Chapter 4 Accelerator Structures I Single-Cavity

The Pillbox Cavity

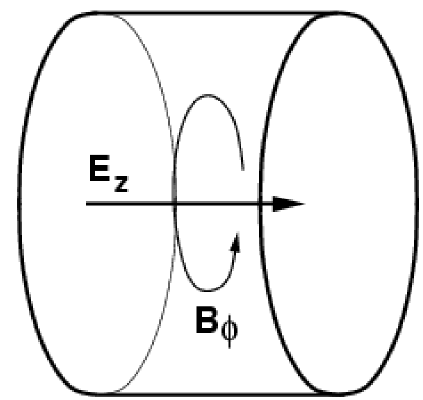
Power Relations

Bunchers

Buncher Cavity - Superfish

Beam Rebunching

Pillbox Cavity



A simple resonant cavity is the pillbox cavity.

The cavity supports an E-field along the axis, and

$$\nabla \times \vec{H} = \epsilon_0 \vec{E}$$

indicates that a B-field circulates around the axis in the azimuthal (f) direction.

The pillbox cavity forms the basis of the Alvarez accelerator cavity and a typical buncher cavity.

We will analyze the fields and their modes in the pillbox cavity.

Boundary Conditions

The electric field vector E and magnetic field vector B are subject to **boundary conditions** on metallic surfaces.

No component of the E vector may be parallel to a metallic surface.
The E field vector is perpendicular to the surface.

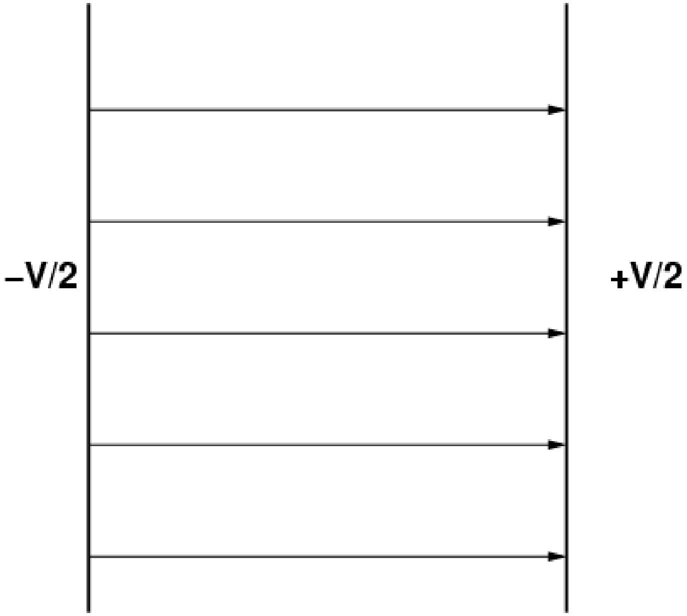
No component of the B vector may be perpendicular to a metallic surface.
The B field vector is parallel to the surface.

The H field at the surface is mirrored by an equivalent current density J in the surface (amps/meter), oriented 90 degrees in the metal to the direction of H at the surface.

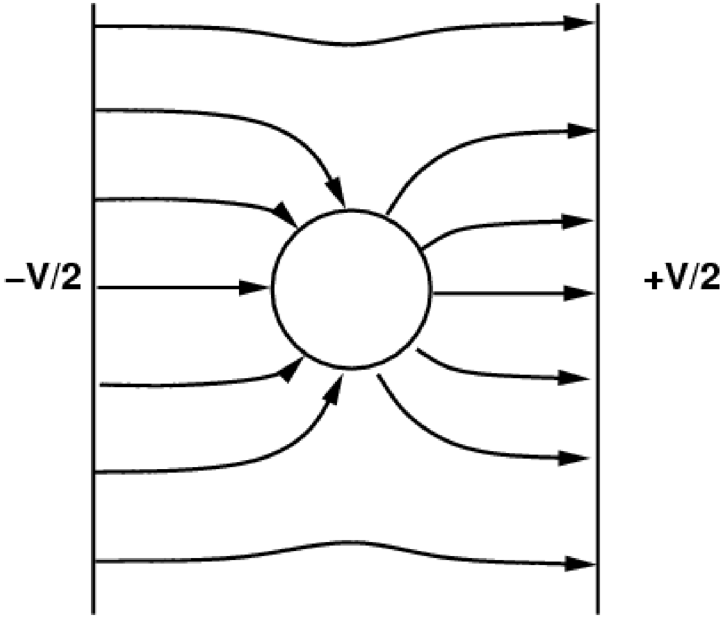
The surface current J will flow in the metal, and if the surface is lossy, will result in power being dissipated in the material.

E-Field on Metallic Boundary

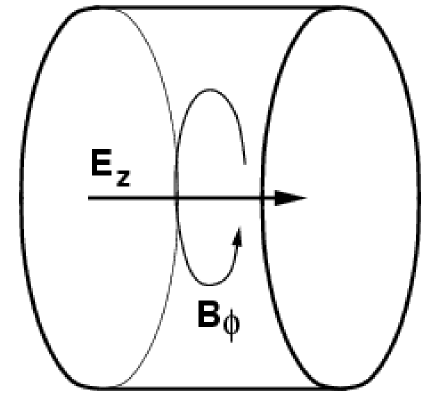
Between two parallel plates, the E-field is perpendicular to the plates. (There may be fringe fields at the edges of the plates, but the E-vector is still perpendicular.)



If a conducting rod or sphere is inserted between the plates, the E-field vector must terminate on the sphere at right angles to the surface.



Analysis of the Pillbox Cavity



We will use cylindrical coordinates $r, \ \phi, \ z$

The E-field vector is everywhere perpendicular to the walls.

The only field component is E_z

E_z = 0 on the sidewalls

E_ϕ = 0 on the sidewalls

The H-field vector has no component perpendicular to any wall.

H_z = 0 on the endwalls

H_r = 0 on the sidewalls

Only H_ϕ is present.

We have not said anything yet about the variation of $E_z(z)$ along the cavity.

The Wave Equation for the Pillbox Cavity

The wave equation in cylindrical coordinates is (Wangler page 28)

$$\frac{\partial^2 E_z}{\partial z^2} + \frac{1}{r}\frac{\partial E_z}{\partial r} + \frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial^2 \phi^2} - \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2} = 0$$

There are two sets of solutions to this equation

Transverse magnetic (TM) solutions: Pillbox cavities, Alvarez linacs

Transverse electric (TE) solutions: Deflecting cavities, RFQ linacs

The wave equation is of the form that has Bessel functions as its solution.

$$z^2 \frac{d^2 x}{dz^2} + z\frac{dx}{dz} + (z^2 - m^2)x = 0$$

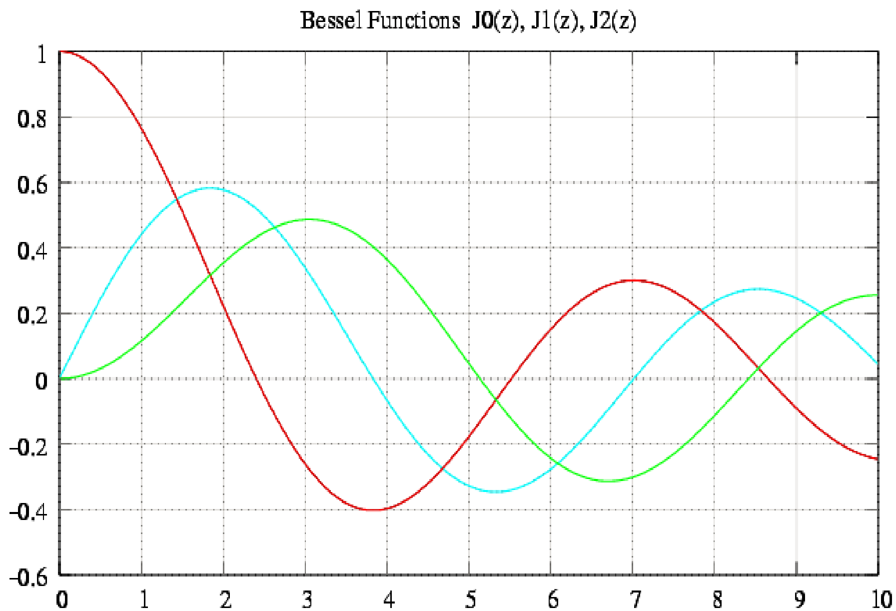
Most accelerators are constructed with some sort of cylindrical symmetry, so we can use the same set of coordinates for both analyses.

