Measuring Bunch Length Using Fluctuations in Synchrotron Radiation

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Introduction

- Bunch lengths are getting shorter:
  - 30 ps in typical rings
  - < 10 ps with special low-momentum-compaction lattices
  - 10 to 100 fs in new linac-based light sources (LCLS at SLAC)
- Fastest streak camera has a resolution of 200 fs/pixel.
  - Also expensive and complex for a routine monitor.
- Various new techniques have been devised.
- A technically simple, but subtle, scheme (Zolotorev and Stupakov, 1996) studies the statistics of single-bunch emission, either examining:
  - Turn-to-turn variations in the energy in a narrow band, or
  - Single-shot variations in the spectrum
Electric Field of a Bunch

- The electrons in the bunch are randomly distributed:
  - Normalized distribution $f(t)$: $\int_{-\infty}^{\infty} f(t)dt = 1$, $f(t)$ is real
  - Characteristic time duration $\sigma_t$
  - Later we will use a Gaussian: $f(t) = \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left(-\frac{t^2}{2\sigma_t^2}\right)$
- Electric field is the sum of the fields of the $N >> 1$ electrons:
  $$E(t) = \sum_{k=1}^{N} e(t - t_k)$$
- Fourier transform of the field:
  $$\hat{E}(\omega) = \int_{-\infty}^{\infty} E(t)e^{i\omega t} dt = \hat{e}(\omega) \sum_{k=1}^{N} e^{i\omega t_k}$$
- The total field is noisy.
  - $\hat{e}$ is smooth; the noise in $\hat{E}$ comes from the random spacing of the $t_k$. 
Energy Radiated by the Bunch

The energy radiated by the bunch is:

\[ W = \int_{-\infty}^{\infty} |E(t)|^2 \, dt \]

\[ = \int_{-\infty}^{\infty} dt \frac{1}{2\pi} \sum_{k=1}^{N} \int_{-\infty}^{\infty} d\omega \hat{e}(\omega)e^{-i\omega(t-t_k)} \frac{1}{2\pi} \sum_{l=1}^{N} \int_{-\infty}^{\infty} d\omega' \hat{e}^*(\omega')e^{i\omega'(t-t_l)} \]

\[ = \frac{1}{(2\pi)^2} \sum_{k,l=1}^{N} \int \int d\omega d\omega' dt \hat{e}(\omega)\hat{e}^*(\omega')e^{i(\omega t_k - \omega t_l)} e^{-i(\omega - \omega')t} \]

\[ = \frac{1}{2\pi} \sum_{k,l} \int d\omega |\hat{e}(\omega)|^2 e^{i\omega(t_k - t_l)} \]

- We used $|E|^2$ for power to simplify notation.

- We made use of the identity: $\int_{-\infty}^{\infty} e^{i\omega t} \, dt = 2\pi \delta(\omega)$
To get the bunch length, we find the mean (1st moment) and variance (2nd moment) of the energy per pulse $W$.

For any distribution $p(t)$ and function $q(t)$, define:

- **Mean**
  \[ m_q = \langle q(t) \rangle = \int q(t) p(t) dt \]

- **Variance**
  \[ \sigma^2_q = \left\langle \left[ q(t) - m_q \right]^2 \right\rangle = \int \left[ q(t) - m_q \right]^2 p(t) dt = \langle q^2(t) \rangle - m_q^2 \]

- **Standard deviation** $= \sigma_q$
The ensemble-averaged (also time-averaged) energy is then:
\[
\langle W \rangle = m_w = \frac{1}{2\pi} \sum_{k,l} \int dt_k dt_l f(t_k) f(t_l) \int d\omega |\hat{\epsilon}(\omega)|^2 e^{i\omega(t_k-t_l)}
\]
\[
= \frac{1}{2\pi} \int d\omega |\hat{\epsilon}(\omega)|^2 \left[ \sum_{k=l} \int dt_k dt_l f(t_k) f(t_l) + \sum_{k\neq l} \int dt_k dt_l f(t_k) f(t_l) e^{i\omega(t_k-t_l)} \right]
\]
\[
= \frac{1}{2\pi} \int d\omega |\hat{\epsilon}(\omega)|^2 \left[ N + N^2 |\hat{f}(\omega)|^2 \right] \quad \text{(for } N \gg 1)\]

- The first term is incoherent radiation from the \( N \) electrons.
- The second term is coherent radiation:
  - The characteristic width of \( \hat{f}(\omega) \) is \( \sigma_\omega = 1/\sigma_i \)
  - Coherent term is insignificant when \( \omega \gg \sigma_\omega \)
Bandwidth and Coherence Time

