# Chapter 1 - Introduction and Basics

Units and Beam Rigidity

Beam Parameters

Emittance

Maxwell's Equations

## Computers and Accelerators

The first accelerators, through the early 70's were designed mainly “by hand”, using slide rules and analytic methods. Now, computers are essential and allow very precise predictions and analysis of the operating characteristics of accelerators.

**Beam Dynamics**

Number of particles is now $10^8$ or more, in some cases equal to the actual number of particles in a bunch. Clusters of nodes with teraflop capability are now available to predict beam halo and chaotic behavior.

**Electrodynamics**

Many 3-D electromagnetic solvers now exist with the ability to reproduce the boundary conditions of complex structures over many scales. Accurate multipactor and wakefield calculations, electron-cloud simulations now possible.

**Mechanical and Electrical Engineering**

CAD layout, time-dependent thermal modeling, structure and vacuum modeling

**Reference Material Database**

Accelerator conferences, texts, monographs

**BUT** computers are a tool, not an end in designing and diagnosing an accelerator. This course will attempt to integrate the use of computers with an understanding of the underlying physics principles of accelerators.

# Units

We will use MKS units in this course.

meter
second
kilogram

volt
ampere
ohm
electron-volt

Electrical units can be confusing.   We will break down some of the units (farad, henry, tesla) and see what's inside them.

# Let's Look at electrical units, with some memory aids

Start with a simple equation that uses that quantity.

$$Q = \boldsymbol{C}\, V \qquad [\,farad\,] = \frac{coulomb}{volt} = \frac{amp\cdot sec}{volt} = \frac{sec}{ohm}$$

$$V = \boldsymbol{L}\, \dot{I} \qquad [\,henry\,] = \frac{volt\cdot sec}{amp} = ohm\cdot sec$$

$$\dot{\boldsymbol{B}}\, A = V \qquad [\,tesla\,] = \frac{volt\cdot sec}{m^2}$$

$$V = I\, \boldsymbol{R} \qquad [\,ohm\,] = \frac{volt}{amp}$$

$$\nabla\times\boldsymbol{H} = J \qquad [\,H\,] = \frac{amp}{m}$$

$$B = \boldsymbol{\mu_0} H \qquad [\mu_0] = \frac{henries}{meter} = \frac{B}{H} = \frac{volt\cdot sec}{amp\cdot m} = \frac{ohm\cdot sec}{m}$$

$$\mu_0\cdot\boldsymbol{\epsilon_0} = \frac{1}{c^2} \qquad [\epsilon_0] = \frac{farads}{meter} = \frac{amp\cdot sec}{volt\cdot m}$$

$$\boldsymbol{Z_0} = \sqrt{\frac{\mu_0}{\epsilon_0}} \qquad [\,Z_0\,] = \sqrt{\frac{volt\cdot sec}{amp\cdot m}\cdot\frac{volt\cdot m}{amp\cdot sec}} = \sqrt{\frac{volt^2}{amp^2}} = ohm$$

$$\mu_0 = 4\pi\times 10^{-7}\, H/m$$
$$\epsilon_0 = 8.854\times 10^{-12}\, F/m$$

*Carry units along in your calculations. It can help catch errors.*

# A Practical Example

I came across the following equation the other day (frequency dependence of single-point multipactoring), where *f* is frequency, *B* is magnetic field and *m* is the mass of the electron.

$$\frac{f}{N} = \frac{e\,B_0}{2\pi\,m}$$

Instead of having to look up the physical constants, and knowing that the units of magnetic field *B* are volt-seconds/meter$^2$, I recast the equation as:

$$\frac{f}{N} = \left(\frac{e}{mc^2}\right)\frac{c^2\,B_0}{2\pi} \qquad \left[\frac{e}{e-volt}\;\frac{m^2}{sec^2}\;\frac{volt-sec}{m^2}\right]$$

The units of the ($e/mc^2$) are 1/volt, and of $c^2\ B_0$ are volt/second, and I know that the value of ($e/mc^2$) is 1/(511000 volts), so the units come out okay (sec$^{-1}$) and I don't have to look up the mass of the electron or its charge.

*f/N* = 2.8x10$^{10}$. for $B_0$ = 1 Tesla. If you use *e* and *m*, e = 1.6x10$^{-19}$ coul, and $m_e$ = 9.11x10$^{-31}$ kg, and you get the same answer, but I had to look it up.

# Particle Kinematics and Dynamics

Most of the time for the linacs we will study, the beam is non-relativistic. This simplifies the dynamical equations slightly. Often, we will use NR analysis, but let's develop the relativistic equations that will apply in higher-energy accelerators.

Normalized velocity $\beta = \frac{v}{c}$

Relativistic factor

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}, \qquad \beta = \frac{\sqrt{\gamma^2-1}}{\gamma} = \frac{\sqrt{(\gamma-1)(\gamma+1)}}{\gamma}$$

Kinetic energy of a particle $KE = (\gamma-1)\cdot mc^2$

Total energy of a particle $E_{tot} = \gamma\cdot mc^2$

Important: we will express energy in units of electron-volts. The rest mass of a proton is approximately 938x$10^6$ e-V or 938 MeV, expressed in energy units $mc^2$.

In the non-relativistic limit, the normalized velocity is $\beta = \sqrt{\frac{2\,KE}{mc^2}}$

# Beam Rigidity

When a beam interacts with magnetic fields, it is useful to define the rigidity of the beam, which is proportional to its momentum.

Momentum is defined as $p = \beta \gamma m c$

It is useful to define momentum in energy units. Often, momentum is displayed with the units MeV/c. $p = \dfrac{\beta \gamma m c^2}{c}, \quad pc = \beta \gamma m c^2$

In the extreme relativistic case (electrons) $\beta \to 1, \quad \gamma \gg 1, \quad pc \sim KE$

The beam rigidity is defined as $R \equiv \dfrac{A}{q} \dfrac{\beta \gamma}{c} \left( \dfrac{m_p c^2}{e} \right)$

Why is this useful? In a magnetic field B, the beam direction is deflected by the angle θ

$B L = R \theta$ where *L* is the path length of the orbit in the magnet

Some new things here:
A is the mass of the particle in units of atomic mass units, = 1/12 mass of $C^{12}$
q is the charge of the particle in units of one electron charge

Note also in the parentheses that the unit of $\left( \dfrac{m_p c^2}{e} \right)$ is volts

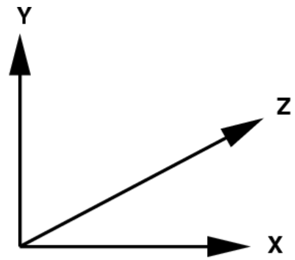
Thus, the unit of rigidity is volt-sec/m. However the unit of magnetic field B is Tesla = volt-sec/$m^2$, so the units of rigidity are also Tesla-meters.