The TM Modes

The TM solution to the wave equation in cylindrical coordinates has the form (Wangler, page 30-31), with the sinusoidal time dependence removed:

$$E_z \propto J_m(k_{mn} r) \quad \cos m\theta \quad \cos k_z z$$
$$E_r \propto p J'_m(k_{mn} r) \quad \cos m\theta \quad \sin k_z z$$
$$E_\theta \propto \frac{p}{r} J_m(k_{mn} r) \quad \sin m\theta \quad \sin k_z z$$

$$B_z = 0$$
$$B_r \propto -\frac{i}{r} J_m(k_{mn} r) \quad \sin m\theta \quad \cos k_z z$$
$$B_\theta \propto -i J'_m(k_{mn} r) \quad \cos m\theta \quad \cos k_z z$$



m, n, p are integers that describe the mode of the solution.
The J_m are Bessel functions of the first kind. J'_m is the derivative of the Bessel function.

$$k_z = \frac{\pi p}{L_{cav}}, \qquad \frac{\omega^2}{c^2} = k_{mn}^2 + k_z^2 \qquad J'_0(z) = -J_1(z), \quad J'_1(z) = J_0(z) - \frac{1}{z} J_1(z), \ldots$$

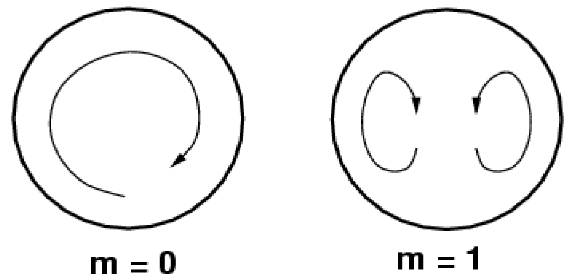
The *i* in the equations for *B* is $\sqrt{-1}$ and indicates that the E and B fields are 90 degrees offset from each other in RF phase.

TM Mode Indices *m, n, p* TM_{mnp}

The modes are described by three indices.

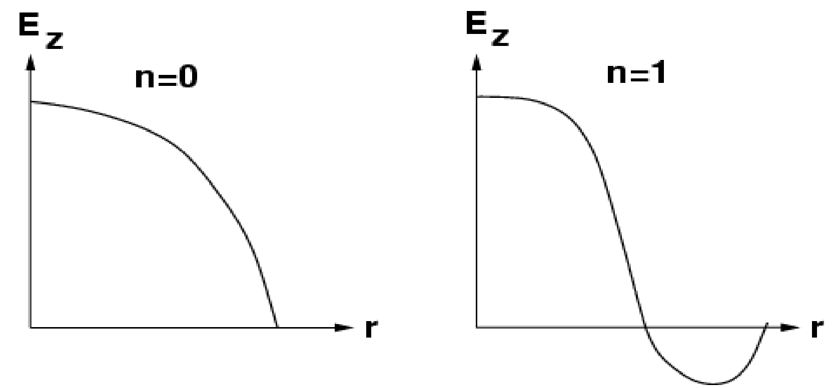
m is the number of variation of field
of the azimuthal variable ϕ:

m = 0, 1, 2, ...



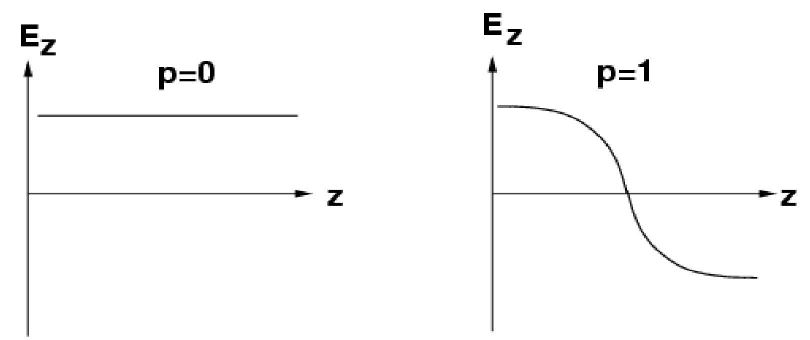
n is the number of nulls in E_z along the
radial direction

n = 1, 2, 3, ...



p is the number of nodes of E_z along
the z-axis.

p = 0, 1, 2, ...



Peak and RMS Voltage, Power

The DC power dissipated in a resistor is

$$P = IV = \frac{V^2}{R} = I^2 R$$

For sinusoidal alternating current, with a period t, the voltage has the form

$$V = V_0 \cos \omega t , \qquad t = \frac{1}{f} = \frac{2\pi}{\omega}$$

And the thermal power *P* deposited in the resistor is

$$P = \frac{1}{Rt} \int V^2(t)\, dt = \frac{1}{2} \frac{V_0^2}{R}$$

(The integral of $\cos^2$ is ½ .)

Here, V_0 is the peak voltage of the sine wave. The ½ in the equation is a result of the definition of the amplitude of the sinusoidal waveform.

The usual definition of the amplitude of a sinusoidal AC waveform is the RMS (root mean square) value, which is

$$V_{rms} = \frac{1}{\sqrt{2}} V_{peak}$$

Then

$$P_{rms} = \frac{V_{rms}^2}{R}$$

Peak fields are of interest in accelerators, as they define the energy gain across a gap, for example, but the conventional definition is the RMS value of the voltage or current.

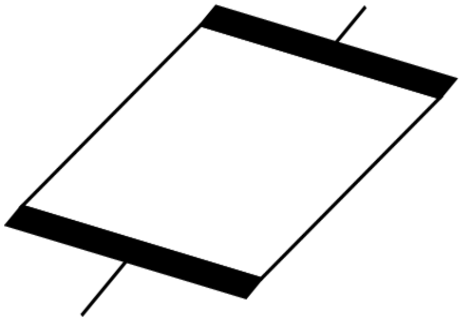
What is the peak and peak-and-peak value of 120 volt AC line voltage?

Skin Effect

Since the RF current flow is confined to the surface of the conductor, not its bulk, the resistance may be expressed as resistance/square.

The sheet resistivity (resistance/square) is

$$R_{sq} = \frac{1}{\delta \sigma}$$



Where δ is the skin depth and σ is the bulk conductivity of the material.

For copper at room-temperature, r = 1/σ = 1.724x10^{-8} ohm-meter.

At a frequency of 200 MHz, δ = 4.7x10^{-6} meter, R_{sq} = 0.0037 ohms/square.

Note that R_{sq} scales as (*frequency*)^{½}.

Skin Depth

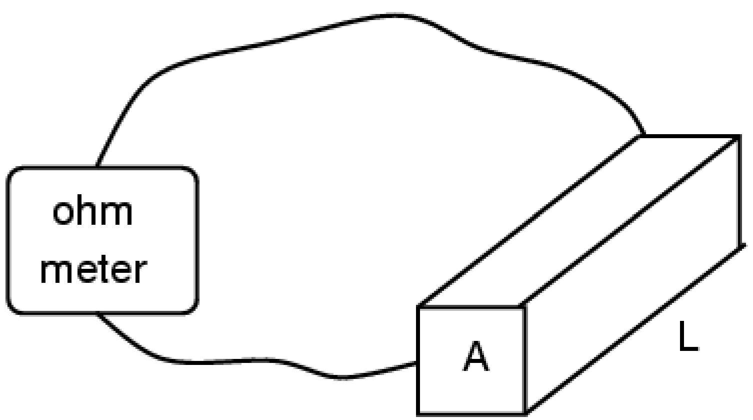
High frequency (RF) current tends to flow along the surface, and not in the bulk of conductors. The apparent RF resistivity of a conductor is higher than the DC resistivity. The current flow through a conductor with finite resistance results in the generation of heat.

The 1/e decay depth d of RF current flowing in a conductor is

$$\delta = \sqrt{\frac{\lambda}{\pi \mu_0 c \sigma}} = \sqrt{\frac{2}{\mu_0 \omega \sigma}} = \sqrt{\frac{1}{\pi \mu_0 f \sigma}}$$

Where λ is the wavelength of the RF, ρ is the bulk (DC) conductivity of the conductor.

The bulk DC resistivity $\sigma = 1/\rho$ of a conductor can be measured on a sample of cross-sectional area *A* and length *L*. The ohmmeter measures a resistance *R*.



$$\rho = R\frac{A}{L}, \qquad R = \frac{\rho L}{A}$$

Conductivity s = $1/\rho$ [ohm^{-1} meter^{-1}]

Power Dissipation on Cavity Walls

We have calculated the RF magnetic field distribution within the cavity. The current density [amps/meter] on the wall is numerically the same as the magnetic field H [amps/meter] at the wall.

The power dissipation over an area element dA of the wall is

$$P_{diss} = \frac{R_{square}}{2} \int_{walls} H_{wall}^2 \; dA$$

using the peak value of the magnetic field H_{wall} and the average (thermal) value of the power P_{diss}. The lumped circuit analogy for a DC current I is

$$P_{diss} = R \; I^2$$

The quality factor Q_0 is purely geometric and is

$$\frac{Q_0}{R_{sq}} = \frac{1}{R_{sq}} \frac{\omega U}{P} = \frac{\omega \frac{1}{4} \int_{cavity} (\epsilon_0 E^2 + \mu_0 H^2) dV}{\frac{R_{sq}}{2} \int_{walls} H^2 \; dA}$$

Power Factor

Let's see how much power it would take to excite an accelerator gap.

