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USPAS, Jan. 2004

Centroid and Envelope Descriptions of Beam Evolution

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§6 Formulation and use of 1st Order Coupled Moment Approaches

- Motivation
- Example illustration - dispersive effects.

References.

Notes:

- Some of this material has been covered by J.J. Barnard and these notes represent a more detailed treatment.
- J.J. Barnard also covers related topics:
  - Longitudinal envelope evolution and the Neuffer distribution
  - 3D modes of a bunched beam (Transverse - Longitudinal modes etc.)

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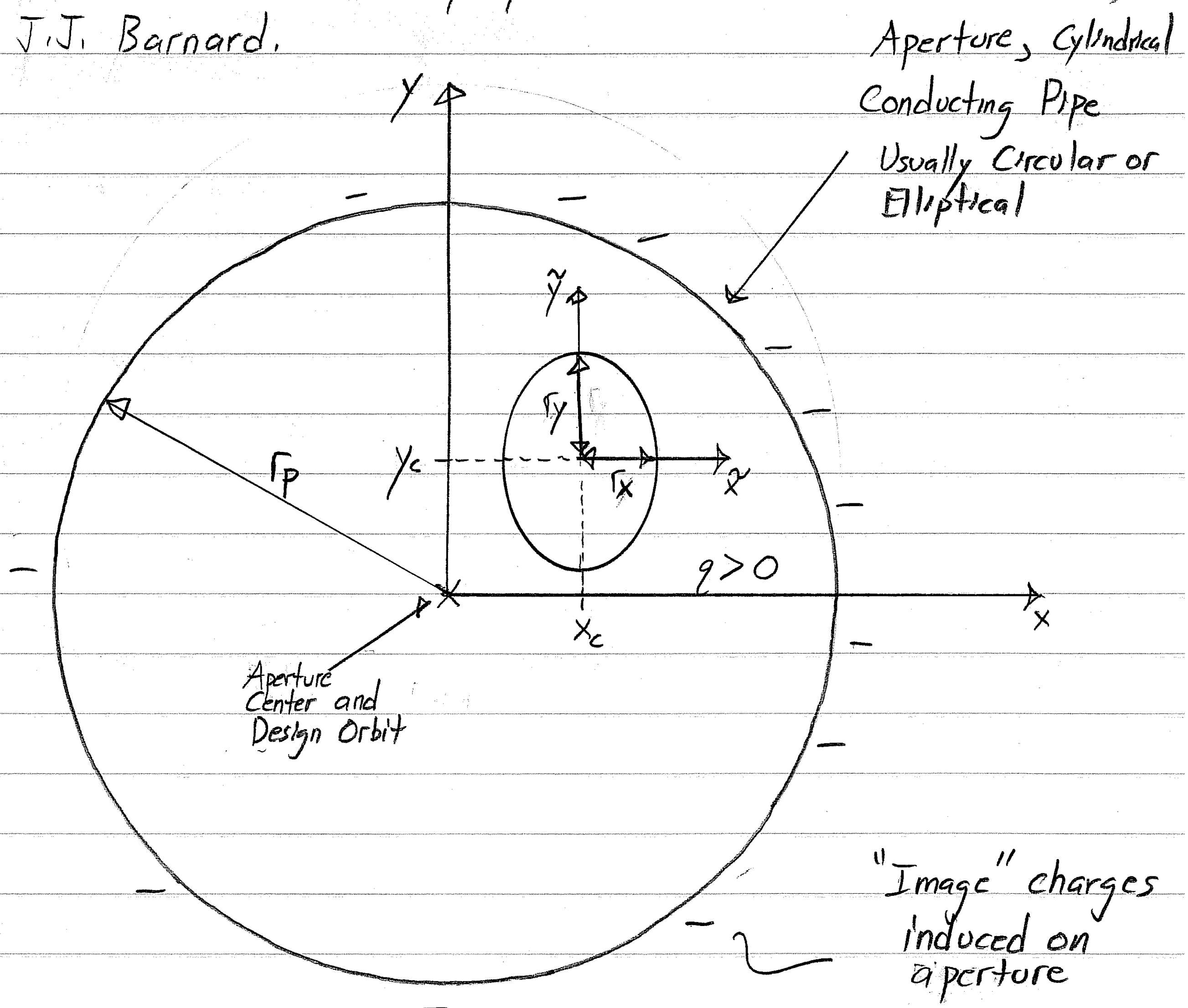
§1 Overview – Transverse Centroid and Envelope Descriptions of Beam Evolution

We analyze transverse centroid and envelope properties of unbunched ($\partial/\partial z = 0$) beams. Longitudinal and bunched beam (3D) properties are covered in lectures by J.J. Barnard.



Centroid

x_c, y_c x- and y- coordinates of transverse beam center of mass

Envelope

r_x, r_y x- and y- principal axis radii of elliptical beam envelope.

Oscillations in the beam centroid and envelope radii are the <u>lowest-order</u> collective oscillations of an intense beam.

<u>Centroid Oscillations</u> – are associated with errors (beam distribution, dipole bending terms in optics, mechanical alignment, ...) and are purposefully suppressed to the level possible.

- Exception: when a beam is kicked (inserted or extracted) into or out of a transport channel as is often done in rings.

<u>Envelope Oscillations</u> – Can have two components in periodic lattices:

1) <u>Matched Envelope</u> – Periodic flutter synchronized to periodic focusing structure to produce net focusing.

2) Mismatched Envelope – Excursions deviate from matched flutter motion and are seeded/driven by errors (beam distribution, focusing, ...)

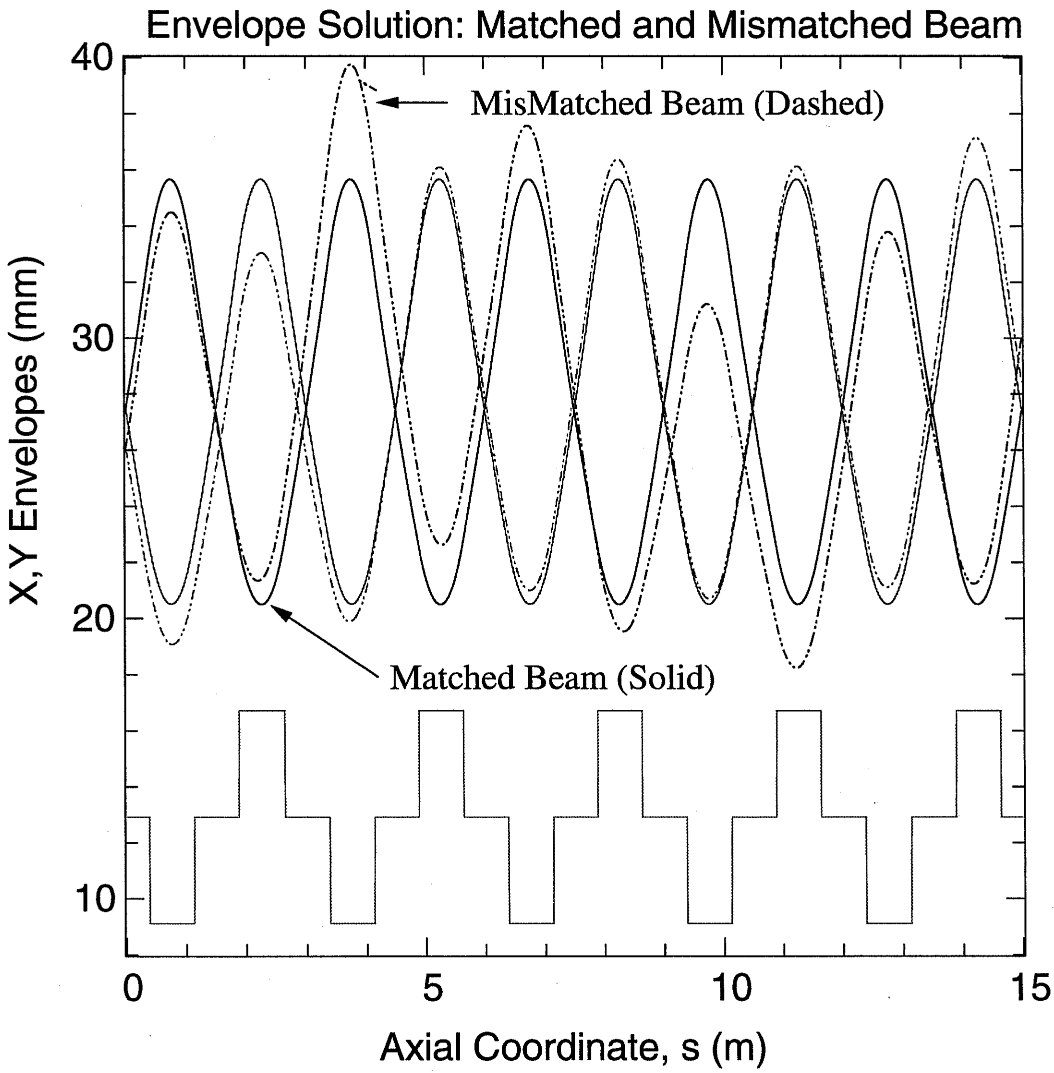
[See plot next page]

Strong radial confinement of maximum beam-edge excursions is desired for economical transport.

- Limits material volume, higher energy density.

Example FODO Quadrupole Transport Channel

sigma/sigma_0 = 9.94/72.0 = 0.138



Black: x–envelope

Red: y–envelope

Centroid and Envelope oscillations are the most important collective modes of a beam.

- Force balances based on the matched beam envelope equation predict scaling of transportable beam parameters and are used to design transport structures.

- Any instabilities in the beam centroid or envelope oscillations prevent reliable transport.
  - Parametric locations of instability regions should be understood and avoided in machine design.

Although it is <u>necessary</u> to design to avoid envelope and centroid instabilities, it is not alone <u>sufficient</u> for effective machine operation.

- Higher-order instabilities not present in low-order envelope models can degrade beam quality and control and must also be evaluated.

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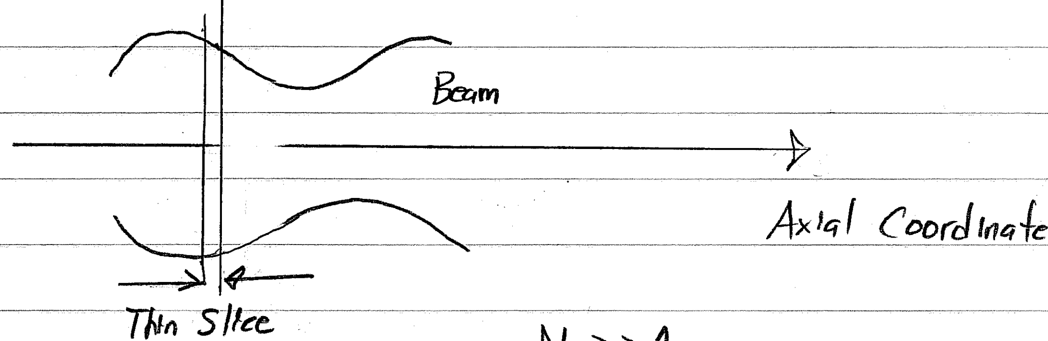
§2 Derivation of Transverse Centroid and Envelope Equations of Motion for an Unbunched Beam

Transverse Statistical Averages

Let N be the number of particles in a thin axial slice of beam at axial coordinate s:



$N >> 1$

$$\boxed{\langle \ldots\ldots \rangle_\perp \equiv \frac{1}{N} \sum_{\substack{i=1 \\ \text{particles}}}^{N} \ldots\ldots}$$

This average can also be defined in terms of a transverse single-particle distribution function $f_\perp(x, y, x', y', s)$ as:

$$\boxed{\langle \ldots\ldots \rangle_\perp = \frac{\int d^2x\, d^2x' \ldots\ldots f_\perp}{\int d^2x\, d^2x'\, f_\perp}}$$

- The distribution and averages can be generalized to include axial momentum spread.

Transverse Particle Equations of Motion

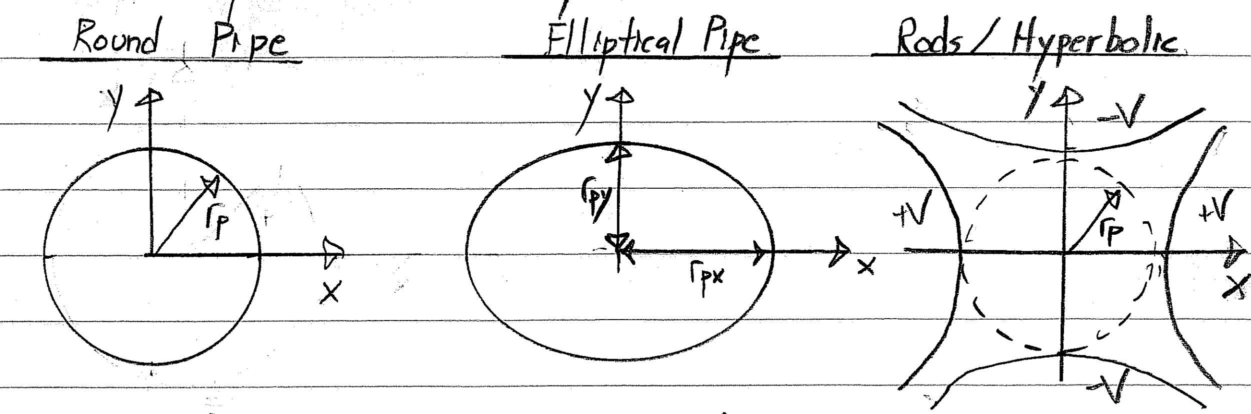
Consistent with earlier analysis we take:

$$\boxed{\begin{gathered} x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} x' + \kappa_x x = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x} \\ y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} y' + \kappa_y y = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y} \\ \nabla_\perp^2\phi = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = -\frac{\rho}{\epsilon_0} \\ \phi = 0 \text{ on Aperture pipe.} \end{gathered}}$$

Here we assume:

- Unbunched beam with $\partial/\partial z = 0$ in the slice.
- No momentum spread.

Various apertures are possible:

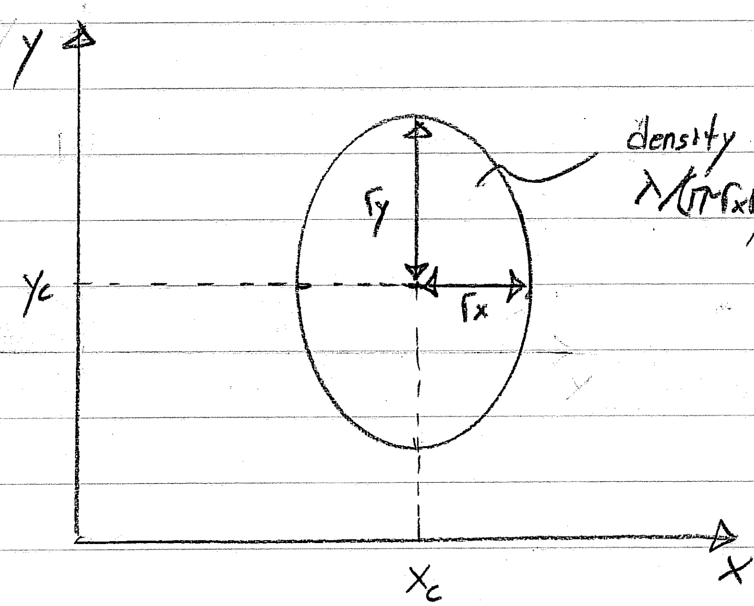


Round Pipe	Elliptical Pipe	Rods / Hyperbolic
Drifts Magnetic Quadrupoles Sextupoles ⋮ Acceleration Cells	In Rings where dispersion expands beam in x. Drifts Magnetic Optics ⋮	Electric Quadrupoles

Distribution Asumptions

To lowest order, linearly focused intense beams are expected to be nearly uniform density in the core out to an edge where the density falls rapidly to zero. We take:

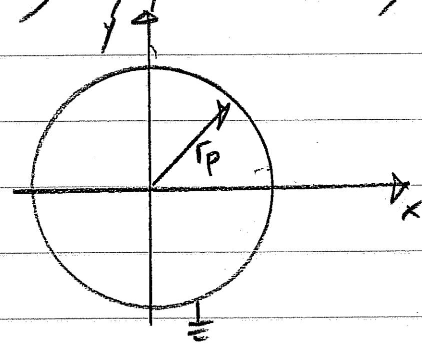
$$\boxed{\begin{aligned} \rho(x,y) &= \begin{cases} \dfrac{\lambda}{\pi r_x r_y} \; ; & \left(\dfrac{x-x_c}{r_x}\right)^2 + \left(\dfrac{y-y_c}{r_y}\right)^2 \leq 1 \\ 0 \; ; & \text{otherwise} \end{cases} \\ \lambda &= q\int d^2x\, d^2x'\, f_\perp = \int d^2x\, \rho(x,y) = \text{const.} \\ &= \text{line-charge density} \end{aligned}}$$



Self-Field Model

(Reiser § 4.4.4 has a small amout of material on this topic)

In these notes we will consider only round conducting pipe boundary conditions:



Other extruded pipe structures, elliptical, hyperbolic, etc., can be treated analogously but will be more complicated.

$$\phi(r = r_p) = 0$$

The method of images can be used to calculate the electric field "generated" by a charge distribution $\rho(x, y)$ within the pipe. First note that the field produced by a test line-charge λ_0 at $\vec{x}_\perp = \vec{x}_0$ in free space is given by:

$$\boxed{\vec{E}_\perp = \frac{\lambda_0}{2\pi\epsilon_0} \frac{(\vec{x}_\perp - \vec{x}_0)}{|\vec{x}_\perp - \vec{x}_0|^2}}$$

Line charge λ_0 at $\vec{x}_\perp = \vec{x}_0$ in free-space.

In the presense of the cylinder the line charge will induce an image:

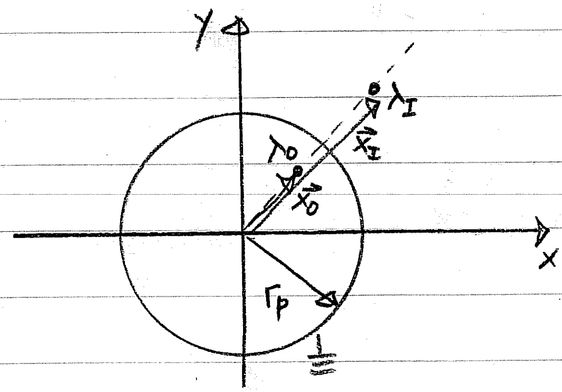


Image charge and coordinate

$$\boxed{\begin{aligned} \lambda_I &= -\lambda_0 \\ \vec{x}_I &= \frac{r_p^2}{|\vec{x}_0|^2} \vec{x}_0 \end{aligned}}$$

$$\Rightarrow \phi(r = r_p) = 0$$

The line charges "direct" and "image" can then be added in free-space to obtain the total field.

$$\vec{E}_\perp = \underbrace{\frac{\lambda_0}{2\pi\epsilon_0}\frac{(\vec{x}_\perp-\vec{x}_0)}{|\vec{x}_\perp-\vec{x}_0|^2}}_{\vec{E}_{d\perp}} - \underbrace{\frac{\lambda_0}{2\pi\epsilon_0}\frac{\left(\vec{x}_\perp-\frac{r_p^2}{|\vec{x}_0|^2}\vec{x}_0\right)}{\left|\vec{x}_\perp-\frac{r_p^2}{|\vec{x}_0|^2}\vec{x}_0\right|^2}}_{\vec{E}_{I\perp}}$$

Direct Field Image Field

valid inside the pipe.

Using linear superposition we take:

$$\boxed{\vec{E}_\perp = \vec{E}_{d\perp} + \vec{E}_{I\perp}}$$

where:

Direct:
$$\vec{E}_{d\perp}(\vec{x}_\perp) = \frac{1}{2\pi\epsilon_0}\int d^2x_0 \, \frac{\rho(\vec{x}_0)(\vec{x}_\perp-\vec{x}_0)}{|\vec{x}_\perp-\vec{x}_0|^2}$$

Image:
$$\vec{E}_{I\perp}(\vec{x}_\perp) = \frac{-1}{2\pi\epsilon_0}\int d^2x_0 \, \frac{\rho(\vec{x}_0)\left(\vec{x}_\perp-\frac{r_p^2}{|\vec{x}_0|^2}\vec{x}_0\right)}{\left|\vec{x}_\perp-\frac{r_p^2}{|\vec{x}_0|^2}\vec{x}_0\right|^2}$$

The direct field can be calculated exactly (see distribution function notes) as carried out previously

<u>Direct Field</u>

$$\boxed{\begin{aligned} E_{dx} &= \frac{\lambda}{\pi\epsilon_0}\frac{(x-x_c)}{(r_x+r_y)r_x} \\ E_{dy} &= \frac{\lambda}{\pi\epsilon_0}\frac{(y-y_c)}{(r_x+r_y)r_y} \end{aligned}}$$

The image charge term can be evaluated more conveniently using complex notation:

$$\underset{\sim}{E} \equiv E_{Iy} + i E_{Ix}$$

$$\underset{\sim}{z} \equiv x + iy$$

$$\underset{\sim}{z}_0 \equiv x_0 + i y_0$$

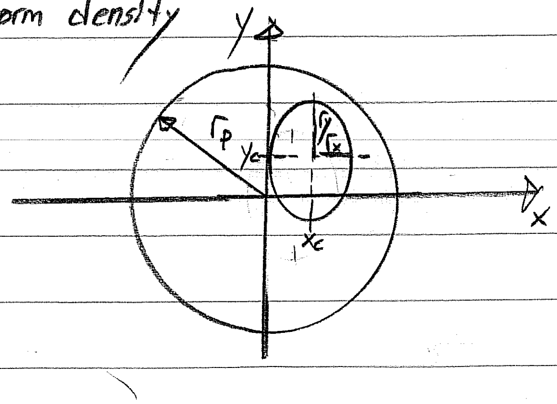
Then some algebra yields:

$$\boxed{\underset{\sim}{E} = E_{Iy} + i E_{Ix} = \frac{1}{2\pi i \varepsilon_0} \int d^2x_0 \, \frac{\rho(\vec{x}_0)}{\underset{\sim}{z} - \frac{r_p^2}{x_0^2 + y_0^2} \underset{\sim}{z}_0}}$$

which can be expanded as:

$$\boxed{\begin{aligned} \underset{\sim}{E} &= \sum_{n=1}^{\infty} \underset{\sim}{C}_n \, \underset{\sim}{z}^{\,n-1} \\ \underset{\sim}{C}_n &= \frac{-1}{2\pi i \varepsilon_0} \int d^2x \, \rho(\vec{x}_\perp) \frac{(x - iy)^n}{r_p^{2n}} \end{aligned}}$$

For our particular distribution model where we take a displaced elliptical beam with uniform density



The field components E_{Ix} and E_{Iy} are difficult to calculate explicitly. E.P. Lee, E. Close, and L. Smith, Nuc. Inst. and Methods, 1126 (1987). calculated E_{Ix} and E_{Iy} perturbatively when the centroid displacement lies along the principal x-axis.