## Homework Problems 1.1

Familiarize with computer, find spread sheet

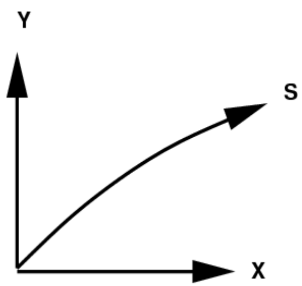
Calculate relativistic factors
Rigidity
Beam in a magnet
Beam deflection

# Coordinate Systems




For a linear trajectory, a rectangular coordinate system is used.

The beam propagates along the z-axis.



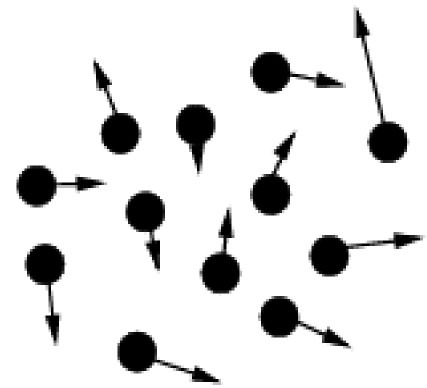

Often, the beam follows a curved trajectory in space.

The instantaneous coordinate the central beam trajectory follows is s.   x and y are perpendicular to s.

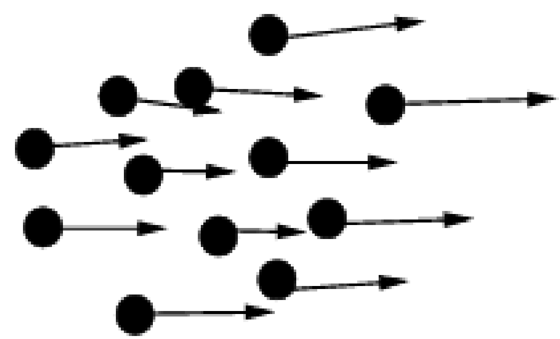
We will often jump between s and z as the longitudinal coordinate.

# Ensembles of Particles

A group of particles, with some thermal energy.



A group of particles, with coherent motion in one direction, and some randomness.



# Particle Coordinates

Each particle in the ensemble can be described with six coordinates

$$x,\ x',\ y,\ y',\ z,\ z'$$

*x, y* and *z* represent the coordinates of the particle with respect to a reference particle.

*x', y'* and *z'* represent derivatives wrt s, the longitudinal trajectory.

The reference particle may exist, as in simulations, or the values may represent deviations from the average of all particles in the ensemble.

The six coordinates representing a particle may be represented by a vector **P**.

$$P = \begin{bmatrix} x \\ x' \\ y \\ y' \\ z \\ z' \end{bmatrix}$$

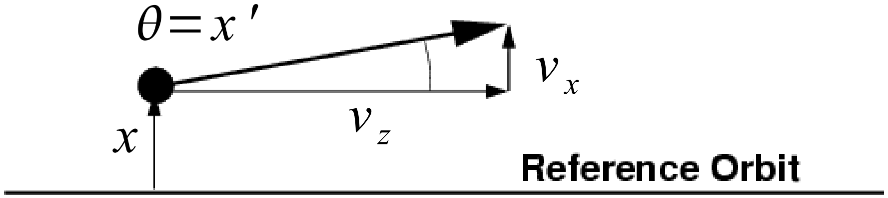
Instead of using *z*', we will use the fractional momentum error to represent motion along the trajectory coordinate.

*x', y'* and *dp/p* are unitless quantities.

$$P = \begin{bmatrix} x \\ x' \\ y \\ y' \\ z \\ \delta p / p \end{bmatrix}$$

# The Paraxial Approximation

A velocity of the particle may not be along the reference orbit *s*.
We may approximate the velocity as the component along *s* and define $x'$, $y'$ of the particle as the (small) angle the trajectory makes relative to the reference orbit.

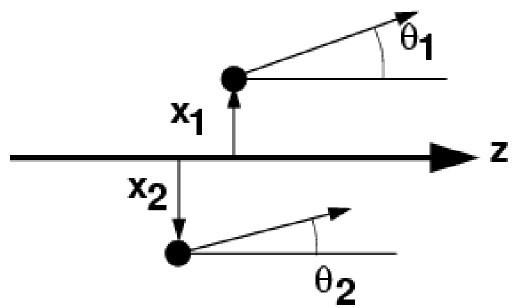


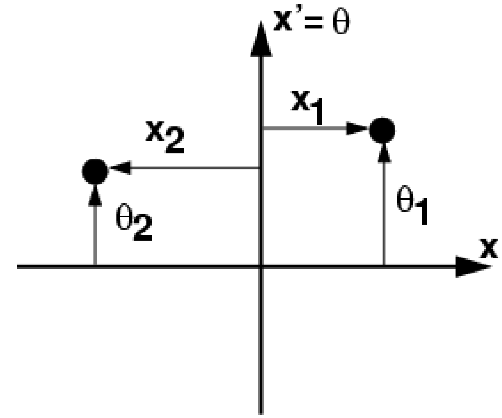

The beam orbit is expanded in a truncated series, with the transverse velocity vector small compared to the longitudinal velocity.

The first-order paraxial approximation is equivalent to the approximations $\sin(x') = x'$, $\cos(x') = 1$, $x'$ the angle the trajectory makes to the reference orbit.

# Phase Space

Let us consider just two coordinates of a particle: x and x' (or y and y').




Each 2-dimensional projection of the particle, x and x', can be represented as a point in x-x' phase space.

