{1}{N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, P_x\, f(\vec{x}, \vec{P}, t)\, d\vec{x}\, d\vec{P}$$

An expression of the form $<x^{n_1}\, y^{n_2}\, z^{n_3}\, P_x^{n_4}\, P_y^{n_5}\, P_z^{n_6}>$ is referred to as the n^{th} order moment, $M_{n_1,\, n_2,\, n_3,\, n_4,\, n_5,\, n_6}$, of the distribution function, where $n = n_1 + n_2 + n_3 + n_4 + n_5 + n_6$:

$$<x^{n_1}\, y^{n_2}\, z^{n_3}\, P_x^{n_4}\, P_y^{n_5}\, P_z^{n_6}> = \frac{1}{N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dy\, dz\, dP_x\, dP_y\, dP_z$$

$$x^{n_1} y^{n_2} z^{n_3} P_x^{n_4} P_y^{n_5} P_z^{n_6} f\,(x,\, y,\, z,\, P_x,\, P_y,\, P_z,\, t).$$

The following combination of second moments of distribution function is called the root-mean-square beam emittance:

$$\text{э}_{rms} = \sqrt{<x^2><x'^2> - <x\ x'>^2}$$

and the normalized root-mean-square beam emittance is given by

$$\varepsilon_{rms} = \frac{1}{mc}\sqrt{<x^2><P_x^2> - <xP_x>^2}$$

By the reasons discussed below, beam emittance is adopted as the value, four times large than rms emittance

$$\text{э} = 4\sqrt{<x^2><x'^2> - <x\ x'>^2}$$

Consider the rms beam emittance concept in more detail. The density of particles in phase space, normalized by the total number of particles *N,* is described by a distribution function $\rho_x(x, x')$, which is an integral of the beam distribution function over the remaining variables:

$$\rho_x(x, x') = \frac{1}{N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, x', y, y', z, z')\, dy\, dy\, dz\, dz'$$

It is convenient to consider distributions in phase space with elliptical symmetry:

$$\rho_x(x, x') = \rho_x(\gamma_x x^2 + 2\,\alpha_x x\, x' + \beta_x x'^2)$$

Such distributions have particle densities, $\rho_x(x, x')$, that are constant along concentric ellipses

$$r_x^2 = \gamma_x x^2 + 2\,\alpha_x x\, x' + \beta_x x'^2$$

but are different from ellipse to ellipse, so one can write $\rho_x(x, x') = \rho_x(r_x^2)$. Namely, equation this describes a family of similar ellipses, which differ from each other by their areas. Using previous transformation the ellipse equation can be rewritten as

$$r_x^2 = (x\sigma_x' - x'\sigma_x)^2 + \left(\frac{x}{\sigma_x}\right)^2$$

Let us calculate rms beam parameters and rms beam emittance for an arbitrary function $\rho_x\,(x, x')$. We begin by changing variables:

$$\begin{cases} \dfrac{x}{\sigma_x} = r_x \; cos\varphi \\ x\sigma_x' - x'\sigma_x = r_x \; sin\varphi \end{cases}$$

Now we rewrite it as

$$\begin{cases} x = r_x \sigma_x cos\varphi \\ x' = r_x \sigma_x' cos\varphi - \dfrac{r_x}{\sigma_x} \; sin\varphi \end{cases}$$

The absolute value of the Jacobian of transformation gives us the volume transformation factor of the phase space element:

$$dx\,dx' = (abs \begin{vmatrix} \dfrac{\partial x}{\partial r_x} & \dfrac{\partial x}{\partial \varphi} \\ \dfrac{\partial x'}{\partial r_x} & \dfrac{\partial x'}{\partial \varphi} \end{vmatrix})\, dr_x \; d\varphi = r_x \; dr_x \; d\varphi$$

Then, the rms values are:

$$<x^2> = \int_o^{2\pi} \int_o^{\infty} (r_x \sigma_x cos\varphi)^2 \rho_x(r_x^2)\, r_x\, dr_x\, d\varphi$$

$$<x'^2> = \int_o^{2\pi} \int_o^{\infty} \left(r_x \sigma_x' cos\ \varphi - \frac{r_x}{\sigma_x} sin\ \varphi\right)^2 \rho_x(r_x^2)\, r_x\, dr_x\, d\varphi$$

$$<xx'> = \int_o^{2\pi} \int_o^{\infty} r_x \sigma_x cos\varphi \left(r_x \sigma_x' cos\varphi - \frac{r_x}{\sigma_x} sin\varphi\right) \rho_x(r_x^2)\, r_x\, dr_x d\varphi$$

Let us take into account previously introduced expressions:

$$\sigma = \sqrt{\beta}$$

$$\sigma' = - \frac{\alpha}{\sqrt{\beta}}$$

$$\beta\gamma - \alpha^2 = 1$$

Calculation of integrals over φ gives:

$$<x^2> = \pi \beta_x \int_o^\infty r_x^3 \rho_x(r_x^2)\, dr_x$$

$$<x'^2> = \pi \gamma_x \int_o^\infty r_x^3 \rho_x(r_x^2)\, dr_x$$

$$<x x'> = - \pi \alpha_x \int_o^\infty r_x^3 \rho_x(r_x^2)\, dr_x$$

Therefore, beam emittance is given by

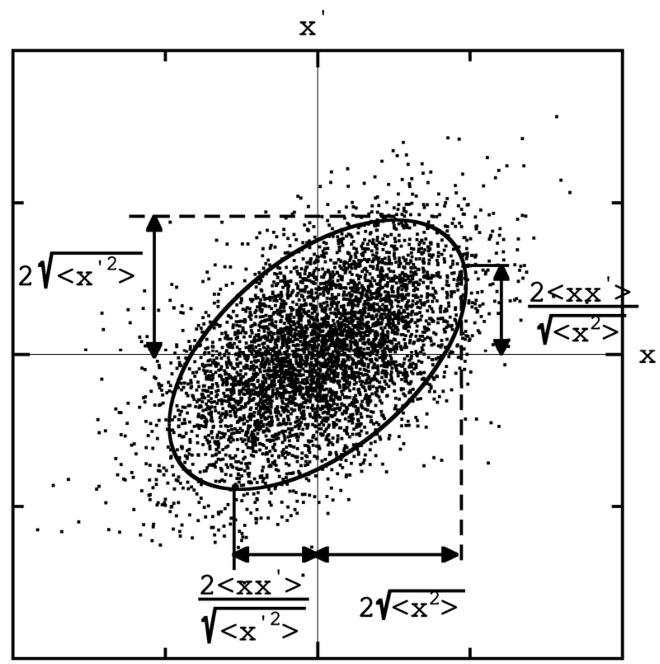
$$э_x = 4\pi \int_o^\infty r_x^3 \rho_x(r_x^2)\, dr_x$$

Twiss parameters

$$\alpha_x = -4\,\frac{<x\,x'>}{\text{э}_x} \qquad \beta_x = 4\,\frac{<x^2>}{\text{э}_x} \qquad \gamma_x = -4\,\frac{<x'^2>}{\text{э}_x}$$

Rms beam ellipse

$$\left(\frac{4<x'^2>}{\text{э}_x}\right)x^2 - 2\left(\frac{4<xx'>}{\text{э}_x}\right)x\,x' + \left(\frac{4<x^2>}{\text{э}_x}\right)x'^2 = \text{э}_x$$



Beam distribution and rms ellipse.

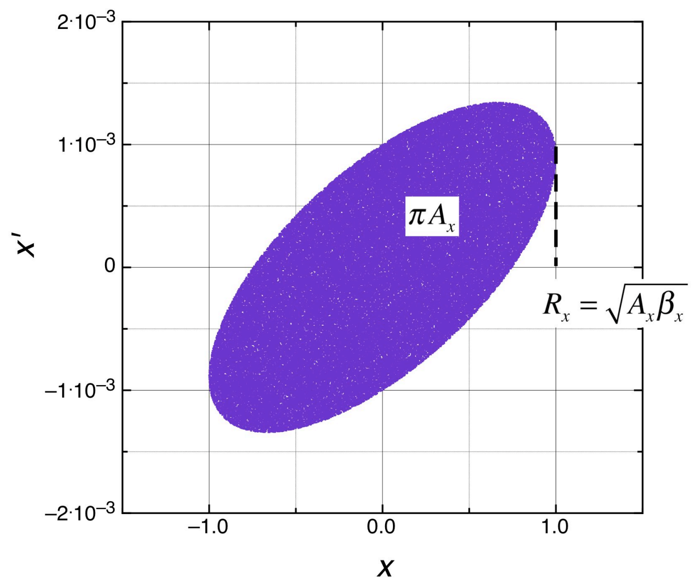
Example: Uniformly populated ellipse

Consider an example, where the beam ellipse has an area o f πA_x, and is uniformly populated by particles. Particle density is constant inside the ellipse $r_x^2 = A_x$:

$$\rho_x(r_x^2) = \frac{1}{\pi A_x}$$

Calculation of the rms value, $<x^2>$, gives:

$$<x^2> = \pi\,\beta_x \int_o^{\sqrt{A_x}} r_x^3\,\rho_x(r_x^2)\,dr_x = \frac{A_x\,\beta_x}{4}$$



Uniformly populated ellipse at phase plane (x, x').

The beam boundary is given by

$$R_x = \sqrt{A_x \beta_x}$$

Radius of the beam represented as a uniformly populated ellipse is equal to twice the rms beam size:

$$R = 2\sqrt{<x^2>}$$

Rms beam emittance:

$$\epsilon_x = \frac{4}{A_x}\int_o^{\sqrt{A_x}} r_x^3\, dr_x = A_x$$

Therefore, the area of an ellipse, uniformly populated by particles, coincides with the 4 x rms beam emittance. This explains the choice of the coefficient 4 in the definition of rms beam emittance.

1.8. Particle distributions in phase space

Consider quadratic from of 4-dimensional phase space variables:

$$I = (\sigma_x x' - \sigma_x' x)^2 + (\frac{x}{\sigma_x})^2 + (\sigma_y y' - \sigma_y' y)^2 + (\frac{y}{\sigma_y})^2$$

Consider different distributions $f = f(I)$ in phase space which depend on quadratic form:

Water Bag:

$$f = \begin{cases} \frac{2}{\pi^2 F_o^2}, & I \leq F_o \\ 0, & I > F_o \end{cases}$$

Parabolic:

$$f = \frac{6}{\pi^2 F_o^2}(1 - \frac{I}{F_o})$$

Gaussian:

$$f = \frac{1}{\pi^2 F_o^2} exp(-\frac{I}{F_o})$$

Normalization: $\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f\, dx\, dx'\, dy\, dy' = 1$

Projection of distributions on phase plane

$$\rho_x(x,x') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, x', y, y')\, dy\, dy'$$

Let us change the variables (y, y') for new variables T, ψ

$$\sigma_y y' - \sigma_y' y = T \cos\psi$$

$$\frac{y}{\sigma_y} = T \sin\psi$$

Phase space element $dy\, dy'$ is transformed as

$$dy\, dy' = \begin{vmatrix} \frac{\partial y}{\partial T} & \frac{\partial y}{\partial \psi} \\ \frac{\partial y'}{\partial T} & \frac{\partial y'}{\partial \psi} \end{vmatrix} dT\, d\psi = T\, dT\, d\psi\, .$$