They obtained:

for $|x_c|/r_p \ll 1$

$$y_c = 0$$

$$E_{Ix} = \frac{\lambda}{2\pi\varepsilon_0 r_p^2}\left[f\cdot(x - x_c) + g\cdot x_c \right] + \mathcal{O}\left(\frac{x_c}{r_p}\right)^3$$

$$E_{Iy} = \frac{-\lambda}{2\pi\varepsilon_0 r_p^2} f\cdot y + \mathcal{O}\left(\frac{x_c}{r_p}\right)^3$$

where

$$f = \frac{r_x^2 - r_y^2}{4r_p^2} + \frac{x_c^2}{r_p^2}\left[1 + \frac{3}{2}\left(\frac{r_x^2 - r_y^2}{r_p^2}\right) + \frac{3}{8}\left(\frac{r_x^2 - r_y^2}{r_p^2}\right)^2\right]$$

$$g = 1 + \frac{r_x^2 - r_y^2}{4r_p^2} + \frac{x_c^2}{r_p^2}\left[1 + \frac{3}{4}\left(\frac{r_x^2 - r_y^2}{r_p^2}\right) + \frac{1}{8}\left(\frac{r_x^2 - r_y^2}{r_p^2}\right)^2\right]$$

This expression and symmetry can be used to calculate the leading order image fields for

$$x_c = 0, \quad |y_c|/r_p \ll 1$$

too. The more general case of simultaneous $x_c \neq 0$ and $y_c \neq 0$ results in more complicated expressions. Also, other less symmetrical geometries (e.g. electric quadrupoles) will produce still more complicated expressions.

We are now ready to derive statistical eqns of motion for the beam centroid and envelope. Note that this same "direct" and "image" resolution of $\vec{E}_\perp$ will apply with any $\perp$ aperture geometry. A circular pipe is just the most simple example.

Statistical Average Equations of Motion

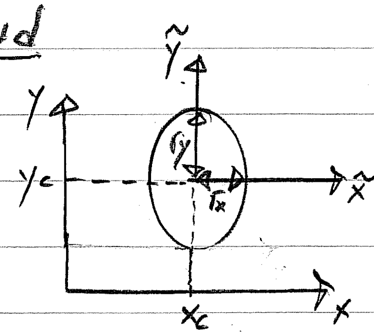
Consistent with the assumed structure of the distribution we denote:

Beam Centroid

$$\boxed{\begin{aligned} \langle x \rangle_\perp &\equiv x_c \\ \langle y \rangle_\perp &\equiv y_c \end{aligned}}$$

Coordinates with respect to centroid

$$\boxed{\begin{aligned} \tilde{x} &\equiv x - x_c \\ \tilde{y} &\equiv y - y_c \end{aligned}}$$



Edge Radii

$$\boxed{\begin{aligned} r_x &= 2\langle \tilde{x}^2 \rangle_\perp^{1/2} \\ r_y &= 2\langle \tilde{y}^2 \rangle_\perp^{1/2} \end{aligned}}$$

Using these results we can express the transverse particle equations of motion in the form:

$$\boxed{\begin{aligned} x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x(s)x - \frac{2Q(x-x_c)}{(r_x+r_y)r_x} &= \frac{q}{m\gamma_b^3\beta_b^2c^2}E_{Ix} \\ y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' + \kappa_y(s)y - \frac{2Q(y-y_c)}{(r_x+r_y)r_y} &= \frac{q}{m\gamma_b^3\beta_b^2c^2}E_{Iy} \end{aligned}}$$

From "direct" terms of $\vec{E}_\perp$ using same steps as developed in previous lectures

Taking the statistical average of this equation and using

$$\langle x'\rangle_\perp = \langle x\rangle_\perp' = x_c'$$

$$\langle y'\rangle_\perp = \langle y\rangle_\perp' = y_c'$$

obtains the centroid equations:

Centroid Equations

$$\boxed{\begin{aligned} x_c'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x_c' + \kappa_x(s)x_c &= \frac{q}{m\gamma_b^3\beta_b^2c^2}\langle E_{Ix}\rangle_\perp \\ y_c'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y_c' + \kappa_y(s)y_c &= \frac{q}{m\gamma_b^3\beta_b^2c^2}\langle E_{Iy}\rangle_\perp \end{aligned}}$$

- Terms from direct field vanish in the average.

Using the Lee et al formula for E_{Ix} we have $\langle E_{Ix}\rangle_\perp = \frac{\lambda}{2\pi\epsilon_0 r_p^2} g X_c$ giving:

$$\boxed{X_c'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} X_c' + \kappa_x(s) X_c = Q g \frac{X_c}{r_p^2}}$$

$$g = 1 + \frac{r_x^2 - r_y^2}{4 r_p^2} + \frac{X_c^2}{r_p^2}\left[1 + \frac{3}{4}\left(\frac{r_x^2 - r_y^2}{r_p^2}\right) + \frac{1}{8}\left(\frac{r_x^2 - r_y^2}{r_p^2}\right)^2\right]$$

where:

$$\boxed{Q = \frac{q\lambda}{2\pi\epsilon_0 m \gamma_b^3 \beta_b^2 c^2} = \text{const}}$$

is the usual dimensionless perveance.

We defer analysis of the centroid equations till later other than to note that g depends on X_c, r_x, r_y, and r_p.

Next, we derive "envelope" equations for the evolution of the elliptical beam edge under response to linear applied focusing, self-field, and image-charge forces.

Envelope Equations

$$r_x = 2\langle \tilde{x}^2\rangle_\perp^{1/2} \xrightarrow{\text{differentiate}} \boxed{r_x' = 4\langle \tilde{x}\tilde{x}'\rangle_\perp / r_x}$$

$$r_y = 2\langle \tilde{y}^2\rangle_\perp^{1/2} \rightarrow \boxed{r_y' = 4\langle \tilde{y}\tilde{y}'\rangle_\perp / r_y}$$

Next define statistical emittances:

$$\boxed{\begin{aligned} \varepsilon_x &= 4\left[\langle \tilde{x}^2\rangle_\perp \langle \tilde{x}'^2\rangle_\perp - \langle \tilde{x}\tilde{x}'\rangle_\perp^2\right]^{1/2} \\ \varepsilon_y &= 4\left[\langle \tilde{y}^2\rangle_\perp \langle \tilde{y}'^2\rangle_\perp - \langle \tilde{y}\tilde{y}'\rangle_\perp^2\right]^{1/2} \end{aligned}}$$

Using these definitions and differentiating r_x' and r_y' again:

$$\boxed{\begin{aligned} r_x'' &= \frac{4\langle \tilde{x}\tilde{x}''\rangle_\perp}{r_x} + \frac{\varepsilon_x^2}{r_x^3} \\ r_y'' &= \frac{4\langle \tilde{y}\tilde{y}''\rangle_\perp}{r_y} + \frac{\varepsilon_y^2}{r_y^3} \end{aligned}}$$

To derive moment equations in a useful form, we need equations of motion for $\langle \tilde{x}\tilde{x}''\rangle_\perp$ and $\langle \tilde{y}\tilde{y}''\rangle_\perp$. To make simplifications, we derive equations of motion for $\tilde{x}$ and $\tilde{y}$:

Using:

$$\tilde{x} \equiv x - x_c$$
$$\tilde{y} \equiv y - y_c$$

and subtracting the centroid equation of motion from the equations for x and y obtain:

$$\boxed{\begin{aligned} &\tilde{x}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{x}' + \kappa_x(s)\tilde{x} - \frac{2Q\,\tilde{x}}{(r_x+r_y)r_x} \\ &\qquad = \frac{q}{m\gamma_b^3\beta_b^2c^2}\left[E_{Ix} - \langle E_{Ix}\rangle_\perp\right] \\ &\tilde{y}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{y}' + \kappa_y(s)\tilde{y} - \frac{2Q\,\tilde{y}}{(r_x+r_y)r_y} \\ &\qquad = \frac{q}{m\gamma_b^3\beta_b^2c^2}\left[E_{Ix} - \langle E_{Ix}\rangle_\perp\right] \end{aligned}}$$

Using these results, in $\langle \tilde{x}\tilde{x}''\rangle_\perp$ and $\langle \tilde{y}\tilde{y}''\rangle_\perp$ we obtain:

$$\boxed{\begin{aligned} &r_x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} r_x' + \kappa_x(s) r_x - \frac{2Q}{r_x+r_y} - \frac{\varepsilon_x^2}{r_x^3} \\ &\qquad = \frac{4q}{m\gamma_b^3\beta_b^2c^2}\langle \tilde{x} E_{Ix}\rangle_\perp \\ &r_y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} r_y' + \kappa_y(s) r_y - \frac{2Q}{r_x+r_y} - \frac{\varepsilon_y^2}{r_y^3} \\ &\qquad = \frac{4q}{m\gamma_b^3\beta_b^2c^2}\langle \tilde{y} E_{Iy}\rangle_\perp \end{aligned}}$$

- Note that $\langle \tilde{x}\langle E_{Ix}\rangle_\perp\rangle_\perp = 0 = \langle \tilde{y}\langle E_{Iy}\rangle_\perp\rangle_\perp$

To solve these formulas expressions must be used to calculate $\langle \tilde{x} E_{Ix}\rangle_\perp$ and $\langle \tilde{y} E_{Iy}\rangle_\perp$ in terms of beam, geometry, and envelope parameters <u>and</u> ε_x and ε_y must be specified.
It can be shown (see problem sets) that in the absense of nonlinear forces (i.e., nonlinear terms in E_{Ix} and E_{Iy} negligeable) that:

$$\boxed{\begin{aligned} \varepsilon_{nx} &= (\gamma_b\beta_b)\left[\langle \tilde{x}^2\rangle_\perp\langle \tilde{x}'^2\rangle_\perp - \langle \tilde{x}\tilde{x}'\rangle_\perp^2\right]^{1/2} = (\gamma_b\beta_b)\,\varepsilon_x = \text{const.} \\ \varepsilon_{ny} &= (\gamma_b\beta_b)\left[\langle \tilde{y}^2\rangle_\perp\langle \tilde{y}'^2\rangle_\perp - \langle \tilde{y}\tilde{y}'\rangle_\perp^2\right]^{1/2} = (\gamma_b\beta_b)\,\varepsilon_y = \text{const.} \end{aligned}}$$

- $\gamma_b\beta_b = \text{const.} \Rightarrow$ no acceleration and $\varepsilon_x = \text{const}$, $\varepsilon_y = \text{const.}$

- Even when E_{Ix} and E_{Iy} have nonzero nonlinear terms ε_{nx} and ε_{ny} typically only have small, adiabatic evolutions and solving the equations over a few periods with $\varepsilon_{nx} = \text{const}$ and $\varepsilon_{ny} = \text{const}$ can be a good approximation. This is particularly true for high space-charge intensities since the ε_x^2/r_x^3 and ε_y^2/r_y^3 terms are of lesser importance (relatively smaller) in such regimes.

For the special case of $\vec{x}_c = x_c \hat{x}$ and $|x_c|/r_p \ll 1$, the formulas by Lee et. al. can be applied to reduce the envelope equation to:

$$\boxed{\begin{aligned} &r_x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} r_x' + \left[\kappa_x(s) - \frac{Qf}{r_p^2}\right] r_x - \frac{2Q}{r_x+r_y} - \frac{\varepsilon_x^2}{r_x^3} = 0 \\ &r_y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} r_y' + \left[\kappa_y(s) + \frac{Qf}{r_p^2}\right] r_y - \frac{2Q}{r_x+r_y} - \frac{\varepsilon_y^2}{r_y^3} = 0 \end{aligned}}$$

where

$$Q = \frac{q\lambda}{2\pi\varepsilon_0 m \gamma_b^3 \beta_b^2 c^2} = \text{const}, \quad \text{Perveance}$$

$$f = \frac{r_x^2 - r_y^2}{4 r_p^2} + \frac{x_c^2}{r_p^2}\left[1 + \frac{3}{2}\left(\frac{r_x^2 - r_y^2}{r_p^2}\right) + \frac{3}{8}\left(\frac{r_x^2 - r_y^2}{r_p^2}\right)^2\right]$$

General Comments:

- Equations for x_c, y_c (centroid) ; r_x, r_y (envelope) must be solved simultaneously when images are included.
- Image fields will generally contain nonlinear terms and therefore there is no self-consistent KV "equilibrium" distribution that does not change form when the beam evolves in the presence of images.
- Derivation of the full coupled equations are complicated and the form of the final equations reflect the aperture structure and the assumed space-charge profile.

§3 Centroid Equations

The centroid equations

$$\boxed{\begin{aligned} x_c'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} x_c' + \kappa_x(s) x_c &= \frac{q}{m\gamma_b^3\beta_b^2c^2} \langle E_{Ix} \rangle_\perp \\ y_c'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} y_c' + \kappa_y(s) y_c &= \frac{q}{m\gamma_b^3\beta_b^2c^2} \langle E_{Iy} \rangle_\perp \end{aligned}}$$

Couple to the envelope radii r_x, r_y via

$$\begin{aligned} &\langle E_{Ix} \rangle_\perp \\ &\langle E_{Iy} \rangle_\perp \end{aligned} \sim \text{Avg Image Fields}$$

and cannot be solved alone. If images are negligeable, then we have:

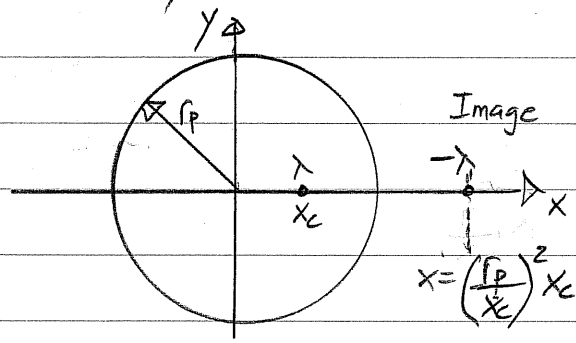
$$\boxed{\begin{aligned} x_c'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} x_c' + \kappa_x(s) x_c &= 0 \\ y_c'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} y_c' + \kappa_y(s) y_c &= 0 \end{aligned}}$$

These equations were analyzed in earlier lectures on transverse single particle equations of motion of a particle moving in linear applied fields.

- Stable for $\sigma_0 < 180°$
  Stop bands when $\sigma_0 > 180°$
- Orbits have Courant-Snyder invarients.

To estimate how large image charge contributions can be model a beam as a line-charge λ at the centroid coordinate

$$x = x_c$$
$$y = 0 \qquad \text{(choose coordinates)}$$



$$E_r = \frac{\lambda}{2\pi\epsilon_0 r}$$

for line-charge at origin.

Then

$$\langle \vec{E}_I \rangle_\perp = \frac{-\lambda}{2\pi\epsilon_0 [r_p^2/x_c - x_c]} \hat{x}$$

and the x-centroid equation becomes:

$$x_c'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} x_c' + \kappa_x(s)\, x_c = \frac{Q}{[r_p^2/x_c - x_c]}$$

where

$$Q \equiv \frac{q\lambda}{2\pi\epsilon_0 m \gamma_b^3 \beta_b^2 c^2}$$

take $x_c^2/r_p^2 \ll 1$:

$$\frac{1}{[r_p^2/x_c - x_c]} \simeq \frac{x_c}{r_p^2}\left[1 + \left(\frac{x_c}{r_p}\right)^2\right]$$

$$\boxed{x_c'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} x_c' + \left[\kappa_x(s) - \frac{Q}{r_p^2}\right] x_c = \frac{Q}{r_p^4} x_c^3}$$

Leading order image correction to focusing

Nonlinear Image correction

Using the image formulas derived by Lee et. al for an elliptical beam we have:

$$x_c'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} x_c' + \kappa_x(s) x_c = \frac{Q}{r_p^2} g\, x_c$$

$$g = 1 + \frac{r_x^2 - r_y^2}{4 r_p^2} + \frac{x_c^2}{r_p^2}\left[1 + \frac{3}{4}\left(\frac{r_x^2 - r_y^2}{r_p^2}\right) + \frac{1}{8}\left(\frac{r_x^2 - r_y^2}{r_p^2}\right)^2\right]$$

Set

$$r_x = r_y$$

$$\Rightarrow g = 1 + \frac{x_c^2}{r_p^2}$$ and we obtain the same equation:

$$\Rightarrow x_c'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} x_c + \left[\kappa_x(s) - \frac{Q}{r_p^2}\right] x_c = \frac{Q}{r_p^4} x_c^3$$ ✓

Some Remarks:

- Fringe field structures in centroid equations are often more important because the centroid terms are typically seeded by beam and alignment errors that do not come with compensating "kicks" as in matched lattices.
- Orbit phases can be influenced over long path lengths by nonlinear terms
  - Often seen in numerical simulations
- If $x_c = 0 = y_c$ $\quad x_c' = 0 = y_c'$ at some s, a distribution asymmetry or alignment error must be present to drive the centroid off-axis.

§4 Envelope Equations

For the envelope equations it is generally found that the "image" charge terms are of lesser importance.

- Neglect image charges in the analysis to simplify treatment
  - Complicated enough for periodic focusing lattices!

The envelope equations are the single most important design equations for space-charge dominated beams.

- If transport is bad at the envelope level of modeling things are likely to deteriorate further when more effects are included.

Summary - Transverse Envelope Equations:

$$r_x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} r_x' + \kappa_x(s) r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} = 0$$

$$r_y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} r_y' + \kappa_y(s) r_y - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_y^2}{r_y^3} = 0$$

Terms: Acceleration; Applied Focus; Space-Charge (Defocus); Emittance/Thermal (Defocus)

$$\gamma_b\beta_b\varepsilon_x = \text{const.}$$
$$\gamma_b\beta_b\varepsilon_y = \text{const.}$$

$\gamma_b\beta_b$ = Specified function of s (Acceleration Schedule)

Comments:

- When x- and y-focusing strengths are equal, it is usually the case that phase-space is "equipartitioned" with

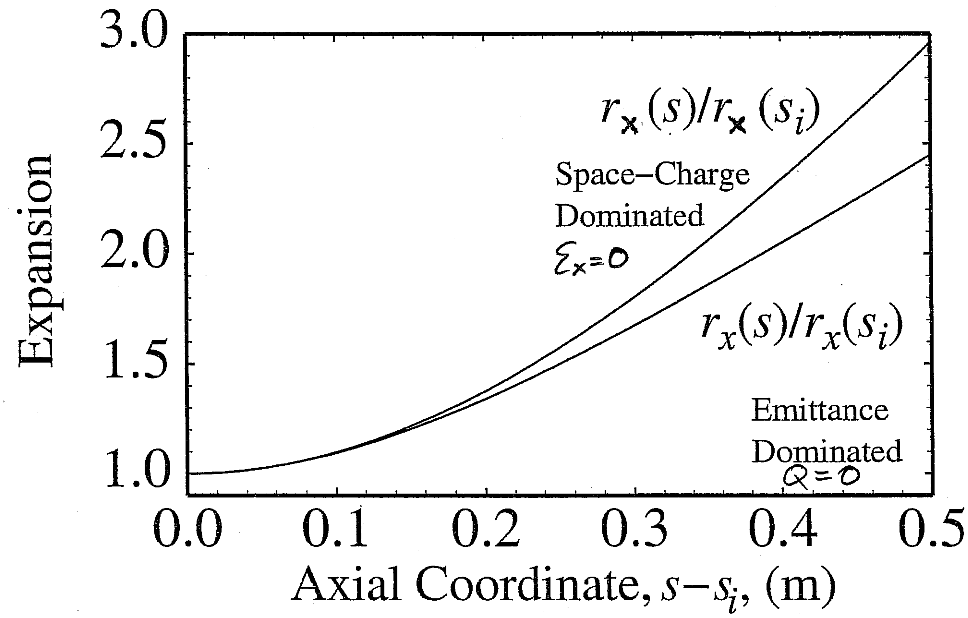
$$\varepsilon_x = \varepsilon_y \equiv \varepsilon$$

- Emittance terms $\sim \varepsilon_x^2 / r_x^3$ lead to strong defocusing for small r_x. This defocusing is very important on target focal spots.
  - Much of accelerator physics centers on control of emittance growth that can occur due to nonideal (nonlinear) forces acting on the beam.

- If the beam radially expands enough, space-charge defocusing forces will dominate emittance defocusing forces:

$$\frac{Q}{r_x + r_y} >> \frac{\varepsilon_x^2}{r_x^3} \quad \text{for } r_x \text{ large.}$$

Example - free expansion rates from space-charge and emittance terms with $r_x = r_y$, and $Q/2r_x = \varepsilon_x^2/r_x^3$ and $r_x' = 0$ at $s = s_i$:



- Acceleration cells typically result in only small fractional energy changes in a high energy beam and acceleration terms

$$\sim \frac{(\gamma_b \beta_b)' r_x'}{(\gamma_b \beta_b)}$$

can often be neglected.

  - Can be important near the injector.

- The Envelope equations are fully self-consistent for a KV distribution as shown in previous lectures on transverse distribution functions. Although this distribution is unphysical (δ-function form produces many high-order instabilities) it's low order properties (envelope) are believed reliable. Moreover, the KV envelope equations have been shown by Sacherer [F.J. Sacherer, IEEE Trans. Nuc. Sci, 18, 1101 (1971)] to apply to any distribution with elliptical charge symmetry:

$$\rho = \rho\left(\left(\frac{x}{r_x}\right)^2 + \left(\frac{y}{r_y}\right)^2 \right)$$

provided that consistent (Vlasov Model) rms emittance evolutions are employed for $\varepsilon_x(s)$ and $\varepsilon_y(s)$. $\varepsilon_x = \text{const}$ and $\varepsilon_y = \text{const}$ only apply to the KV distribution for elliptic symmetry beams.

In this "equivalent beam" sense the KV envelope eqns are applied to non-KV beams. See Reiser for more detailed discussions.

- In practice, applied fields in good transport channels are nearly linear, emittance evolutions are adiabatic, and the linear space-charge model is reasonable for a wide range of real distributions. Thus the envelope eqns can be applied with good accuracy to many practical transport problems.