We will first assume that the three projections (x-x'), (y-y') and (z-dp/p) are uncorrelated. This is a good approximation for systems dealing with linacs, except where solenoidal magnetic fields are found, such as in low-energy beam transport systems from the ion source to the linac entrance. We can treat that special case separately.

# Emittance

The phase-space representation of a beam can be useful, but also misleading. First, let's calculate the statistics of each 2-dimensional projection. These will be the 2-dimensional emittances. Assume N particles we will sum over:

First moments

$$\bar{x} = \frac{1}{N}\sum x_i \qquad \bar{x}' = \frac{1}{N}\sum x_i'$$

Second moments

$$\overline{x^2} = \frac{1}{N}\sum x_i^2 \qquad \overline{x'^2} = \frac{1}{N}\sum x_i'^2$$

Cross-term

$$\overline{xx'} = \frac{1}{N}\sum x_i x_i'$$

Subtract off first moments, get standard deviations

$$\sigma_x^2 = \overline{x^2} - (\bar{x})^2$$

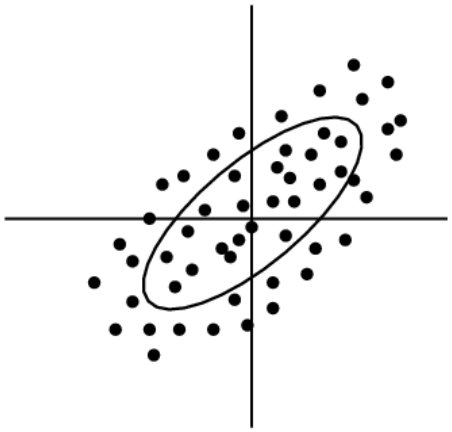
$$\sigma_{x'}^2 = \overline{x'^2} - (\bar{x}')^2$$

$$\sigma_{xx'} = \overline{xx'} - \bar{x}\,\bar{x}'$$

Definition of root-mean-square (rms) emittance

$$\epsilon = \sqrt{\sigma_x^2 \sigma_{x'}^2 - (\sigma_{xx'})^2}$$

The ellipse area drawn is an rms area, and many points will lie outside it.



# Canonical Variables, Normalized Emittance

Emittance is preserved in linear systems – meaning that external restoring force terms are linear and no non-linear fields are present in the beam acting on itself.

But what happens if the beam is accelerated?

The variables we have been using, (x-x'), etc, are convenient, but do not preserve the numerical value of the emittance if the beam energy (velocity) is changed.

The canonical variables $(x,p_x)$, $(y,p_y)$ variables do.

The emittance defined using (x-x') is called the *unnormalized* emittance.

The normalized emittance is preserved under acceleration and includes a normalization proportional to the momentum of the beam, effectively transforming the angle coordinate into a transverse momentum coordinate.

$$\epsilon_{normalized} = \beta \gamma \epsilon_{unnormalized}$$

The normalized emittance is useful in measuring the degree of beam blow-up in accelerators due to non-linear effects.

# Twiss Parameters

Also called betatron parameters. The emittance plot contains only the lowest-order (rms) statistics of the beam. The shape of the ellipse may be represented by three independent variables, the ellipse area, its tilt, and a projection along one axis.

$$\alpha \equiv -\frac{\sigma_{xx'}}{\epsilon}$$

$$\beta \equiv \frac{\sigma_x^2}{\epsilon}$$

A fourth parameter may be defined, but it is not an independent variable

$$\gamma \equiv \frac{\sigma_{x'}^2}{\epsilon}, \qquad \beta\gamma = 1 + \alpha^2$$

Note an unfortunate choice of representation. The beta and gamma used here are different from the relativistic parameters, and frequently both appear in the same equation.

The ellipse drawn in the phase space plot is represented by

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon$$

# RMS Beam Size from Twiss Parameters

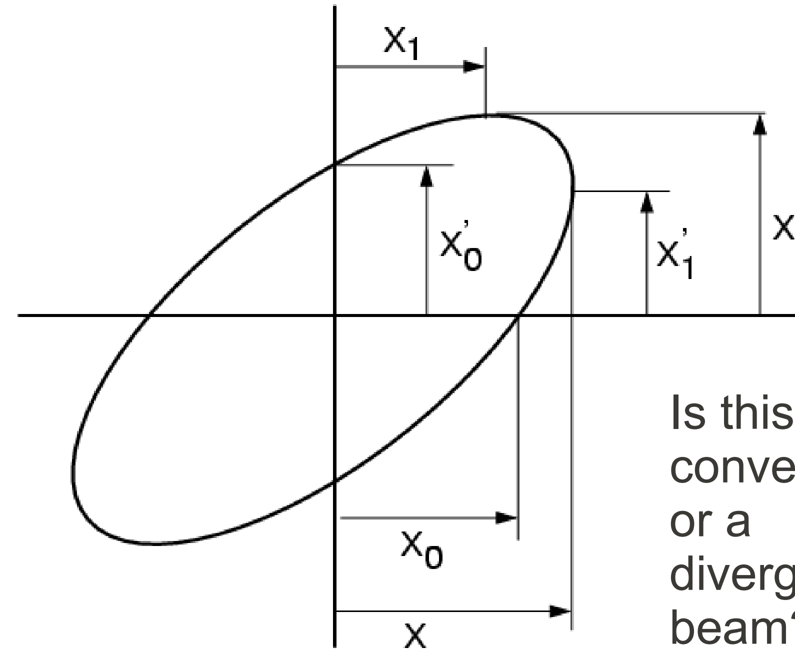
$$x = \sqrt{\beta\epsilon} \qquad x' = \sqrt{\gamma\epsilon}$$

$$x_0 = \sqrt{\frac{\epsilon}{\gamma}} \qquad x'_0 = \sqrt{\frac{\epsilon}{\beta}}$$

$$x_1 = -\alpha\sqrt{\frac{\epsilon}{\gamma}} \qquad x'_1 = -\alpha\sqrt{\frac{\epsilon}{\beta}}$$

