Chapter 3 Beam Dynamics II - Longitudinal

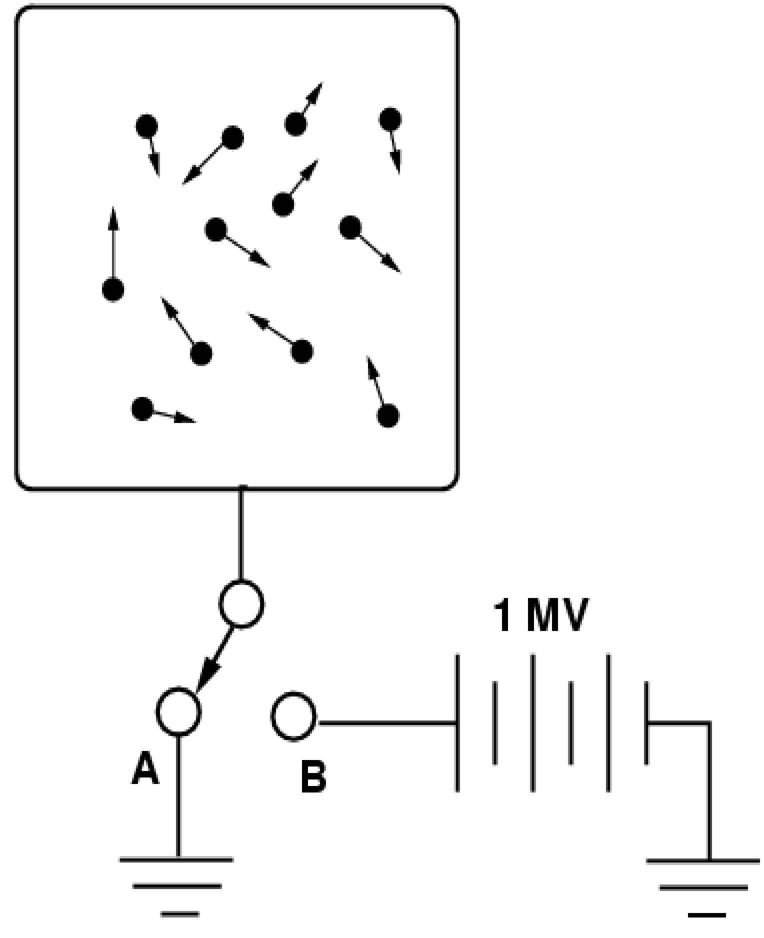
Sequences of Gaps

Transit Time Factor

Phase Stability

The Faraday Cage

Protons are confined in a conducting box, at low energy. Assume they can bounce off the walls with no energy loss.

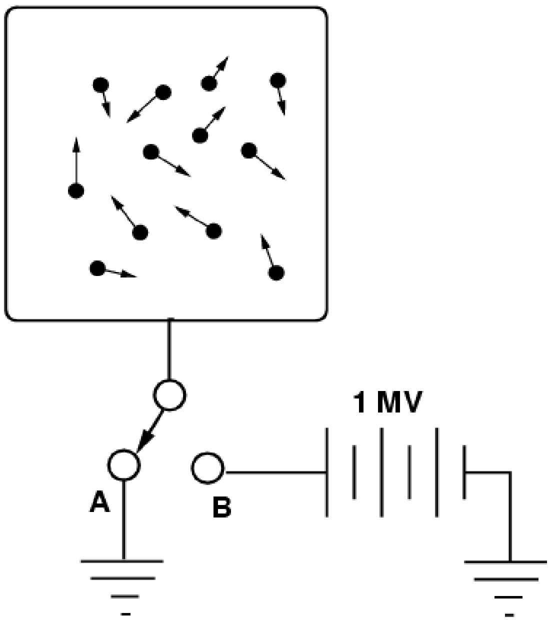


Move the switch from position A to B. The potential on the box rises from 0 to 1 MV.

What is the proton energy now?

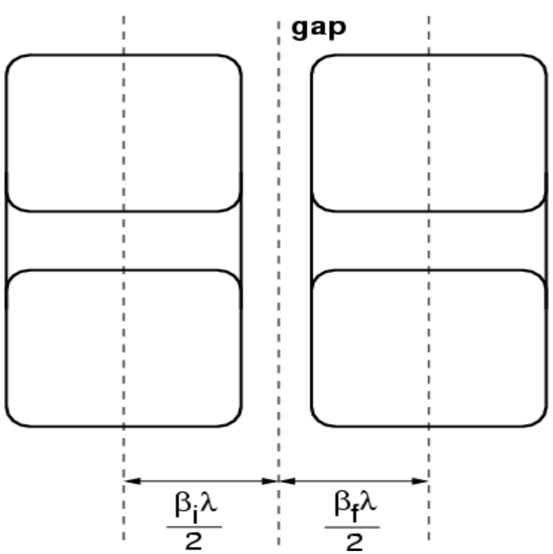
The Linac Drift Tube

A linear accelerator (linac) is comprised of a succession of drift tubes. These drift tubes have holes in their ends so the particles can enter and exit, and when particles are inside the drift tube, a Faraday Cage, the potential of the drift tube may vary without changing the energy of the particle.



Acceleration takes place when a charged particle is subjected to a **field**. The field inside the Faraday cage is not affected by the **potential** outside.

(Aside from fields generated by the protons themselves, the field inside the Faraday cage is zero.)



The drift tubes are arranged in a sequence with a passage through their middle for the particles to pass. The field in the gap between the drift tubes accelerates the particles.

Some Actual Linac Configurations

We will look at:

- Sloan-Lawrence Structure (Ising, Wideroe)
- Alvarez Structure
- RFQ Structure
- Coupled-Cavity Structure

Some Kinematics

For simplicity, we will assume the particles are non-relativistic. The normalized velocity is

$$\beta = \sqrt{\frac{2T}{mc^2}}$$

T is the kinetic energy of the particle, mc^2 is the rest mass, 938 MeV for protons.

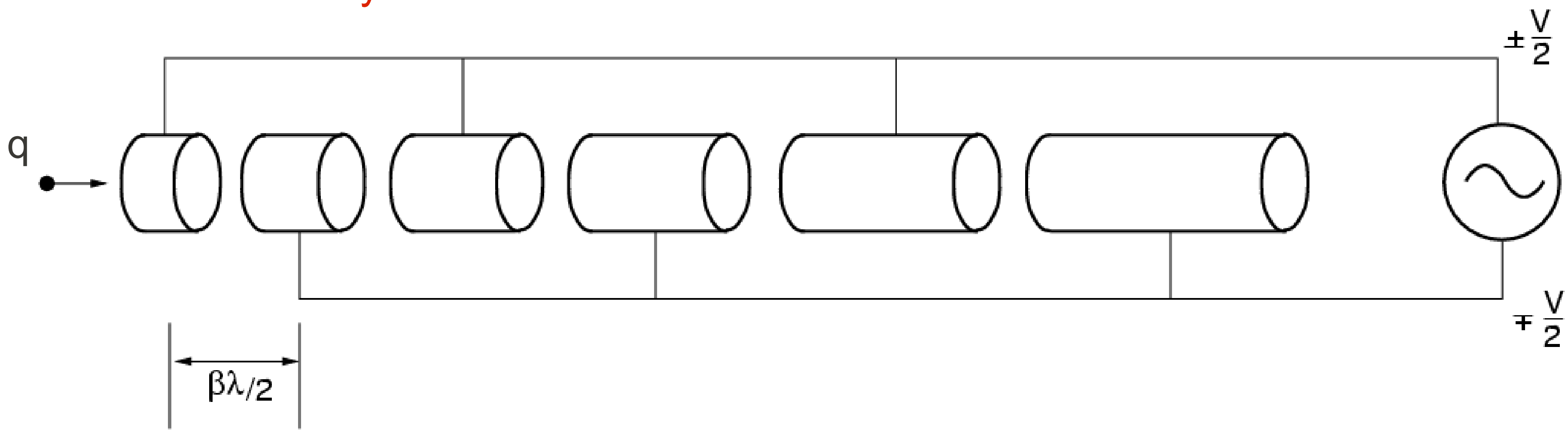
The wavelength λ of an RF frequency f is $\lambda = \dfrac{c}{f}$

For a particle traveling with velocity β, the distance Δz traveled in one period of the frequency f is $\beta\lambda$.

$$\Delta z = \beta c \times t = \beta c \times \frac{1}{f} = \beta c \times \frac{\lambda}{c} = \beta \lambda$$

The Sloan-Lawrence Structure

This is the conceptually the easiest to understand, and the oldest. It is also the basis of the cyclotron.