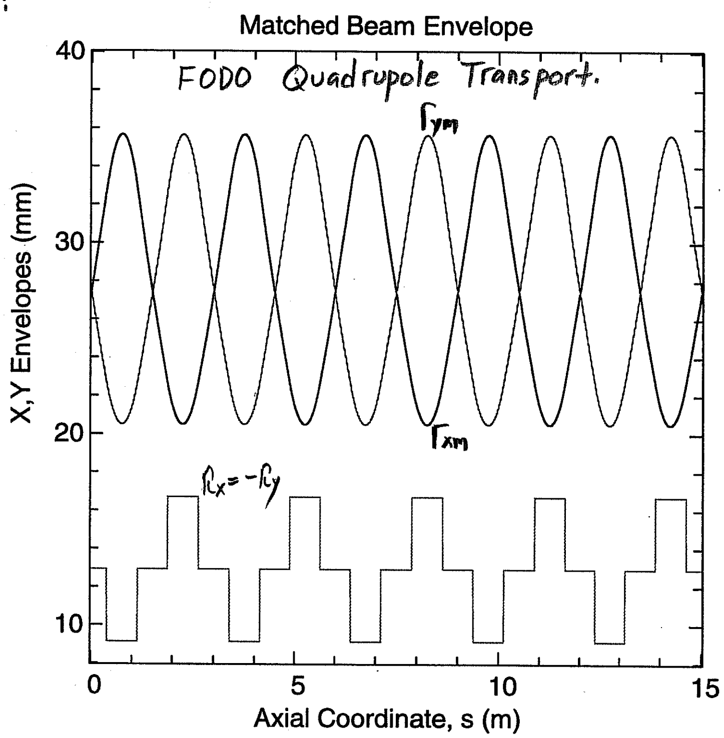
Matched Envelope:

For a periodic lattice, the best radial beam confinement in terms of the smallest possible max beam envelope excursions is obtained for the periodic "matched" beam envelope with the same periodicity as the lattice:

Matching condition:

$$\boxed{\begin{array}{ll} r_x \equiv r_{xm}: & r_{xm}(s+L_p) = r_{xm}(s) \\ r_y \equiv r_{ym}: & r_{ym}(s+L_p) = r_{ym}(s) \\ & L_p = \text{Lattice Period} \end{array}}$$

Example:



Matching requires that initial values of r_x, r_y, r_x', r_y' have values consistent with the lattice structure.

- Values of r_x, r_y, r_x', r_y' needed constrain the distribution of beam particles $f_\perp$.

Mismatched Envelope

Envelope evolution in a periodic focusing lattice can be resolved as:

$$\boxed{\begin{aligned} r_x(s) &= \underbrace{r_{xm}(s)}_{\text{Matched Terms}} + \underbrace{\delta r_x(s)}_{\text{"Mismatch" Terms}} \\ r_y(s) &= r_{ym}(s) + \delta r_y(s) \end{aligned}}$$

Usually the amplitude of mismatch terms are controlled with

$$\boxed{\begin{aligned} r_{xm} &\gg |\delta r_x| \\ r_{ym} &\gg |\delta r_y| \end{aligned}}$$

Mismatch oscillations can be created several ways:

1/ Initial envelope coordinates deviate from from matched beam values, with (one or more):

$$\boxed{\begin{aligned} r_x(s_i) &\neq r_{xm}(s_i) & \quad r_x'(s_i) &\neq r_{ym}(s_i) \\ r_y(s_i) &\neq r_{ym}(s_i) & \quad r_y'(s_i) &\neq r_{ym}(s_i) \\ & & s = s_i \quad &\text{initial coordinate.} \end{aligned}}$$

- Result from structure of particle distribution.

2/ Driving Perturbations from focusing and beam distribution errors:

Focusing: $\kappa_x(s) \longrightarrow \kappa_x(s) + \delta\kappa_x(s)$

$\kappa_y(s) \longrightarrow \kappa_y(s) + \delta\kappa_y(s)$

Distribution: $Q \longrightarrow \overset{= \text{const}}{Q} + \delta Q(s)$

$\varepsilon_x \longrightarrow \varepsilon_x + \delta\varepsilon_x(s)$

$\varepsilon_y \longrightarrow \varepsilon_y + \delta\varepsilon_y(s)$

Ideal — Driving Terms

- Focusing errors $\delta\kappa_x(s)$ and $\delta\kappa_y(s)$ result from imperfect optics (excitation errors, mechanical position, ...) and can cause the lattice to be aperiodic.

- Perveance errors $\delta Q(s)$ result from lost particles and/or halo particles entering and leaving the "core" distribution.

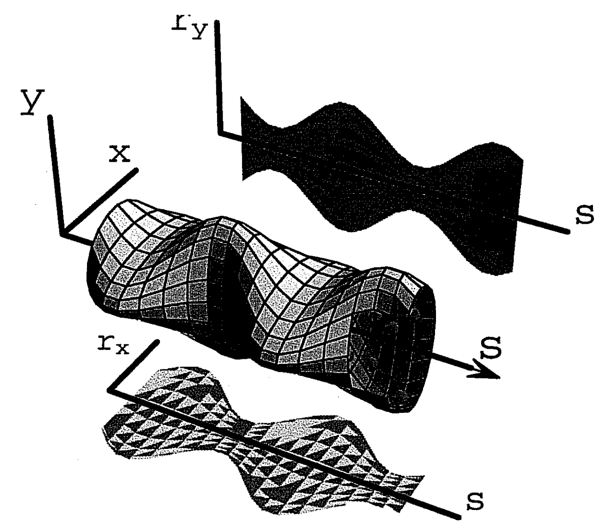
$$\delta Q(s) \leq 0 \qquad \text{long time}$$

$$\delta Q(s) \quad \text{can locally increase}$$

- Emittance errors $\delta\varepsilon_x(s)$ and $\delta\varepsilon_y(s)$ result from acceleration errors and/or nonlinear fields and higher-order effects growing effective (statistical) beam phase-space area.

Envelope Equations

Uniform density elliptical beam in a linear focusing channel: $\gamma_b\beta_b = \text{const.}$



$$\boxed{\begin{aligned} r_x''(s) + \kappa_x(s) r_x(s) - \frac{2Q}{r_x(s)+r_y(s)} - \frac{\varepsilon_x^2}{r_x^3(s)} &= 0 \\ r_y''(s) + \kappa_y(s) r_y(s) - \frac{2Q}{r_x(s)+r_y(s)} - \frac{\varepsilon_y^2}{r_y^3(s)} &= 0 \end{aligned}}$$

Beam edge radii

$$r_x(s) = 2\sqrt{\langle x^2\rangle_\perp}$$
$$r_y(s) = 2\sqrt{\langle y^2\rangle_\perp}$$

Rms emittances

$$\varepsilon_x = 4\left[\langle x^2\rangle_\perp \langle x'^2\rangle_\perp - \langle xx'\rangle_\perp^2\right]^{1/2} = \text{const}$$
$$\varepsilon_y = 4\left[\langle y^2\rangle_\perp \langle y'^2\rangle_\perp - \langle yy'\rangle_\perp^2\right]^{1/2} = \text{const}$$

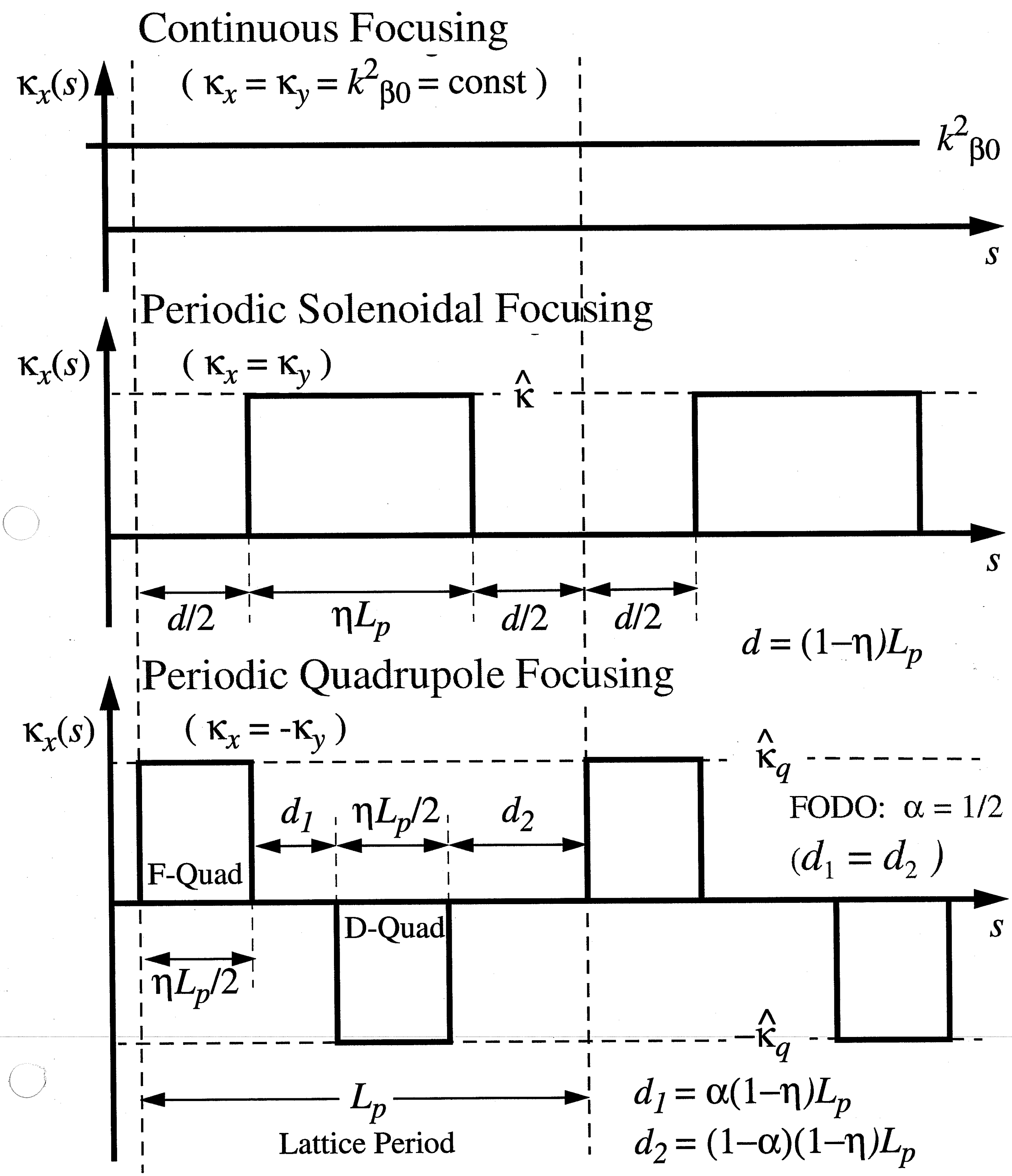
Lattice focusing functions

$$\kappa_x(s), \quad \kappa_y(s)$$

Dimensionless perveance

$$Q = \frac{q\lambda}{2\pi\epsilon_0 m c^2 \gamma_b^3 \beta_b^2} = \text{const}$$

Applied Focusing Lattices



Matched Envelope and Parameters

Matched solution has periodicity of the lattice:

$$\kappa_x(s+L_p) = \kappa_x(s) \qquad \boxed{r_x(s) = r_{xm}(s): \quad r_{xm}(s+L_p) = r_{xm}(s)}$$
$$\kappa_y(s+L_p) = \kappa_y(s) \qquad \boxed{r_y(s) = r_{ym}(s): \quad r_{ym}(s+L_p) = r_{ym}(s)}$$

Single-particle phase advances ($\varepsilon_x = \varepsilon_y = \varepsilon$):

Depressed: $$\sigma = \varepsilon \int_0^{L_p} \frac{ds}{r_{xm}^2(s)}$$

Undepressed: $$\sigma_0 = \lim_{Q \to 0} \sigma$$

Parameters of matched Beam:

Direct:

Solenoid: $L_p,\ Q,\ \varepsilon,\ \hat{\kappa},\ \eta$

Quadrupole: $L_p,\ Q,\ \varepsilon,\ \hat{\kappa},\ \eta,\ \alpha$

Indirect:

Solenoid: $\sigma_0,\ \sigma,\ \eta$

Quadrupole: $\sigma_0,\ \sigma,\ \eta,\ \alpha$

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Undepressed Phase Advance σ_0 Constrains Lattice Parameters

Using a transfer matrix approach on undepressed single-particle orbits

Solenoidal Focusing:

$$\cos\sigma_0 = \cos(2\Theta) - \frac{1-\eta}{\eta}\Theta\sin(2\Theta)$$

$$\Theta \equiv \frac{\sqrt{\hat{\kappa}}L_p}{2}$$

Quadrupole Focusing:

$$\cos\sigma_0 = \cos\Theta\cosh\Theta + \frac{1-\eta}{\eta}\Theta(\cos\Theta\sinh\Theta - \sin\Theta\cosh\Theta) - 2\alpha(1-\alpha)\frac{(1-\eta)^2}{\eta^2}\Theta^2\sin\Theta\sinh\Theta$$

$$\Theta \equiv \frac{\sqrt{|\widehat{\kappa_q}|}L_p}{2}$$

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Perturbed Envelope Equations

Envelope Perturbations:

$$\begin{aligned} r_x(s) &= r_{xm}(s) + \delta r_x(s) \\ r_y(s) &= r_{ym}(s) + \delta r_y(s) \end{aligned}$$

Driving Perturbations:

$$\begin{aligned} \kappa_x(s) &\to \kappa_x(s) + \delta\kappa_x(s) \\ \kappa_y(s) &\to \kappa_y(s) + \delta\kappa_y(s) \\ Q &\to Q + \delta Q(s) \\ \varepsilon_x &\to \varepsilon_x + \delta\varepsilon_x(s) \\ \varepsilon_y &\to \varepsilon_y + \delta\varepsilon_y(s) \end{aligned}$$

Linear perturbation equations:

$$\begin{aligned} \delta r_x'' + \kappa_x \delta r_x + \frac{2Q}{(r_{xm}+r_{ym})^2}(\delta r_x + \delta r_y) + \frac{3\varepsilon_x^2}{r_{xm}^4}\delta r_x \\ = -r_{xm}\delta\kappa_x + \frac{2}{r_{xm}+r_{ym}}\delta Q + \frac{2\varepsilon_x^2}{r_{xm}^3}\delta\varepsilon_x \\ \delta r_y'' + \kappa_y \delta r_y + \frac{2Q}{(r_{xm}+r_{ym})^2}(\delta r_x + \delta r_y) + \frac{3\varepsilon_y^2}{r_{ym}^4}\delta r_y \\ = -r_{ym}\delta\kappa_y + \frac{2}{r_{xm}+r_{ym}}\delta Q + \frac{2\varepsilon_y^2}{r_{ym}^3}\delta\varepsilon_y \end{aligned}$$

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Perturbed Envelope Equations

Homogeneous Solution:

- Describes normal mode oscillations
- Original analysis by Struckmeier and Reiser [Part. Accel. **14**, 227 (1984)]

Particular Solution:

- Describes action of driving terms

Homogeneous solution expressible as a map:

$$\delta\vec{R}(s) = \mathbf{M}_e(s|s_i) \cdot \delta\vec{R}(s_i)$$

$$\delta\vec{R}(s) = (\delta r_x, \delta r_x', \delta r_y, \delta r_y')$$

$$\mathbf{M}_e(s|s_i) = 4 \times 4 \text{ transfer map.}$$

Eigenvalues and eigenvectors of map through one period describe mode:

$$\boxed{\mathbf{M}_e(s_i + L_p|s_i) \cdot \vec{E}_n(s_i) = \lambda_n \vec{E}_n(s_i)}$$

Stability

$$\lambda_n = \gamma_n e^{i\sigma_n} \qquad \begin{array}{ll} \sigma_n \rightarrow & \text{mode phase advance (real)} \\ \gamma_n \rightarrow & \text{mode growth factor (real)} \end{array}$$

Launching

$$\delta\vec{R}(s_i) = \sum_{n=1}^{4} \alpha_n \vec{E}_n(s_i)$$

$$\alpha_n = \text{const}$$

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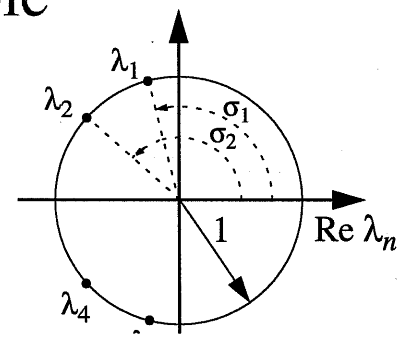
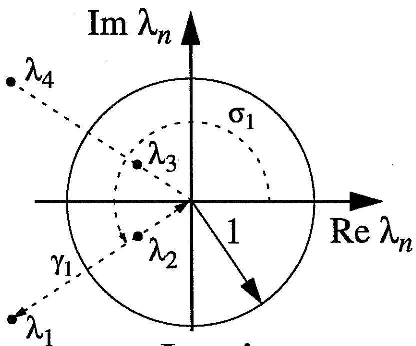
Eigenvalue Symmetry Classes

a) Stable



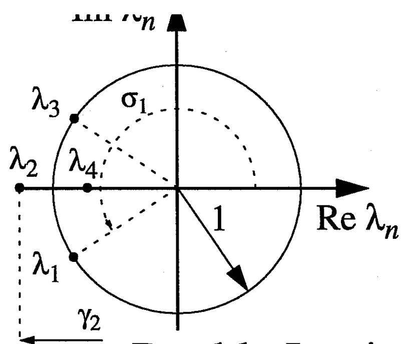
Eigenvalues	Eigenvectors
$\lambda_1 = e^{i\sigma_1}$	$\vec{E}_1$
$\lambda_2 = e^{i\sigma_2}$	$\vec{E}_2$
$\lambda_3 = 1/\lambda_1 = \lambda_1^* = e^{-i\sigma_1}$	$\vec{E}_3 = \vec{E}_1^*$
$\lambda_4 = 1/\lambda_2 = \lambda_2^* = e^{-i\sigma_2}$	$\vec{E}_4 = \vec{E}_2^*$

b) Unstable Confluent Resonance



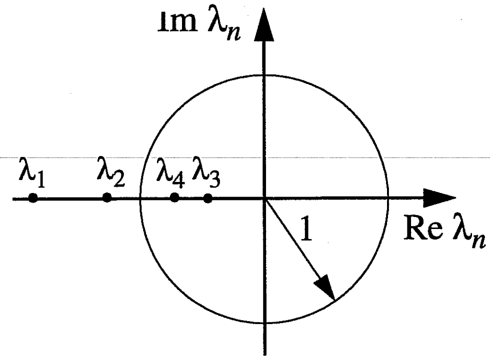
Eigenvalues	Eigenvectors
$\lambda_1 = \gamma_1 e^{i\sigma_1}$	$\vec{E}_1$
$\lambda_2 = 1/\lambda_1^* = (1/\gamma_1)e^{i\sigma_1}$	$\vec{E}_2$
$\lambda_3 = 1/\lambda_1 = (1/\gamma_1)e^{-i\sigma_1}$	$\vec{E}_3 = \vec{E}_2^*$
$\lambda_4 = \lambda_1^* = \gamma_1 e^{-i\sigma_1}$	$\vec{E}_4 = \vec{E}_1^*$

c) Unstable Lattice Resonance



Eigenvalues	Eigenvectors
$\lambda_1 = e^{i\sigma_1}$	$\vec{E}_1$
$\lambda_2 = \gamma_2 e^{i\pi}$	$\vec{E}_2$ (real)
$\lambda_3 = \lambda_1^* = e^{-i\sigma_1}$	$\vec{E}_3 = \vec{E}_1^*$
$\lambda_4 = 1/\lambda_2 = (1/\gamma_2)e^{i\pi}$	$\vec{E}_4$ (real)

d) Unstable Double Lattice Resonance



Eigenvalues	Eigenvectors
$\lambda_1 = \gamma_1 e^{i\pi}$	$\vec{E}_1$ (real)
$\lambda_2 = \gamma_2 e^{i\pi}$	$\vec{E}_2$ (real)
$\lambda_3 = 1/\lambda_1 = (1/\gamma_1)e^{i\pi}$	$\vec{E}_3$ (real)
$\lambda_4 = 1/\lambda_2 = (1/\gamma_2)e^{i\pi}$	$\vec{E}_4$ (real)

Pure Mode Launching Conditions

Launching conditions for distinct normal modes corresponding to the eigenvalue classes illustrated:

$$A_\ell = \text{ mode amplitude (real)}$$
$$\psi_\ell = \text{ mode launch phase (real)}$$

Case	Mode	Launching Condition	Lattice Period Advance
(a) Stable	1 - Stable Osc.	$\delta\mathbf{R}_1 = A_1 e^{i\psi_1}\mathbf{E}_1 + \text{C.C.}$	$\mathbf{M}_e\delta\mathbf{R}_1(\psi_1) = \delta\mathbf{R}_1(\psi_1+\sigma_1)$
	2 - Stable Osc.	$\delta\mathbf{R}_2 = A_2 e^{i\psi_2}\mathbf{E}_2 + \text{C.C.}$	$\mathbf{M}_e\delta\mathbf{R}_2(\psi_2) = \delta\mathbf{R}_2(\psi_2+\sigma_2)$
(b) Unstable Confluent Res.	1 - Exp. Growth	$\delta\mathbf{R}_1 = A_1 e^{i\psi_1}\mathbf{E}_1 + \text{C.C.}$	$\mathbf{M}_e\delta\mathbf{R}_1(\psi_1) = \gamma_1\delta\mathbf{R}_1(\psi_1+\sigma_1)$
	2 - Exp. Damping	$\delta\mathbf{R}_2 = A_2 e^{i\psi_2}\mathbf{E}_2 + \text{C.C.}$	$\mathbf{M}_e\delta\mathbf{R}_2(\psi_2) = (1/\gamma_1)\delta\mathbf{R}_2(\psi_2+\sigma_1)$
(c) Unstable Lattice Res.	1 - Stable Osc.	$\delta\mathbf{R}_1 = A_1 e^{i\psi_1}\mathbf{E}_1 + \text{C.C.}$	$\mathbf{M}_e\delta\mathbf{R}_1(\psi_1) = \delta\mathbf{R}_1(\psi_1+\sigma_1)$
	2 - Exp. Growth	$\delta\mathbf{R}_2 = A_2\mathbf{E}_2$	$\mathbf{M}_e\delta\mathbf{R}_2 = -\gamma_2\delta\mathbf{R}_2$
	3 - Exp. Damping	$\delta\mathbf{R}_3 = A_3\mathbf{E}_4$	$\mathbf{M}_e\delta\mathbf{R}_3 = -(1/\gamma_2)\delta\mathbf{R}_3$
(d) Unstable Double Lattice Resonance	1 - Exp. Growth	$\delta\mathbf{R}_1 = A_1\mathbf{E}_1$	$\mathbf{M}_e\delta\mathbf{R}_1 = -\gamma_1\delta\mathbf{R}_1$
	2 - Exp. Growth	$\delta\mathbf{R}_2 = A_2\mathbf{E}_2$	$\mathbf{M}_e\delta\mathbf{R}_2 = -\gamma_2\delta\mathbf{R}_2$
	3 - Exp. Damping	$\delta\mathbf{R}_3 = A_3\mathbf{E}_3$	$\mathbf{M}_e\delta\mathbf{R}_3 = -(1/\gamma_1)\delta\mathbf{R}_3$
	4 - Exp. Damping	$\delta\mathbf{R}_4 = A_4\mathbf{E}_4$	$\mathbf{M}_e\delta\mathbf{R}_4 = -(1/\gamma_2)\delta\mathbf{R}_4$

$$\delta\vec{R}_\ell \equiv \delta\vec{R}_\ell(s_i), \quad \vec{E}_\ell \equiv \vec{E}_\ell(s_i), \quad \text{and} \quad \mathbf{M}_e \equiv \mathbf{M}_e(s_i+L_p|s_i)$$