A typical 200 MHz accelerator has an average field strength of 2 MV/m. A linac is injected with by a 460 keV proton source, with a velocity β = 0.031. Since λ = 1.5 meters, the cell length is 4.7 cm. The accelerating gap is typically ¼ of the cell length, or 1.17 cm. The diameter of a typical drift tube is 21 cm.

We can calculate the capacitance of the gap.

$$C = \epsilon_0 \frac{A}{d} = 26\, pF$$

The voltage on the gap is 2 MV/m times the cell length, or 94 kV.

The capacitive reactance of the gap is

$$X_{gap} = \frac{1}{j\omega C} = -j30.6\, ohms$$

And the peak current at 94 kV to charge this capacitance is V/X_{gap} = 3070 amperes.

The RMS power is ½ $V\,I$ = 144 Megawatts! (A lot of power! But it is mostly reactive)

Power Factor

In our example of driving a 26 picoFarad gap, notice that the driving voltage and current are 90 degrees out of phase.

The power delivered by a voltage source supplying a current is actually the vector product. For voltage and current expressed as RMS quantities,

$$P = I\,V\cos\phi$$

where ϕ is the phase difference between the voltage and current waveforms.

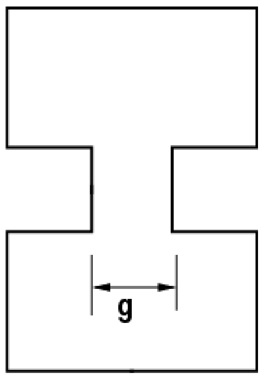
If ϕ = 90 degrees, no actual power is delivered to the load. However, the power company is still supplying volts, and the wires are still carrying current, which spin the wattmeter. The term *cos* ϕ is the power factor of the load.

Power Factor = 100% *cos* ϕ

And the units are volt-amp, (KVA, MVA). The most efficient load has a 100% power factor: the voltage and current are in phase.

Resonant Energy Storage

To realize the benefit of a resonant structure, we will calculate the same configuration, but now included in a single-cell linac cavity. The SUPERFISH code will calculate the actual parameters of the cavity.



For the same peak voltage across the gap, 94 kV, including the power loss on all the walls, only 29 kilowatts is required.

If the end walls are removed, as in a longer structure of several cells, this cell will require only 7.74 kW.

This is a huge reduction in power, compared to exciting a capacitive gap in a non-resonant system, a savings of about 18000.

The drive power to the linac cell is stored and built up over a period of time, the filling time, to produce a high gap field.

Resonant Cavities

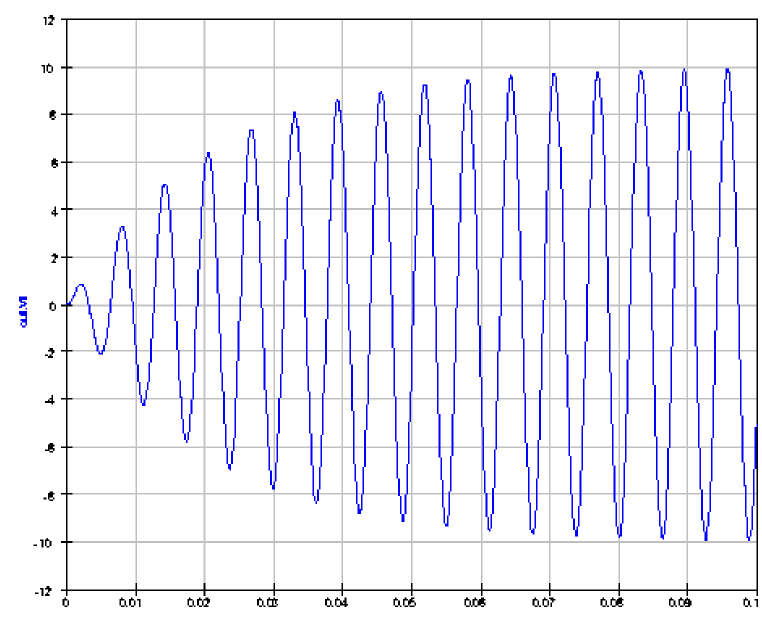
We saw that at resonance, a system can be driven to large amplitude with less power than a non-resonant system, as energy is stored in the oscillator during the build-up.

If the oscillator has no dissipation (loss), the stored energy will increase indefinitely. If there is energy loss in the structure, it will be proportional to the stored energy in the structure, which is proportional to the square of the amplitude of the fields, and the fields will approach an asymptotic value.

At the asymptotic field level, the energy loss per cycle is equal to the energy from the source per cycle.

These structures are **narrow band** and support fields at one particular frequency.

Some accelerator systems are **wide band**, similar to the charged capacitor example where special waveforms are required. They are much less efficient.

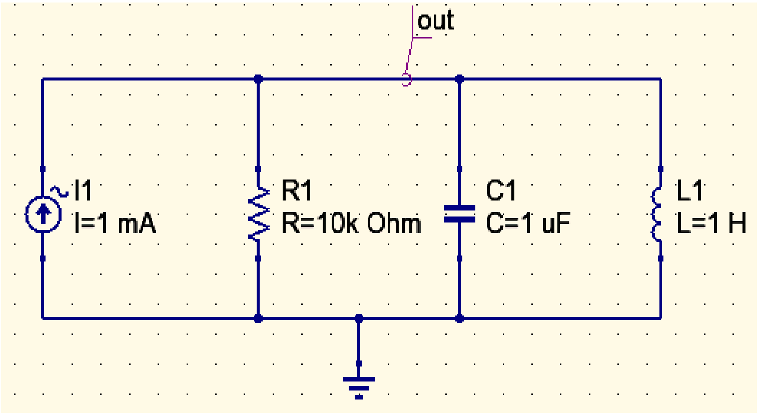


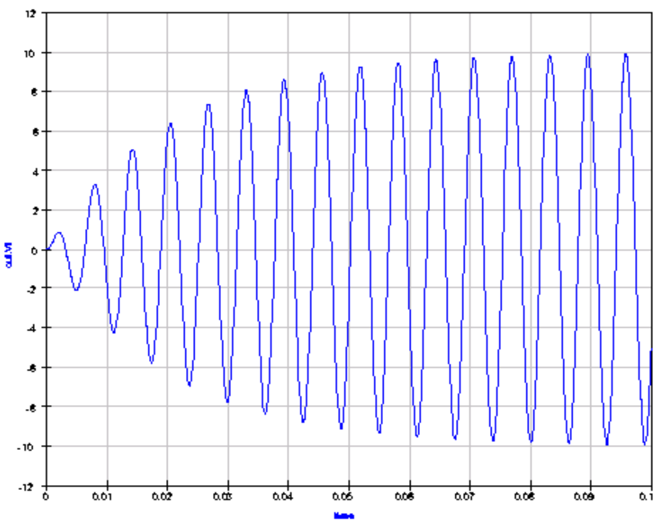
Shunt Impedance of a Lumped Circuit

Consider a lumped-circuit model of a lossy resonantor.

I1 is a current source with infinite internal impedance. It feeds energy into the LC resonant circuit.

A resistor R1 shunts the circuit, with a loss V^2/R.





When the generator is turned on, the voltage in the circuit builds up to an asymptotic limit. At this limit, the energy supplied to the circuit equals the dissipation in the resistor.

The stored energy in the resonant circuit is

$$U = \tfrac{1}{2} C V^2 = \tfrac{1}{2} L I^2$$

Which is larger than the energy delivered by the generator in one RF cycle.

Shunt Impedance of an Accelerator Cavity

The shunt impedance of an accelerator measures the effectiveness of transforming input RF power to accelerating voltage.

$$Z_{sh} = \frac{V_{peak}^2}{P_{rms}}$$

The shunt impedance Z_{sh} relates the peak voltage across the gap to the rms RF power suppled from the power source.

The shunt impedance is directly related to the Q of the cavity. $Q = \frac{\omega U}{P}$

The value Z_{sh}/Q_0 is independent of the wall resistivity of the cavity (assuming that it is constant over the entire cavity inner surface). Z_{sh}/Q_0 is entirely dependent on the geometry of the cavity.

The subscript "0" of Q_0 refers to the quality factor of the cavity that does not have any external circuit elements. Since the cavity must have an RF source connected to it, the actual Q is a function of the intrinsic cavity Q_0 and the external circuit elements, such as the RF coupler and source. We will cover coupled systems later.

Circuit Q

The quality factor, or Q of a resonant circuit is proportional to the total stored energy of the circuit divided by the power lost per cycle.

$$Q = \frac{\omega U}{P}$$

The Q of a resonant cavity is a measure of the power loss in the walls of the cavity due to the current flowing through walls of finite resistivity.