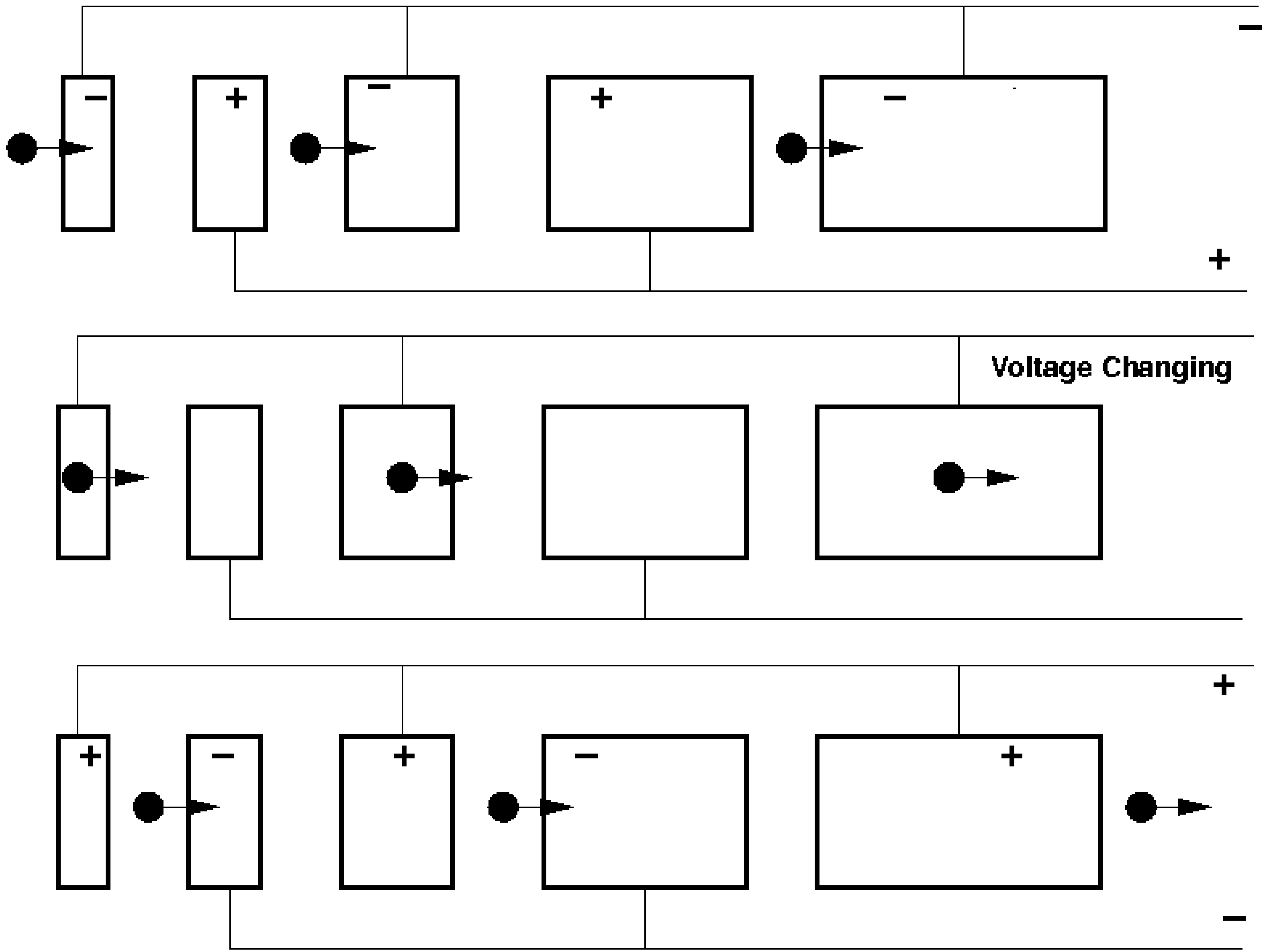
Alternate drift tubes are connected to a voltage generator providing a peak-to-peak voltage V across the gaps with angular frequency $\omega = 2\pi f$.

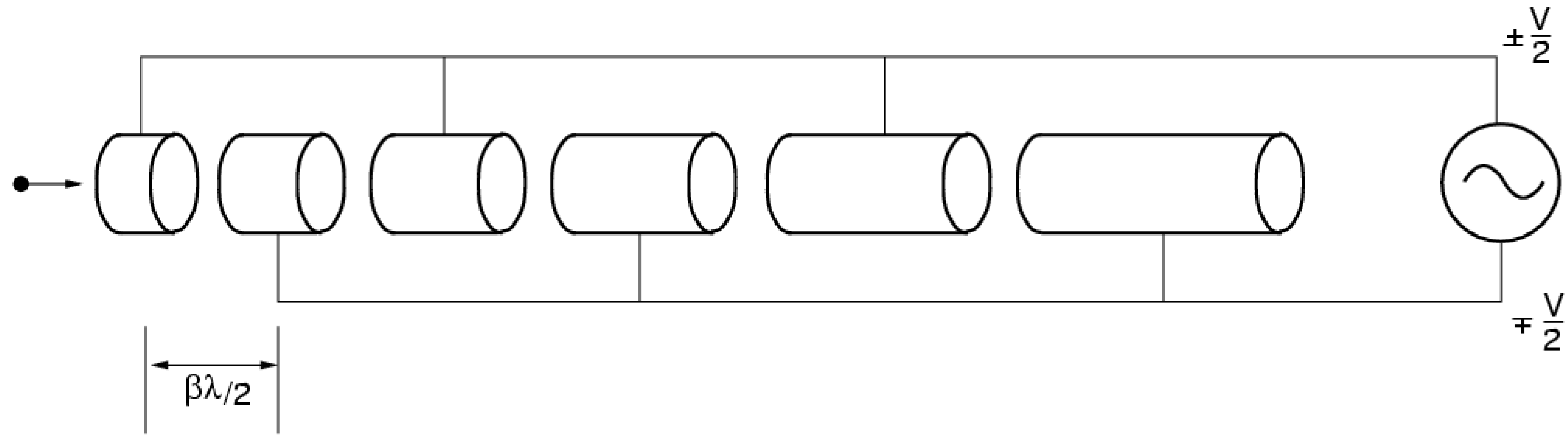
When ion with charge *qe* enters the first gap, the voltage across the gap is *V* and with a polarity that accelerates the ion. Right then the next gap will have the opposite polarity.

However, by the time the ion gets to the next gap, the polarity of the generator will have reversed, and the ion will again be accelerated.

Each drift tube comprises a Faraday Cage. Its potential may vary, but the energy of the particles in the drift tube is not affected (neglecting fringe fields).

Positive Ions in a Sloan-Lawrence Accelerator at Various Phases





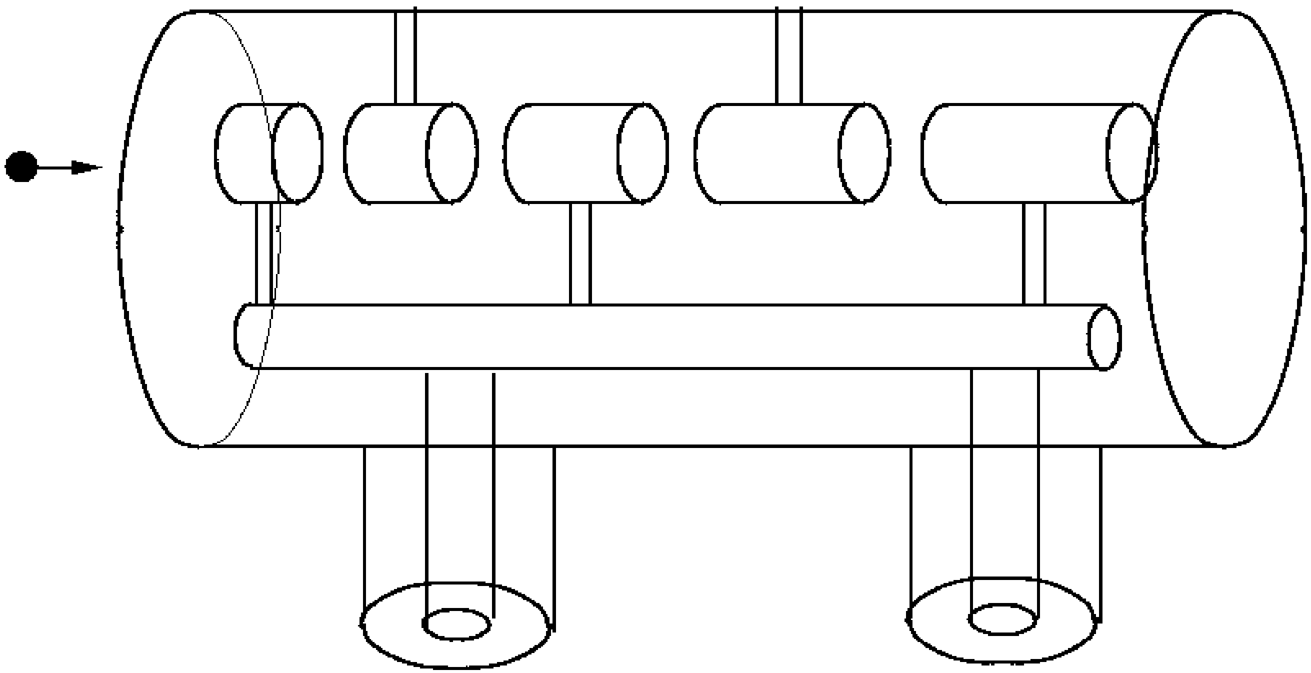
For the ion to remain synchronous as it accelerates, the distance between the gaps must be ½ an RF cycle, or more generally, (n + ½) cycles, where n is an integer.

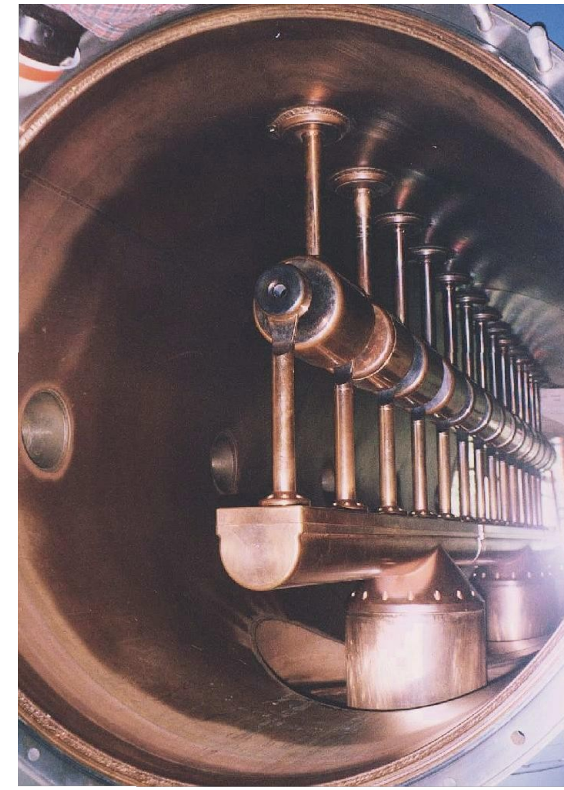
For an RF frequency of f, the free-space wavelength is λ, and the ion must travel (n + ½) bl between the gaps, n = 0, 1, 2,... .