Decoupled Modes

In a continuous or solenoidal focusing channel

$$\kappa_x(s) = \kappa_y(s) = \kappa(s)$$

with a round matched-beam solution

$$\varepsilon_x = \varepsilon_y = \varepsilon = \text{const}$$
$$r_{xm}(s) = r_{ym}(s) = r_m(s)$$

envelope perturbations are simply decoupled with:

$$\delta r_+(s) = \frac{\delta r_x(s) + \delta r_y(s)}{2}$$
$$\delta r_-(s) = \frac{\delta r_x(s) - \delta r_y(s)}{2}$$

$$\delta r_+'' + \kappa\,\delta r_+ + \frac{2Q}{r_m^2}\delta r_+ + \frac{3\varepsilon^2}{r_m^4}\delta r_+ = -r_m\left(\frac{\delta\kappa_x + \delta\kappa_y}{2}\right) + \frac{1}{r_m}\delta Q + \frac{2\varepsilon^2}{r_m^3}\left(\frac{\delta\varepsilon_x + \delta\varepsilon_y}{2}\right)$$
$$\delta r_-'' + \kappa\,\delta r_- + \frac{3\varepsilon^2}{r_m^4}\delta r_- = -r_m\left(\frac{\delta\kappa_x - \delta\kappa_y}{2}\right) + \frac{2\varepsilon^2}{r_m^3}\left(\frac{\delta\varepsilon_x - \delta\varepsilon_y}{2}\right)$$

Decoupled Mode Properties

Space charge terms ~ Q only directly expressed in equation for $\delta r_+(s)$

- Indirectly present in both equations from matched envelope $r_m(s)$

Homogeneous Solution:

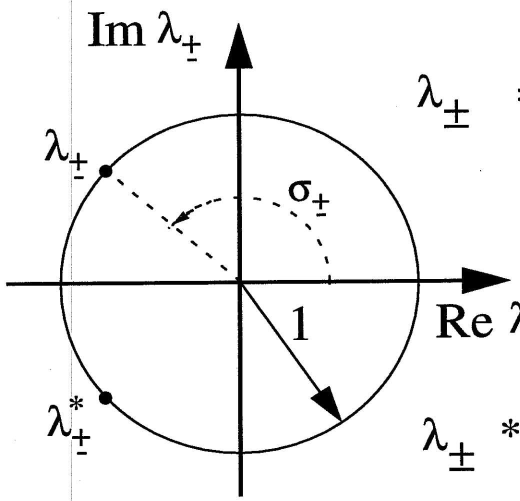
- Restoring term for $\delta r_+(s)$ larger than for $\delta r_-(s)$
  - → breathing mode oscillates faster than quadrupole mode

Particular Solution:

- Misbalances in focusing and emittance driving terms can project onto either mode
  - → nonzero $\kappa_x(s) + \kappa_y(s)$ and $\varepsilon_x(s) + \varepsilon_y(s)$ project onto breathing mode
  - → nonzero $\kappa_x(s) - \kappa_y(s)$ and $\varepsilon_x(s) - \varepsilon_y(s)$ project onto quadrupole mode
- Perveance driving perturbations project only on breathing mode

Decoupled Modes: Eigenvalue Symmetries

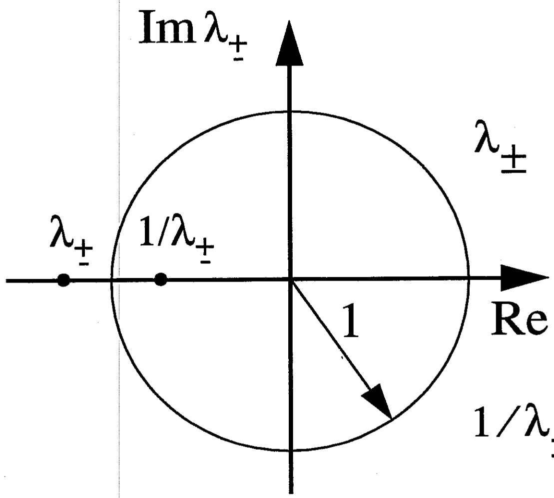
Stable



$$\lambda_{\pm} = e^{i\sigma_{\pm}}$$

$$\lambda_{\pm}{}^{*} = 1/\lambda_{\pm} = e^{-i\sigma_{\pm}}$$

Unstable, Lattice Resonance



$$\lambda_{\pm} = \gamma_{\pm}\, e^{-i\pi}$$

$$1/\lambda_{\pm} = (1/\gamma_{\pm})\, e^{-i\pi}$$

Homogeneous solution map:

$$\delta\vec{R}(s) = \mathbf{M}_e(s|s_i)\cdot\delta\vec{R}(s_i)$$
$$\delta\vec{R}(s) = (\delta r_x, \delta r_x', \delta r_y, \delta r_y')$$
$$\mathbf{M}_e(s|s_i) = 4\times 4 \text{ transfer map.}$$

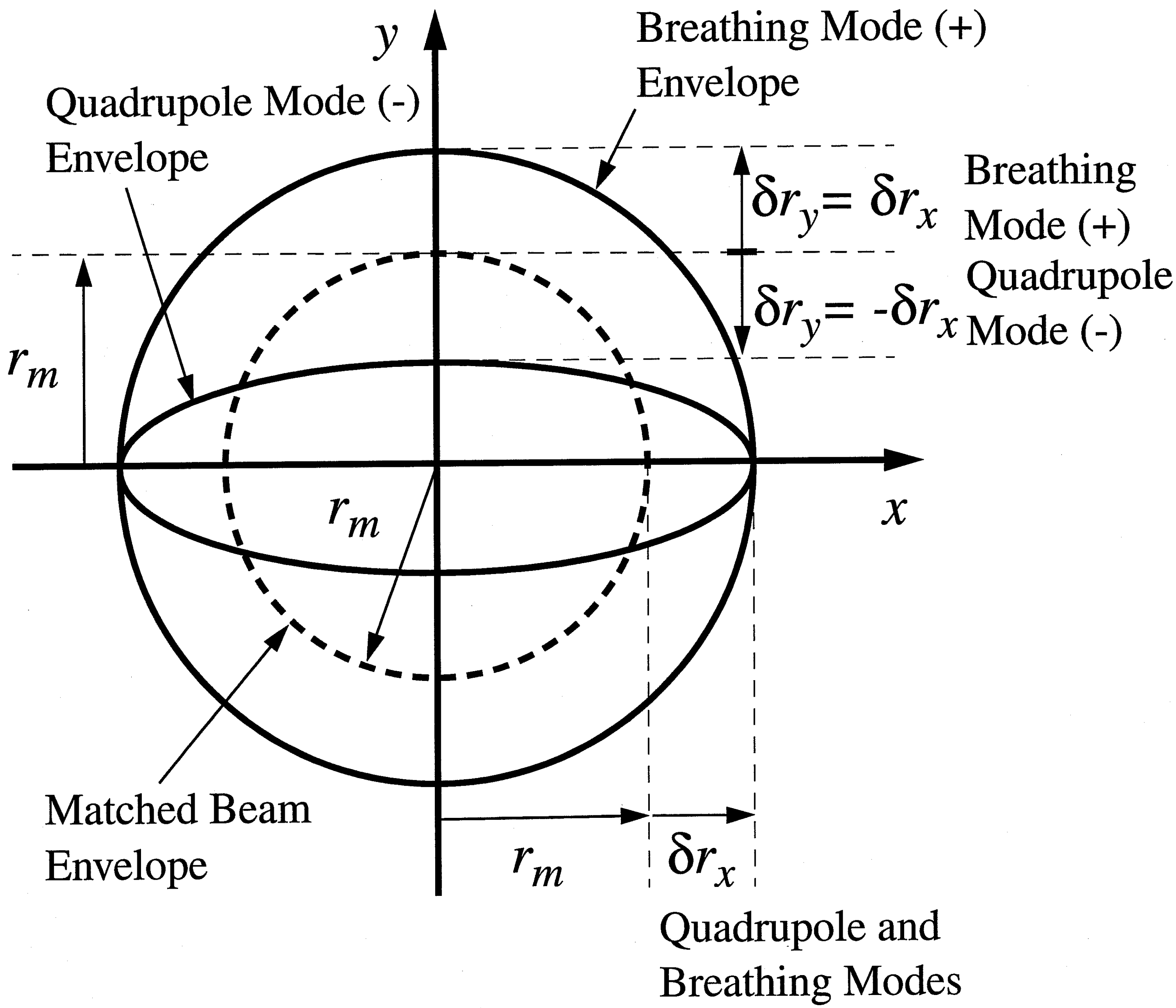
Reduced eigenvalue problem:

$$\mathbf{M}_e(s_i + L_p|s_i)\cdot\vec{E}_n(s_i) = \lambda_n\vec{E}_n(s_i)$$

Mode decoupling simplifies:

- Eigenvalue classes
- Mode symmetries
- Launching conditions

Decoupled Modes: Launching Conditions



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General Mode Limits

Using phase-amplitude analysis can show for any linear focusing lattice:

1) Phase advance of any normal mode satisfies the zero space-charge limit:

$$\boxed{\lim_{Q \to 0} \sigma_\ell = 2\sigma_0}$$

2) Pure normal modes evolve with a quadratic phase-space (Courant-Snyder) invariant in the normal coordinates of the mode

Simply expressed for decoupled modes:

$$\boxed{\left[\frac{\delta r_\pm(s)}{w_\pm(s)}\right]^2 + \left[w'_\pm(s)\delta r_\pm(s) - w_\pm(s)\delta r'_\pm(s)\right]^2 = \text{const}}$$

where

$$w''_+(s) + \kappa(s)\, w_+(s) + \frac{2Q}{r_m^2(s)} w_+(s) + \frac{3\varepsilon^2}{r_m^4(s)} w_+(s) - \frac{1}{w_+^3(s)} = 0$$

$$w''_-(s) + \kappa(s)\, w_-(s) + \frac{3\varepsilon^2}{r_m^4(s)} w_-(s) - \frac{1}{w_-^3(s)} = 0$$

$$w_\pm(s + L_p) = w_\pm(s)$$

Analogous for coupled modes [theory of Edwards and Teng applies, IEEE Trans Nuc. Sci. **20**, 885 (1973)]

Continuous Focusing

Focusing:

$$\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \left(\frac{\sigma_0}{L_p}\right)^2 = \text{const}$$

Matched beam:

symmetric beam:

$$\varepsilon_x = \varepsilon_y = \varepsilon = \text{const}$$
$$r_{xm}(s) = r_{ym}(s) = r_m(s)$$

match condition:

$$k_{\beta 0}^2 r_m - \frac{Q}{r_m} - \frac{\varepsilon^2}{r_m^3} = 0$$

depressed phase advance:

$$\sigma = \sqrt{\sigma_0^2 - \frac{Q}{(r_m/L_p)^2}} = \frac{\varepsilon L_p}{r_m^2}$$

one parameter needed for scaled solution:

$$\frac{k_{\beta 0}^2 \varepsilon^2}{Q^2} = \frac{\sigma_0^2 \varepsilon^2}{Q^2 L_p^2} = \frac{(\sigma/\sigma_0)^2}{[1-(\sigma/\sigma_0)^2]^2}.$$

$0 \le \frac{\sigma}{\sigma_0} \le 1$

space-charge lim

No space-charge

Decoupled Modes:

$$\delta r_\pm(s) = \frac{\delta r_x(s) \pm \delta r_y(s)}{2}$$

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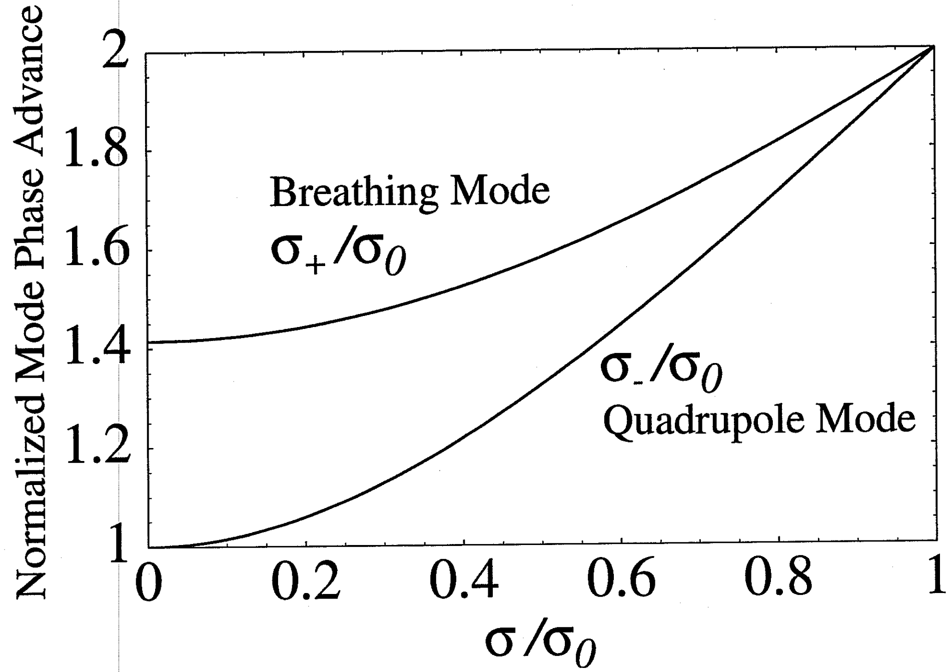
Continuous Focusing - can be solved completely

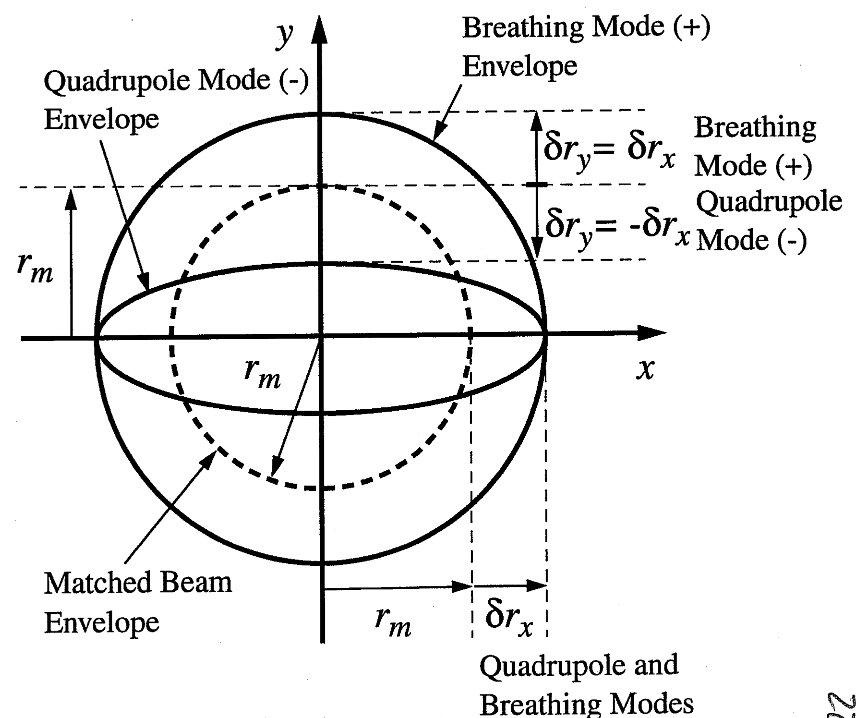
Homogeneous Solution (normal modes):

$$\boxed{\delta r_\pm(s) = \delta r_\pm(s_i)\cos\left(\sigma_\pm \frac{s - s_i}{L_p}\right) + \frac{\delta r'_\pm(s_i)}{\sigma_\pm / L_p}\sin\left(\sigma_\pm \frac{s - s_i}{L_p}\right)}$$

$\sigma_+ \equiv \sqrt{2\sigma_0^2 + 2\sigma^2}$ "breathing" mode phase advance

$\sigma_- \equiv \sqrt{\sigma_0^2 + 3\sigma^2}$ "quadrupole" mode phase advance





22/

Particular Solution (driving perturbations):

Green's function form of solution:

$$\frac{\delta r_\pm(s)}{r_m} = \frac{1}{L_p^2}\int_{s_i}^{s} d\tilde{s}\; G_\pm(s,\tilde{s})\delta p_\pm(\tilde{s})$$

$$\delta p_+(s) = -\frac{\sigma_0^2}{2}\left[\frac{\delta\kappa_x(s)}{k_{\beta 0}^2} + \frac{\delta\kappa_y(s)}{k_{\beta 0}^2}\right] + (\sigma_0^2 - \sigma^2)\frac{\delta Q(s)}{Q} + \sigma^2\left[\frac{\delta\varepsilon_x(s)}{\varepsilon} + \frac{\delta\varepsilon_y(s)}{\varepsilon}\right]$$

$$\delta p_-(s) = -\frac{\sigma_0^2}{2}\left[\frac{\delta\kappa_x(s)}{k_{\beta 0}^2} - \frac{\delta\kappa_y(s)}{k_{\beta 0}^2}\right] + \sigma^2\left[\frac{\delta\varepsilon_x(s)}{\varepsilon} - \frac{\delta\varepsilon_y(s)}{\varepsilon}\right]$$

$$G_\pm(s,\tilde{s}) = \frac{1}{\sigma_\pm/L_p}\sin\left(\sigma_\pm\frac{s-\tilde{s}}{L_p}\right)$$

Green's function solution is fully general. Insight gained from simplified solutions for specific classes of driving perturbations:

- Adiabatic
- Sudden

→covered here

- Ramped
- Harmonic

→covered in full paper

23/

Continuous Focusing – adiabatic particular solution

For driving perturbations $\delta p_+(s)$ and $\delta p_-(s)$ slow on qudrupole mode wavelength ~ $2\pi L_p/\sigma_-$ the solution is:

Focusing

Perveance

$$\frac{\delta r_+(s)}{r_m} = \frac{\delta p_+(s)}{\sigma_+^2}$$

$$= -\left[\frac{1}{2}\frac{1}{1+(\sigma/\sigma_0)^2}\right]\frac{1}{2}\left(\frac{\delta\kappa_x(s)}{k_{\beta 0}^2} + \frac{\delta\kappa_y(s)}{k_{\beta 0}^2}\right) + \left[\frac{1}{2}\frac{1-(\sigma/\sigma_0)^2}{1+(\sigma/\sigma_0)^2}\right]\frac{\delta Q(s)}{Q}$$

$$+ \left[\frac{(\sigma/\sigma_0)^2}{1+(\sigma/\sigma_0)^2}\right]\frac{1}{2}\left(\frac{\delta\varepsilon_x(s)}{\varepsilon} + \frac{\delta\varepsilon_y(s)}{\varepsilon}\right)$$

Emittance

Coefficients of adiabatic terms in square brackets"[ ]"

Focusing

$$\frac{\delta r_-(s)}{r_m} = \frac{\delta p_-(s)}{\sigma_-^2}$$

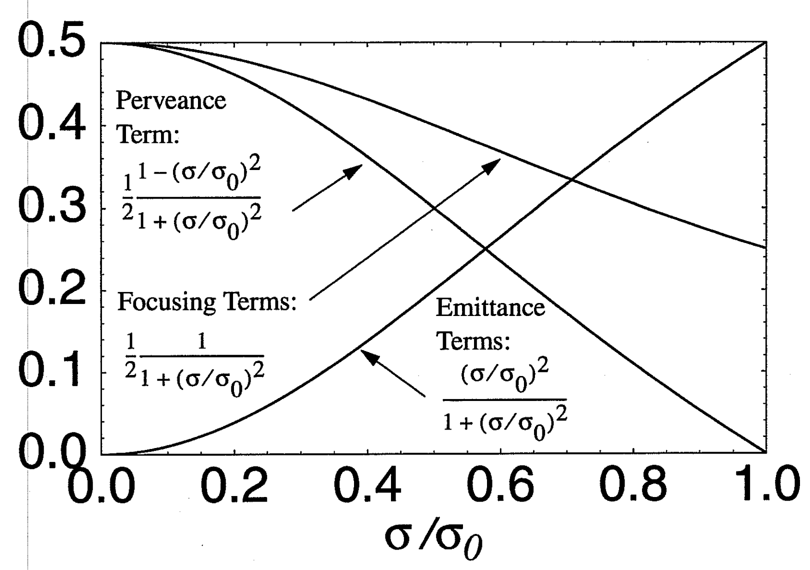
$$= -\left[\frac{1}{1+3(\sigma/\sigma_0)^2}\right]\frac{1}{2}\left(\frac{\delta\kappa_x(s)}{k_{\beta 0}^2} - \frac{\delta\kappa_y(s)}{k_{\beta 0}^2}\right)$$

$$+ \left[\frac{2(\sigma/\sigma_0)^2}{1+3(\sigma/\sigma_0)^2}\right]\frac{1}{2}\left(\frac{\delta\varepsilon_x(s)}{\varepsilon} - \frac{\delta\varepsilon_y(s)}{\varepsilon}\right)$$

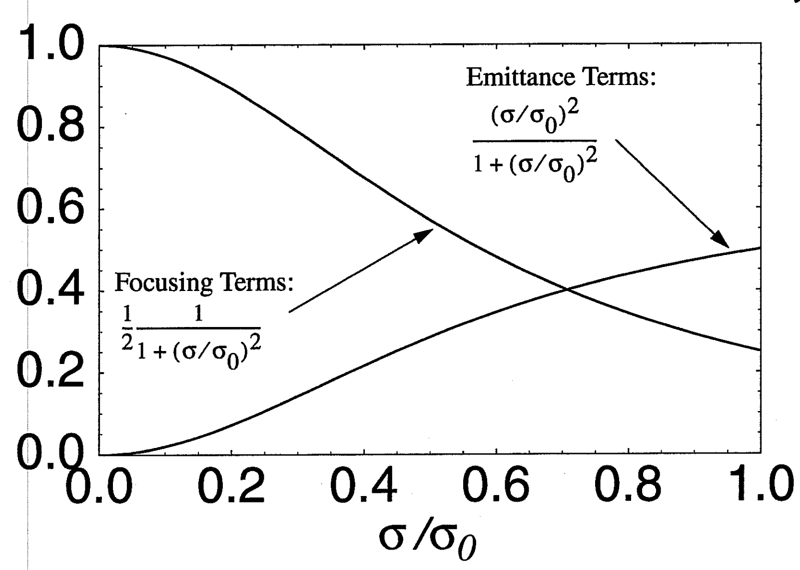
Emittance

Continuous Focusing – adiabatic solution coefficients

a) Adiabatic Solution Coefficients for $\delta r_+ = (\delta r_x + \delta r_y)/2$



b) Adiabatic Solution Coefficients for $\delta r_- = (\delta r_x - \delta r_y)/2$



Relative strength of:

- Space-Charge (Perveance)
- Applied Focusing
- Emittance

terms vary with space-charge depression (σ/σ_0) for both breathing and quadrupole modes.