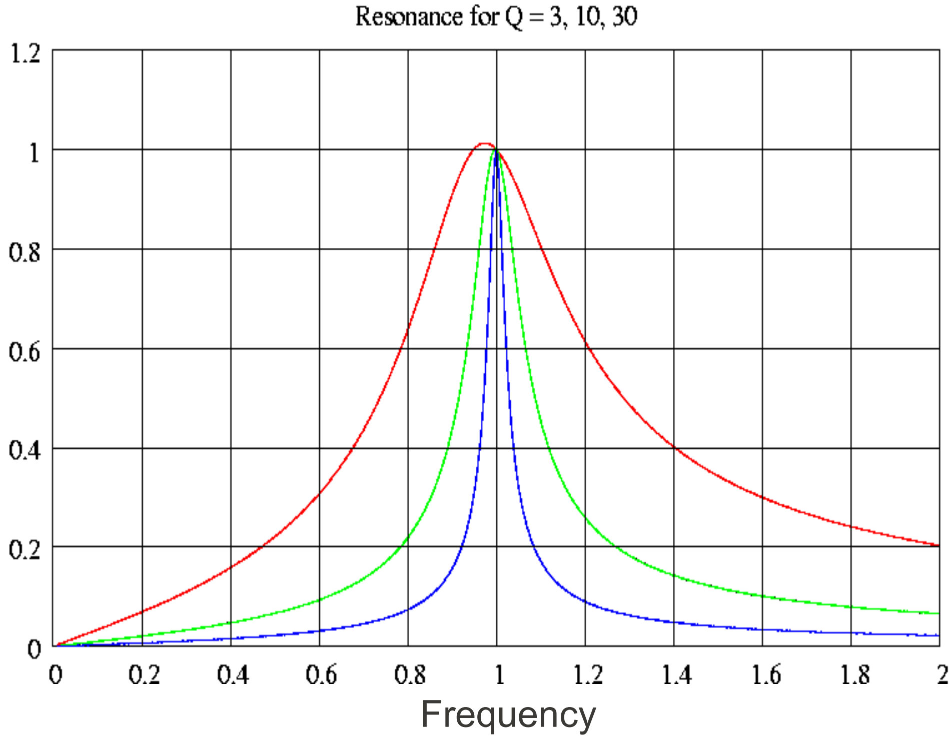
The reactance of the L and C in the lumped-element circuit are

$$X_L = \omega L, \qquad X_C = \frac{1}{\omega C}$$

At resonance, $X_L = X_C$. For a shunt resistor R, the Q of the circuit is

$$Q = \frac{R}{X_C} = \frac{R}{X_L}$$

The width of the resonance widens with lower Q. The approximate bandwidth is f/Q.



Kilpatrick Criterion

High surface electric fields in a cavity can lead to electron emission and sparking. Experiments carried out 50 years ago led to an empirical criterion of the safe surface field limit in a cavity, above which electrical breakdown was probable. The cavities were provided with oil-pumped vacuum systems, unlike the clean organics-free vacuum system we use today.

Kilpatrick established his formula for a safe surface field as a function of RF frequency. The equation is in implicit form:

$$f\,[MHz] = 1.64\,E^2\,e^{-8.5/E}$$

Freq [MHz]	E [MV/m]
200	14.7
400	19.4
1300	32.1

where the electric field *E* is expressed in units of MV/m.

It is known that this formula significantly underestimates the sparking limit at frequencies lower than 200 MHz. At higher frequencies, the criterion scales approximately as $f^{0.4}$.

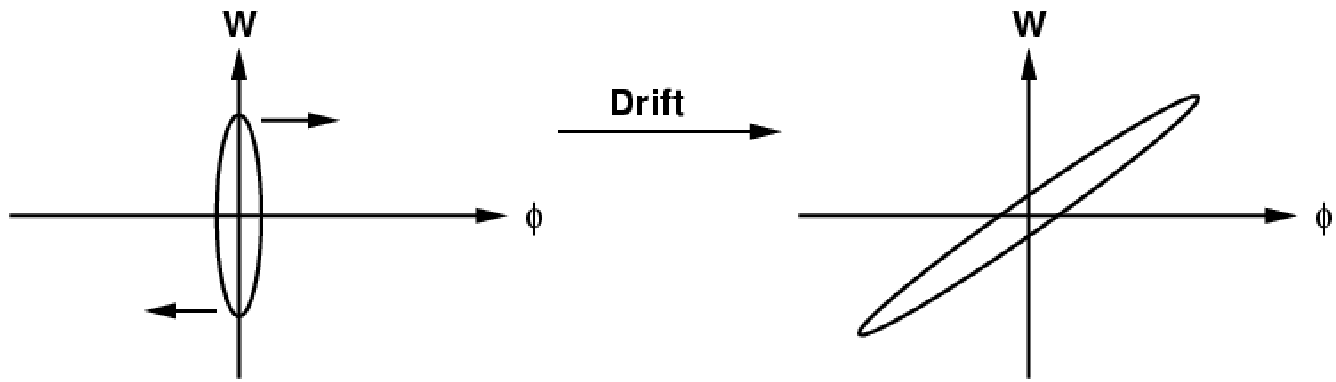
Today, with much cleaner vacuum systems, cavities are operated at much higher surface electric field levels. Fields in accelerators are still sometimes expressed in units of kilpatrick. Now, better physics models are known to predict the surface breakdown field, but kilpatrick is here to stay.

Longitudinal Focusing

In a beam transport system, a bunched beam must be periodically refocused in both transverse and longitudinal planes.

We have already covered alternating-gradient (AG) transverse focusing provided by quadrupoles. Longitudinal focusing is provided by **rebunching cavities**.

The energy spread and space charge defocusing forces spread the beam out along its length. In phase space representation,



The phase spread of the bunch $\Delta\phi$ is related to its physical length $\Delta\phi = \Delta z \cdot \frac{2\pi}{\beta\lambda}$

The energy spread ΔW will cause the ions with higher energy than the synchronous particle to speed ahead, and low energy ions to lag behind. If this continues unabated, the bunches, separated by $\beta\lambda$, will eventually merge.

Debunching Distance

If a bunched beam travels far enough, the bunches will merge and the beam will debunch and the bunch structure will be lost.

The bunch half-length grows to $\Delta z = \beta\lambda/2$ to merge with the adjacent bunch, which will also have grown to a half-length $\beta\lambda/2$.

For a bunch with energy $W = \frac{1}{2} mc^2 \beta^2$, the half-energy spread is $\Delta W = mc^2 \beta \Delta \beta$

The bunch half-length grows to $\frac{\beta\lambda}{2}$ in time t given by $\Delta \beta c t = \frac{\beta\lambda}{2}$

In time *t* the bunch has traveled the debunching distance $L_{debunch} = \beta c t$

Eliminate t and find the debunching distance for a bunched beam with energy spread DW

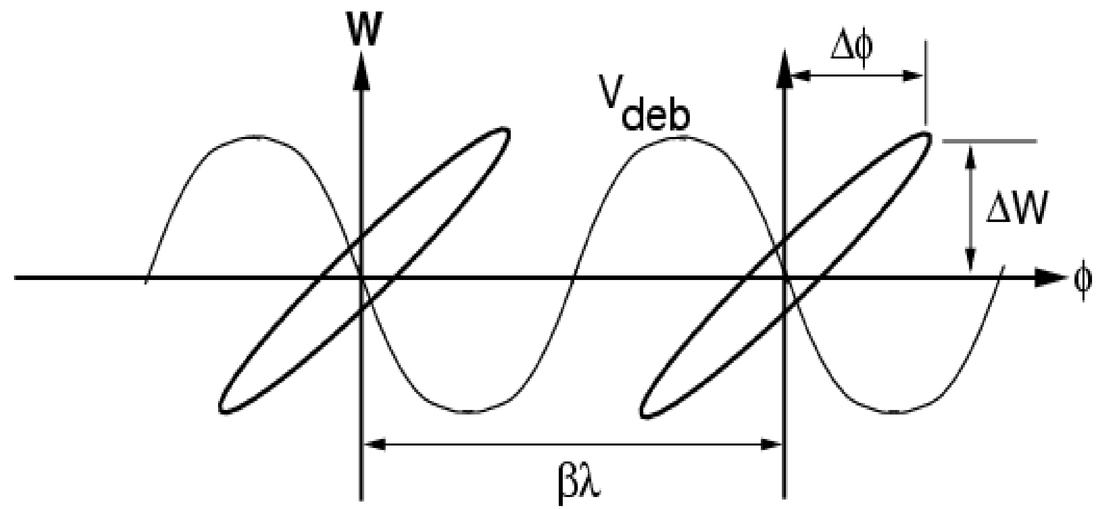
$$L_{debunch} = \beta \lambda \frac{W}{\Delta W}$$