Notice that the length of the drift tubes grows so that the distance between the gaps is given by the above formula.

The RFQ (covered later) is an example of a Sloan-Lawrence structure.

The Wideroe Structure





Drift tubes alternate between an excited coaxial line and the grounded outer wall.

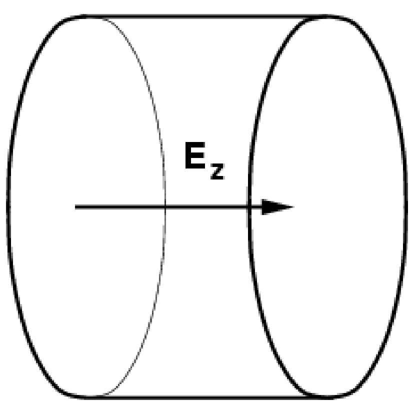
The coaxial line is supported by quarter-wave stubs that contain the cooling lines and support the excited drift tubes. Usually, only the grounded drift tubes contain focusing devices.

These machines were built at GSI and LBNL for the acceleration of uranium beams, which required very low frequency operation. (Why?)

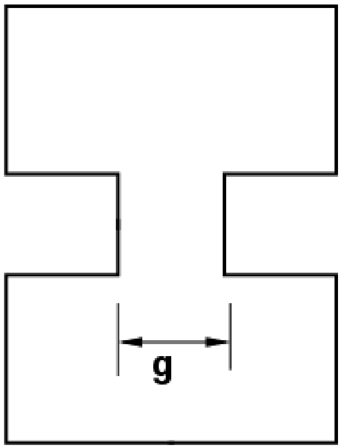
The Alvarez Structure

Almost all hadron linacs are of this variety. It was invented in the 1940's at LBNL, by Luis Alvarez, at the Radiation Laboratory, later LBNL.

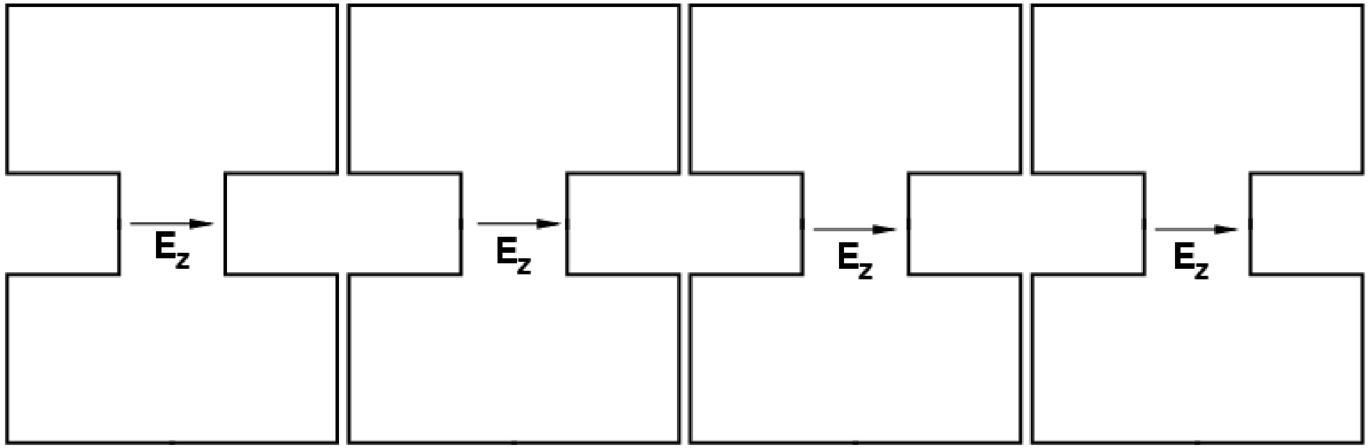
The structure is based on a resonant pillbox cavity operating at the lowest resonance, where the E-field is uniform along the axis (z-direction) of the cavity.



In order to have a high accelerating efficiency (high TTF), we introduce a re-entrant gap in the structure.



To form an accelerator, take a set of single-cell cavities and concatenate them.



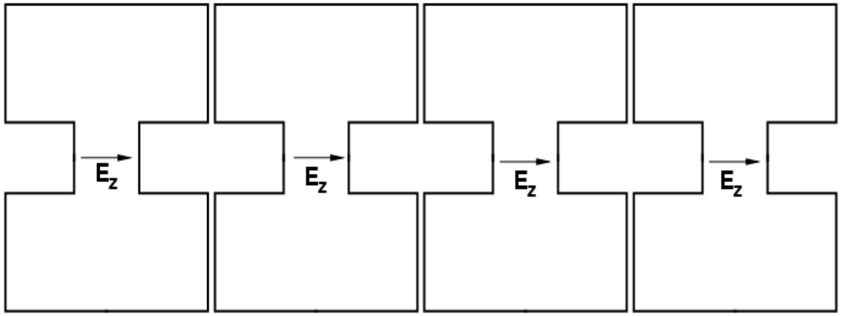
If we phase the RF fields in all the cavities to be identical in time, then the spacing between the gaps is $\beta\lambda$, or more generally, $n\beta\lambda$, where n is an integer: n = 1, 2, 3,... .

By the time the ion gets from one gap to the next, the field has progressed one (or multiple) RF cycles.

This structure would work, and has been used. It has the advantage that if the cavities are individually excited with arbitrary phase, then the energy of the ion can be varied, as λ is a constant, but β can be chosen for the ion in question.

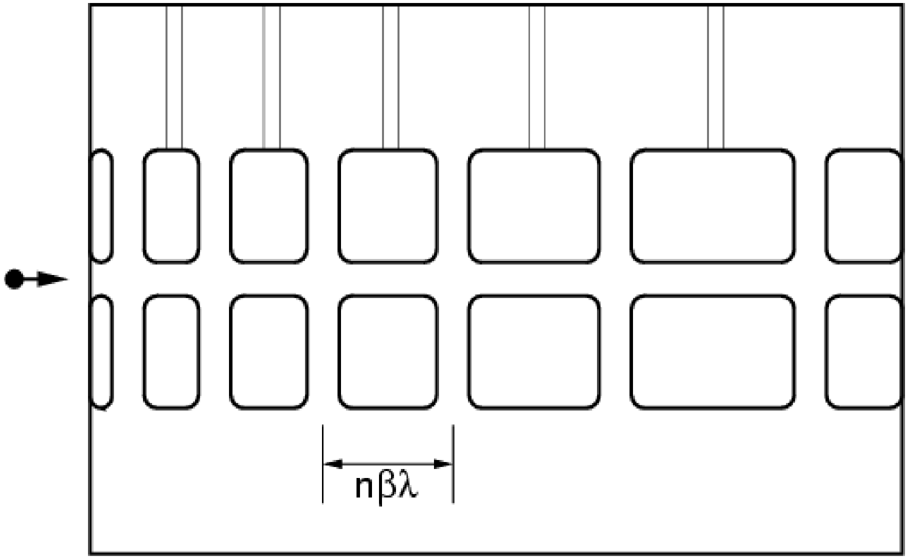
Note, however, if the cavities are all in synchronism, then the velocity at each location is fixed, thereby the energy is fixed. The Alvarez structure is inherently a fixed-velocity structure.

We can improve on this configuration.



Currents flow along the walls, heating them and requiring RF power. The fields on each side of the walls separating each cell are identical in amplitude and direction, so the walls separating the cells may be removed. The drift tubes are suspended on small rods (stems).

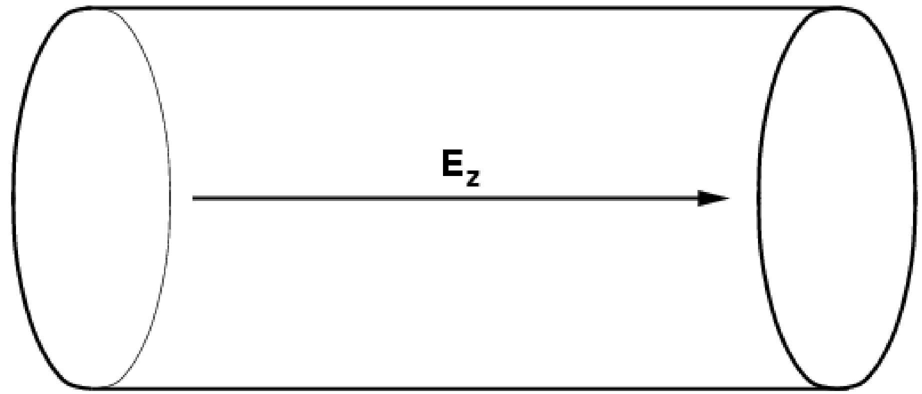
Here, the beam aperture is shown.



This is one way to look at the Alvarez structure: a series of single-cell cavities, all in phase, where the separating septa (walls) are removed without altering the field configuration.