The quadratic form is $I = r_x^2 + T^2.$

where the following notation is used: $r_x^2 = (\sigma_x x' + \sigma_x' x)^2 + (\frac{x}{\sigma_x})^2.$

With new variables, the projection on phase space is

1. $$\rho_x(x, x') = \pi \int_o^{\infty} f(r_x^2 + T^2)\, dT^2.$$

Water Bag distribution

$$f = \begin{cases} \dfrac{2}{\pi^2 F_o^2}, & I = r_x^2 + T^2 \le F_o \\ 0, & I > F_o \end{cases}$$

is restricted by surface

$$r_x^2 + T_1^2 = F_o \,, \quad T_1^2 = F_o - r_x^2$$

Projection of *Water Bag* distribution on (x, x')

$$\rho_x(x, x') = \frac{2}{\pi F_o^2} \int_o^{T_1^2} dT^2 = \frac{2}{\pi F_o} (1 - \frac{r_x^2}{F_o})$$

For *Parabolic* distribution, projection on *x, x′* plane is

$$\rho_x(x, x') = \frac{6}{\pi F_o^2} \int_0^{T_1^2} (1 - \frac{r_x^2 + T^2}{F_o})\, dT^2 = \frac{3}{\pi F_o} (1 - \frac{r_x^2}{F_o})^2$$

For *Gaussian* distribution projection on *x, x′* plane is

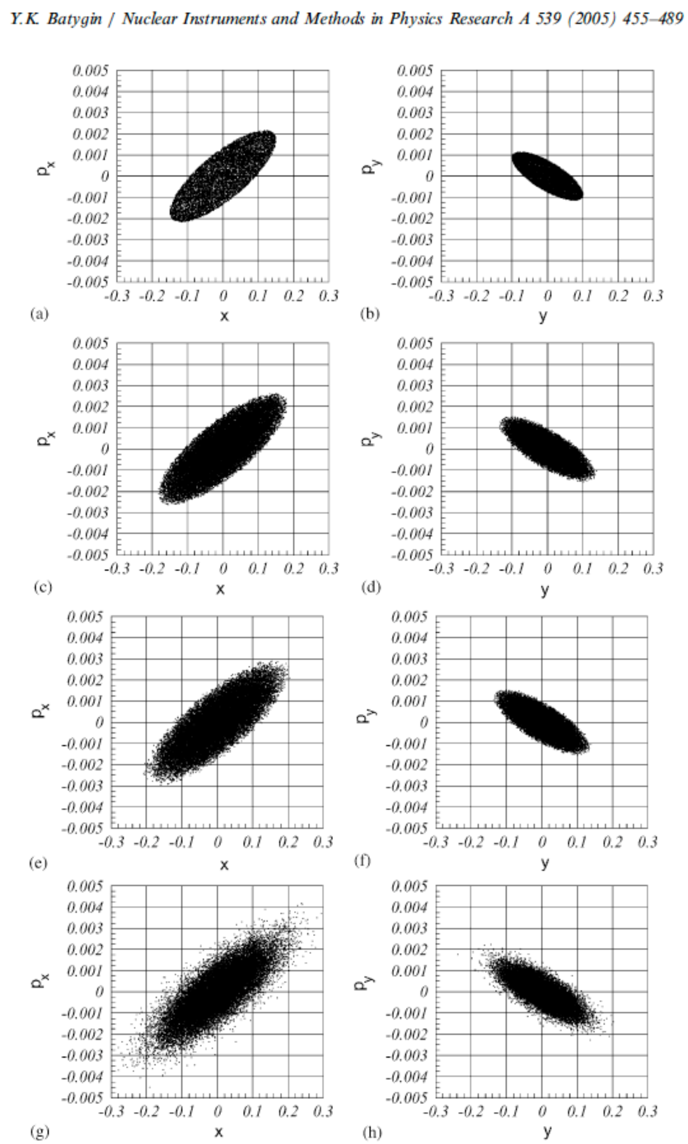
$$\rho_x(x, x') = \frac{1}{\pi F_o^2} \int_o^{\infty} exp(- \frac{r_x^2 + T^2}{F_o})\, dT^2 = \frac{1}{\pi F_o} exp(- \frac{r_x^2}{F_o}).$$

KV

Water Bag

Parabolic

Gaussian



$\varepsilon_{\max} = 4\varepsilon_{rms}$

$\varepsilon_{\max} = 6\varepsilon_{rms}$

$\varepsilon_{\max} = 8\varepsilon_{rms}$

$\varepsilon_{\max} = \infty$

Particle distributions with equal values of ε_{rms} .

Root mean square emittance

Four rms beam emittance $Э_x = 4\pi \int_o^{\infty} r_x^3 \rho_x (r_x^2)\, dr_x$

Water bag distribution, is limited by the surface

$$r_x^2 + r_y^2 \leq F_o, \qquad \text{or}$$

$$r_x^2 \leq F_o - r_y^2$$

Maximum value of r_x^2 is achieved when $r_y^2 = 0$ and vise versa. Therefore, projection of water bag distribution, is limited by $r_{x,\,max}^2 = F_o$. Substituion of expressions for $\rho_x(r_x^2)$, and integration gives:

$$Э_x = \frac{8}{F_o} \int_o^{\sqrt{F_o}} r_x^3 \left(1 - \frac{r_x^2}{F_o}\right) dr_x = \frac{2}{3} F_o \,,$$