- The light is filtered to a narrow bandwidth \( \sigma_{\text{filt}} \) centered at \( \omega_{\text{filt}} \)
  - The characteristic coherence time for oscillations of the filtered electric field is:
    \[ \tau_{\text{coh}} = 1/\sigma_{\text{filt}} \]
  - We are interested in the statistics of the incoherent part of the emission.
  - The filter is chosen so that \( M = \sigma_t/\tau_{\text{coh}} >> 1 \), or \( 1/\sigma_t = \sigma_\omega <= \sigma_{\text{filt}} \)
    - We can neglect the coherent-radiation term.
    - The filter band is also narrow compared to \( \omega_{\text{filt}} \), and so \( \sigma_\omega < \sigma_{\text{filt}} << \omega_{\text{filt}} \)
    - Since the bunch duration is many coherence times, it can be pictured as \( M \) independently radiating modes, each with random amplitude.
    - The filtered power \( |\hat{e}(\omega)|^2 \) from each electron, which is not random (but has random timing), has a characteristic width of \( \sigma_{\text{filt}} \).
Variance of the Energy

\[ \sigma_w^2 = \left\langle |W|^2 \right\rangle - \left| \left\langle W \right\rangle \right|^2 \]

\[ = \frac{1}{(2\pi)^2} \sum_{k,l,m,n} \iiint d\omega dt_k dt_l \iiint d\omega' dt_m dt_n |\hat{e}(\omega)|^2 |\hat{e}(\omega')|^2 f(t_k) f(t_l) f(t_m) f(t_n) e^{i\omega(t_k-t_l)-i\omega'(t_m-t_n)} \]

\[ - \left[ \frac{N}{2\pi} \int d\omega |\hat{e}(\omega)|^2 \right]^2 \]

\[ = \frac{1}{(2\pi)^2} \left[ \sum_{k=l,m=n} (...) + \sum_{k=m,l=n} (...) \right] - \left[ \frac{N}{2\pi} \int d\omega |\hat{e}(\omega)|^2 \right]^2 \]

\[ = \frac{1}{(2\pi)^2} \sum_{k,l} \iint d\omega d\omega' |\hat{e}(\omega)|^2 |\hat{e}(\omega')|^2 \iint dt_k dt_l f(t_k) f(t_l) e^{i(\omega-\omega')(t_k-t_l)} \]

\[ = \left( \frac{N}{2\pi} \right)^2 \iint d\omega d\omega' |\hat{e}(\omega)|^2 |\hat{e}(\omega')|^2 \left| \hat{f}(\omega-\omega') \right|^2 \]

\[ = \left( \frac{N}{2\pi} \right)^2 \int d\omega |\hat{e}(\omega)|^4 \int d\omega' \left| \hat{f}(\omega') \right|^2 \]

The next slide explains some steps used here.
Wait…How Was That Done?

- As before, we kept only the significant combinations:
  - $k = l, m = n$: Canceled by the last term (the mean-squared term).
  - $k = n, l = m$: Gives $\hat{f}(\omega + \omega')$ terms, which are small.
  - Neglect coherent-radiation terms.
- $\hat{f}(\omega - \omega')$ has width $\sigma_\omega$, much narrower than width of $\hat{e}(\omega')$.
  - We can set $\hat{e}(\omega') \approx \hat{e}(\omega)$ when integrating over $\omega'$. 
Ratio to Mean

- Ratio of the variance to the mean squared:
  \[
  \frac{\sigma_w^2}{m_w^2} = \frac{\int d\omega |\hat{e}(\omega)|^4}{\left[ \int d\omega |\hat{e}(\omega)|^2 \right]^2} \int d\omega \left| f(\omega) \right|^2
  \]

- A beam in a storage ring is Gaussian in time (slide 3). In the frequency domain, the distribution becomes:
  \[
  \hat{f}(\omega) = \frac{1}{\sqrt{2\pi\sigma_t}} \int_{-\infty}^{\infty} \exp\left( -\frac{t^2}{2\sigma_t^2} + i\omega t \right) dt = \exp\left( -\frac{\omega^2 \sigma_t^2}{2} \right)
  \]