$$\beta\gamma = 1+\alpha^2$$

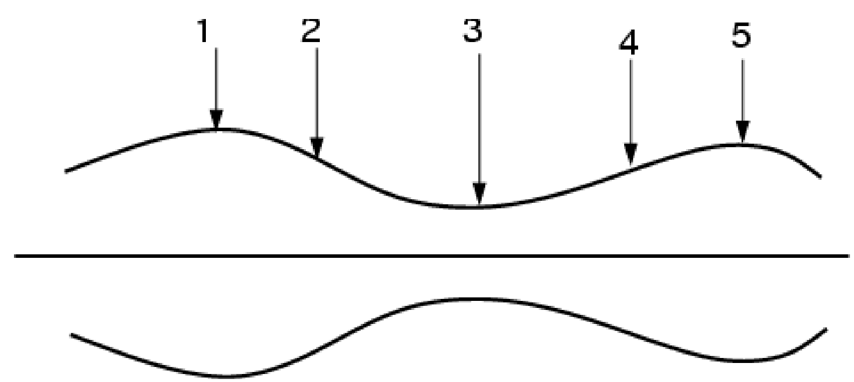
$$\alpha = -\frac{1}{2}\frac{d\beta}{ds}$$




Is this a converging or a diverging beam?

The beam envelope along z

1: waist, $\alpha = 0$
2: converging, $\alpha > 0$
3: waist, $\alpha = 0$
4: diverging, $\alpha < 0$
5: waist, $\alpha = 0$



# Transport Variables

The TRANSPORT code, one of the oldest, uses a different set of variables to specify beam correlations. Beam correlations correspond to the tilt of the beam ellipse.

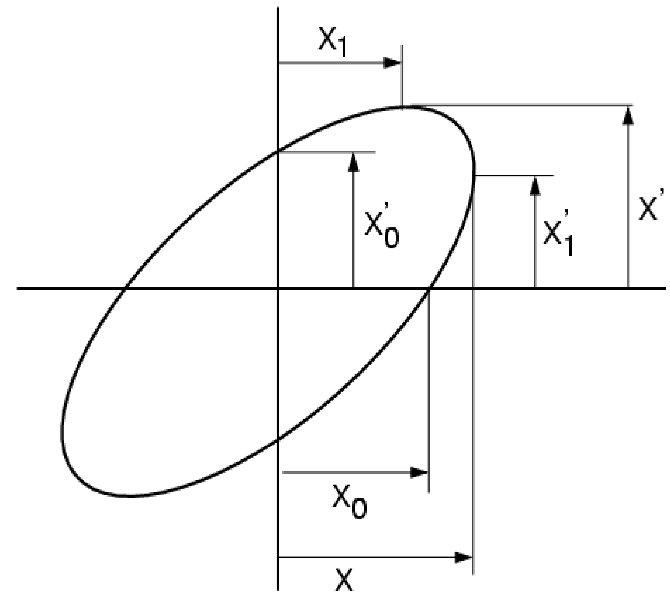
$$x = \sqrt{\beta\epsilon} \qquad x' = \sqrt{\gamma\epsilon}$$

$$x_0 = \sqrt{\frac{\epsilon}{\gamma}} \qquad x'_0 = \sqrt{\frac{\epsilon}{\beta}}$$

$$x_1 = -\alpha\sqrt{\frac{\epsilon}{\gamma}} \qquad x'_1 = -\alpha\sqrt{\frac{\epsilon}{\beta}}$$

$$\beta\gamma = 1+\alpha^2$$

$$\alpha = -\frac{1}{2}\frac{d\beta}{ds}$$ This is the correlation




Equivalent TRANSPORT variables

$$x = \sqrt{\beta_x\epsilon_x}, \qquad y = \sqrt{\beta_y\epsilon_y}$$

$$x' = \sqrt{\gamma_x\epsilon_x}, \qquad y' = \sqrt{\gamma_y\epsilon_y}$$

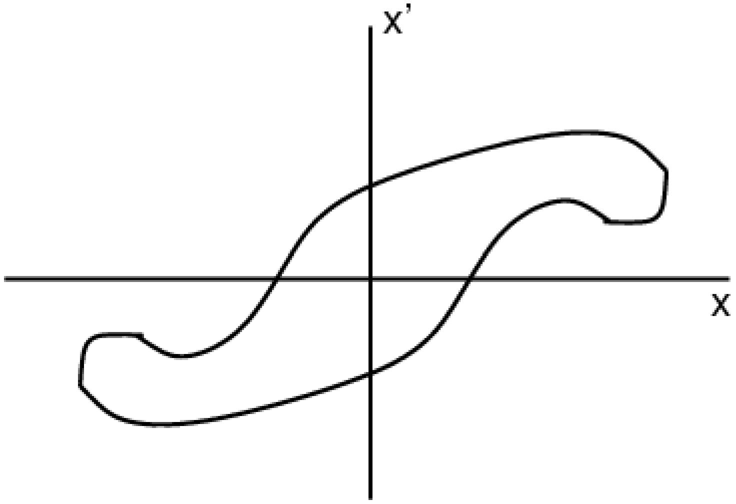
$$r_{12} = -\frac{\alpha_x}{\sqrt{1+\alpha_x^2}}, \qquad r_{34} = -\frac{\alpha_y}{\sqrt{1+\alpha_y^2}}$$

$$1 < r_{12}, r_{34} < +1$$

$$-\infty < \alpha_x, \alpha_y < +\infty$$

# Limitations of This Approach

What do we do with something like this? This beam shows a lot of spherical aberration, a common aberration.