Rebuncher Cavity

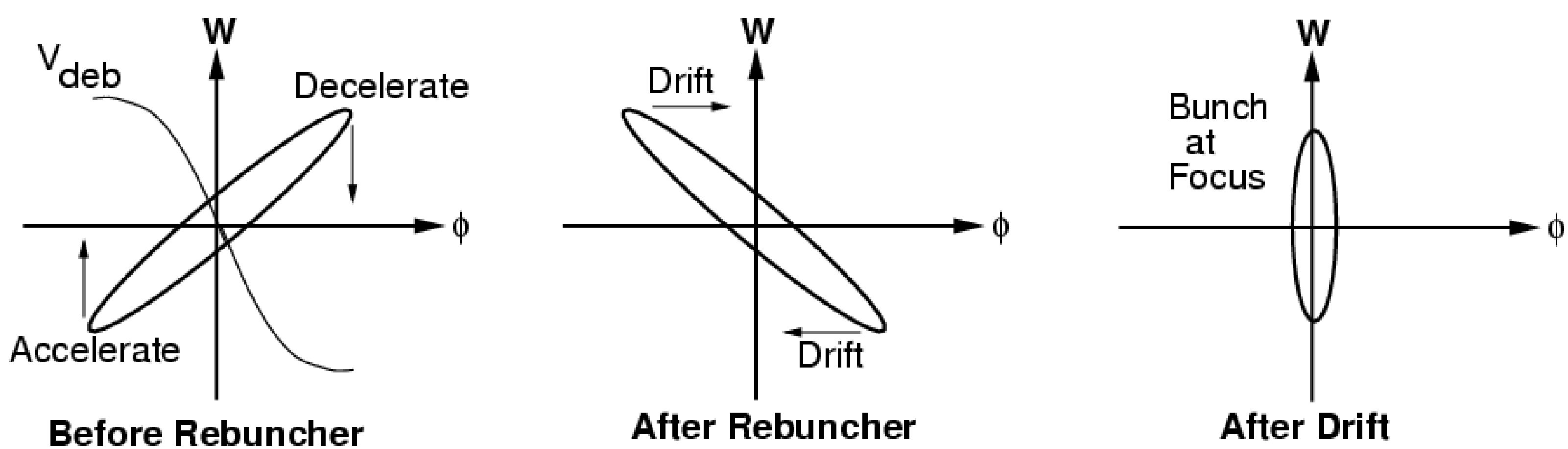
As the bunch spreads in phase (or length) a rebuncher cavity may be used to speed lagging ions up and slow down leading ions. The voltage on the rebuncher cavity gap is phased not to accelerate the center of the bunch, and to act on the ends of the bunch.

The gap voltage V_{deb} accelerates the lagging ions and decelerates the leading ions.

The bunch has spread out in ϕ due to the energy spread ΔW.



The energy of the ends of the bunch are inverted, and the ions at the ends drift back into the center region of the bunch.



Rebuncher Gap Voltage

The rebuncher cavity introduces a coherent energy spread to the bunch, related to the phase spread. For a small longitudial emittance (so we can treat it simply), the cavity induces a new coherent energy spread.

For a particle in the bunch with an energy offset ΔW at a phase offset $\Delta\phi$, the energy offset is changed by the rebuncher cavity to ΔW' by the cavity voltage V(ϕ). The cavity voltage is

$$V_{gap} = V_0 \sin\omega t$$

The required voltage at phase offset $\Delta\phi$ is

$$V_{gap}(\phi) = \frac{\Delta W' - \Delta W}{e} = V_0 \sin\Delta\phi \sim V_0 \Delta\phi$$

Where we restrict the bunch length to the linear part of the sinusoid (< 45 degrees).

$$V_0 = \frac{\Delta W' - \Delta W}{e\Delta\phi}$$

And, using the same formalism as the kick buncher, the bunch will drift back together after drifting $L_{rebunch}$

$$L_{rebunch} = \beta\lambda \frac{W}{\Delta W_{buncher}}$$

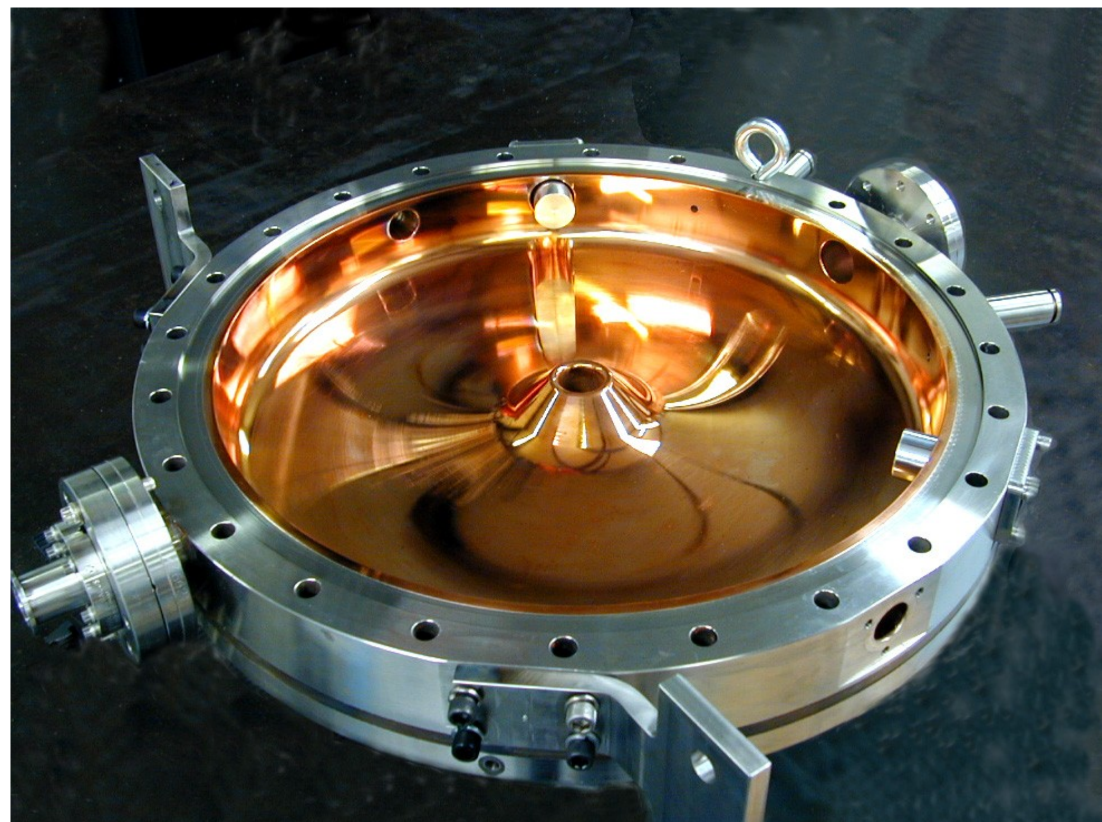
Rebuncher Cavity

We will use 2-D electromagnetics codes such as SUPEFISH to calculate the parameters of a rebuncher cavity.

A rebuncher (or a buncher) cavity is typically a pillbox cavity operating in the TM_{010} mode with the beam traveling along its axis in the region of maximum longitudinal E field. The cavity frequency is usually the same as the bunch or the linac frequency with a precise phase relationship to the bunches passing through it.

The rebuncher phase passes through zero at the center of the bunch, for no net energy change of the bunch (otherwise it would be an accelerating cavity).

This is one-half of the SNS rebuncher, showing the beam aperture and tuning pistons.

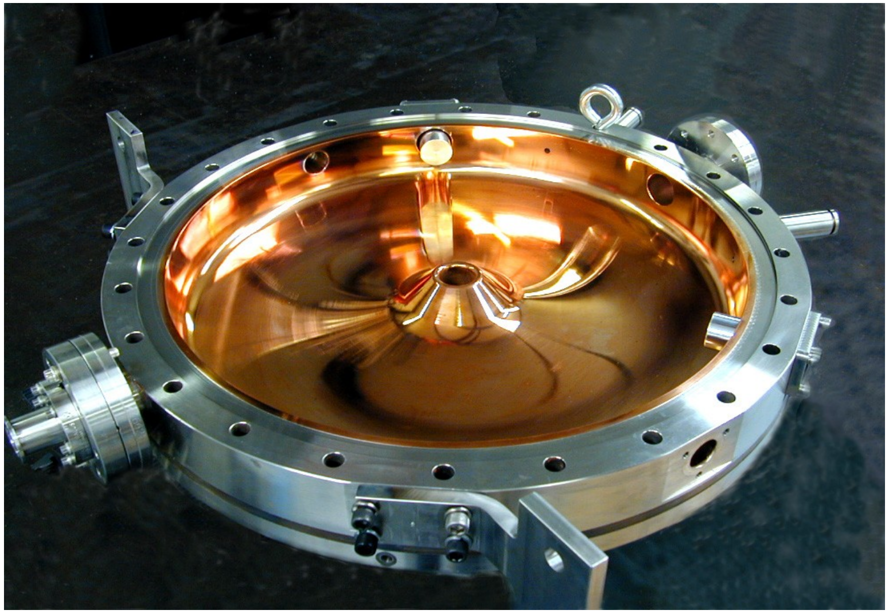


Rebuncher Cavity Optimization

The rebuncher cavity must resonate at the desired frequency.   Further optimization is need in the following areas:

Shunt Impedance	minimize required RF power
Length	to fit into a crowded transport system
Multipactoring	to ease conditioning
Peak surface fields	to minimize tendency of sparking
Atmospheric pressure	to minimize barometric frequency changes
Tuner configuration	maximize effectiveness, minimize RF loss
Vacuum	configure vacuum ports, ultimate pressure
Thermal control	minimize frequency shift
Profile	ease of construction, minimize peak fields

and many others



Computational Modeling of Accelerators

Early accelerator were designed without computers at all

Fields were found analytically, or with “rubber sheet” or electrolytic tank simulations, or with scale-models.