Analogously, for *parabolic* distribution

$$Э_x = \frac{12}{F_o} \int_o^{\sqrt{F_o}} r_x^3 \left(1 - \frac{r_x^2}{F_o}\right)^2 dr_x = \frac{F_o}{2} ,$$

For *Gaussian* distribution

$$Э_x = \frac{4}{F_o} \int_o^{\infty} r_x^3 \, exp\left(- \frac{r_x^2}{F_o}\right) dr_x = 2 \, F_o ,$$

Fraction of particles residing within a specific emittance

The distribution $\rho(r_x^2)$ is the particle density in the phase plane (x, x'), divided by the total number of particles, N. Fraction of particles

$$\eta = N(\ni_x)/N_o$$

within the emittance $\ni_x$ is an integral of $\rho(r_x^2)$ over an ellipse area of $\ni_x$:

$$\eta = \frac{N(\ni)}{N_o} = \int\int \rho_x(r_x^2)\ dx\,dx' = \int_o^{2\pi}\int_o^{\sqrt{\ni}} \rho_x(r_x^2)\, r_x\, dr_x\, d\varphi = \pi\int_o^{\ni} \rho_x(r_x^2)\ dr_x^2$$

Distributions on phase plane are:

Water bag $$\rho_x(r_x^2) = \frac{4}{3\pi \text{э}_x}\left(1 - \frac{2}{3}\frac{r_x^2}{\text{э}_x}\right)$$

Parabolic $$\rho_x(r_x^2) = \frac{3}{2\pi \text{э}_x}\left(1 - \frac{r_x^2}{2\,\text{э}_x}\right)^2$$

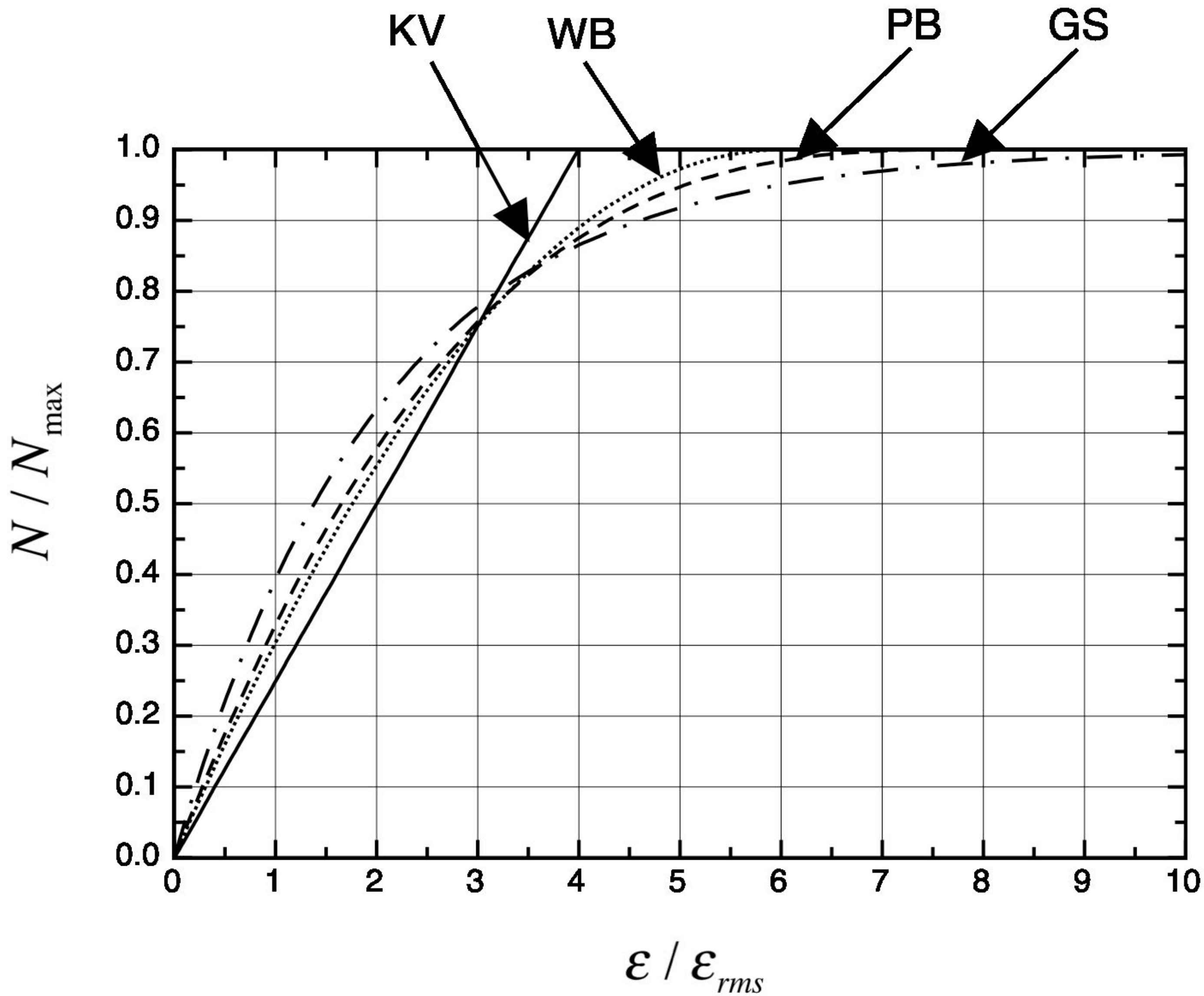
Gaussian $$\rho_x(r_x^2) = \frac{2}{\pi \text{э}_x}\, exp\left(-2\frac{r_x^2}{\text{э}_x}\right)$$

Fraction of particles within phase space area is:

Water bag $$\frac{N(\text{э})}{N_o} = \frac{4}{3}\left(\frac{\text{э}}{\text{э}_x}\right)\left(1 - \frac{1}{3}\frac{\text{э}}{\text{э}_x}\right)$$

Parabolic $$\frac{N(\text{э})}{N_o} = \frac{3}{2}\left(\frac{\text{э}}{\text{э}_x}\right)\left[1 - \frac{1}{2}\frac{\text{э}}{\text{э}_x} + \frac{1}{12}\left(\frac{\text{э}}{\text{э}_x}\right)^2\right]$$

Gaussian $$\frac{N(\text{э})}{N_o} = 1 - exp\left(-2\frac{\text{э}}{\text{э}_x}\right)$$



Fraction of particles versus phase space area for different particle distributions.