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Continuous Focusing – sudden particular solution

For step function driving perturbations of form:

$$\boxed{\delta p_\pm(s) = \widehat{\delta p_\pm}\Theta(s - s_p)}$$

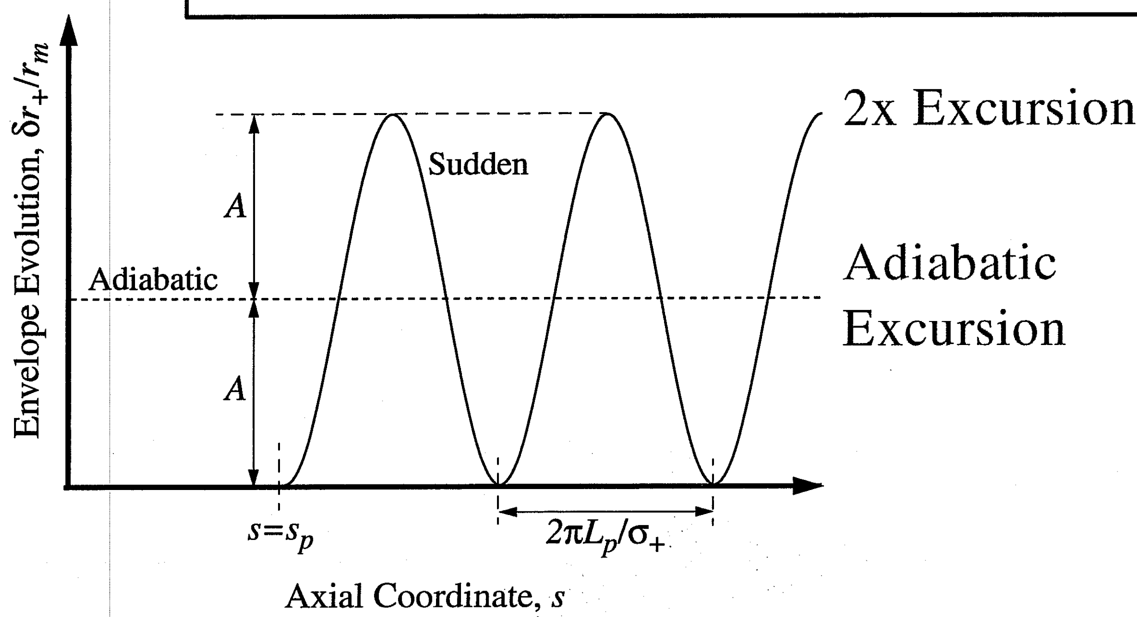
with

$$\widehat{\delta p_+} = -\frac{\sigma_0^2}{2}\left[\frac{\widehat{\delta\kappa_x}}{k_{\beta 0}^2} + \frac{\widehat{\delta\kappa_y}}{k_{\beta 0}^2}\right] + (\sigma_0^2 - \sigma^2)\frac{\widehat{\delta Q}}{Q} + \sigma^2\left[\frac{\widehat{\delta\varepsilon_x}}{\varepsilon} + \frac{\widehat{\delta\varepsilon_y}}{\varepsilon}\right] = \text{const}$$

$$\widehat{\delta p_-} = -\frac{\sigma_0^2}{2}\left[\frac{\widehat{\delta\kappa_x}}{k_{\beta 0}^2} - \frac{\widehat{\delta\kappa_y}}{k_{\beta 0}^2}\right] + \sigma^2\left[\frac{\widehat{\delta\varepsilon_x}}{\varepsilon} - \frac{\widehat{\delta\varepsilon_y}}{\varepsilon}\right] = \text{const}$$

The solution is given by the substitution in the expression for the adiabatic solution:

$$\boxed{\delta p_\pm(s) \to \widehat{\delta p_\pm}\left[1 - \cos\left(\sigma_\pm \frac{s - s_p}{L_p}\right)\right]\Theta(s - s_p)}$$



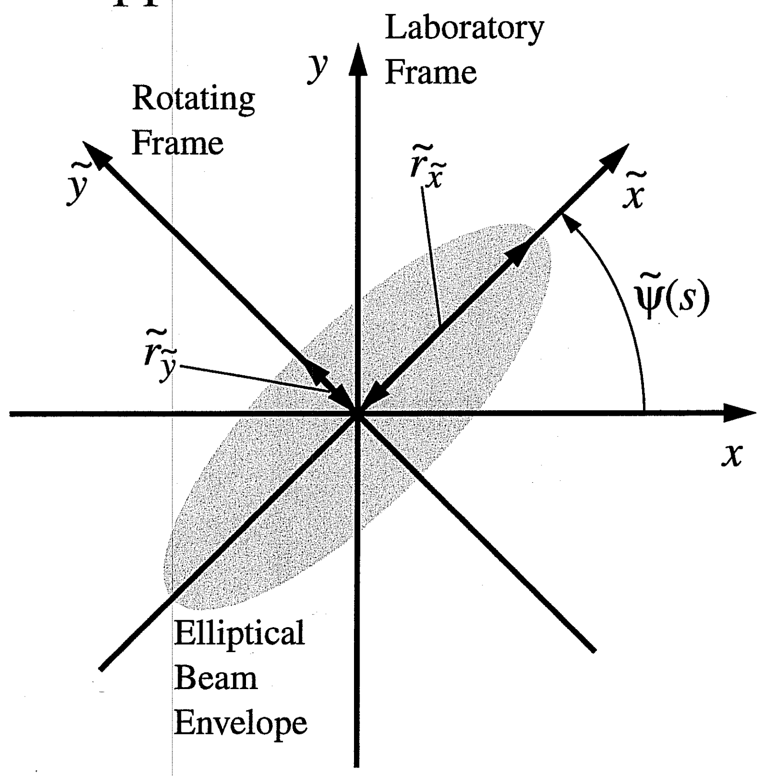
For the same amplitude of total driving perturbations, sudden perturbations result in 2x the envelope excursion that adiabatic perturbations produce.

Solenoidal Focusing – must be analyzed in the rotating Larmor frame

Envelope equations for an elliptical beam with zero canonical angular momentum

$$P_\theta = \langle xy' - yx' \rangle_\perp + k_L(s)\langle x^2 + y^2 \rangle_\perp = 0$$

are mapped to standard form when interpreted in the Larmor frame



$$\tilde{x}(s) = x(s)\cos\tilde{\psi}(s) + y(s)\sin\tilde{\psi}(s)$$
$$\tilde{y}(s) = -x(s)\sin\tilde{\psi}(s) + y(s)\cos\tilde{\psi}(s)$$

$$\tilde{\psi}(s) = -\int_{s_i}^{s} d\bar{s}\; k_L(\bar{s})$$

$$k_L(s) = \frac{qB_z(s)}{2m\gamma_b\beta_b c}$$ Larmor wavenumber

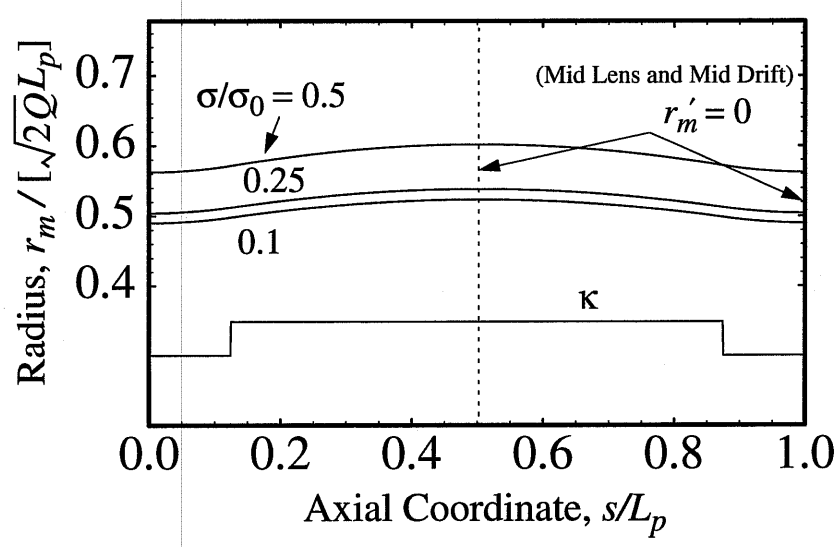
$$\kappa_x(s) = \kappa_y(s) = \kappa(s) = k_L^2(s)$$

A self-consistent elliptical beam KV distribution has been derived for $P_\theta = 0$.

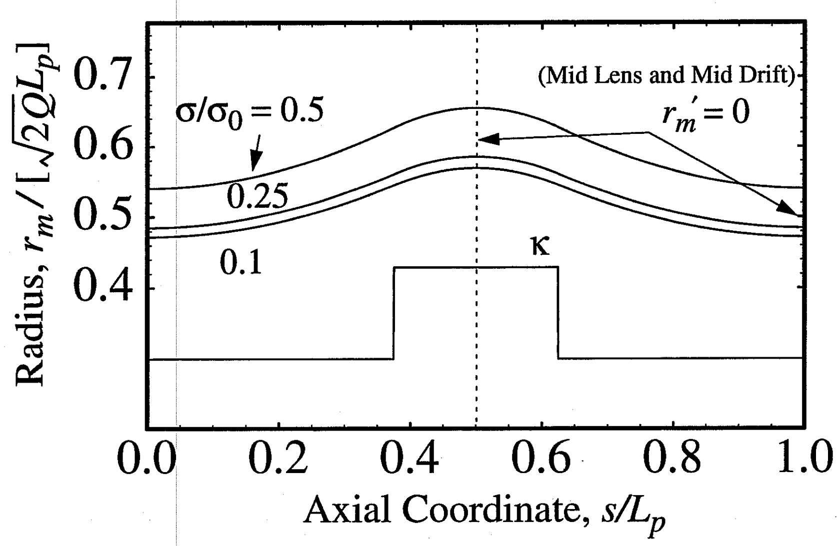
For nonzero P_θ the envelope equations cannot be mapped to the form analyzed.

Solenoidal Focusing – matched beam flutter increases at lower occupancies

a) Matched Envelope for $\sigma_0 = 80^\circ$ and $\eta = 0.75$



b) Matched Envelope for $\sigma_0 = 80^\circ$ and $\eta = 0.25$



Focusing:

$$\kappa_x(s) = \kappa_y(s) = \kappa(s)$$

$$\kappa(s + L_p) = \kappa(s)$$

Matched Beam:

$$\varepsilon_x = \varepsilon_y = \varepsilon = \text{const}$$
$$r_{xm}(s) = r_{ym}(s) = r_m(s)$$

$$r_m(s + L_p) = r_m(s)$$

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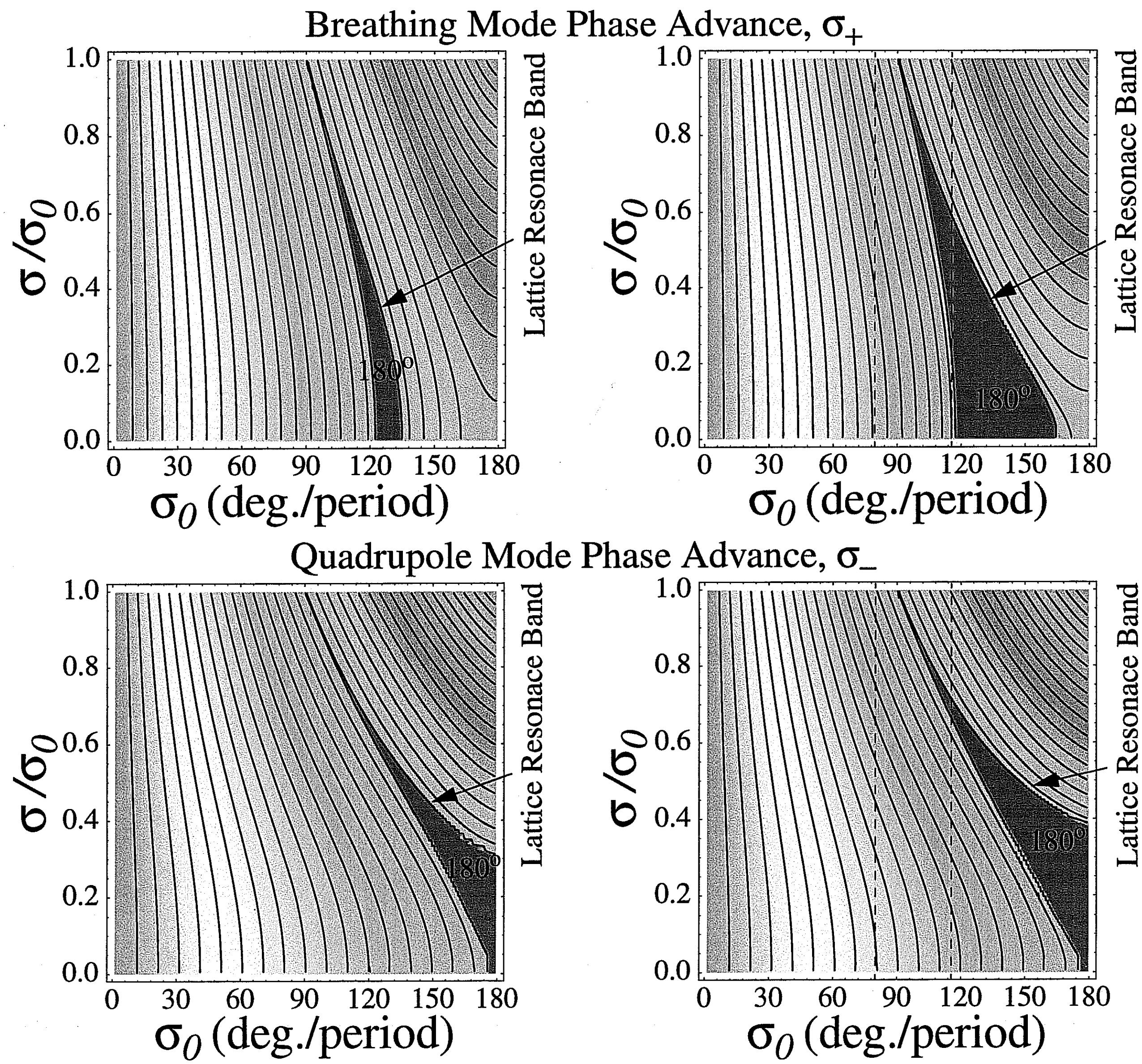
Solenoidal Focusing – parametric plots of breathing and quadrupole envelope mode phase advances for two occupancies

$\eta = 0.75$ $\eta = 0.25$



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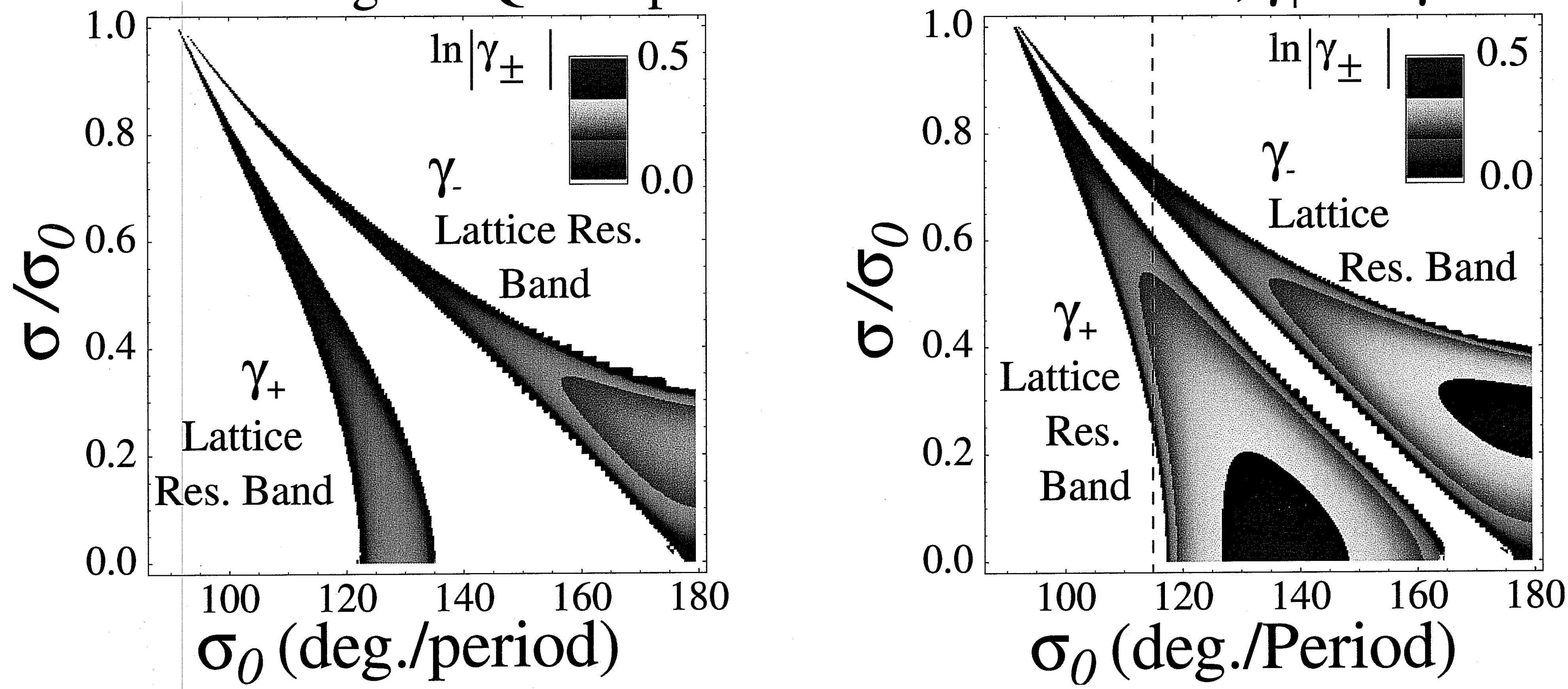
Solenoidal Focusing – parametric plots of breathing and quadrupole envelope mode instability bands for two occupancies