- Assume that the filter is also Gaussian.
  - A filter’s RMS width \( \sigma_{\text{filt}} \) is generally expressed in terms of intensity \( (E^2) \), not field. So, after the filter, the single-electron spectrum is:
    \[
    |\hat{e}(\omega)|^2 = \frac{p_1}{\sqrt{2\pi\sigma_{\text{filt}}}} \exp\left[ -\frac{(\omega - \omega_{\text{filt}})^2}{2\sigma_{\text{filt}}^2} \right]
    \]
Finding the Length of a Gaussian Bunch

\[
\int d\omega |\hat{e}(\omega)|^2 = p_1
\]

\[
\int d\omega |\hat{e}(\omega)|^4 = \frac{p_1^2}{2\sqrt{\pi} \sigma_{\text{filt}}}
\]

\[
\int d\omega |\hat{f}(\omega)|^2 = \frac{\sqrt{2\pi}}{\sigma_t}
\]

\[
\frac{\sigma_w^2}{m_w^2} = \frac{1}{\sqrt{2}\sigma_t \sigma_{\text{filt}}} = \frac{\tau_{\text{coh}}}{\sqrt{2}\sigma_t} = \frac{1}{\sqrt{2}M}
\]

**Conclusion:** The bunch length \(\sigma_t\) can be determined by finding the mean and variance of many measurements of the radiated energy \(W\) through a narrow filter of known bandwidth \(\sigma_{\text{filt}}\).
Example

- View 550-nm light through a filter with a 1-nm bandwidth (in intensity):
  - \( \omega_{\text{filt}} = \frac{2\pi c}{\lambda_{\text{filt}}} = 3.425 \times 10^{15} \text{ s}^{-1} \)
  - \( \sigma_{\text{filt}} = \frac{\omega_{\text{filt}} \sigma_\lambda}{\lambda_{\text{filt}}} = 6.227 \times 10^{12} \text{ s}^{-1} \)
  - \( \tau_{\text{coh}} = \frac{1}{\sigma_{\text{filt}}} = 0.16 \text{ ps} \)
- Measure the statistics:
  - \( \sigma_w/m_w = 0.08 \)
- The bunch length \( \sigma_t = 18 \text{ ps} \).
What if the Pulse isn’t Gaussian?

- Interferometric method
  - Split the pulse, delay one part by a time $\tau$, and recombine at the detector, for a total field:
    \[ E_{\text{total}}(t, \tau) = E(t) + \alpha E(t - \tau) \]
  - A Michelson interferometer can be used.
  - In the frequency domain:
    \[ \hat{E}_{\text{total}}(\omega, \tau) = \hat{e}(\omega) \sum_k e^{i\omega t_k} \left(1 + \alpha e^{i\omega \tau}\right) \]
  - When $\tau = 0$, this is the same as the previous approach.
  - We will see that the result is the autocorrelation of the distribution $f(t)$ as a function of the delay $\tau$:
    \[ \int f(t) f(t - \tau) dt \]
  - When $f(t)$ is real and symmetric, the autocorrelation can normally be inverted to find $f$. 
Energy in the Pulse

\[ W(\tau) = \frac{1}{(2\pi)^2} \sum_{k,l} \int \int d\omega d\omega' \hat{\epsilon}(\omega)\hat{\epsilon}^*(\omega')(1 + \alpha e^{i\omega\tau})(1 + \alpha^* e^{-i\omega'\tau}) e^{-i\omega(t-t_k)+i\omega'(t-t_l)} \]

\[ = \frac{1}{2\pi} \sum_{k,l} \int d\omega |\hat{\epsilon}(\omega)|^2 |1 + \alpha e^{i\omega\tau}|^2 e^{i\omega(t_k-t_l)} \]

\[ \langle W(\tau) \rangle = m_w(\tau) = \frac{1}{2\pi} \sum_{k,l} \int d\omega |\hat{\epsilon}(\omega)|^2 |1 + \alpha e^{i\omega\tau}|^2 \int \int dt_k dt_l f(t_k) f(t_l) e^{i\omega(t_k-t_l)} \]

\[ = \frac{N}{2\pi} \int d\omega |\hat{\epsilon}(\omega)|^2 |1 + \alpha e^{i\omega\tau}|^2 + \frac{N^2}{2\pi} \int d\omega |\hat{\epsilon}(\omega)|^2 |1 + \alpha e^{i\omega\tau}|^2 |\hat{f}(\omega)|^2 \]

- As before, we neglect the second term, for coherent radiation, because the filter passes light only at a high frequency \( \omega_{\text{filt}} \).
The next slide explains some steps used here.
The Usual Tricks