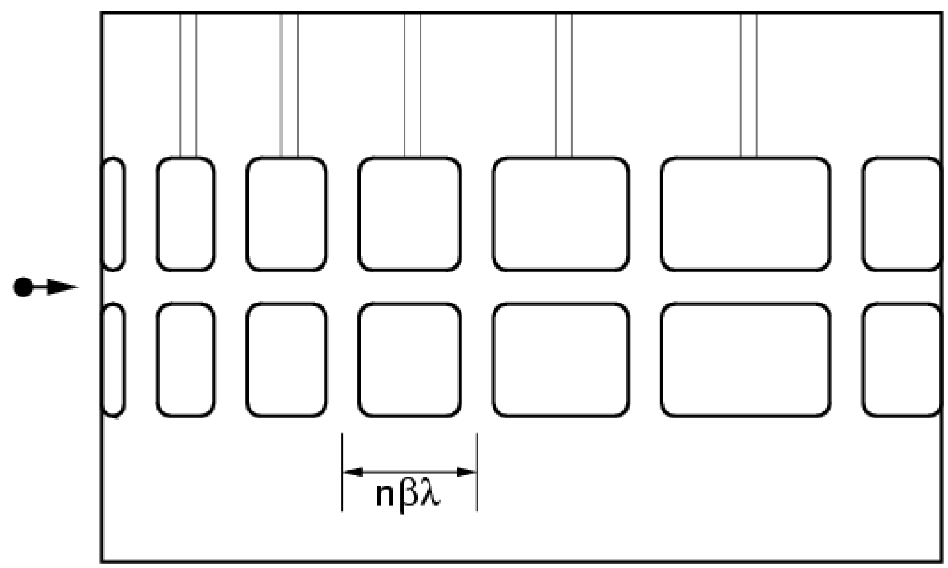
An Alternate Way of Considering the Alvarez Structure.

Start off with a long pillbox cavity with the electric field along the axis.



An ion in this field will just oscillate along z, but not be accelerated.

If drift tubes are introduced into the cavity with the right spacing, the ion will be inside a **Faraday Cage**. When the field reverses the ion is shielded from the field and is not decelerated. The ion comes to the accelerating gap at the time when the field will accelerate the beam.



Note the half drift tubes at the ends.

Homework Problems 3.1

1. Explain why the distance between the gaps in a Sloan-Lawrence structure is $\beta\lambda/2$ and for an Alvarez structure is $\beta\lambda$.

2
Can the distance between gaps in the two types of structures be any other multiple of $\beta\lambda/2$ or $\beta\lambda$?

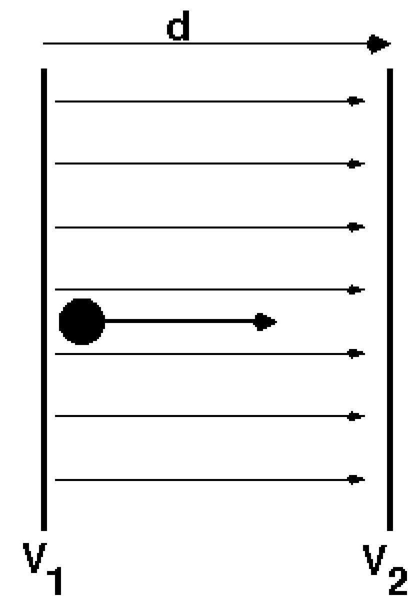
3
Using a spreadsheet, calculate the drift tube sequence for a linac of 5 drift tubes. Assume TTF=0.9.

Charged Particle in a Static Electric Field

Two parallel plates are at potentials V_1 and V_2, and are separated by distance *d*.

The electric field between the plates is

$$E_z = \frac{V_2 - V_1}{d}$$



If a particle with charge *q* leaves the plate with potential V_1, when it reaches the plate with potential V_2 has gained (or lost) energy

$$\Delta W = q E_z d$$

Units: Energy is expressed in units of electron-volts. The unit of *q* is electron, the unit of $E_z\, d$ is (Volts/meter x meter = Volts).

Electron vs. Proton Energy

Will an electron, with a mass of 1/1838 of a proton, have the same energy as a proton when accelerated through the same voltage drop? (Don't forget to reverse the polarity of the voltage source.)

Yes, it will.   But it will have a different **momentum** and a different **velocity**.

For a 1 MeV proton and a 1 MeV electron, the relativistic factors are:

	γ	β	$\beta\gamma$
p^+	1.00107	0.04614	0.04619
e^-	2.95695	0.94108	2.78272

$$KE = (\gamma - 1)mc^2$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\frac{p_p}{p_e} = \frac{(\beta\gamma mc)_p}{(\beta\gamma mc)_e} = 30.47$$

(Old CRT-type) color TV sets have an accelerating potential of about 26 kV.
Is the electron beam that hits the screen relativistic?

Heavy Ions

Ions are characterized by the number of electron charges q and the mass A in units of AMU (Atomic Mass Units), 1/12 of the mass of a carbon-12 atom.

For protons, q = -1 and A = 1.0073.
For an alpha particle, q = -2 and A = 4.0026. (We won't pay attention to the sign.)

There are two conventions when specifying the kinetic energy of an ion:

the total kinetic energy
the kinetic energy per nucleon.

Neither of these is the total energy, which is the kinetic energy + the rest mass.

Accelerator physicists tend to use the kinetic energy per nucleon, as this directly gives the velocity of the ion.

Nuclear physicists tend to use total kinetic energy, as this relates to the energy transferred in a nuclear reaction.

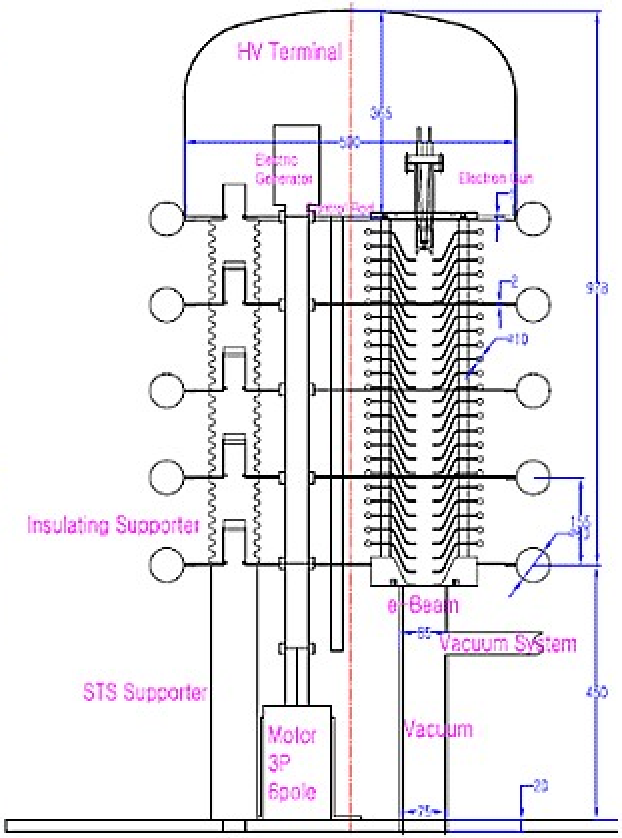
Non-relativistically: $W = \frac{1}{2}\beta^2 m c^2$

If *m* is the total mass, *W* is the total KE,
if *m* is the mass of 1 AMU, *W* is the KE/n.