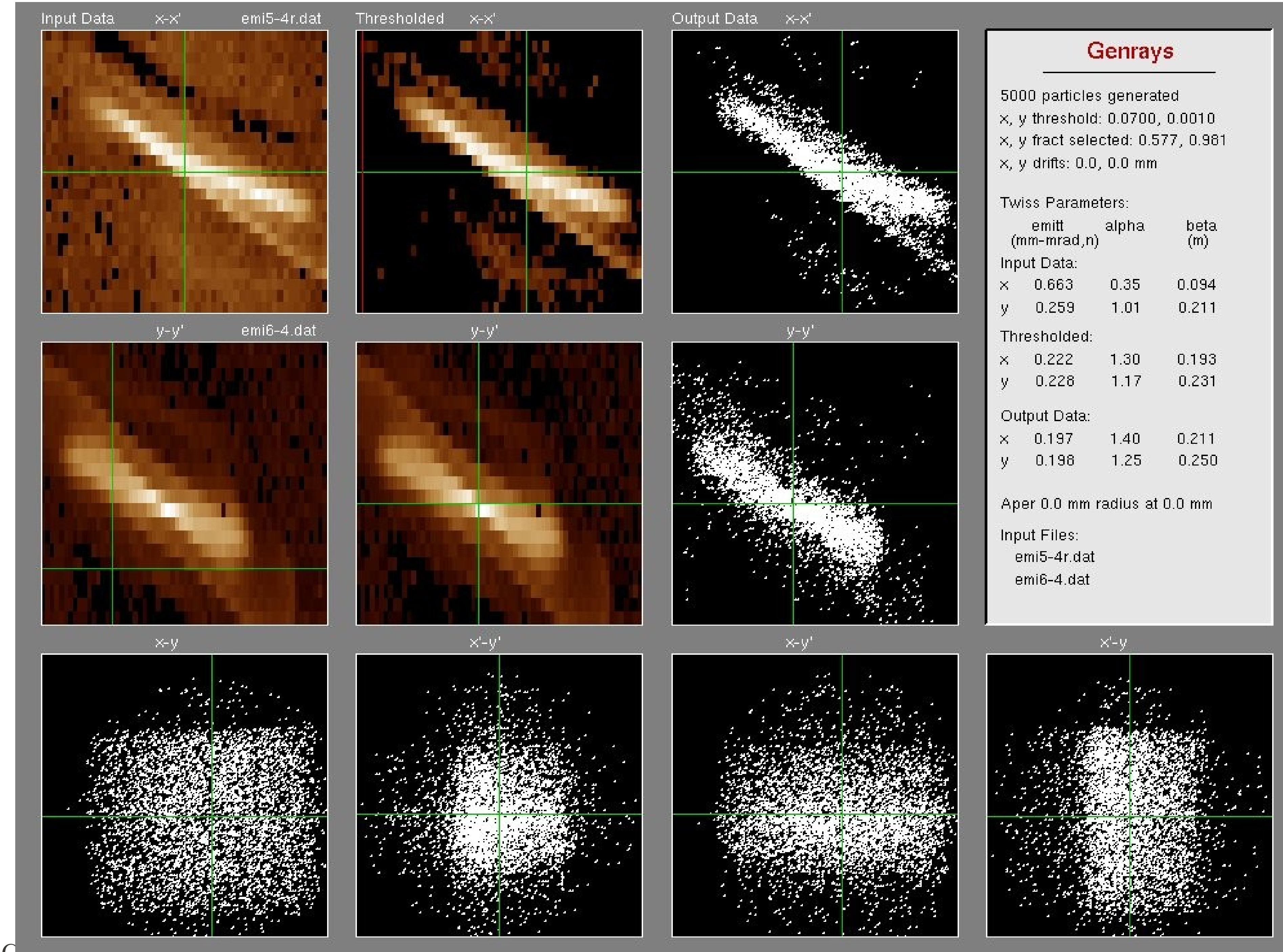



The first-order Twiss parameter formulation only accurately models beams with a regular distribution and an elliptical emittance boundary.

Real beams are far more complex in their distribution functions. However, beams, after many manipulations, tend to revert to gaussian distributions, and the simple formulation it not too bad a representation.

When beam losses become significant, as in high-power linacs, other approaches are used: ensembles of individual particles are tracked to observe the generation of beam halo and its loss on accelerator components.

# An example of a real emittance plot of an ion source.



## Homework Problems 1.2

Use a spreadsheet

Calculate the emittance of a small ensemble of particles
Calculate the Twiss parameters and rms beam parameters

Transport the ensemble a distance L = 10 meters downstream

Calculate the emittance, Twiss and rms parameters again

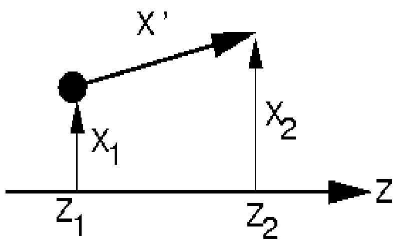
plot the ensemble at both locations.

# Transport Elements

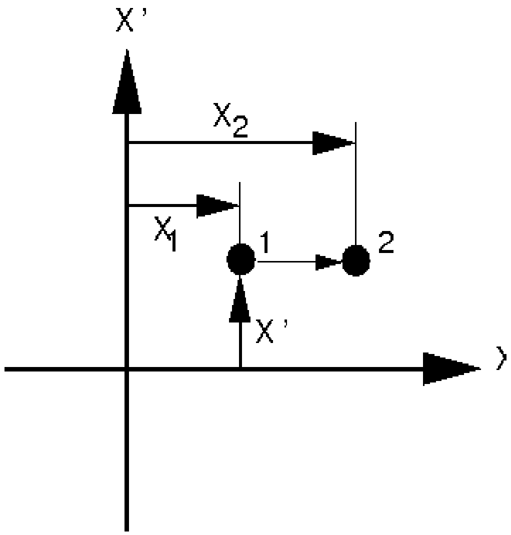
We will consider drift sections, focusing elements and dipole bends.

On the left, a particle moves from position $z_1$ to $z_2$ and its trajectory has the angle x'.

On the right is the phase space diagram of the particle at position $z_1$ and $z_2$. The coordinates of the phase space diagram are x and x'.




A particle with slope x' moves from $z_1$ to $z_2$.