1.9. Emittance of the beam in particles sources

The ultimate goal of accelerator designers is to minimize emittance as much as possible. An intrinsic limitation of beam emittance in particle sources comes from the finite value of plasma temperature in an ion source, or the finite value of cathode temperature in an electron source. Equilibrium thermal particle momentum distribution in these sources is in fact, close to the Maxwell distribution:

$$f(p) = n\left(\frac{m}{2\pi kT}\right)^{3/2} exp\left(-\frac{p^2}{2mkT}\right)$$

Rms value of mechanical momentum is

$$\langle p_x^2 \rangle = mkT$$

Beam radius is usual ly adopted to be double the root-mean-square beam size, $R = 2\sqrt{\langle x^2 \rangle}$. Fortunately, for particle sources, one can assume that $\langle xP_x \rangle = 0$ because there is n o correlation between particle position and particle momentum. Therefore, the normalized emittance of a beam, extracted from a particle source, is

$$\varepsilon = 2R\sqrt{\frac{kT}{mc^2}}$$

Some sources can be operated only in presence of a longitudinal magnetic field, which produces an additional limitation on the value of the beam emittance. For instance, in an electron-cyclotron-resonance (ECR) ion source, charged particles are born in a longitudinal magnetic field B_z, fulfilling the ECR resonance condition $2\omega_L = \omega_{RF}$, where ω_L is the Larmor frequency of electrons and w_{RF} is the microwave frequency. Canonical momentum of an ion, $P_x = p_x - qA_x$, in a longitudinal magnetic field B_z is:

$$P_x = p_x - q\frac{B_z y}{2}$$

The rms value of canonical momentum is given by:

$$<P_x^2> = <p_x^2> - q B_z <p_x y> + \frac{q^2 B_z^2}{4} <y^2>$$

The first term describes the thermal spread of mechanical momentum of ions in plasma, and is given by $<p_x^2> = mkT$. The middle term equals zero because there is no correlation between p_x and y inside the source. The last term is proportional to the rms value of the transverse coordinate $<y^2> = R^2/4$. As a result, we can rewrite $<P_x^2>$ as follows:

$$<P_x^2> = <p_x^2> + (\frac{q B_z R}{4})^2$$

The normalized beam emittance ε, extracted from the source is

$$\varepsilon = 2R\sqrt{\frac{kT_i}{mc^2} + \left(\frac{qB_z R}{4\,mc}\right)^2}$$

Therefore, the presence of a longitudinal magnetic field at the source acts to increase the value of the beam emittance.

1.10. Space charge effects in the extraction region of particle sources: Child-Langmuir Law

2.5.2 *

Planar Diode with Space Charge (Child–Langmuir Law)

Let us now include the effect of the space charge of the electron current in the diode on the potential distribution and electron motion. To simplify our analysis, we assume that all electrons are launched with initial velocity $\mathbf{v}_0 = 0$ from the cathode (i.e., they are moving on straight lines in the x-direction). This is an example of *laminar flow* where electron trajectories do not cross and the current density is uniform. We try to find the steady-state solution ($\partial/\partial t = 0$) in a self-consistent form. The electrostatic potential is determined from the space-charge density ρ via Poisson's equation, with $\phi = 0$, at $x = 0$ and $\phi = V_0$ at $x = d$, as in the previous case. The relationship between ρ, the current density $\mathbf{J}$, and the electron velocity $\mathbf{v}$ follows from the continuity equation ($\nabla \cdot \mathbf{J} = 0$ or $\mathbf{J} = \rho\mathbf{v} =$ const). The velocity in turn depends on the potential ϕ and is found by integrating the equation of motion. Thus we have the following three equations:

$$\nabla^2\phi = \frac{d^2\phi}{dx^2} = -\frac{\rho}{\epsilon_0} \quad \text{(Poisson's equation)}, \tag{2.129}$$

$$J_x = \rho\dot{x} = \text{const} \quad \text{(continuity equation)}, \tag{2.130}$$

$$\frac{m}{2}\dot{x}^2 = e\phi(x) \quad \text{(equation of motion)}. \tag{2.131}$$

*From M.Reiser, Theory and Design of Charged Particle Beams, Wiley, 1994

Substituting $\dot{x} = [2e\phi(x)/m]^{1/2}$ from (2.131) into (2.130) and $\rho = J_x/\dot{x}$ from (2.130) into (2.129) yields

$$\frac{d^2\phi}{dx^2} = \frac{J}{\epsilon_0(2e/m)^{1/2}}\frac{1}{(\phi)^{1/2}}, \tag{2.132}$$

where the current density $J = -J_x$ is defined as a positive quantity. After multiplication of both sides of Equation (2.132) with $d\phi/dx$, we can integrate and obtain

$$\left(\frac{d\phi}{dx}\right)^2 = \frac{4J}{\epsilon_0(2e/m)^{1/2}}\phi^{1/2} + C. \tag{2.133}$$