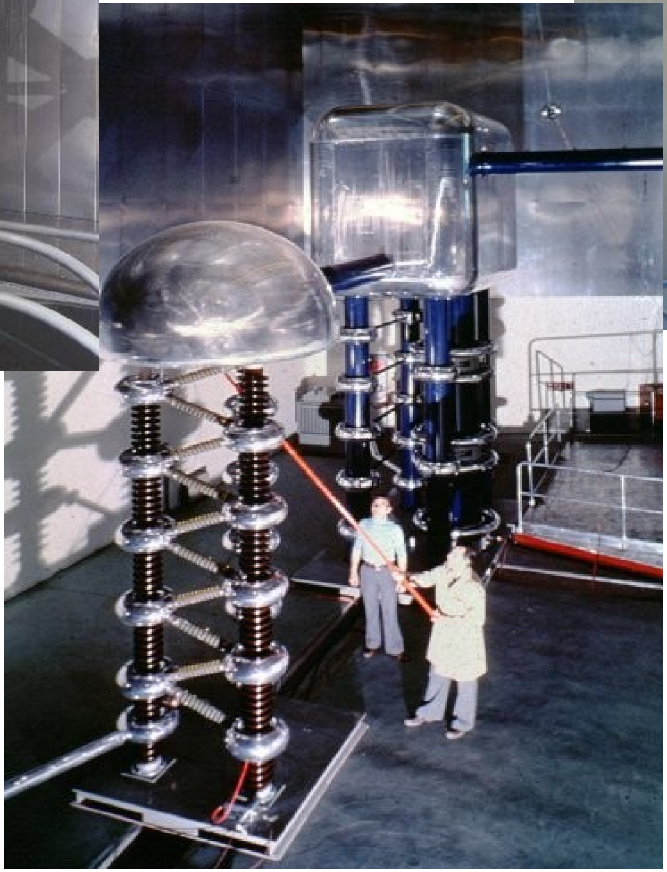
The Cockcroft-Walton Accelerator

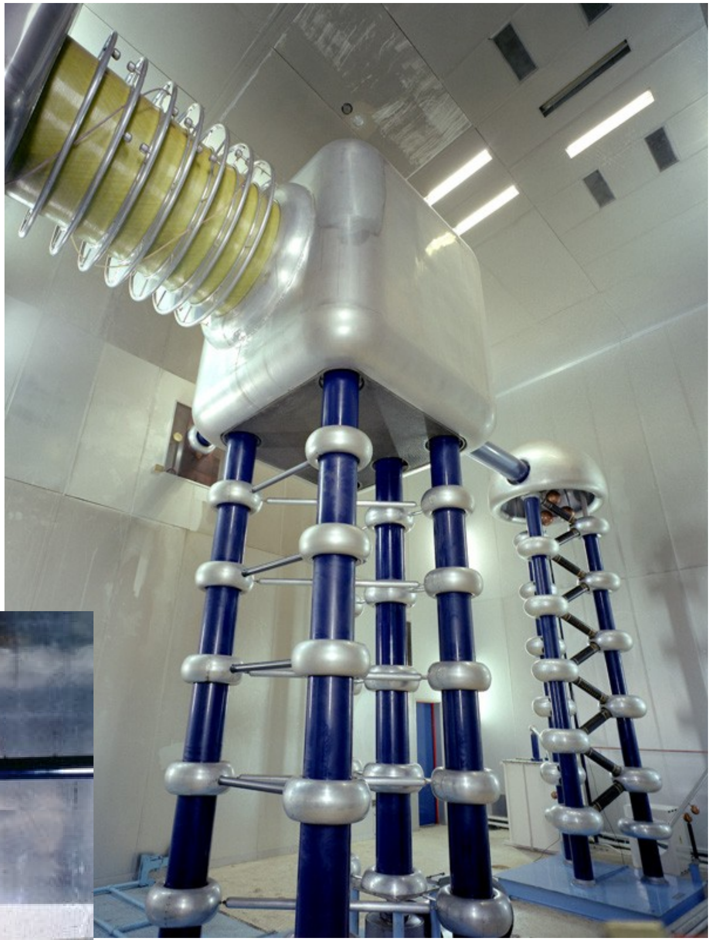
The Cockcroft-Walton (CW) accelerator works on the principle of acceleration of charged particles through a field formed by a high-voltage terminal where an ion source is located, to a different potential, usually ground.

Almost all accelerators start with a d.c. voltage drop accelerator to get the ion started, and then other types of accelerators usually follow to accelerate to higher energy.



More Cockcroft-Waltons





One problem with C-W's is the ion source must be at high voltage. (How is the power transmitted to the ion source house?)

C-W's are still in use at FNAL and LANL, where major RFQ development has taken place.

The Transit Time Factor

For a very narrow gap and small aperture, the field is nearly constant in the gap, and the ion picks up a kinetic energy almost equal to the voltage across the gap.

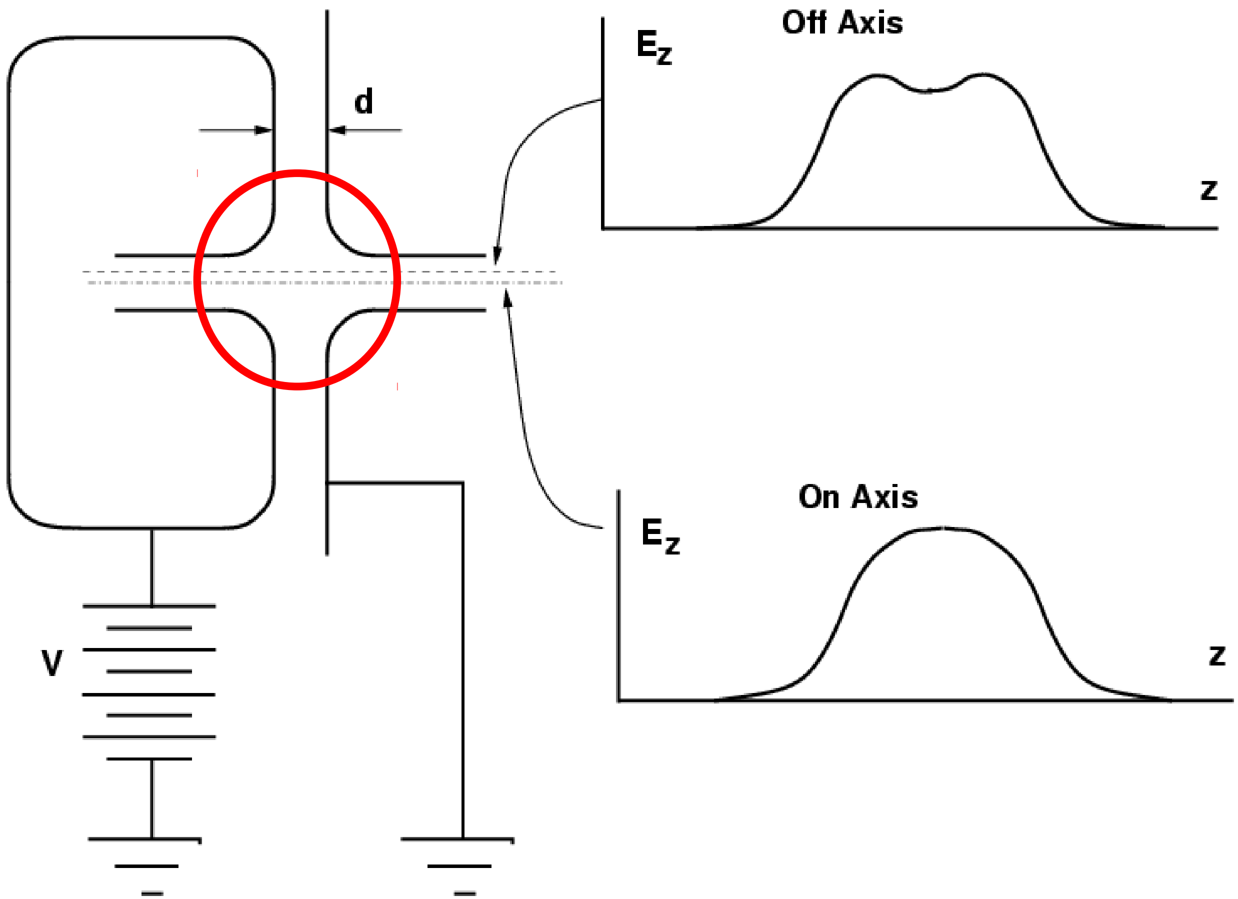
For wider gaps, necessary to hold the gap voltage without sparking, the ion is in the gap for a longer time, and spends more time at a field level less than the peak field for a short gap. The field is changing in time, and eventually even reverses. For a very long gap, the ion may spend more time in the reverse field polarity and actually be decellerated. In this case $T < 0$.

An idealized calculation assumes a square field profile from $-g/2$ to $+g/2$. Real fields not so ideal. We also assume no velocity change due to acceleratio in the gap and that the field is at a maximum when the ion was in the center of the gap. For longitudinal focusing, the ion will enter the gap as the field is still rising.

A more accurate calculation integrate the actual field in the gap, including the fringe field in the drift tube bore along the beam axis.

The relationship between the bunch centroid and the peak of the gap field is the stable phase, and the sign is negative for longitudinal focusing.

DC Acceleration Across Physical Gap



Total voltage is the integral of the field. This is a generalization of the simple formula.

The on-axis and off-axis integrals will be identical.

The off-axis field $E_z(z)$ is enhanced near the outer edges of the gap.

This produces a significant effect on the beam dynamics in the gap.

$$V = \int E(z)\,dz$$

$$W = W_0 + \int e\,E(z)\,dz$$

The kinetic energy is increased by the voltage drop across the gap. (Is this relativistically correct?)

Acceleration by Time-Varying Fields: Transit Time Factor

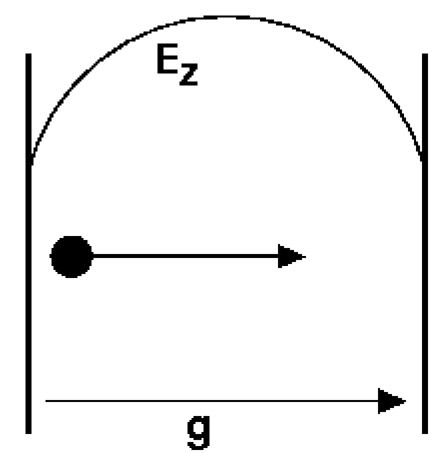
Let the field in the gap vary sinusoidally with angular frequency u

$$E(r,z,t) = E(r,z)\ cos(\omega t)$$

We will choose *t* so that the ion cross the center of the gap at *t*=0 where the field is at a maximum value.

For a particle traveling at a velocity β, and the field varies with angular frequency ω,

$$z = \beta c t \qquad \lambda = \frac{c}{f} = \frac{2\pi c}{\omega}$$



so the spatial variation of the field is

$$E(r,z,t) = E(r,z)\ cos(\frac{\omega z}{\beta c}) = E(r,z)\cos(\frac{2\pi}{\beta\lambda} z)$$

Acceleration by Time-Varying Fields: Transit Time Factor

$$\Delta W = qe\int_{-\infty}^{+\infty} E(r,z)\cos(\omega t)\,dz = qe\int_{-\infty}^{+\infty} E(r,z)\cos\left(\frac{\omega z}{\beta c}\right)dz$$

For a very idealized case we can estimate the transit time factor by assuming that the field isflat in the gap, and zero outside of the gap region and there is no radial dependence.

$$E(z) = 0, \quad z < -\frac{g}{2}, \quad z > \frac{g}{2}$$

Then

$$\Delta W = qe\int_{-\frac{g}{2}}^{+\frac{g}{2}} \frac{V}{g}\cos\left(\frac{2\pi}{\beta\lambda}z\right)dz = qV\frac{\sin\left(\frac{\pi g}{\beta\lambda}\right)}{\frac{\pi g}{\beta\lambda}} \equiv qVT$$

This defines the transit-time factor *T* or *TTF.* The ion, passing through the gap sees a varying field which is not always at its peak value. The TTF takes into account the fact that the field is not flat in the gap.