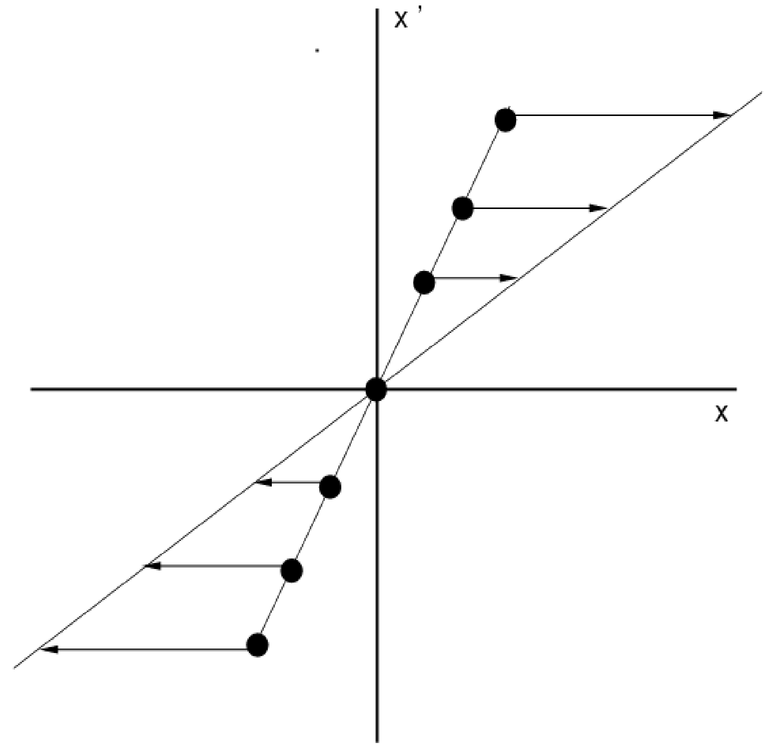

In phase space, the particle moves from point 1 to point 2. The angle x' to the axis remains constant, and the distance from the axis increases from $x_1$ to $x_2$.

# Displacement for Different Angles x'

Particles with larger values of angle x' move further away from their original position.

This shows the evolution of particles in a drift space.

The emittance of the beam is preserved.



# Vector Representation of a Particle

$$P = \begin{bmatrix} x \\ x' \\ y \\ y' \\ dz \\ dp/p \end{bmatrix}$$

One particle in an ensemble has 6 coordinates:

displacement and angle in the x-plane relative to a reference particle
displacement and angle in the y-plane relative to a reference particle

Longitudinal position dz relative to a reference particle
Difference in momentum relative to a reference particle

We will often look at just a 2-dimensional projection.

Here, we extract just the x and x' coordinates.

$$P_x = \begin{bmatrix} x \\ x' \end{bmatrix}$$

# Matrix Representation of a Drift

First-order beam transport elements are represented by N x N matrices: this matrix multiplies a particle vector, producing a new particle vector which has been acted on by the transport matrix.

2-D matrix for a drift of length *L*

$$M_{drift} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}$$

$$P_x^{final} = M_{drift} P_x^{initial} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix} = \begin{bmatrix} x + x' L \\ x' \end{bmatrix}$$

The displacement increment of the final vector depends linearly on the drift distance *L* and the (unchanged) angle *x′* to the trajectory vector.

Note that for zero drift length L=0, or zero angle x'=0 the vector is unchanged.

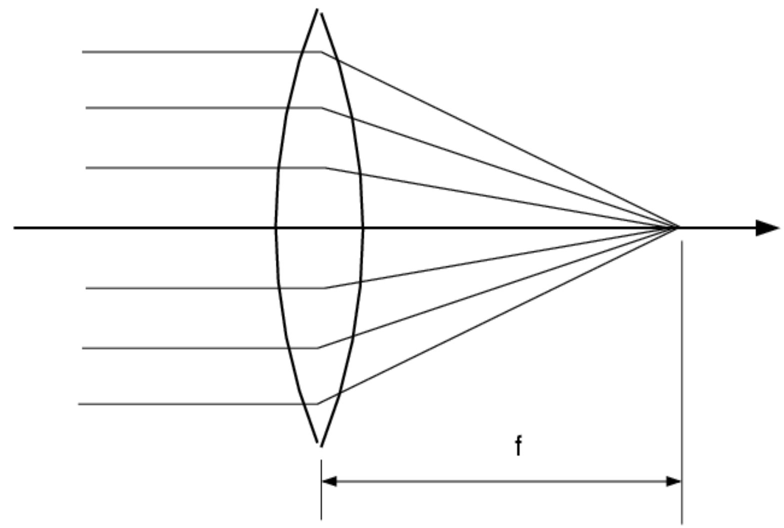
# Matrix Multiplication

Matrix multiplication is non-commutative, that is, the product depends on the order of the matrices. For a beam transport line with components that are represented by matrices *M1 M2 ... Mn*, the multiplication proceeds from the first to the last beam line element from the right to the left.

$$M_{tot} = M_n M_{n-1} M_{n-2} \cdots M_2 M_1$$

# Linear Lens

The angle a ray is deflected is proportional to the displacement from the axis.




For incoming rays parallel to the trajectory axis, the focal length of the lens is the distance to the crossing point of the axis.

$$M_{lens} = \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{bmatrix}$$

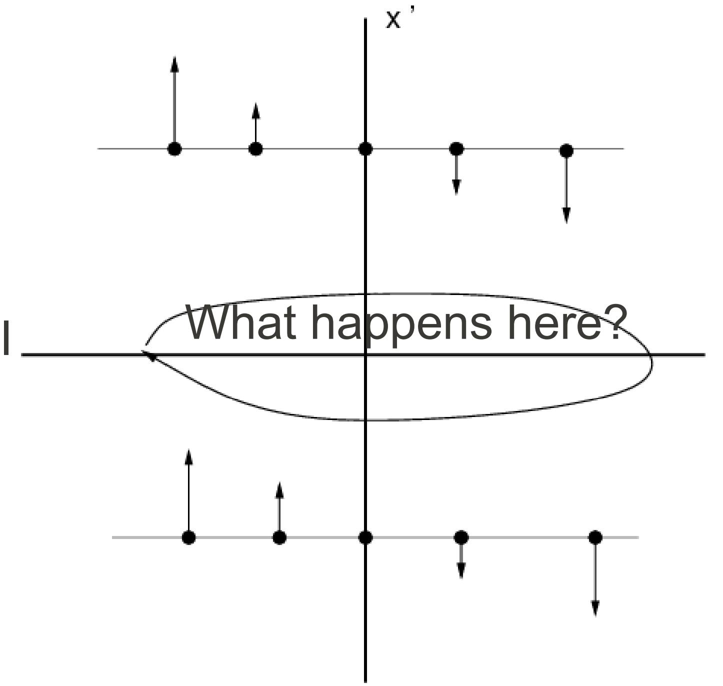
Proof: follow a lens with focal length *f* by a drift of length *f*. The incoming beam is parallel to the axis.