This resulted in the drift tube shapes in the early linacs, which were worked out analytically by Bob Gluckstern. The drift tubes at the low-energy end looked like fat pancakes, and like footballs at the high-energy end.

In the 1960's, the first codes were written.

Electromagnetics codes to find resonant frequencies of simple structures
Particle simulation codes to calculate the beam dynamics

The first simulation runs were run with a few hundred particles. The simulation may take hours. Now, 10^8-10^9 particles in the bunch may be run, equaling the actual number of particles in the bunch. These simulations also may take hours, showing that software bloat keeps pace with computation speed.

Advances simulations now are very sophisticated: e-cloud, multipactoring, wakefield, chaotic behavior are accurately modeled.

Computers now cheaply model expensive accelerators accurately.

Types of Simulations

Beam Dynamics
- Number of macroparticles 10^8 or more
- Using clusters with Teraflop capability

Electrodynamics
- Accurate calculation of complex 3-D structures
- New approaches to boundary conditions over wide scale of detail
- Multipactoring – must iterate over large number of initial conditions

Mechanical CAD
- Detailed modeling of complex structures
- Thermal, atmospheric deformation, vacuum, weight, alignment

Other
- Improved graphical interfaces
- Attempts made at universal language to define accelerator components
- The Web facilitates searching and communications

BUT: computers are not enough. They are a tool, an essential tool, in designing and diagnosing accerators.

Electromagnetics Solvers

EM solvers may be divided into 2-D and 3-D categories. The 2-D solvers may be used in a majority of cases, and are easier and faster to use. The 3-D solvers are just now becoming good enough to give quantatively accurate results.

2-D solvers (partial list)

Superfish - Oldie but goodie. The standard.
Urmel-T - The European choice, finds many modes in one run

3-D solvers (partial list)

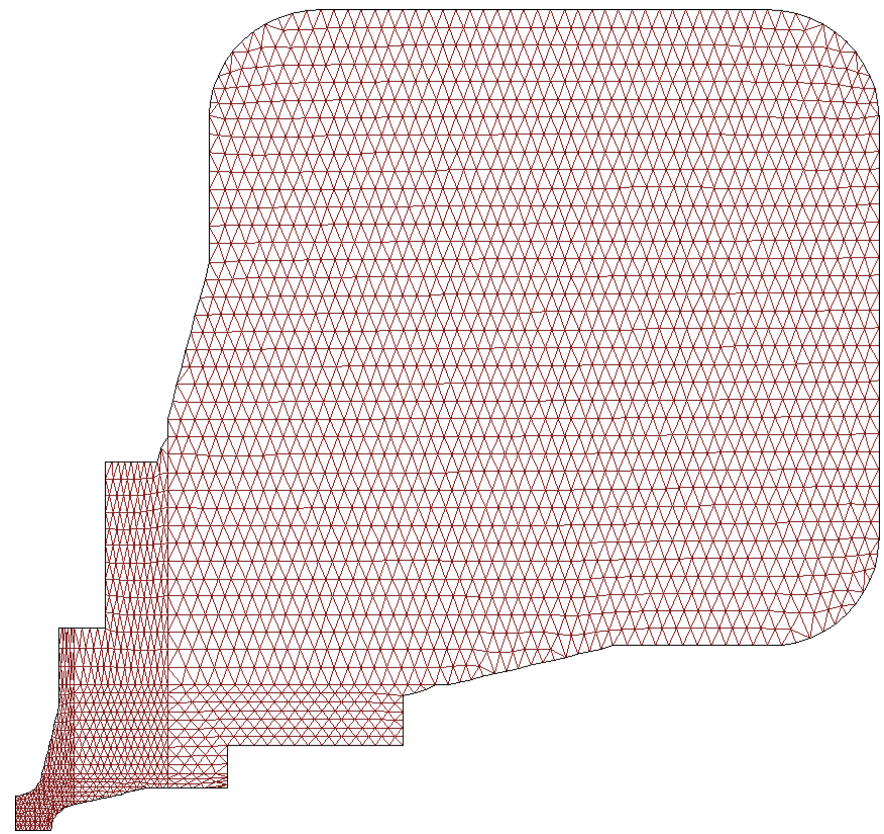
Mafia - The standard. Very difficult to use, and is now obsolete.
gdfidl - Similar to Mafia
Microwave Studio CST - Replacing Mafia, very good boundary condx
HFSS
Ansys electromagnetics module – part of an industry standard CAD code
Field Precision – several codes by Stan Humphries
Omega3 - Kwok Ko's SLAC group, now commercialized, finite element
Superlans - Russian code

Superfish

In this course, we will use the 2-D solver Superfish.   (Derivation of name?)

Superfish is part of the Poisson package developed at LBNL by Halbach and Holsinger and now distributed by the Los Alamos Accelerator Code Group. It is now free to users and may be downloaded from LANL.   This group of codes is also used in the 2-D design of electromagnets and has an optimizer package to tune a design.

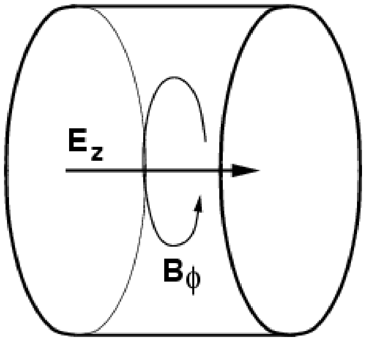
Superfish has had nearly 40 years of existence and is well debugged and reliable.   The triangular mesh follows boundaries well, and the mesh can be rescaled within the problem area.



Components Modeled with Superfish

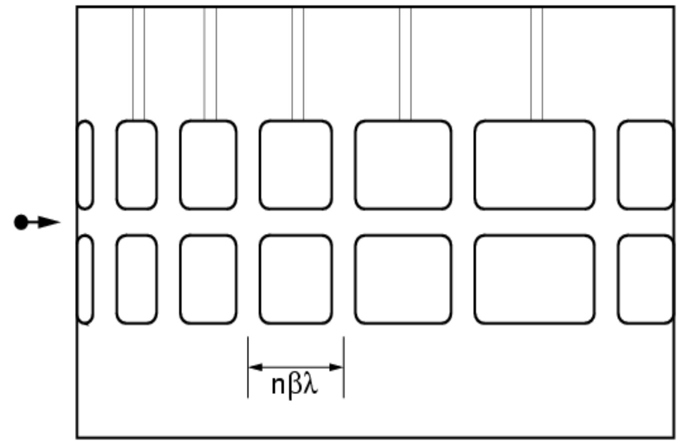
Superfish calculates TM waves in structures
- B_ϕ magnetic component
- E_r, E_z electric field component
- A gap exists on the z-axis to support E_z field

Cavities with rotational symmetry
- Pillbox cavities
- Elliptical Cavities



Cavities with approximate rotation symmetry
- Drift tube linacs with support stems

"Trick" solutions
- RFQs, which support TE modes



General Approach Using 2-D Codes for Accelerator Structures

We will optimize a buncher cavity, DTL cells, and an RFQ cavity

The specifications fall into hierarchical categories

Highest
- Frequency - to match other frequencies in the accelerator

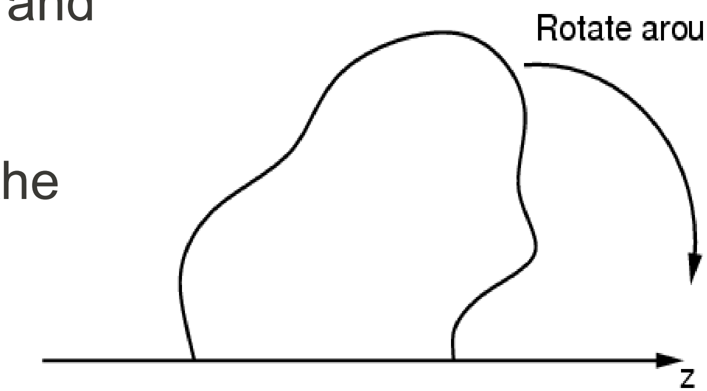
Next
- Maximize shunt impedance
- Minimize peak E-fields
- Minimize thermal hot spots