Typically, for a gap ¼ the length of the cell, in a bl cell, T = 0.9.

Note that this calculation does not take into account the effect of a finite bore radius of the beam aperture. We will add that in later.

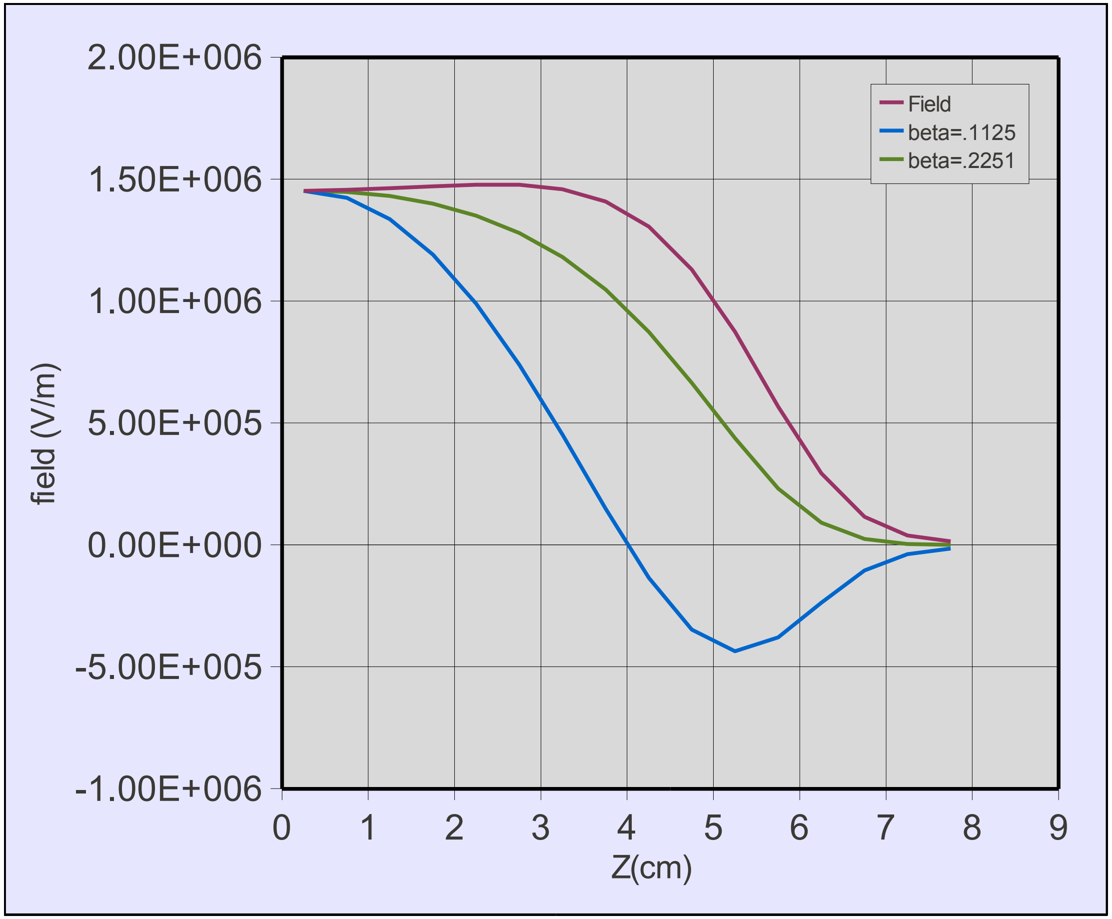
Transit Time Factor Depends on Particle Velocity

The ion integrates the electric field along the axis as the field is changing in time. A slow particle may see the field actually reverse and decelerate the particle. The energy gain in a gap with average axial field of E_0 is

$$\Delta W = q E_0 T \cos(\phi)$$

The plot shows the field calculated by Superfish (red), and the field experienced by a particle crossing the cavity for an RF phase advance of 180 degrees (green) and 360 degrees (blue). This is for the final buncher cavity geometry (with nosecones).

The energy corresponds to 5.9 and 24.7 MeV, and the TTFs are 0.364 and 0.804.



Homework Problems 3.2

1. Trace out the equipotentials for a gap.

2. Trace out the field vectors for a gap.

3,4,5. Trace out the trajectories for an on- and off-axis proton, and an electron

Riding the Surf: Acceleration and Phase Stability

Acceleration takes place because the ion is synchronous with a standing or traveling wave in the structure of a polarity that causes acceleration.

Other waves may be present, but if the ion velocity is not synchronous with them, they will, on the average, not affect the ion's energy.

The fundamental objective of RF accelerators is to design a structure that contains a component of the electric field that moves in synchronous with the ion. Additional consideration is then given to focusing the ions, both longitudinally and transversely. The ions surf on an electric wave.

Longitudinal Focusing

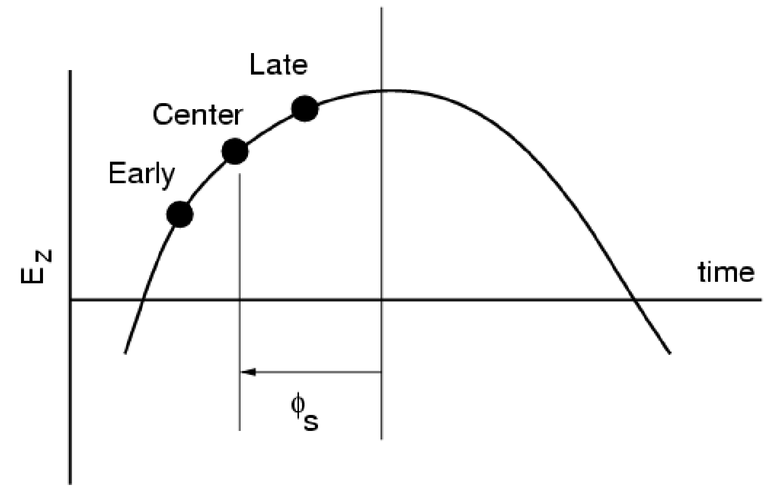
The beam energy is changed by the integral of the field: $\Delta W = q e \int E_z dz$

In contrast to linear transverse restoring fields, the longitudinal fields are **nonlinear**. Also, the longitudinal field is used to both **accelerate** and **focus** the beam.

The longitudinal focus is brought about by a rising field as the bunch enters the accelerating gap. Late particles are given an extra kick to bring them back to the bunch, early particles receive less acceleration and fall back into the bunch.

In the time domain, later ions experience a larger acceleration, earlier ions a smaller acceleration.

As the time variation of the field is sinusoidal, and not linear in time, the restoring force on the bunch is non-linear.

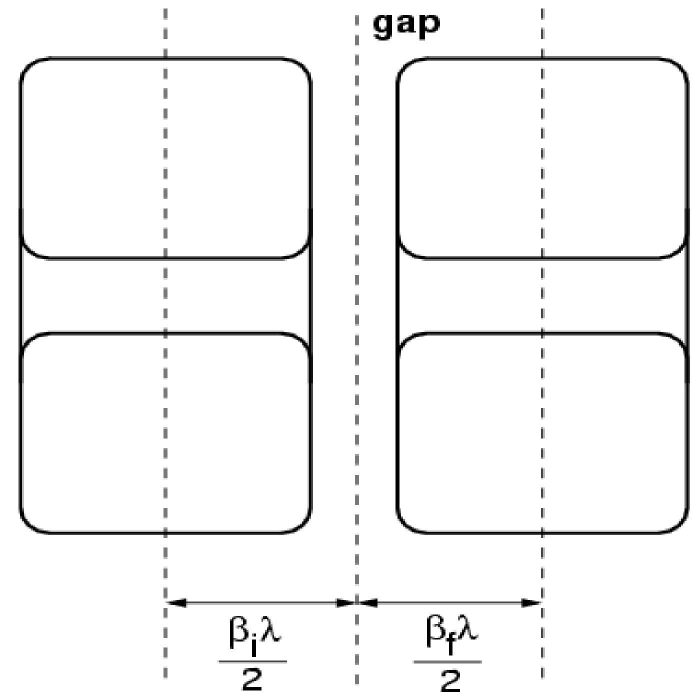


By convention, the phase of the center of the bunch is referred to the point in the RF cycle where the field is maximum. The phase which produces longitudinal focusing is negative.

(In synchrotron terminology, the phase is measured to the zero crossing.)

Nonlinear Phase and Energy Equations

The basic Alvarez accelerating cell has two halves: before and after the accelerating gap. The initial beam velocity b_i is increased to the final beam velocity b_f after the gap.



The cell length is $$L_{cell} = \frac{\beta_i + \beta_f}{2}\lambda$$

We will derive two first-order difference equations, one for the energy after the accelerating gap, and the other for the change in phase relative to the stable phase after the gap. From these difference equations, we can derive the equation of motion of particles in the bunch.

These equations describe the motion of a particle within the bunch, relative to the “synchronous particle”, the ideal particle that follows the initial design of the accelerator. The equation of motion of the synchronous particle is determined by the drift tube sequence and field amplitude of the ideal accelerator.

We first define the phase slip of a particle relative to the synchronous phase as a consequence of a energy error in a unit cell of length $L_c = \beta_s \lambda$.