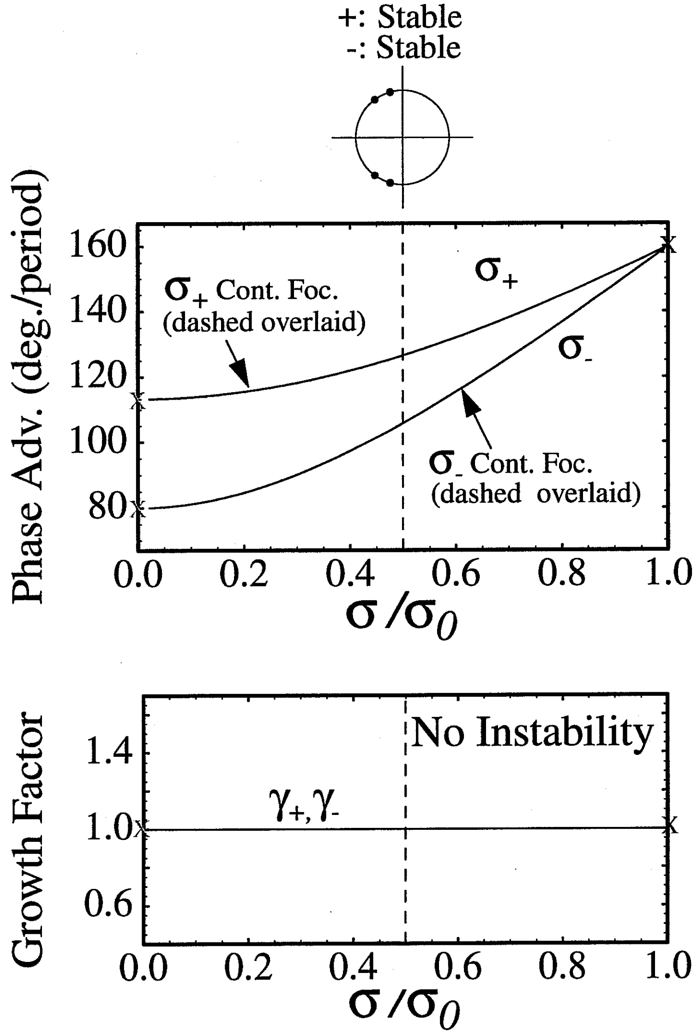
$\eta = 0.75$ $\eta = 0.25$

Breathing and Quadrupole Mode Growth Factors, γ_+ and γ_-

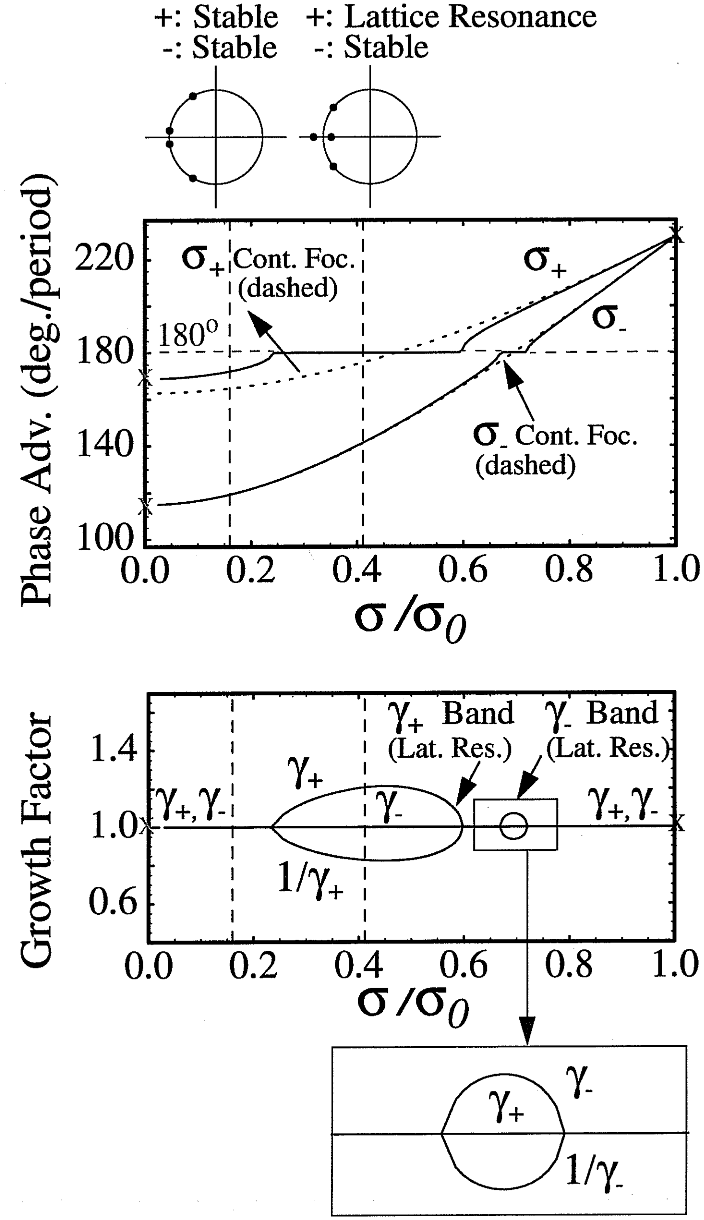


Solenoidal Focusing – cross-section of mode band structure at fixed σ_0

a) $\eta = 0.25$, $\sigma_0 = 80^o$



b) $\eta = 0.25$, $\sigma_0 = 115^o$



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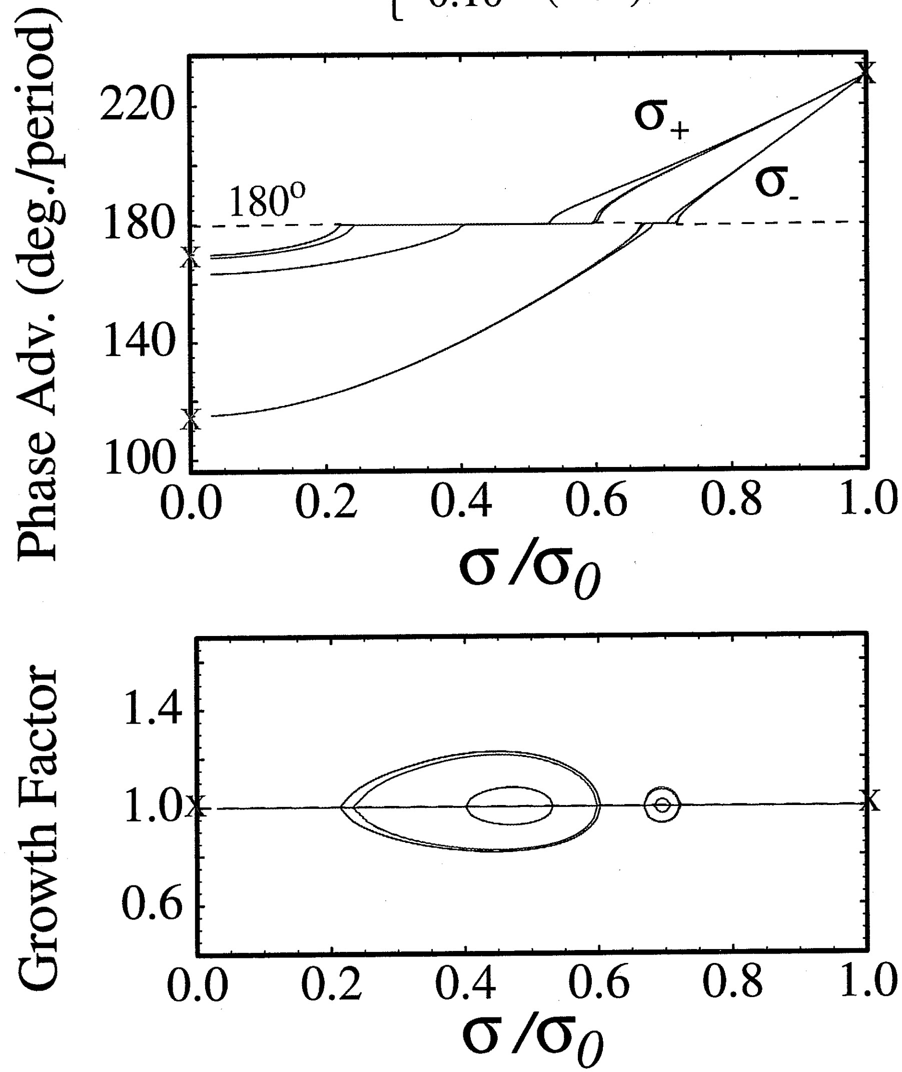
Solenoidal Focusing – instability bands of modes become wider and stronger for smaller occupancy

$$\eta = \begin{cases} 0.75 & \text{(Blue)} \\ 0.25 & \text{(Green)} \\ 0.10 & \text{(Red)} \end{cases}$$



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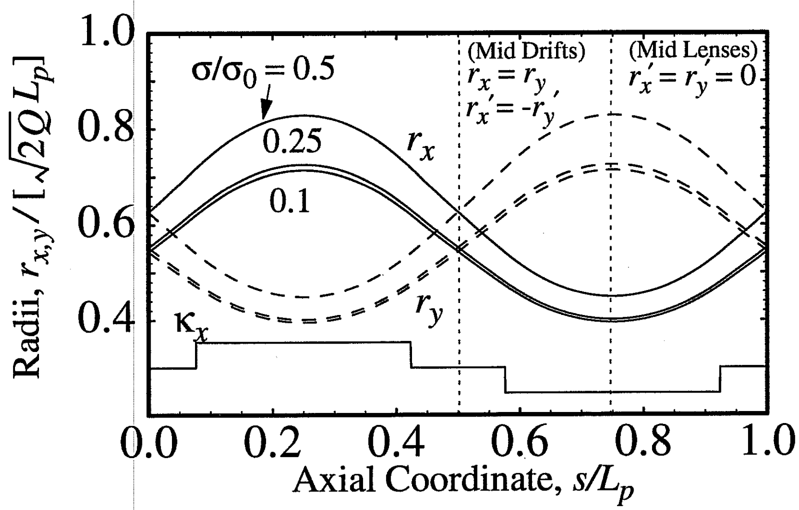
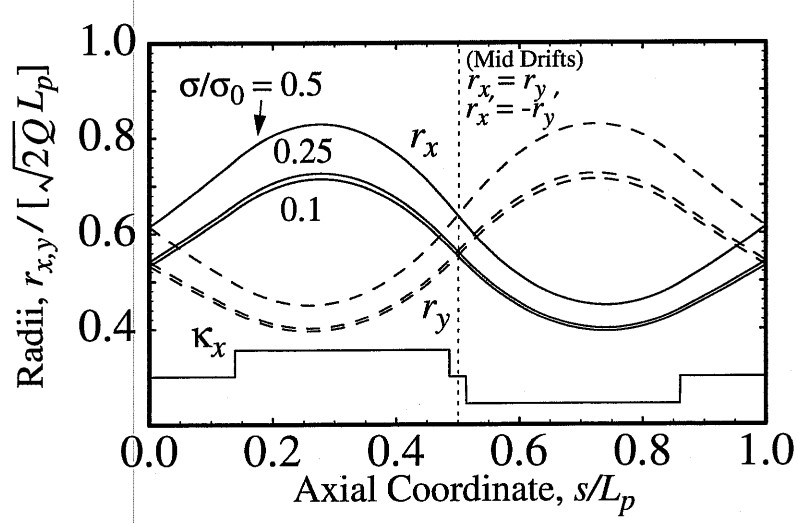
Quadrupole Focusing – matched beam envelopes for FODO and syncopated lattices

a) Matched Envelope for $\sigma_0 = 80^{\circ}$, $\eta = 0.6949$, and $\alpha = 1/2$



b) Matched Envelope for $\sigma_0 = 80^{\circ}$, $\eta = 0.6949$, and $\alpha = 0.1$



Focusing:

$$\kappa_x(s) = -\kappa_y(s) = \kappa_q(s)$$

$$\kappa_q(s + L_p) = \kappa_q(s)$$

Matched Beam:

$$r_{xm}(s + L_p) = r_{xm}(s)$$

$$r_{ym}(s + L_p) = r_{ym}(s)$$

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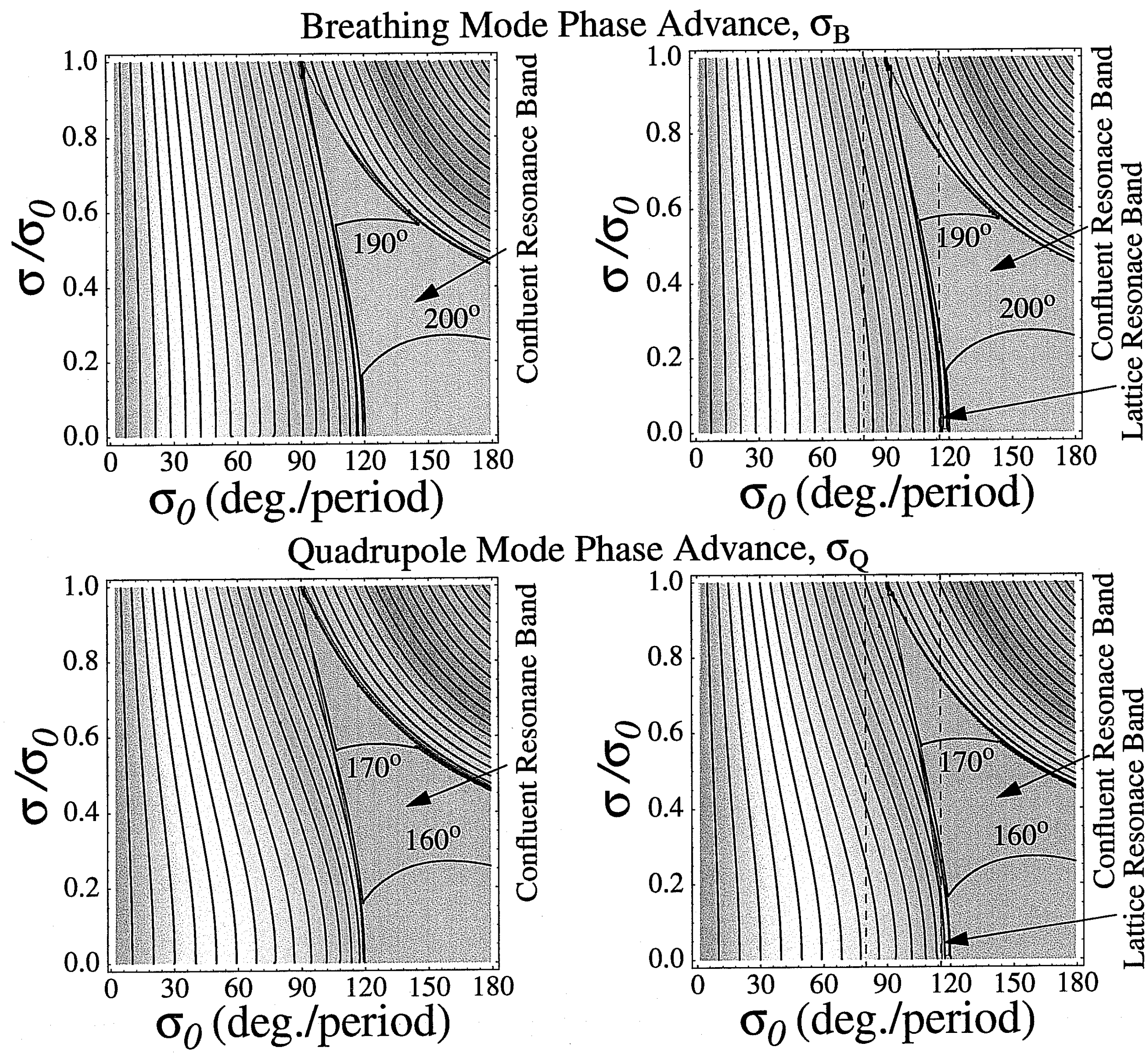
Quadrupole Focusing – parametric plots of breathing and quadrupole mode phase advances for FODO and syncopated lattices

$\eta = 0.6949,\ \alpha = 1/2$ (FODO) $\qquad\qquad \eta = 0.6949,\ \alpha = 0.1$



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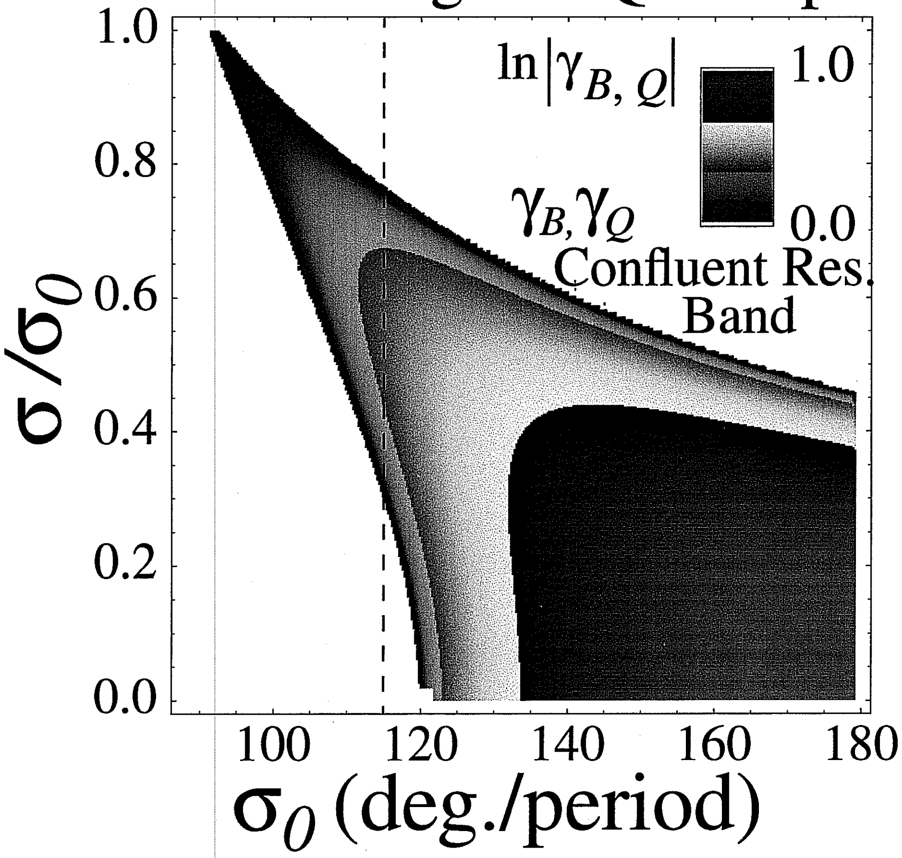
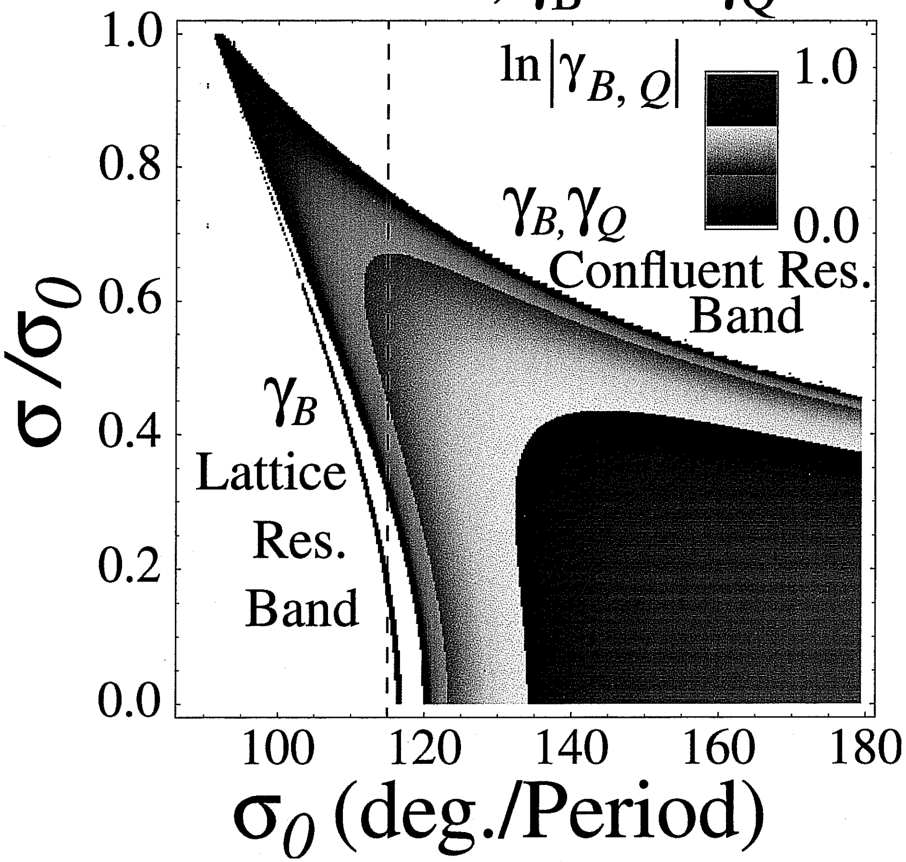
Quadrupole Focusing – parametric plots of breathing and quadrupole mode instability bands for FODO and syncopated lattices

$\eta = 0.6949$, $\alpha = 1/2$ (FODO) $\qquad\qquad$ $\eta = 0.6949$, $\alpha = 0.1$

Breathing and Quadrupole Mode Growth Factors, γ_B and γ_Q





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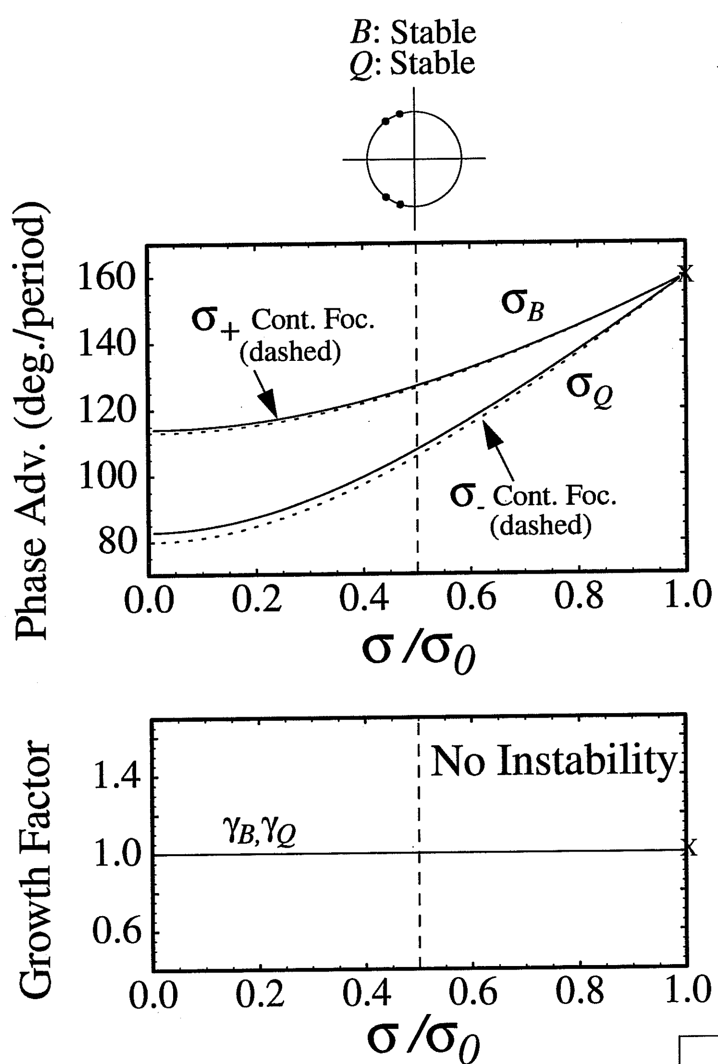
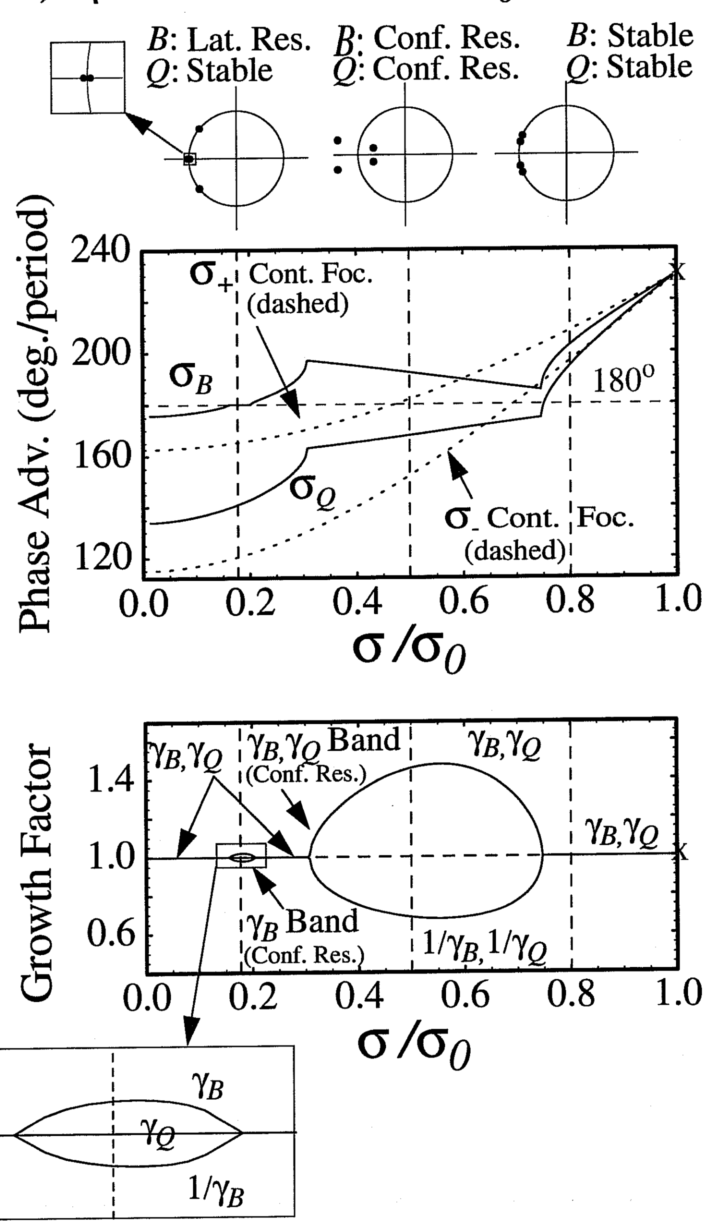
Quadrupole Focusing – cross-section of mode band structure at fixed σ_0