Now $\phi = 0$ at $x = 0$, and if we consider the special case where $d\phi/dx = 0$ at $x = 0$, we obtain $C = 0$. A second integration then yields (with $\phi = V_0$ at $x = d$)

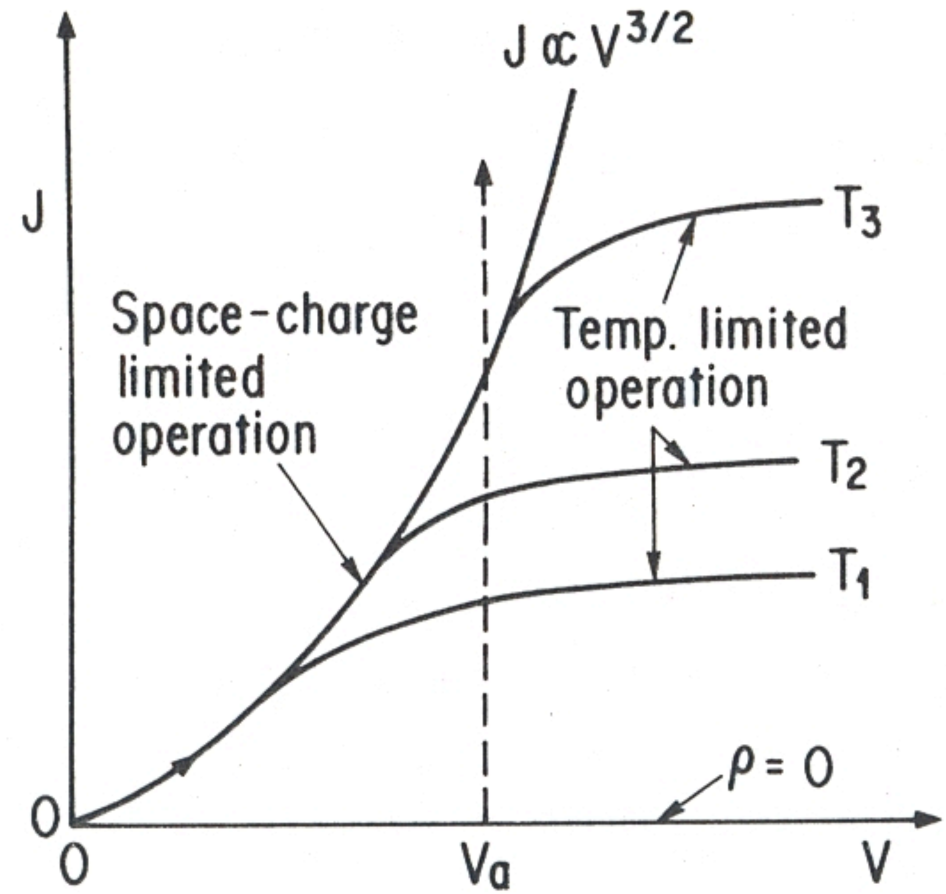
$$\frac{4}{3}\phi^{3/4} = 2\left(\frac{J}{\epsilon_0}\right)^{1/2}\left(\frac{2e}{m}\right)^{-1/4}x,$$

or

$$\phi(x) = V_0\left(\frac{x}{d}\right)^{4/3}. \tag{2.134}$$

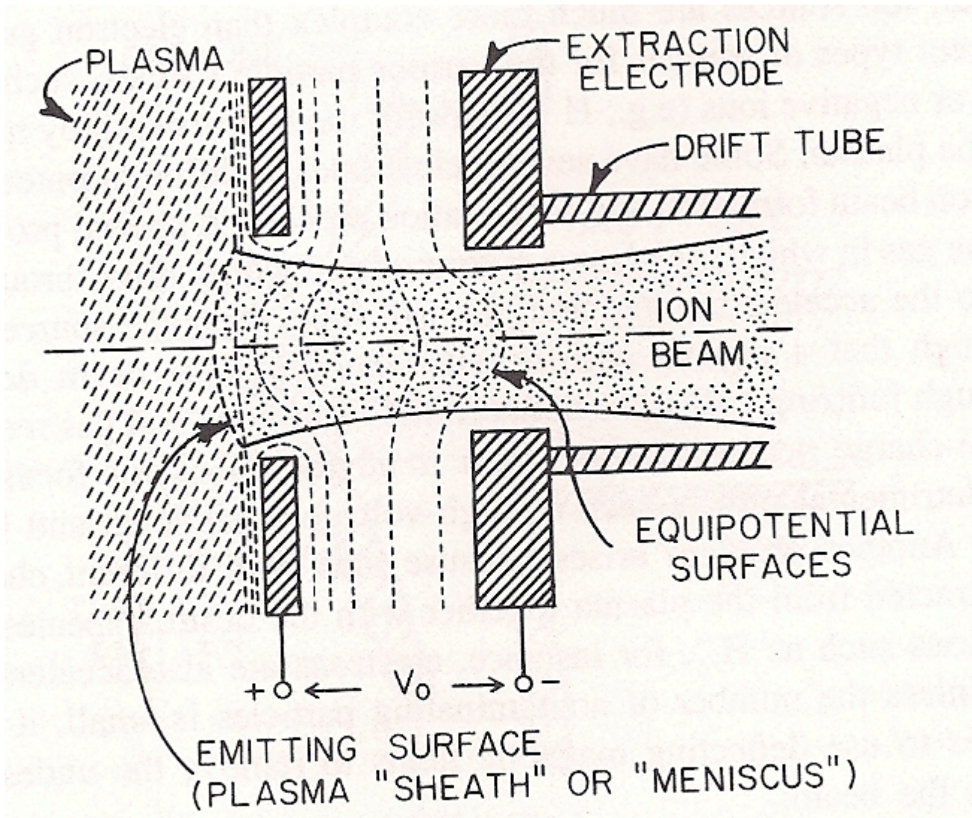
with the relation

$$J = \frac{4}{9}\epsilon_0\left(\frac{2e}{m}\right)^{1/2}\frac{V_0^{3/2}}{d^2}. \tag{2.135}$$



Current-voltage relation at constant cathode temperature (from S.Isagawa, Joint Accelerator School, 1996).