The relationship between the phase slip and the energy error is

$$d\phi = -\frac{2\pi}{\beta\lambda}\Delta z = -\frac{2\pi}{\beta\lambda}\frac{L_c}{\beta}\Delta\beta$$

so

$$d\phi = -\frac{2\pi}{\beta\lambda}\frac{L_c}{\beta}\frac{dW}{mc^2\beta} = -\frac{2\pi L_c}{\beta^3\lambda}\frac{dW}{mc^2}$$

Put in terms of a deviation from the synchronous phase f_s and energy W_s

$$\beta^3\frac{(\phi-\phi_s)}{L_c} = -\frac{2\pi}{\lambda}\frac{(W-W_s)}{mc^2}$$

This (linear) difference equation gives the difference in phase from the stable phase for a given energy difference from the synchronous energy in an acceleration cell of length L_c.

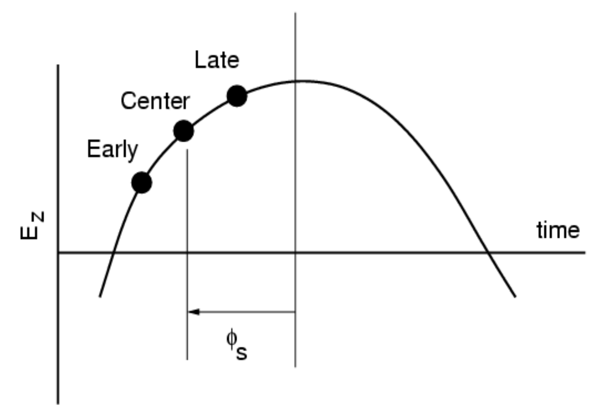
Next, we define the change in energy gain in a cell relative to the synchronous energy for a given phase error. This relationship is non-linear.

The energy gain is related to the phase of the particle relative to the phase of the RF.

$$Energy\,Gain \;=\; qe\int E(z)\,T\cos\phi\,dz \;=\; q\,e\,E_0 L_c T\cos\phi$$

The difference in energy gain from the synchronous particle is then

$$\frac{(W-W_s)}{L_c} \;=\; q\,e\,E_0 T\,(\cos\phi-\cos\phi_s)$$



The term "synchrotron motion" here is a holdover from the original formulation of longitudinal beam dynamics that was first derived for synchrotrons by McMillan and Veksler. Also, we refer to transverse beam oscillations as "betatron oscillations", because the theory was first worked out by Kerst and Serber for betatrons.

Difference Equations of Synchrotron Motion

We now have two difference equations (non-relativistic) in the parameters ϕ and W

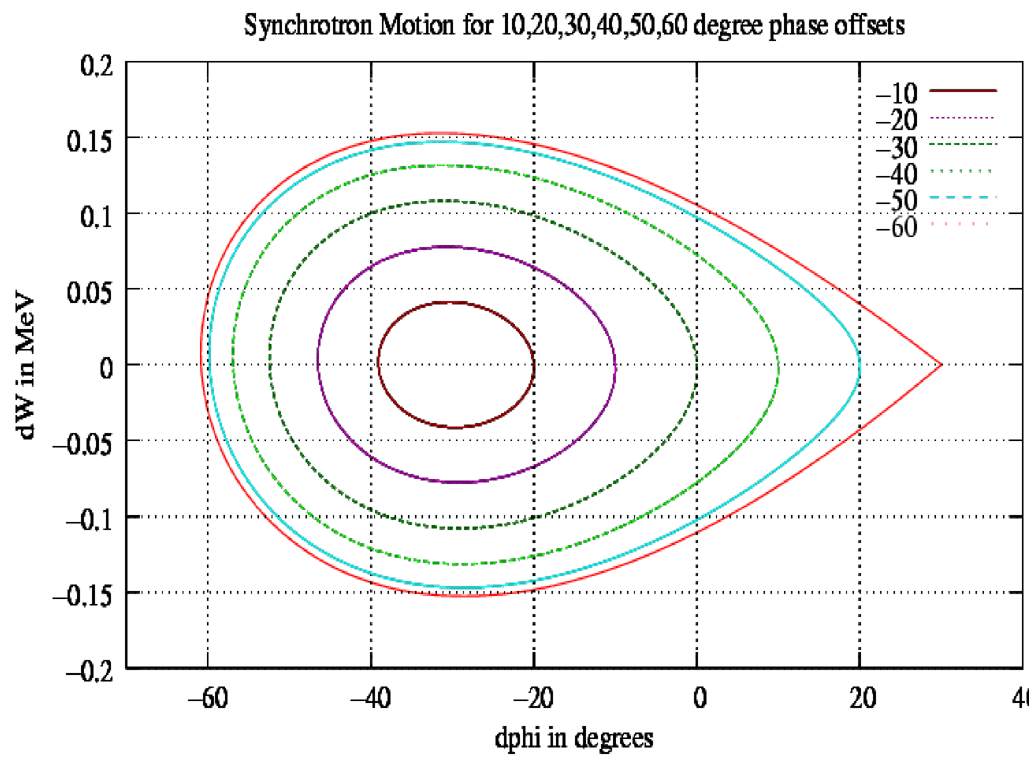
$$\beta^3 \frac{(\phi-\phi_s)}{L_c} = -\frac{2\pi}{\lambda}\frac{(W-W_s)}{mc^2} \qquad \text{and} \qquad \frac{(W-W_s)}{L_c} = qeE_0T(\cos\phi-\cos\phi_s)$$

These equations can be iterated numerically, with different initial conditions of the phase error from the synchronous phase.

The synchronous phase here is -30 degrees. The oscillation is launched with different initial phases 10 to 60 degrees from the stable phase.

Note that small phase oscillations produce a smooth rotation in phase space, but in the limiting case, a cusp occurs at +30 degrees.

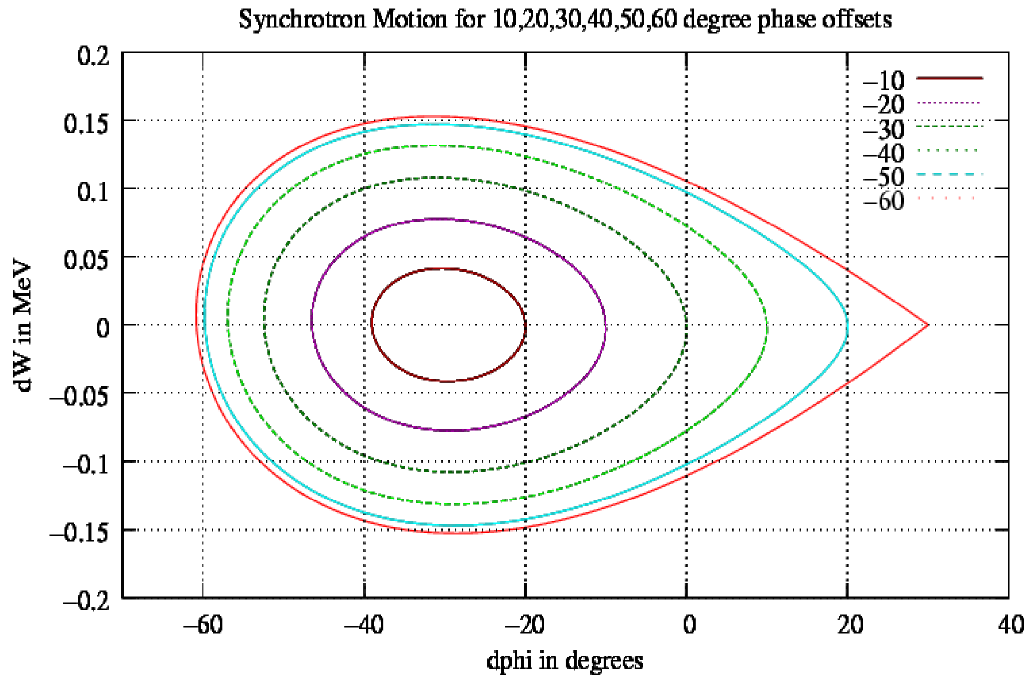
The boundary of the diagram is the separatrix. Motion outside the separatrix is unstable.



Orbits in Longitudinal Phase Space

The particles rotate around the **stable fixed point** at ϕ_s = -30 degrees and dW = 0.

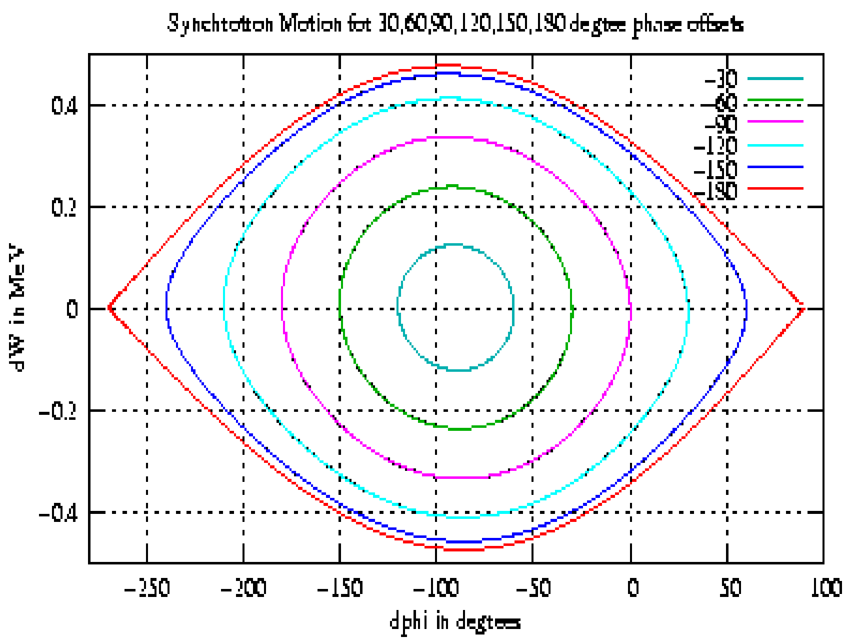
As the oscillation amplitude increases, the orbits become non-linear up to the point where the orbit intersects the **unstable fixed point** at ϕ_s = +30 degrees. This orbit also crosses the energy axis at -60 degrees.