Other
- Optimize geometry to be manufactureable – avoid strange geometries
- Minimize tendency for multipactoring – avoid parallel surfaces
- Locate places for pumping and tuner ports
- Minimize sensitivity to atmospheric pressure and pressure differences
- Make sure it will fit in the beam line

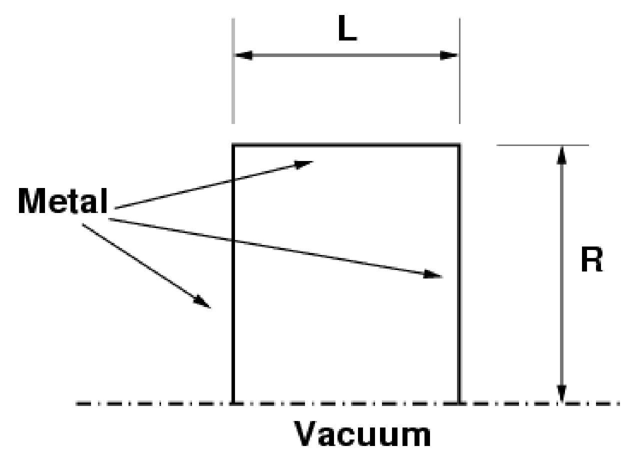
Superfish Input Requirements

The cavity is a solid of revolution. The outline of the solid is given as a list of straight line segments and circular arcs. (Some allow elliptical arcs.)

A gap exist on the z-axis for the generation of the electric field vector E_z.

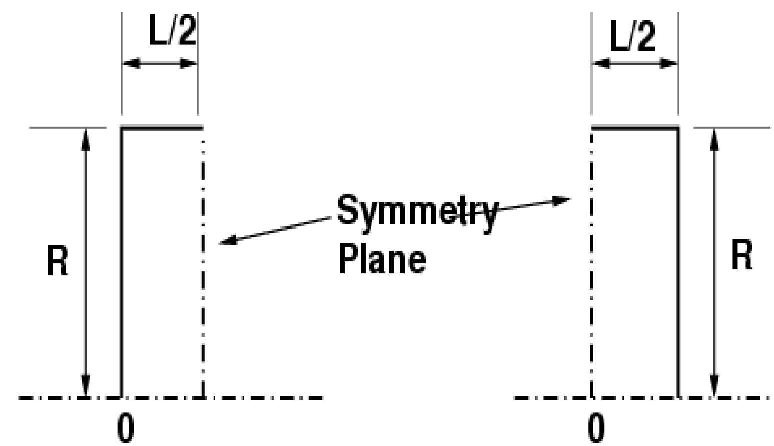


A pillbox cavity is entered as three straight lines. The cavity has the radius R and the length along z of L.



Boundary Conditions and Symmetry Planes

The pillbox cavity, like many cavities, has a symmetry plane at z = L/2. The computational effort is halved by computing only half the cavity volume and applying a symmetry boundary condition.



In a symmetric TM_{010} cavity, the electric field is parallel to the axis at the symmetry point ($E_r = 0$). If the symmetry plane were a conductor, the fields would be unaffected as the electric field is perpendicular to the symmetry plane.

We can specify the symmetry plane as an electric (metallic) boundary, except that no power is dissipated on this surface.

Additional Required Input

The input data deck requires:

A guess at resonant frequency
Designation of metallic surfaces for
power calculation
Mesh size controls
Location of drive point
and some other directives

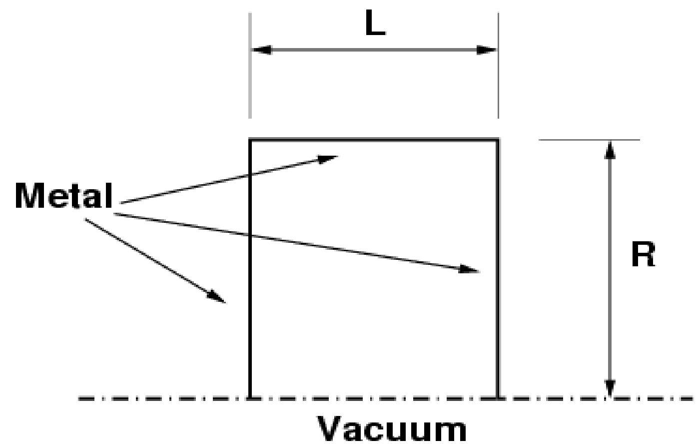
Many of these are defaulted.

Here is the input file for a pillbox cavity 4.7 cm radius, 3 cm long.

The geometry is defined by four points (3 line segments), specified by
$po x = #, y= # $

A fourth line segment closes the region.

```
2.4-GHz TM010 Short Pillbox Cavity
In this problem, Kmax < Lmax

; Copyright 1998, by the University of California.
; Unauthorized commercial use is prohibited.

$reg kprob=1,              ; Superfish problem
dx=.2,                     ; X mesh spacing
freq=2400.,                ; Starting frequency in MHz
xdri=1.,ydri=4.7 $         ; Drive point location

$po x=0.0,y=0.0 $          ; Start of the boundary points
$po x=0.0,y=4.7 $
$po x=3.0,y=4.7 $
$po x=3.0,y=0.0 $
$po x=0.0,y=0.0 $
```



Superfish Output for Pillbox Problem

The resonant frequency for this cavity is 1148.3 MHz:
Superfish finds 1147.13 MHz.

$$\frac{f_c}{2\pi} = \omega_c = \frac{2.405\,c}{R_c}$$

The analytic values of the wall power are 526.8 watts (SF: 523.59) on the outer wall, and 263.26 watts on each end (SF: 261.77), for a total of 1053 watts (SF: 1047.1).

Note again that while specifying peak electric and magnetic fields, the calculated power is an rms quantity.

The calculated magnetic field on the outer wall is 1377 A/m (SF: 1378).

```
SUPERFISH DTL summary
Problem name =.1 m radius, .1 m tall pillbox
Diameter =   20.0000 cm
Mesh problem length [L]  =   10.0000 cm
Frequency [f] (starting value =1150.000) =              1147.423828 MHz
Stored energy [U] for mesh problem only =                   3.74867 mJ
Power dissipation [P] for mesh problem only =               1047.13 W
Q (2.0*pi*f(Hz)*U(J)/P(W)) =                                  25809
Magnetic field on outer wall =                                 1378 A/m
Hmax for wall and stem segments at z=   .00,r=  7.65 cm =      1545 A/m
Emax for wall and stem segments at z= 10.00,r=   .04 cm =     1.000 MV/m

ISEG   zbeg     rbeg     zend     rend  Emax*epsrel   Power     df/dz    df/dr
       (cm)     (cm)     (cm)     (cm)    (MV/m)       (W)         (MHz/mm)
Wall-----------------------------------------------------------------------Wall
 2    .0000    .0000    .0000  10.0000   1.0000    261.7737    .0002    .0000
 3    .0000  10.0000  10.0000  10.0000    .0006    523.5867    .0000 -11.4749
 4  10.0000  10.0000  10.0000    .0000   1.0000    261.7739    .0002    .0000
Wall---------------------------------- Total =    1047.1343 -------------Wall
```

$$H_\theta = -\frac{E_0}{\mu_0 c} J_1(2.405)$$

$$J_1(2.405) = 0.5191$$

Detailed Example: Calculation of a Buncher Cavity

We will design a 200 MHz cavity with a gap voltage of 100 kV. The beam pipe radius is 1 cm. Several steps of optimization will be shown.

First iteration: design a pillbox cavity, no beam pipe, to resonate at 200 MHz. Find the RF power needed to establish a 100 kV peak gap voltage.

The resonant frequency pillbox cavity in the TM010 mode is independent of length. The radius is

$$R = \frac{2.405}{2\pi} \lambda$$

where 2.405 is the first zero of the Bessel function $J_0(k_{01}R)$. Recall that the field is

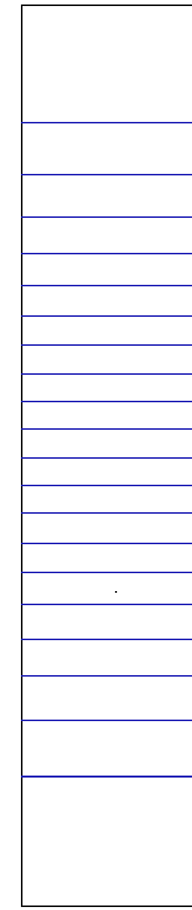
$$E_z \propto J_0(k_{01} R) = 0, \qquad k_{01} = \frac{2.405}{R} = \frac{\omega}{c}$$

We will select a length of 10 cm. For a resonant frequency of 200 MHz, R = 57.4 cm.