In ion sources, the shape of plasma meniscus is determined by the balance between plasma pressure and applied electrostatic voltage for ion extraction.



1. Schematics of a plasma ion source (from M.Reiser, 1994.)

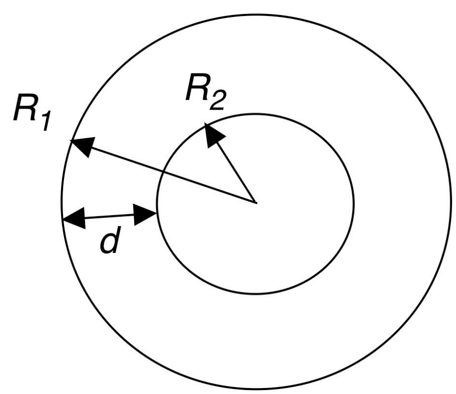
To determine shape of plasma memiscus, let us consider self-consistent problem for the beam extracted from spherical emitter of radius R_1 (plasma) and spherical collector of radius R_2 ($R_2 < R_1$). Saturated current density extracted from the plasma

$$j = n_i e \sqrt{\frac{kT_e}{m_i}}$$

We will assume that all particle have the same extracted velocities, so the current density is $j = \rho\, v_r$ and particle velocity is

$$v_r = \sqrt{\frac{2qU}{m}}$$

where U is the potential between two spheres. Therefore, beam space charge density is



$$\rho = \frac{j}{\sqrt{\frac{2qU}{m}}}$$

On derivation of Child-Langmuir law between spherical surfaces.

Let us substitute space charge density into Poisson's equation in spherical coordinates:

$$\frac{1}{r^2}\frac{d}{dr}(r^2\frac{dU}{dr}) = -\frac{1}{\varepsilon_o}\frac{j}{\sqrt{\frac{2qU}{m}}}$$

Solution of Poisson's equation for concentric spheres is

$$\frac{j}{U^{3/2}} = \frac{4\sqrt{2}}{9}\varepsilon_o\sqrt{\frac{q}{m}}\frac{1}{R_1^2\alpha^2}$$

where $\alpha = Y - 0.3Y^2 + 0.075Y^3$, $Y = \ln\frac{R_2}{R_1}$

This is the Child-Langmuir law for spherical surfaces. When the distance between emitter and collector is much smaller than the raduses $d = R_1 - R_2 << R_1$, the following approximations can be used:

$$Y = \ln(\frac{R_1 - d}{R_1}) \approx -\frac{d}{R_1} - \frac{1}{2}(\frac{d}{R_1^2})^2 - \frac{1}{2}(\frac{d}{R_1^2})^3$$

$$\frac{1}{R_1^2\alpha^2} \approx \frac{1}{d^2}(1 - 1.6\frac{d}{R_1})$$

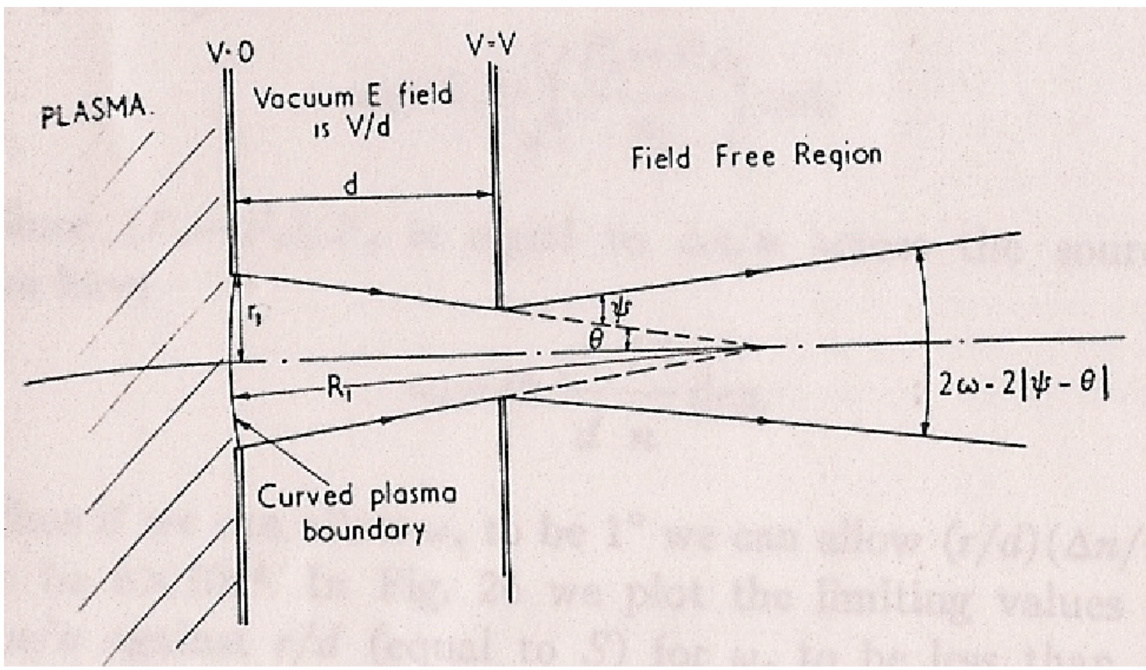
With this approximation, Child-Langmuir law is expressed as

$$\frac{j}{U^{3/2}} = \frac{4\sqrt{2}}{9}\varepsilon_o\sqrt{\frac{q}{m}}\frac{1}{d^2}(1 - 1.6\frac{d}{R_1})$$

Let us apply now obtained result to the problem of plasma beam extraction from small extraction hole of the radius R_{ext}. From Fig the relationship between extraction radius R_{ext} and radius R_1 is

$$R_1 = \frac{r_1}{\sin\theta} \approx \frac{r_1}{\theta}$$

where θ is associated with initial beam slope.