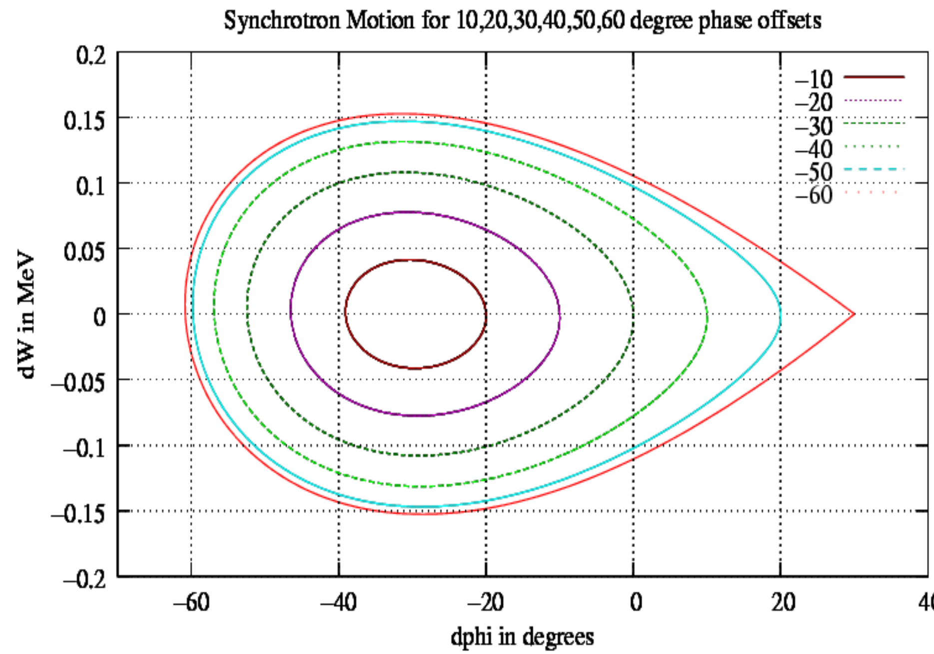


The limiting orbit defines the **separatrix**, outside of which the orbit is unstable and diverges in phase space away from the stable fixed point. The separatrix defines the maximum energy deviation any particle in the bunch may have and still stay captured within the bunch.

The frequency of the oscillation depends on the amplitude of the oscillation. This is the hallmark of a nonlinear oscillation where the restoring (focusing) force is a nonlinear function of the amplitude of the oscillation.

Separatrices for Stationary and Moving Bucket





We refer to the bunch as being in an accelerating "bucket".

The stationary bucket, on the left, corresponds to ϕ_s = -90 degrees, or no energy gain in the gap, which goes as cos ϕ. However, the beam is still focused longitudinally, and the ions exhibit longitudinal motion. The stable fixed point in the stationary bucket is at -90 degrees, the unstable fixed point is at +90 degrees, 180 degrees ahead, and a comparable fixed point is at -270 degrees, 180 degrees behind. There will be a series of separatrices that just touch along the phase axis.

When an unbunched beam is first injected into a series of accelerating cells, the will tend to start bunching and develop an energy spread. This is how the bunching process starts in an RFQ accelerator, for example.

Differential Equation of Longitudinal Motion

The two difference equations can be transformed by eliminating the energy variable W producing a second-order non-linear differential equation of motion. The equation for the evolution of the phase is

$$\frac{d^2\phi}{ds^2} = -\frac{2\pi}{\beta^3\lambda}\frac{qE_0T}{mc^2}(\cos\phi - \cos\phi_s)$$

The equations for energy and phase can be integrated twice, giving

$$\frac{\pi}{\beta^3\lambda}\left(\frac{W-W_s}{mc^2}\right)^2 + \frac{qE_0T}{mc^2}(\sin\phi - \phi\cos\phi_s) = -\frac{qE_0T}{mc^2}(\sin\phi_s - \phi_s\cos\phi_s)$$

Where the term on the right is a constant of integration, the energy in the system.

The second-order differential equation for a harmonic oscillator is $\frac{d^2\phi}{ds^2} = -k^2\phi$ which is linear and has a sinusoidal solution.

Here,

$$\frac{d^2\phi}{ds^2} = -k'^2(\cos\phi - \cos\phi_s)$$

which exhibits not only amplitude-limited behavior, but the oscillation frequency depends on the amplitude of the oscillation. If the amplitude is large enough so that the particle follows along the separatrix, the oscillation frequency goes to zero. (The particle gets stuck at the unstable fixed point.)

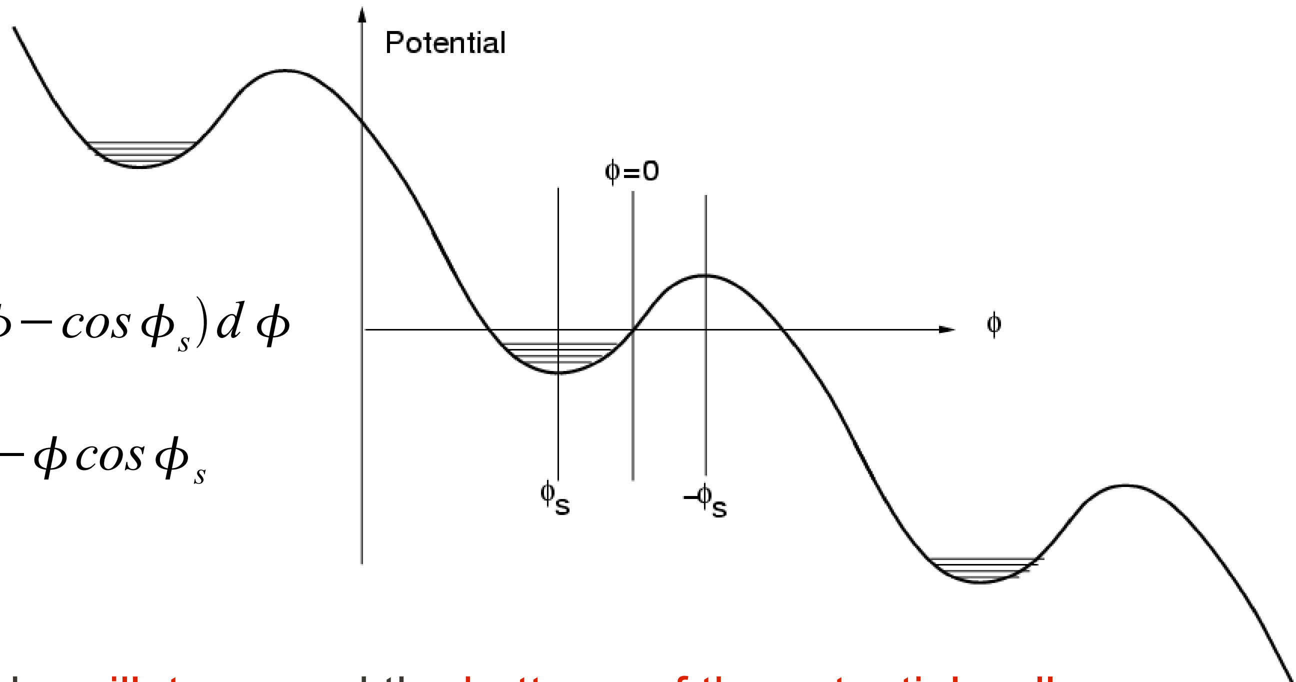
Potential Function of a Series of Separatrices

Since the energy gain is given by $W - W_s = L_c q e E_0 (\cos\phi - \cos\phi_s)$

we can integrate this to get a potential function $\Phi = \int W \, dz$

$$potential \sim \int (\cos\phi - \cos\phi_s) \, d\phi$$

$$= \sin\phi - \phi\cos\phi_s$$



The ions reside in and oscillate around the bottoms of the potential wells.
Ions with energy greater than the well depth can spill out of the well (separatrix), and will have the wrong energy to settle down in an adjacent potential well.

Limiting Energy Spread: Height of the Separatrix

$$\frac{\pi}{\beta^3\lambda}\left(\frac{W-W_s}{mc^2}\right)^2+\frac{q\,e\,E_0T}{mc^2}(\sin\phi-\phi\cos\phi_s)=-\frac{q\,e\,E_0T}{mc^2}(\sin\phi_s-\phi_s\cos\phi_s)$$

By choosing a phase, the the locus of the energy may be mapped out. The maximum energy deviation *(W-W_s)^2* is maximized at $\phi = \phi_s$. This is the half-height of the separatrix, and the maximum energy that will allow a particle to be confined to the bunch.

$$\left(\frac{W-W_s}{mc^2}\right)^2 = -\frac{2\beta^3\lambda}{\pi}\frac{q\,e\,E_0T}{mc^2}(\sin\phi_s-\phi_s\cos\phi_s)$$

or

$$\frac{\Delta W}{mc^2} = \sqrt{\frac{2\,q\,e\,E_0T\beta^3\lambda}{\pi\,mc^2}(\phi_s\cos\phi_s-\sin\phi_s)}$$

(non-relativistic)

Homework Problems 3.3

1
Why is it necessary for longitudinal focusing that the stable phase of the center of the bunch be negative? What would happen if the stable phase were zero? Would this result in a higher accelerating rate?

2
If the accelerating field E_0 and the stable phase ♫ϕ_s is constant, how does the separatrix height ΔW scale with particle energy? How does the relative energy spread $\Delta W/W$ scale with energy?