$$P_x^{final} = \underbrace{\begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix}}_{\text{drift}} \underbrace{\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}}_{\text{lens}} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & f \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -x/f \end{bmatrix}$$

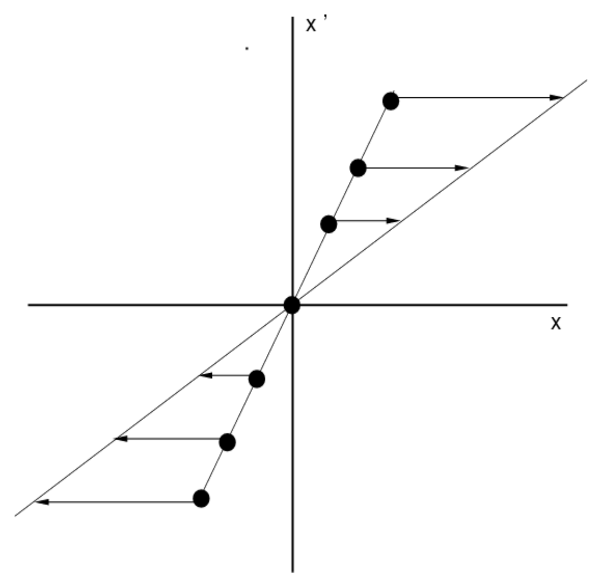
The displacement of the beam at distance f is zero, and the each ray is given an angle -x/f, proportional to the initial displacement.

# Phase-Space Representation of Linear Lens

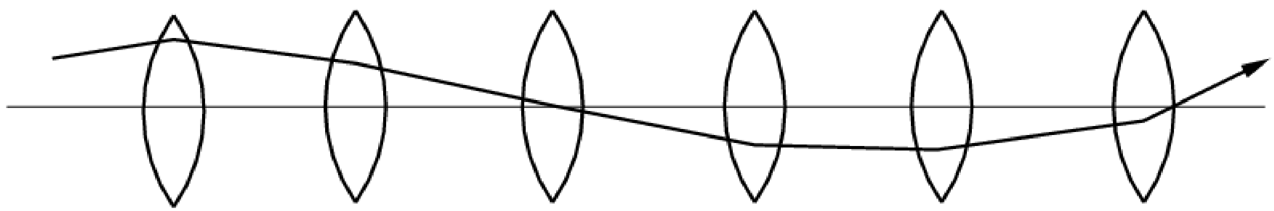
The change in angle is proportional
to the distance from the axis.



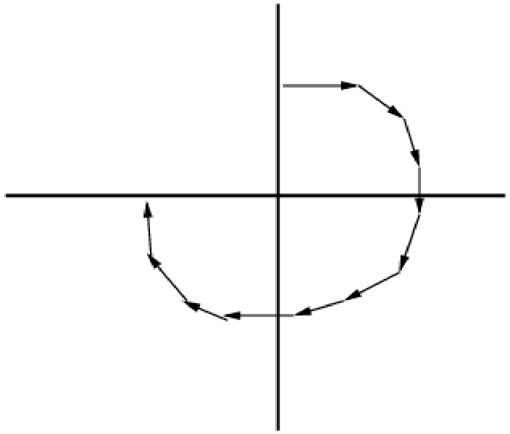

Recall that for a drift, the change
displacement is proportional
to the angle.



# Sequences of Lenses and Drifts

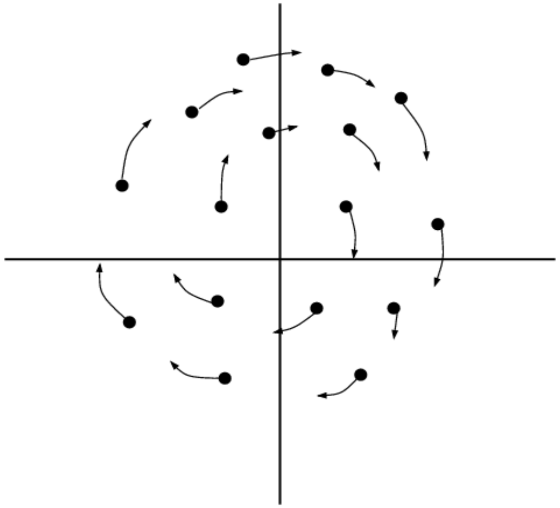


The ray traces out an orbit in phase space. For a regular sequence, the orbit has a period, or tune.



If the orbit repeats after N focusing periods, it has a tune of 1/N per focusing period.

The phase advance per focusing period is $2\pi/N$.



In this system of lenses, the emittance of an ensemble of particles is preserved.

# Symplectic Condition

The determinant of non-accelerating first-order transport matrices is unity. This has the property of preserving the emittance area of the beam. This is a fundamental property of linear transport systems.

$$\det M = 1$$

This is why the Twiss parameters have a deeper meaning than the transport parameters (x, x', etc), which do not contain conserved properties.

# Homework Problems 1.3

Take initial ensemble from Homework2, transport 10 meters, pass through a lens with focal length of 5 meters, and then transport another 5 meters.

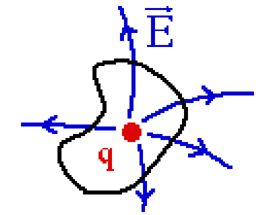
Calculate all the beam parameters right after the lens, and after the second drift. Plot the beam particles.