Scheme of simplified ion optics in beam extraction region (J.R.Coupland et al., Rev. Sci. Instruments, Vol. 44, No 9, (1973), p.1258.

Beam current density $j = \frac{I}{\pi r_1^2}$

Substitution of expression for beam current density into Child-Langmuir law reads:

$$\frac{I}{U^{3/2}} = \frac{4\sqrt{2}\pi}{9}\varepsilon_o\sqrt{\frac{q}{m}}(\frac{r_1}{d})^2(1-1.6\frac{d}{r_1}\theta)$$

Beam perveance: $P_b = \frac{I}{U^{3/2}}$

Child-Langmuir perveance of one dimensional diode $P_o = \frac{4\sqrt{2}\pi}{9}\varepsilon_o\sqrt{\frac{q}{m}}(\frac{r_1}{d})^2$

Extracted beam slope (plasma meniscus): $\theta = 0.625\frac{r_1}{d}(\frac{P_b}{P_0}-1)$

If $P_b << P_o$, it corresponds to the extracted beam with negligible intensity, and initial convergence of the beam is defined by extraction geometry only

$$\theta = -0.625\frac{r_1}{d}$$

1.

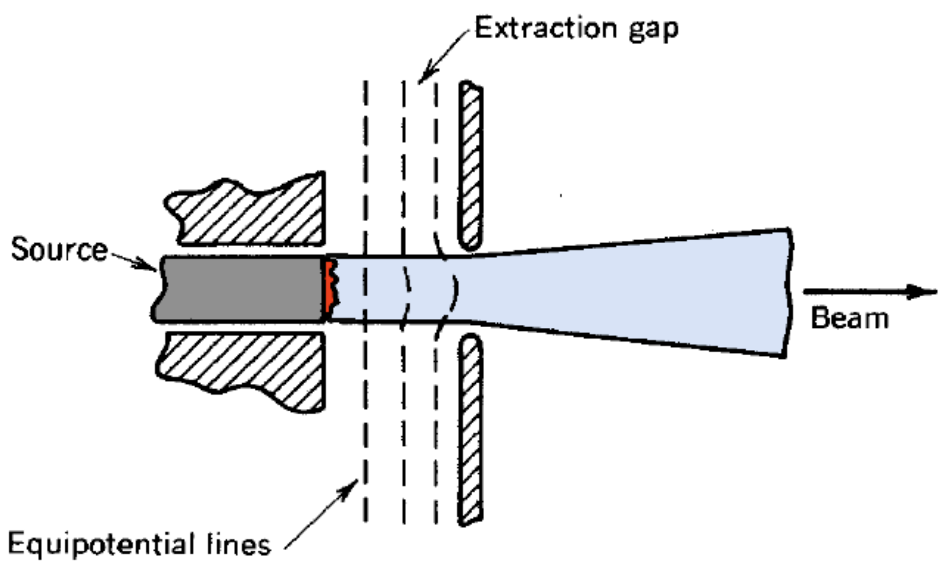
According to Child-Langmuir law, the potential inside extraction gap has the following z-dependence:

$$U(z) = U_{ext} \left(\frac{z}{d_{ext}} \right)^{4/3}$$

Inside extraction gap particles move in the field, which, in the first approximation, has only longitudinal component

$$E_z = \frac{4}{3} U_{ext} \frac{z^{1/3}}{d_{ext}^{4/3}}$$

Outside extraction gap the field drops to zero.



Extraction gap showing defocusing effect (S.Humphries, 1999).

Due to equation $\quad \mathrm{div}\ \vec{E} = \frac{1}{r}\frac{\partial}{\partial r}(r\ E_r) + \frac{\partial E_z}{\partial z} = 0$

any change in longitudinal field results in appearance of transverse field component, which (in this case) defocuses beam:

$$E_r = -\frac{1}{r}\int_o^r \frac{\partial E_z}{\partial z}\ r'\ dr' \approx -\frac{r}{2}\frac{\partial E_z}{\partial z}$$

Eqiuation of particle motion: $\quad \frac{d^2 r}{dz^2} = -\frac{q}{m v_z^2} r \frac{1}{2}\frac{\partial E_z}{\partial z}$

Slope of particle trajectory at the exit of the gap:

$$\psi = \Delta\left(\frac{dr}{dz}\right) = -\frac{q}{2 m v_z^2} r \int \frac{\partial E_z}{\partial z} dz = \frac{q}{2 m v_z^2} r E_z = \frac{r E_z}{4 U_{ext}} = \frac{r}{3d}$$

Finally, divergence of the extracted beam is as follows:

$$\omega = |\theta + \psi| = \left| 0.625 \frac{r_1}{d}\left(\frac{P_b}{P_o} - 1\right) + \frac{r_1}{3d} \right| = 0.29 \frac{r_1}{d}\left(1 - 2.14 \frac{P_b}{P_o}\right)$$