a) $\eta = 0.6949,\ \alpha = 0.1,\ \sigma_0 = 80^o$



b) $\eta = 0.6949,\ \alpha = 0.1,\ \sigma_0 = 115^o$



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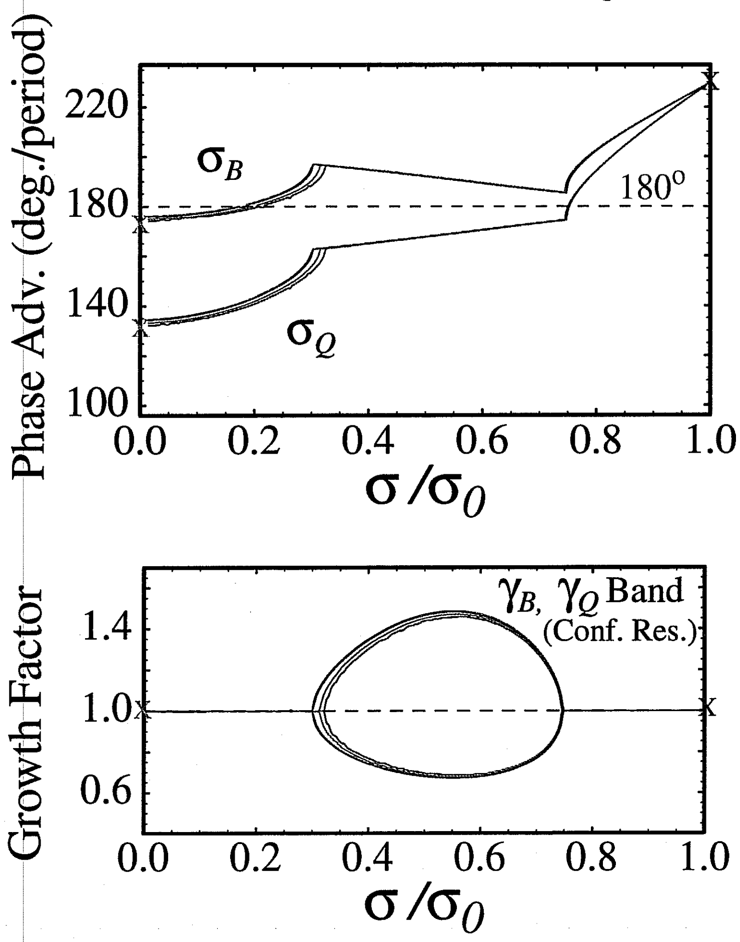
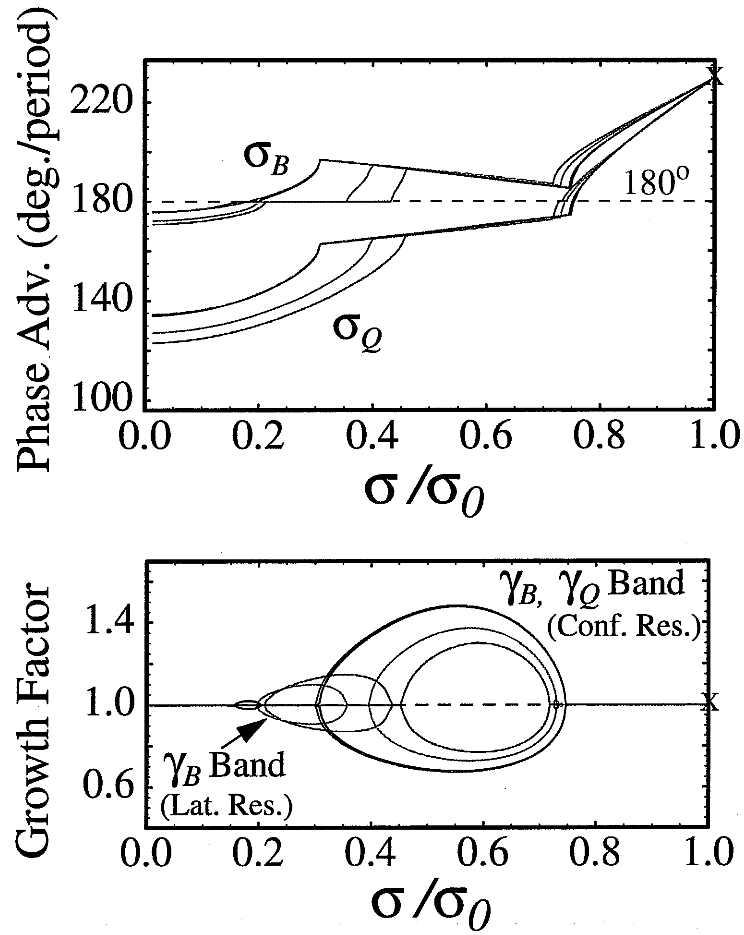
Quadrupole Focusing – instability bands of modes vary little/strongly with occupancy for FODO/syncopated lattices

a) $\alpha = 1/2$ (FODO), $\sigma_0 = 115^{\circ}$ b) $\alpha = 0.1$, $\sigma_0 = 115^{\circ}$

$$\eta = \begin{cases} 0.90 & \text{(Blue)} \\ 0.6949 & \text{(Black)} \\ 0.25 & \text{(Green)} \\ 0.10 & \text{(Red)} \end{cases}$$





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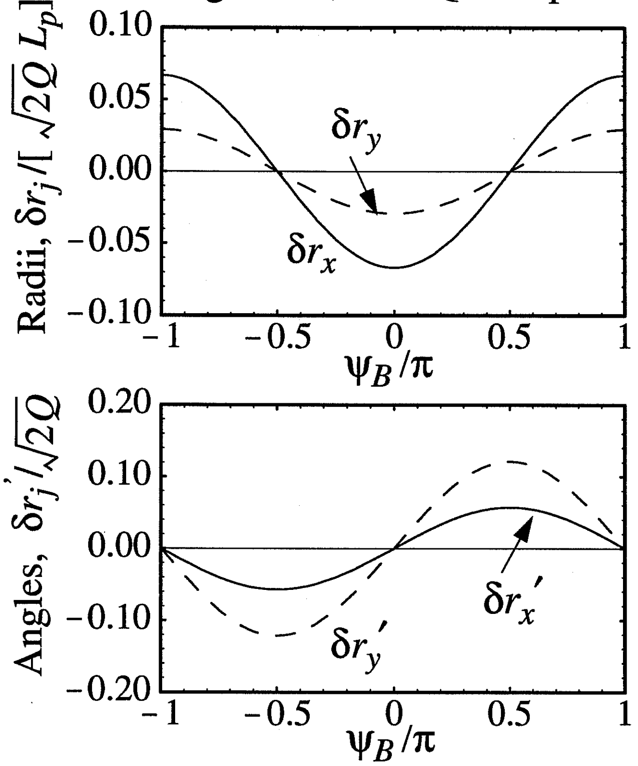
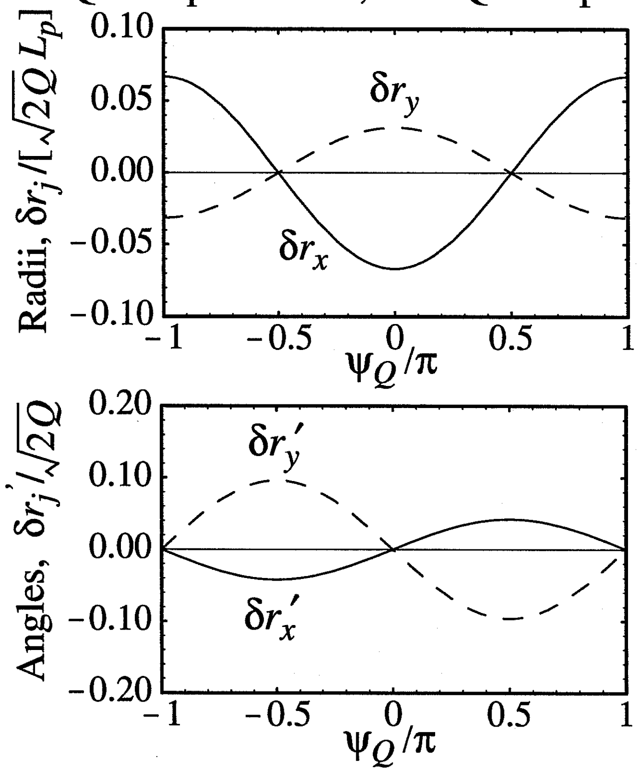
Quadrupole Focusing - mode launching conditions vary with phase in lattice 38/

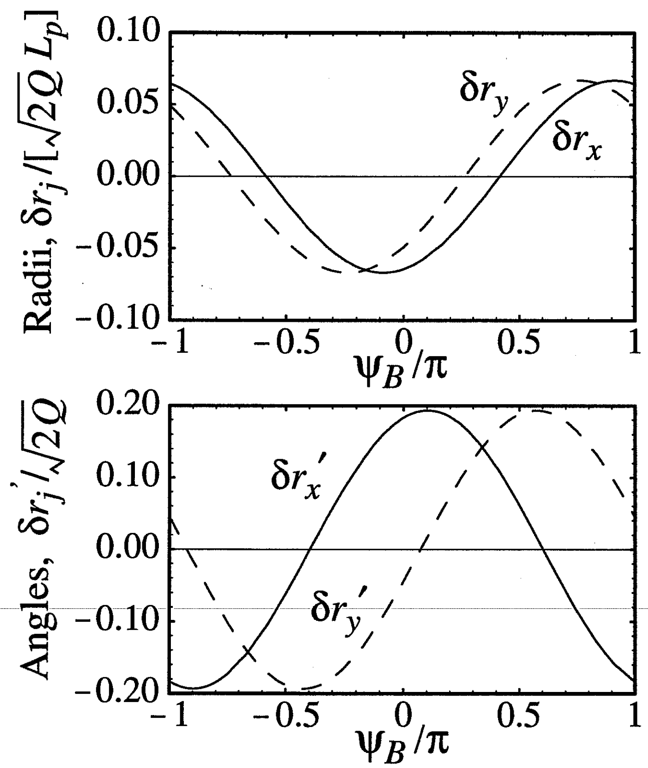
Breathing Mode, Mid-Quadrupole



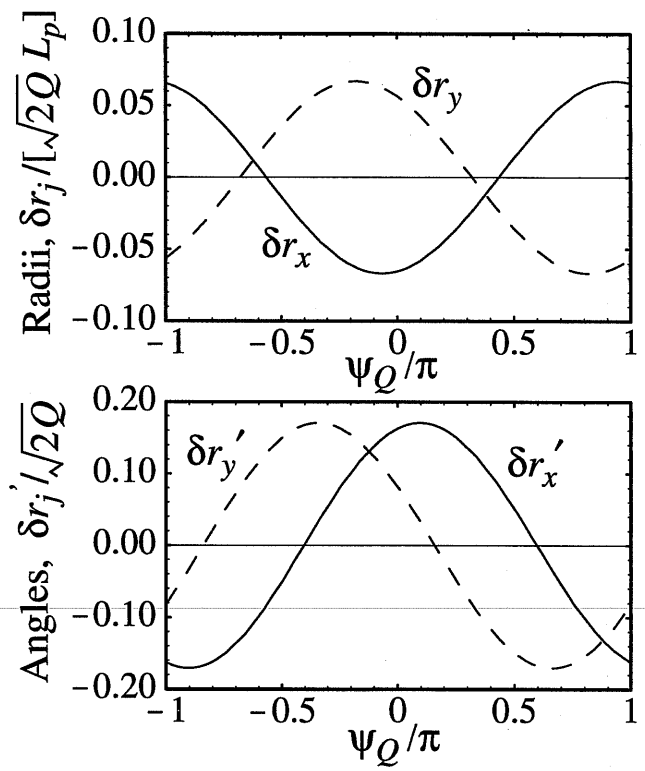
Quadrupole Mode, Mid-Quadrupole



Breathing Mode, Mid-Drift

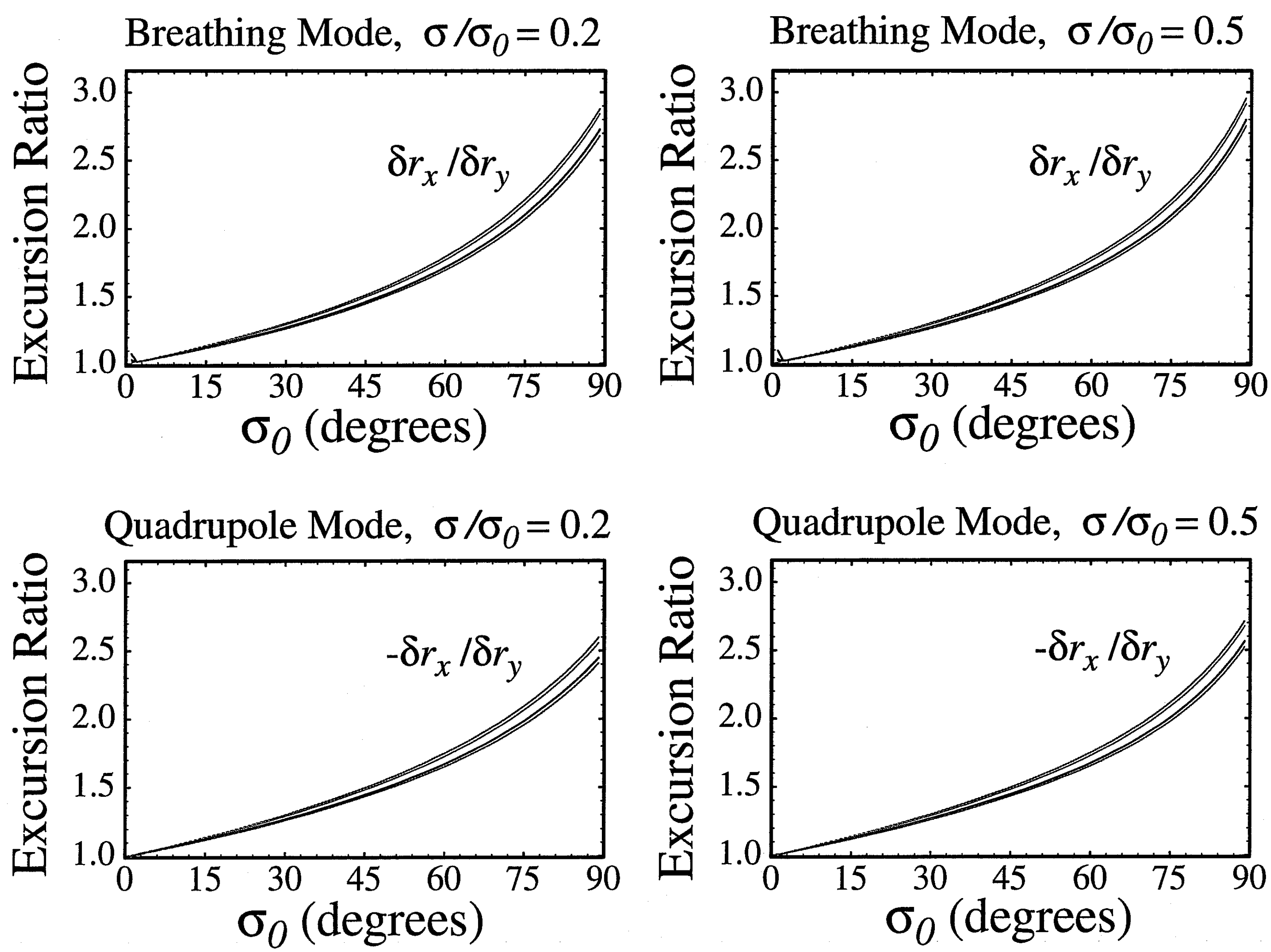


Quadrupole Mode, Mid-Drift



Quadrupole FODO Focusing – simplified launch conditions apply at the axial mid-plane of a quadrupoles ($\delta r_x' = 0 = \delta r_y'$)

$$\eta = \begin{cases} 0.90 & \text{(Blue)} \\ 0.6949 & \text{(Black)} \\ 0.25 & \text{(Green)} \\ 0.10 & \text{(Red)} \end{cases}$$



Thin lenses provide the simplest model of periodic transport channels

Lattice is a sequence of free-drifts and focusing kicks

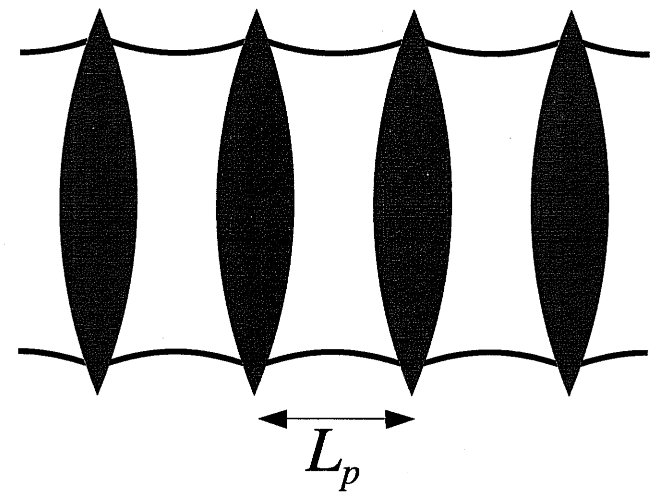
$$\begin{bmatrix} r_{after} \\ r'_{after} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} r_{before} \\ r'_{before} \end{bmatrix}$$

where f = const is the focal length of the lens

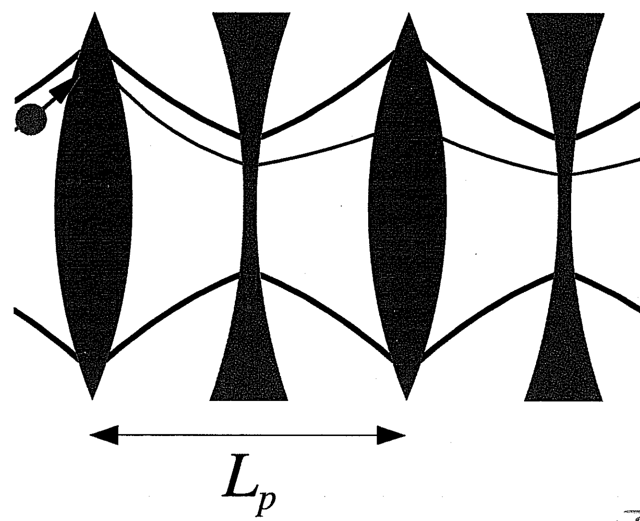
Undepressed phase advances:

$$\cos\sigma_0 = \begin{cases} 1 - \frac{L_p}{2f}, & \text{thin-lens solenoids} \\ 1 - 2\alpha(1-\alpha)\left(\frac{L_p}{f}\right)^2, & \text{thin-lens quadrupoles} \end{cases}$$

Solenoids:



Quadrupoles:



Free expansion of a fully space-charge depressed beam

Analytical solution:

$$\ln \frac{R_+(s)}{R_+(0)} = -R_+'^2(0) + \left(\mathrm{erfi}^{(-1)}\left(\mathrm{erfi}\, R_+'(0) + e^{R_+'^2(0)} \frac{s}{\sqrt{\pi} R_+(0)}\right)\right)^2$$

$$R_-(s) = R_-(s) + R_-'(0)s$$

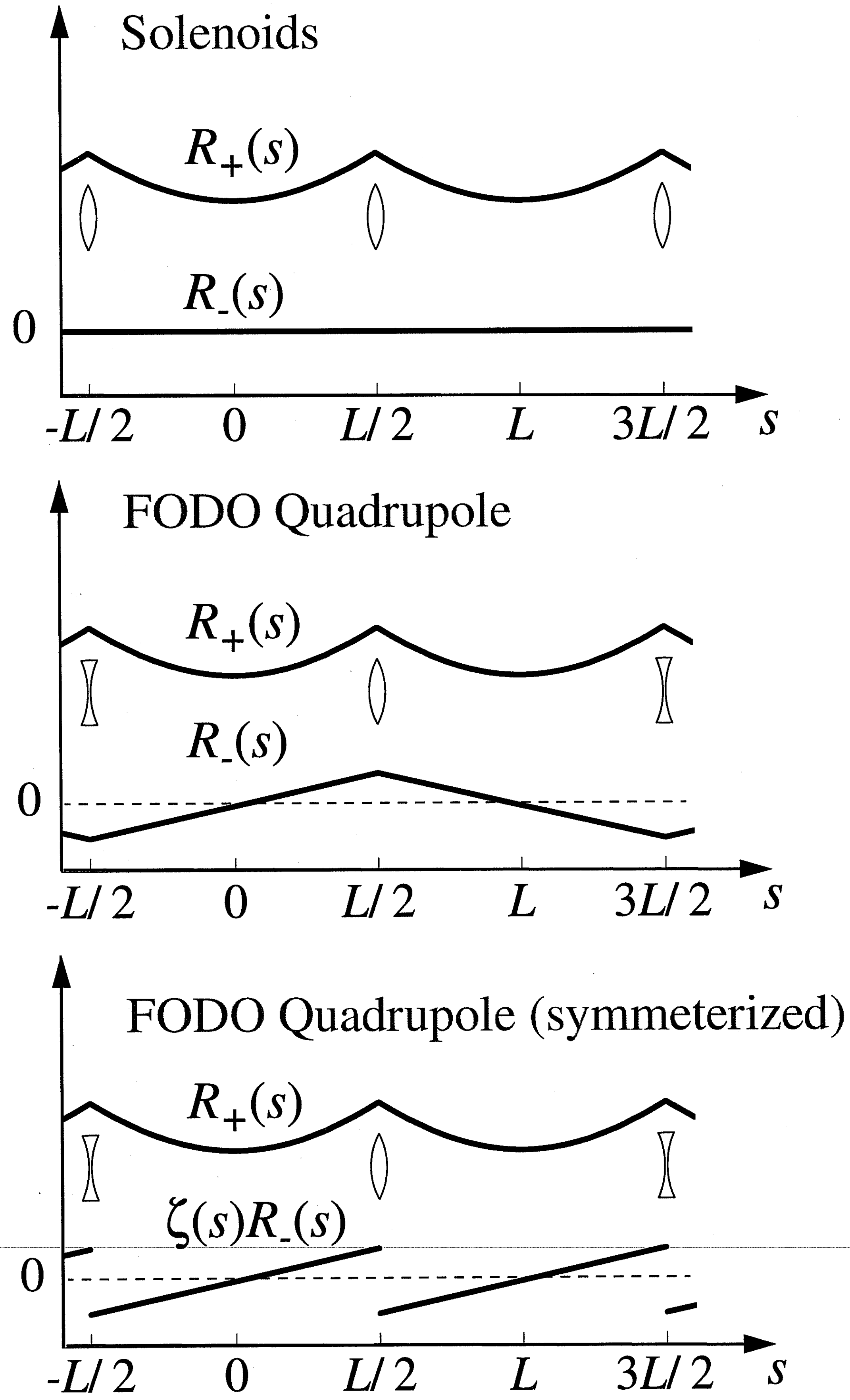
where

$$R_\pm = \frac{r_x \pm r_y}{\sqrt{8Q}} \qquad \text{and} \qquad \mathrm{erfi}(z) = \mathrm{erf}(iz)/i = \frac{2}{\sqrt{\pi}} \int_0^z e^{t^2}\, dt$$

Applications:

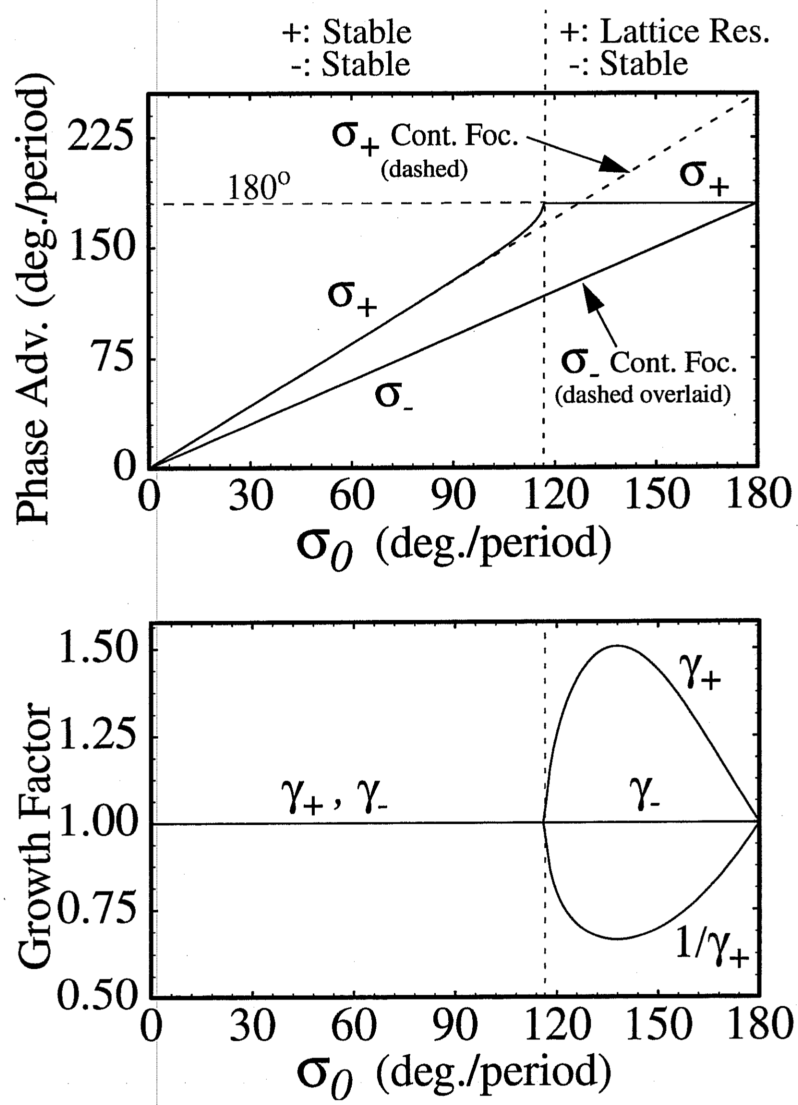
- Fully analytical studies
- Fast design estimates for intense beam

Thin Lens Lattices and Matched Beam

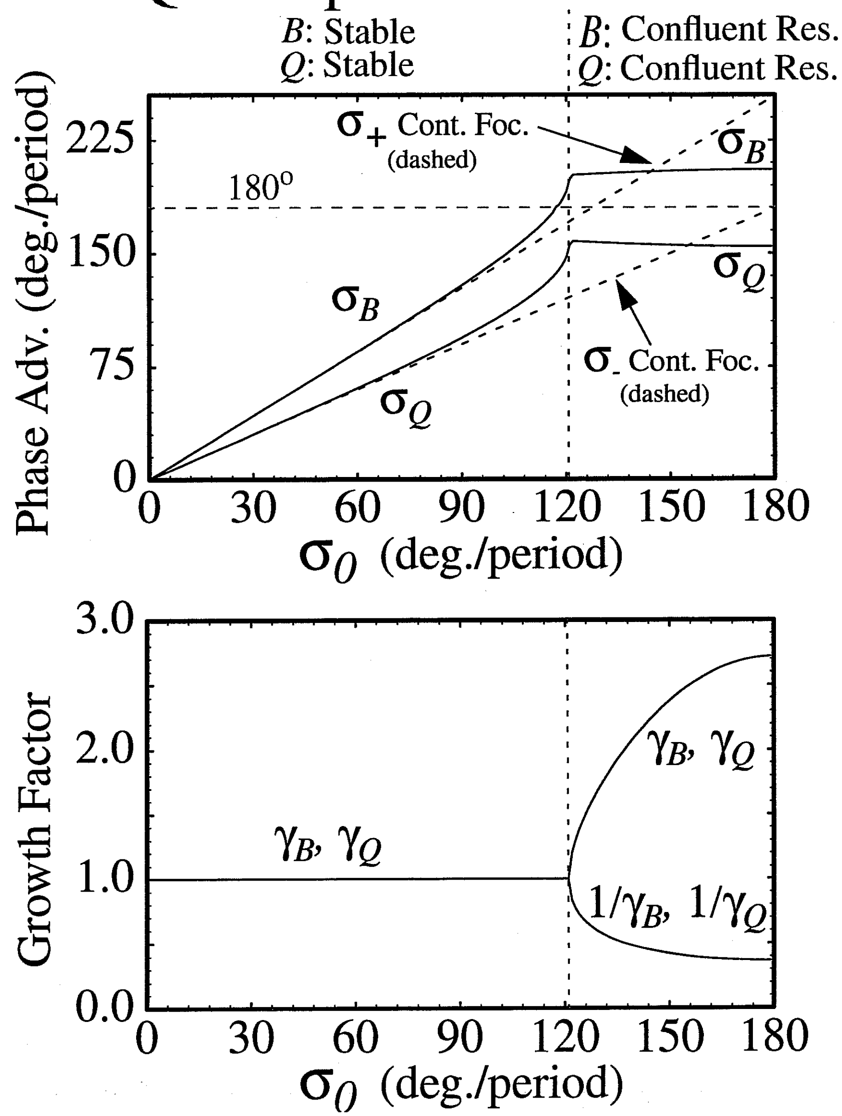


Thin-lens model for periodic focusing channels allows complete analytical solution of envelope mode stability at full space-charge depression ($\sigma = 0$)

Solenoids: 116.715°



FODO Quadrupoles: 121.055°



J.J. Barnard and S.M. Lund
USPAS, Jan. 2004

§5 **Transport Limit Scaling Based on the Matched Beam Envelope Equations for Periodic Focusing Channels**

The scaling of the maximum beam current, or equivalently, the maximum perveance Q that can be transported at a given energy with a specified focusing technology and lattice is of critical importance in designing optimal transport and acceleration channels. Needed equations can be derived from approximate analytical solutions to the matched beam envelope equations for a given lattice. Alternatively, numerical solutions of the envelope equations can be evaluated. But analytical solutions are preferable to understand scaling and enable rapid evaluation of design tradeoffs.

As a practical matter, equations derived must be applied to regimes where technology is feasible.

- Magnet Field Limits
- Electron breakdown
- Vacuum

⋮

Transport limits are inextricably linked to technology. Moreover, higher order stability constraints etc. must also be respected. Treatments of these topics are beyond the scope of this class. Here we present simplified treatments to highlight issues and methods.

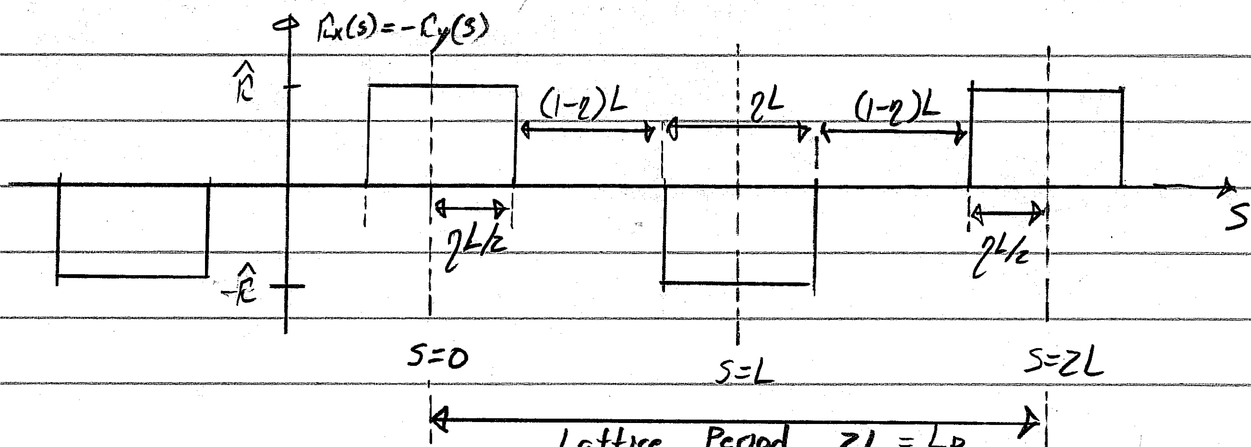
First review an example sketched by J. J. Barnard in the Intro. lectures.

Transport Limits of a Periodic FODO Quadrupole Transport Channel

$$r_{xm}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} r_{xm}' + \kappa_x r_{xm} - \frac{2Q}{r_{xm}+r_{ym}} - \frac{\varepsilon_x^2}{r_{xm}^3} = 0$$

$$r_{ym}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} r_{ym}' - \kappa_x r_{ym} - \frac{2Q}{r_{xm}+r_{ym}} - \frac{\varepsilon_y^2}{r_{ym}^3} = 0$$

$$r_{xm}(s+L_p) = r_{xm}(s) \quad ; \quad r_{ym}(s+L_p) = r_{ym}(s)$$



L = Half-Period $\qquad L = L_p/2$

η = Quadrupole "occupancy" $\quad 0 < \eta \leq 1$

κ = Focus strength $\qquad \kappa = \begin{cases} \dfrac{qE_x'(s)}{m\gamma_b\beta_b^2c^2} \text{ ; Electric} \\ \dfrac{qB_y'(s)}{m\gamma_b\beta_b c} \text{ ; Magnetic} \end{cases}$

Expand $\kappa_x(s)$ as a Fourier Series:

$$\boxed{\begin{aligned} \kappa_x(s) &= \sum_{n=1}^{\infty} \kappa_n \cos\left(\frac{n\pi s}{L}\right) \\ \kappa_n &= \frac{1}{L}\int_0^{2L} \kappa_x(s)\cos\left(\frac{n\pi s}{L}\right) = \frac{2\hat{\kappa}}{n\pi}\left[1-(-1)^n\right]\sin\left(\frac{n\pi\eta}{2}\right) \end{aligned}}$$

And expand the periodic matched beam envelope by:

$$r_{xm} = r_b\left[1 + \Delta\cos\left(\frac{\pi s}{L}\right)\right] + \sum_{n=2}^{\infty} \Delta_{xn}\cos\left(\frac{n\pi s}{L}\right)$$

$$r_{ym} = r_b\left[1 - \Delta\cos\left(\frac{\pi s}{L}\right)\right] + \sum_{n=2}^{\infty} \Delta_{yn}\cos\left(\frac{n\pi s}{L}\right)$$

$r_b = \text{const} =$ avg. beam radius.

$|\Delta| = \text{const} \ll 1$

$\Delta_{\substack{x\\y}n}$ constants with $|\Delta_{\substack{x\\y}n}| \ll |\Delta|$

Take:

- $(\gamma_b\beta_b)' = 0 \quad \Rightarrow$ coasting beam
- $\varepsilon_x = \varepsilon_y = \varepsilon \quad \Rightarrow$ isotropic beam

and insert these expansions in the envelope equations.

Neglect:

- All terms $\mathcal{O}(\Delta^2)$ and higher
- Fast oscillation terms $\sim \cos\left(\frac{n\pi s}{L}\right)$ with $n \geq 2$.

to obtain two independent constraint equations:

Avg (const):
$$\frac{2\Delta\hat{\kappa}}{\pi} r_b \sin\left(\frac{\pi\eta}{2}\right) - \frac{Q}{r_b} - \frac{\varepsilon^2}{r_b^3} = 0$$

Fundamental $\propto \cos\left(\frac{\pi s}{L}\right)$:
$$-\Delta\left(\frac{\pi}{L}\right)^2 r_b + \frac{4\hat{\kappa} r_b}{\pi}\sin\left(\frac{\pi\eta}{2}\right) + \frac{3\Delta\varepsilon^2}{r_b^3} = 0$$

These equations can be solved to express the maximum beam edge excursion as

$$\boxed{\mathrm{Max}[r_{xm}] = \mathrm{Max}[r_{ym}] \simeq r_b(1+|\Delta|) = r_b\left\{1 + \frac{4|\hat{\kappa}|L^2}{\pi^3} \frac{\sin\left(\frac{\pi\eta}{2}\right)}{\left(1 - \frac{3L^2\varepsilon^2}{\pi^2 r_b^4}\right)}\right\}}$$

and the beam Perveance as:

$$\boxed{Q = \frac{2}{\pi^2}\left[\frac{\sin\left(\frac{\pi\eta}{2}\right)}{\left(\frac{\pi\eta}{2}\right)}\right]^2 \frac{\eta^2\hat{\kappa}^2 L^2 r_b^2}{\left(1 - \frac{3L^2\varepsilon^2}{\pi^2 r_b^4}\right)} - \frac{\varepsilon^2}{r_b^2}}$$

Design Strategy:

1) Choose a lattice period $2L$, quadrupole occupancy η, and clear machine "pipe" radius r_p consistent with focusing technology employed.

2) Choose the largest possible focus strength $\hat{\kappa}$ (quadrupole current or voltage excitation) for beam energy with undepressed particle phase advance:

$$\boxed{\sigma_0 \lessapprox 80^\circ / \text{period}, \quad \text{"Tiefenback Limit"}}$$

- Larger phase advances correspond to stronger focus and smaller beam cross-sectional area for given values of Q, ε.

- Weaker phase advance suppresses various particle, envelope, and collective instabilities for reliable transport: [Ref: M.G. Tiefenback, "Space-Charge Limits on the Transport of Ion Beams," UC. Berkeley Ph.d Thesis, 1986 LBL-22465

3) Choose a suitable beam-edge to aperture clearance factor:

$$\boxed{\begin{aligned} r_p &= \mathrm{Max}[r_{xm}] + \Delta_p \\ \Delta_p &= \text{Clearance}. \end{aligned}}$$

to allow for misalignments, limit scraping of halo particles outside the beam core, reduce image charges, gas propagation times from the aperture to the beam, and other nonideal effects.

4) Evaluate choices made using higher-order theory, numerical simulations etc.. Iterate choices made to reoptimize when evaluating cost.

Effective application of this formulation requires extensive practical knowledge:

- Nonideal effects: collective instabilities, halo, electron and gas interactions (species contamination),....
- Technology limits: Voltage breakdown, vacuum, superconducting magnets, ...

In practice, for intense beam transport, the emittance terms $\varepsilon_x, \varepsilon_y$ can often be neglected for the purpose of obtaining simpler scaling relations that are more easily understood.

$$\lim_{\varepsilon_x \to 0} \delta_x = 0$$

$$\lim_{\varepsilon_y \to 0} \delta_y = 0$$

$\Rightarrow$ Full space charge depression

In this limit $Q \to Q_{max}$, the maximum transportable perveance.

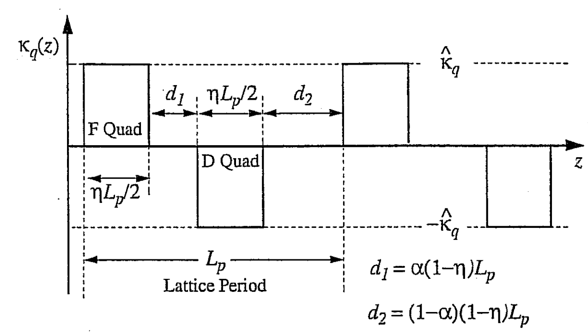
For our previous example for FODO quadrupoles, the $\varepsilon \to 0$ limit obtains:

$$\lim_{\varepsilon \to 0} \mathrm{Max}[r_{xm}] = r_b \left\{ 1 + \frac{4|\hat{\kappa}|L^2}{\pi^3} \sin\left(\frac{\pi}{2}\eta\right) \right\}$$

$$\lim_{\varepsilon \to 0} Q = Q_{max} = \frac{2}{\pi^2} \left[\frac{\sin\left(\frac{\pi}{2}\eta\right)^2}{\left(\frac{\pi}{2}\eta\right)} \right] \eta^2 \hat{\kappa}^2 L^2 r_b^2$$

Unfortunately, the method introduced before are inadequate for lattices with lesser degrees of symmetry such as syncopated quadrupole doublet lattices. However, methods introduced by Lee [E. P. Lee, Physics of Plasmas 9, 4301 (2002)] can be applied in this situation and also obtain more accurate results. It is beyond the scope of this class to carry out derivations with these methods, but we summarize results derived.

Quadrupole Doublet Lattice



Denote:

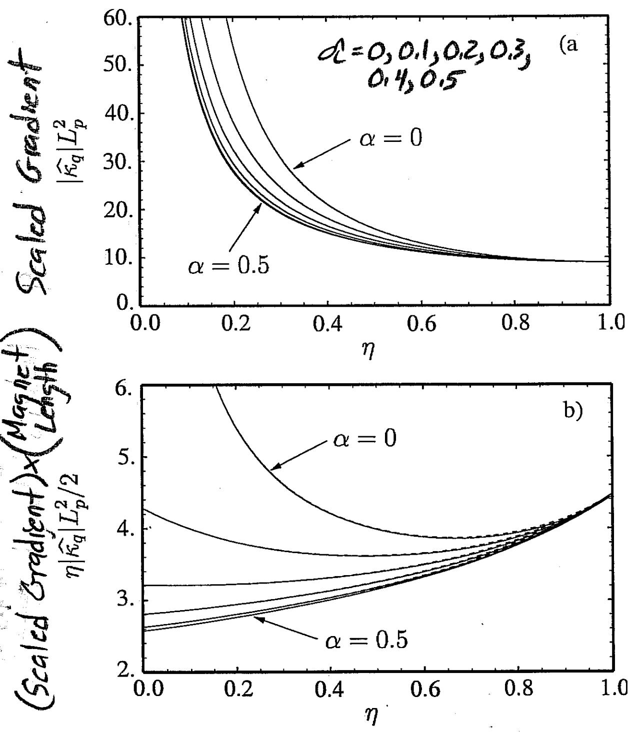
Avg Radius: $\overline{r_m} = \int_0^{L_p} \frac{ds}{L_p}\, r_{xm}(s) = \int_0^{L_p} \frac{ds}{L_p}\, r_{ym}(s)$

Max Excursion in period: $\text{Max}[r_m] \equiv \text{Max}[r_{xm}, r_{ym}]$

Phase Advance

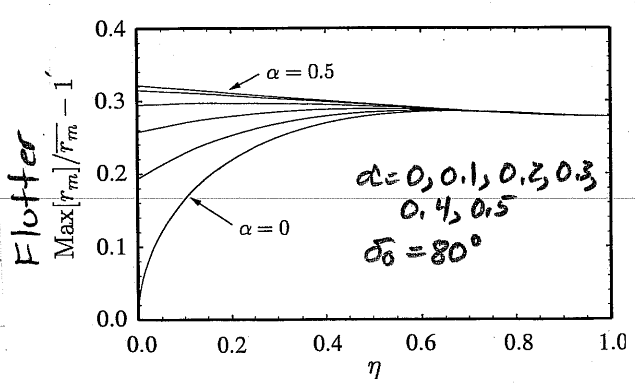
$$\cos\sigma_0 = 1 - \frac{(\eta\widehat{\kappa_q}L_p^2)^2}{32}\left[\left(1-\frac{2}{3}\eta\right) - 4\left(\alpha-\frac{1}{2}\right)^2(1-\eta)^2\right].$$

For $\sigma_0 = 80^\circ$



Envelope Flutter

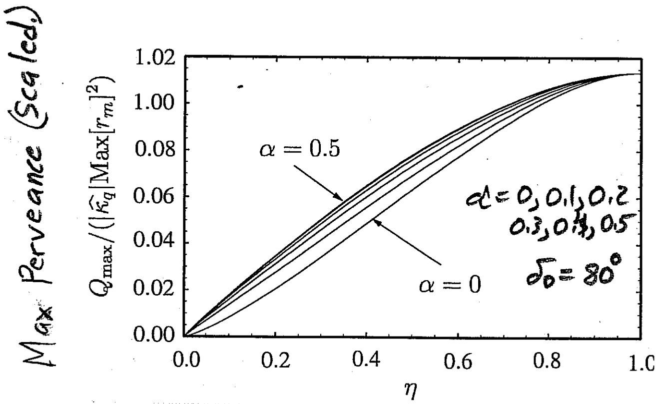
$$\frac{\mathrm{Max}[r_m]}{\overline{r_m}} - 1 = \frac{(1-\cos\sigma_0)^{1/2}(1-\eta/2)[1-4(\alpha-1/2)^2(1-\eta)^2]}{2^{3/2}\left[(1-2\eta/3)-4(\alpha-1/2)^2(1-\eta)^2\right]^{1/2}}.$$



Relations Connecting Max Transportable Perveance Q_{max} and Lattice Parameters

$$
\begin{aligned}
Q_{\max} &= \frac{(1-\cos\sigma_0)^{1/2}}{2^{3/2}} \frac{\eta[(1-2\eta/3) - 4(\alpha-1/2)^2(1-\eta)^2]^{1/2}}{(\mathrm{Max}[r_m]/\overline{r_m})^2} |\widehat{\kappa_q}|\,\mathrm{Max}[r_m]^2 \\
&= \frac{(1-\cos\sigma_0)^{1/2}}{2^{3/2}} \frac{\eta[(1-2\eta/3) - 4(\alpha-1/2)^2(1-\eta)^2]^{1/2}}{\left\{1+\frac{(1-\cos\sigma_0)^{1/2}(1-\eta/2)[1-4(\alpha-1/2)^2(1-\eta)^2]}{2^{3/2}[(1-2\eta/3)-4(\alpha-1/2)^2(1-\eta)^2]^{1/2}}\right\}^2} |\widehat{\kappa_q}|\,\mathrm{Max}[r_m]^2 .
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{Max}[r_m]}{L_p} &= \sqrt{\frac{Q_{\max}}{2(1-\cos\sigma_0)}}\left(\frac{\mathrm{Max}[r_m]}{\overline{r_m}}\right) \\
&= \sqrt{\frac{Q_{\max}}{2(1-\cos\sigma_0)}}\left\{1+\frac{(1-\cos\sigma_0)^{1/2}(1-\eta/2)[1-4(\alpha-1/2)^2(1-\eta)^2]}{2^{3/2}\,[(1-2\eta/3)-4(\alpha-1/2)^2(1-\eta)^2]^{1/2}}\right\},
\end{aligned}
$$



Derivation and application of scaling relations can be complicated. They are often applied in systems codes to generate plots that can be interpreted more readily.

Further Information

Seminal work on envelope modes:

J. Struckmeier and M. Reiser, *Theoretical Studies of Envelope Oscillations and Instabilities of Mismatched Intense Charged-Particle Beams in Periodic Focusing Channels*, Part. Accel. **14,** 227 (1984)

see also

M. Reiser, *Theory and Design of Charged Particle Beams* (John Wiley, 1994)

Extensive paper detailing envelope modes and related material:

S. M. Lund and B. Bukh, *Stability Properties of the KV Envelope Equations Describing Intense Ion Beam Transport* (to appear, PRSTAB, 2004)
Vol. 7 024801 2004

Useful material on symmetries and phase-amplitude methods:

A. Dragt, *Lectures on Nonlinear Orbit Dynamics* in *Physics of High Energy Particle Accelerators,* (American Institute of Physics, 1982), AIP Conf. Proc. No. 87, p. 147

E. D. Courant and H. S. Snyder, *Theory of the Alternating-Gradient Synchrotron*, Annals of Physics **3**, 1 (1958)