# Maxwell's Equations

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}$$

$$\oiint \vec{E} \bullet \hat{\mathbf{n}}\, dS = \frac{q}{\varepsilon_o}$$

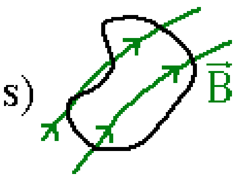
Gauss's Law




$$\nabla \cdot B = 0$$

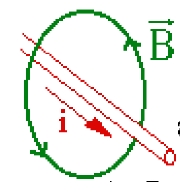
$$\oiint \vec{B} \bullet \hat{\mathbf{n}}\, dS = 0$$

(no monopoles)




$$\nabla \times B = \mu_0 (J + \epsilon_0 \dot{E})$$

$$\oint \vec{B} \bullet d\vec{l} = \mu_\circ \left( i + \varepsilon_\circ \frac{d}{dt} \Phi_E \right)$$

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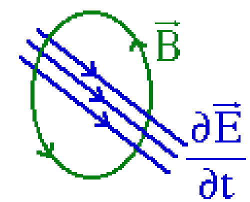
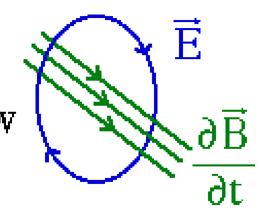



Ampère's Law

$$\nabla \times E = -\dot{B}$$

$$\oint \vec{E} \bullet d\vec{l} = -\frac{d}{dt} \Phi_B$$

Faraday's Law




$$\vec{\nabla} \bullet \vec{E} = \frac{\rho}{\varepsilon_o} \qquad \vec{\nabla} \times \vec{B} = \mu_\circ \left( \vec{j} + \varepsilon_\circ \frac{\partial \vec{E}}{\partial t} \right)$$

$$\vec{\nabla} \bullet \vec{B} = 0 \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

(Differential Forms)

$$B = \mu_0 H, \quad D = \epsilon_0 E$$

# Maxwell's Equations in MKS units.

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot B = 0$$

Why does this equation = 0?

$$\nabla \times E = -\frac{dB}{dt}$$

$$\nabla \times B = \mu_0 \left( J + \epsilon_0 \frac{dE}{dt} \right)$$

$$\mu_0 = 4\pi \cdot 10^{-7}\ Henries/meter$$
$$\epsilon_0 = 8.85 \cdot 10^{-12}\ Farads/meter$$

$$D = \epsilon_0 E + P$$
$$B = \mu_0 H + M$$

$$Force = q(E + v \times B)$$

on a particle of charge *q* moving at velocity *v*

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377\,\Omega$$

Impedance of free space. (**What does this mean?**)

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

# Wave Equation in Free Space

Let's determine the relationship between the electric and magnetic field vector in free space.

$$\nabla \times E = -\frac{dB}{dt}$$

$$\nabla \times (\nabla \times E) = -\frac{d}{dt}(\nabla \times B) = \frac{1}{c^2}\frac{dE}{dt}$$

For a plane wave in x

Remember: $\nabla \times H = \dot{D} \rightarrow \nabla \times B = \frac{1}{c^2}\dot{E}$

$$\frac{d^2 E_x}{dz^2} = \frac{1}{c^2}\frac{d^2 E_x}{dt^2}$$

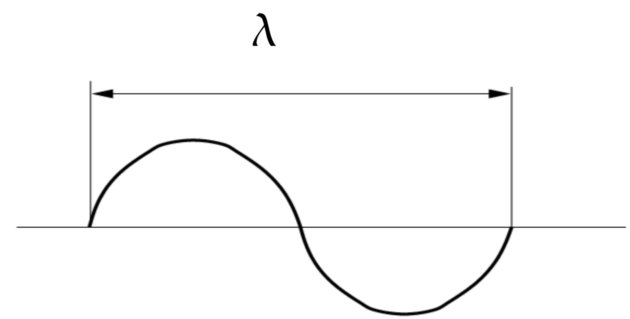
And a similar equation for *B*. These equations can be solved by

$$E_x = E_0 e^{i(kz-\omega t)} \qquad \dot{E}_x = -i\omega E_x \qquad \ddot{E}_x = -\omega^2 E_x$$

$$B_y = B_0 e^{i(kz-\omega t)} \qquad \dot{B}_y = -i\omega B_y \qquad \ddot{B}_y = -\omega^2 B_y$$

# Plane Wave

Define a wavenumber *k,* increases by $2\pi$ for each wave of length $\lambda$.




$$k = \frac{2\pi}{\lambda}, \qquad f\lambda = c, \qquad \frac{1}{\lambda} = \frac{f}{c} = \frac{\omega}{2\pi c}$$

We can rewrite the plane wave as

$$E_x = E_0 e^{i(kz - \omega t)} = E_0 e^{i\left(\frac{\omega}{c} z - \omega t\right)} = E_0 e^{2\pi i\left(\frac{z}{\lambda} - \frac{t}{\tau}\right)} \qquad \tau = \frac{1}{f}$$

For a constant time t, moving along z gives one oscillation period per wavelength $\lambda$.

Sitting at a particular location z, one oscillation occurs for every period t = $\tau$.

One can ride along a particular phase at velocity c (in the lab) as time progresses.

# Ratio of *E* to *H* in a plane wave

Back to Maxwell's equations $\dot{B}_y = -\nabla \times E = \frac{dE}{dz}$

$$E_x = E_0 e^{i\left(\frac{\omega}{c} z - \omega t\right)} \qquad \frac{dE_x}{dz} = i \frac{\omega}{c} E_x$$

$$B_y = B_0 e^{i\left(\frac{\omega}{c} z - \omega t\right)} \qquad \dot{B}_y = -i \omega B_y$$

$$-i \omega B_y = i \frac{\omega}{c} E_x$$

$$B = \mu_0 H$$

$$\frac{E}{H} = \mu_0 c = Z_0$$

The ratio of *E* to *H* fields in free space is $Z_o$, the free-space impedance.
The units are an indication:

volts/meter / amps/meter = volts/amps = ohms

# The Poynting Vector

The Poynting vector represents the power per unit area of an electromagnetic wave.

$$S = E \times H$$

with the SI units of watts/m$^2$. If the E and H fields are sinusoidal, then the time-averaged power per unit area is

$$S_{avg} = \frac{1}{2} E_0 \cdot H_0$$

where *E* and *H* are the peak electric and magnetic field intensity of the plane wave. As the ratio of *E* to *H* in a plane wave are related by $Z_0$,

$$S_{avg} = \frac{1}{2\mu_0 c} E_0^2 = \frac{E_0^2}{2 Z_0} = \frac{\epsilon_0 c}{2} E_0^2$$