- Only certain combinations of the sum are significant:
  - \( k = l, m = n \): Canceled by the last term (mean squared).
  - \( k = m, l = n \): Gives the \( \omega - \omega' \) terms that provide our result.
- \( \hat{f}(\omega - \omega') \) has width \( \sigma_\omega \), much narrower than width of \( \hat{e}(\omega') \).
  - We can set \( \hat{e}(\omega') \approx \hat{e}(\omega) \) when integrating over \( \omega' \).
- Recall that:
  - \( \hat{e}(\omega) \) is centered at a high frequency \( \omega_\text{filt} \)
  - The delay \( \tau \) is comparable to the pulse width \( \sigma_i \)
  - \( \omega \tau \sim \omega_\text{filt} \sigma_i \gg 1 \)
- As a result:
  - In expanding the \( |1 + \alpha e^{i\omega \tau}| \) factors, all but the constant terms and those involving \( \omega - \omega' \) oscillate rapidly, vanishing in the \( \omega' \) integral.
  - But for \( \tau = 0 \) this argument does not apply, and we simply pull the \( |1 + \alpha| \) factors out of the integral.
Variance and Autocorrelation

\[
\sigma_w^2(\tau) = \left(\frac{N}{2\pi}\right)^2 \left(1 + |\alpha|^2\right)^2 \iint d\omega d\omega' |\hat{e}(\omega)|^4 \left|\hat{f}(\omega - \omega')\right|^2 \\
+ \left(\frac{N}{2\pi}\right)^2 |\alpha|^2 \iint d\omega d\omega' |\hat{e}(\omega)|^4 \left|\hat{f}(\omega - \omega')\right|^2 \left(e^{i(\omega - \omega')\tau} + e^{-i(\omega - \omega')\tau}\right) \\
= \left(\frac{N}{2\pi}\right)^2 \left(1 + |\alpha|^2\right)^2 \int d\omega |\hat{e}(\omega)|^4 \int d\omega' \left|\hat{f}(\omega')\right|^2 \\
+ 2\left(\frac{N}{2\pi}\right)^2 |\alpha|^2 \text{Re} \left[\int d\omega |\hat{e}(\omega)|^4 \int d\omega' \left|\hat{f}(\omega')\right|^2 e^{-i\omega'\tau}\right] \\
= \left(\frac{N}{2\pi}\right)^2 \int |\hat{e}(\omega)|^4 d\omega \left[1 + |\alpha|^2\right] \int f(t)^2 dt + 2|\alpha|^2 \int f(t)f(\tau - t)dt
\]

Again, see the next slide for some steps used here.
And More Tricks

- We used a theorem of Fourier transforms: The product of two transforms is an autocorrelation in the time domain.

\[
\int g(t)h(\tau - t)dt = \frac{1}{(2\pi)^2} \int d\omega \int d\omega' \hat{g}(\omega)\hat{h}(\omega') \int dt e^{-i\omega t - i\omega'(\tau - t)}
\]

\[
= \frac{1}{2\pi} \int \hat{g}(\omega)\hat{h}(\omega)e^{-i\omega \tau} d\omega
\]

- We also used a special case of this, Parseval’s theorem:

\[
\int |g(t)|^2 dt = \frac{1}{2\pi} \int |g(\omega)|^2 d\omega
\]

- And we used the fact that \(f(t)\) is real and assumed to be symmetric.

- When we look at the change in the variance as \(\tau\) is scanned, we can ignore the first, \(\tau\)-independent term.

- For \(\alpha = 0\) (no interference), the result reverts to the prior case.
As $\tau$ is scanned, the ratio of the variance to the central (peak) value of the mean gives a constant and a varying term.

When $f(t)$ is real and symmetric, the autocorrelation from the varying term can be inverted to determine $f$. 
Additional Complications

- **Transverse beam size**
  - If the beam is too wide for transversely coherent emission, or if there is diffraction at a limiting aperture, then the measured variance is reduced.

- **Detector noise**
  - Detector noise adds to the measured fluctuations, and must be accounted for to find the correct bunch length.

- **Photon count**
  - If the number of photons on the detector is too low, shot noise will increase the measured fluctuations.
Another Variation

- Spectrographic method
  - Use a spectrometer to make *many* narrow filters.
  - The fluctuations from one wavelength bin to the next then give the bunch length in a single measurement of the pulse.
Conclusion

- We can find the length $\sigma_t$ of a short bunch using a simple statistics of many measurements of the radiated energy $W$ through a narrow filter.
- A more elaborate setup can provide more information about the temporal profile.