Iteration 0: Pillbox Cavity

```
title
buncher 0
run 1
fish
rmax 60.
zmax 11.

nseg 4

rseg 0. 57.4 57.4 0.
zseg 0. 0. 10. 10.
cseg 0. 0. 0. 0.

dz 1.0
rdrive 20.
zdrive 5.
freq 200.
power 2 3 4
boundary 1 0 1 1
begin
end
```

```
SUPERFISH DTL summary
Problem name =buncher 0
SUPERFISH calculates the frequency [f] to at most six place accuracy
depending on the input mesh spacing.


Full cavity length [2L]  =   20.0000 cm         Diameter =  114.8000 cm
Mesh problem length [L]  =   10.0000 cm
Full drift-tube gap [2g] =     .0000 cm
Beta                     =  .1333583       Proton energy =     8.456 MeV
Frequency [f] (starting value = 200.000) =                199.899033 MHz
Eo normalization factor (CON(74)=ASCALE) for  1.000 MV/m =    3833.1
Stored energy [U] for mesh problem only =                  123.52630 mJ
Power dissipation [P] for mesh problem only =                8455.30 W
Q (2.0*pi*f(Hz)*U(J)/P(W)) =                                   18349
Transit time factor [T] =                                     .00000
Shunt impedance [Z] mesh problem only, ((Eo*L)**2/P) =       1.18269 Mohm
Shunt impedance per unit length [Z/L] =                       11.827 Mohm/m
Effective shunt impedance per unit length [Z/L*T*T] =           .000 Mohm/m
Magnetic field on outer wall =                                  1378 A/m
Hmax for wall and stem segments at z= 10.00,r= 44.35 cm =       1545 A/m
Emax for wall and stem segments at z= 10.00,r=   .43 cm =      1.000 MV/m

   Beta        T         Tp        S        Sp       g/L        Z/L
 .13335826   .00000   .15784   .63137   -.10216   .000000   11.826895


ISEG   zbeg     rbeg     zend     rend   Emax*epsrel   Power      df/dz    df/dr
       (cm)     (cm)     (cm)     (cm)    (MV/m)        (W)          (MHz/mm)
Wall-------------------------------------------------------------------------Wall
  2    .0000    .0000    .0000  57.4000    .9999   3600.2800    .0001    .0000
  3    .0000  57.4000  10.0000  57.4000    .0011   1254.7433    .0000   -.3483
  4  10.0000  57.4000  10.0000    .0000   1.0000   3600.2815    .0001    .0000
Wall---------------------------------- Total =    8455.3047 --------------Wall
```



Superfish assumes that the input is a half-cavity and states the full cavity length is 20 cm. However, the calculated power is for the mesh problem, 10 cm. Superfish normalizes the longitudinal field to 1 MV/m, so the 10 cm mesh sustains a voltage of 100 kV. The power of 8455 watts corresponds to 100 kV along the 10 cm axial gap.

Iteration 1: Introduce a beam pipe

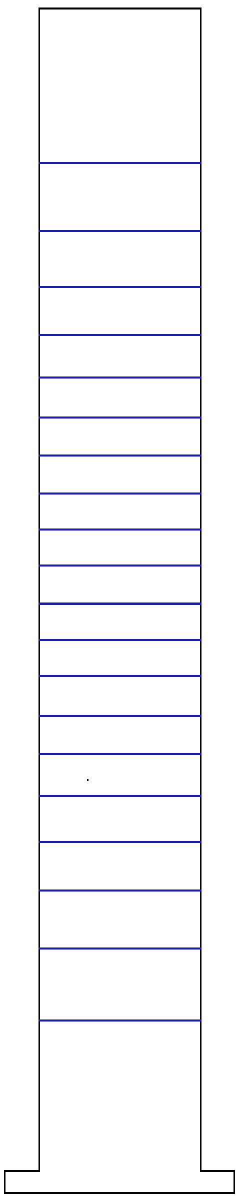
A beam pipe with 1 cm radius is added. The beam pipe extends 2 cm from each end of the 10 cm gap.

This cavity extends over 14 cm, so the field integral

$$\int E_z(z)dz = 140\,kV$$

The power must be normalized for a field integral of 100 kV. Power scales as the square of the field (or voltage integral), and the calculated power of 16621 watts for the 140 kV integral scales to 8480 watts, slightly higher than the 8455 watts of Iteration 0.

The peak E-field is 1.172 MV/m, scaled linearly from 1.641 MV/m, higher than the 1.0 MV/m in Iteration 0. This is due to the sharp corners of the beam pipe inside the cavity.

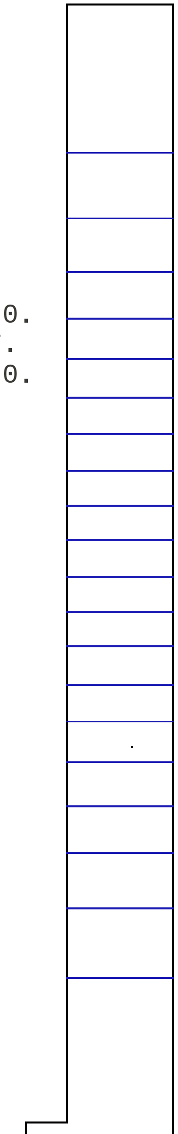


Iteration 2: Use Symmetry Condition

We will calculate the first half of the cavity, as it is symmetric around its middle.

```
1SUPERFISH DTL. summary
 Problem name =buncher 2
 SUPERFISH calculates the frequency [f] to at most six place accuracy
 depending on the input mesh spacing.


 Full cavity length [2L]  =   14.0000 cm         Diameter =  114.8000 cm
 Mesh problem length [L]  =    7.0000 cm
 Full drift-tube gap [2g] =    4.0000 cm
 Beta                     =  .0933538       Proton energy =     4.115 MeV
 Frequency [f] (starting value = 200.000) =                199.905411 MHz
 Eo normalization factor (CON(74)=ASCALE) for  1.000 MV/m =    5361.0
 Stored energy [U] for mesh problem only =                  120.84946 mJ
 Power dissipation [P] for mesh problem only =                8272.39 W
 Q (2.0*pi*f(Hz)*U(J)/P(W)) =                                   18349
 Transit time factor [T] =                                    -.31946
 Shunt impedance [Z] mesh problem only, ((Eo*L)**2/P) =        .59233 Mohm
 Shunt impedance per unit length [Z/L] =                        8.462 Mohm/m
 Effective shunt impedance per unit length [Z/L*T*T] =           .864 Mohm/m
 Magnetic field on outer wall =                                  1927 A/m
 Hmax for wall and stem segments at z=  2.00,r= 44.05 cm =       2161 A/m
 Emax for wall and stem segments at z=  2.00,r=  1.22 cm =      1.637 MV/m

    Beta        T        Tp       S        Sp       g/L        Z/L
  .09335376  -.31946   .19966   .70491  -.16273   .285714    8.461886


 ISEG   zbeg     rbeg     zend     rend  Emax*epsrel   Power     df/dz    df/dr
        (cm)     (cm)     (cm)     (cm)    (MV/m)       (W)         (MHz/mm)
 Wall----------------------------------------------------------------------Wall
   2    .0000    .0000    .0000   1.0000    .0226       .0000    .0000    .0000
   3    .0000   1.0000   2.0000   1.0000    .6160       .0006    .0000    .0005
   4   2.0000   1.0000   2.0000  57.4000   1.6365   7044.9067   -.0004    .0000
   5   2.0000  57.4000   7.0000  57.4000    .0007   1227.4795    .0000   -.3483
 Wall---------------------------------- Total =    8272.3867 -------------Wall
```

```
title
buncher 2
run 1
fish
rmax 60.
zmax 8.

nseg 6

rseg 0. 1. 1. 57.4 57.4 0.
zseg 0. 0. 2. 2.   7.  7.
cseg 0. 0. 0. 0.   0.  0.

dz 0.5
rdrive 20.
zdrive 5.
freq 200.
power 2 3 4 5
boundary 1 0 1 1
begin
end
```



The problem length is 7 cm, for a voltage of 70 kV.   Normalized the power to 100 kV, it is 4220.5 watts for the *half* cavity, or 8441 watts for a full cavity.

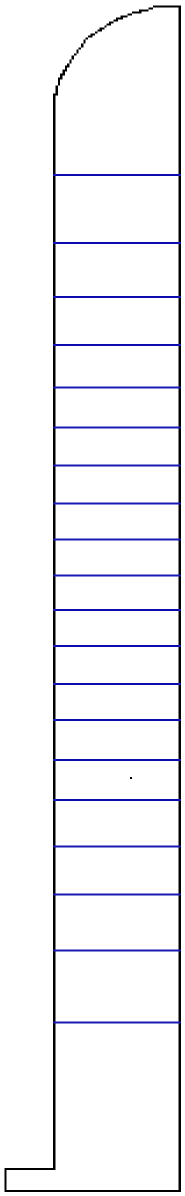
Iteration 3: Round the Outer Radius, half-cavity

A rounded outer radius is included.   The frequency is 203.758 MHz, up from 200, as some H-field volume has been removed.

The scaled power is 7957 watts, a little less, as the current path has been reduced in length, providing less surface for power dissipation.

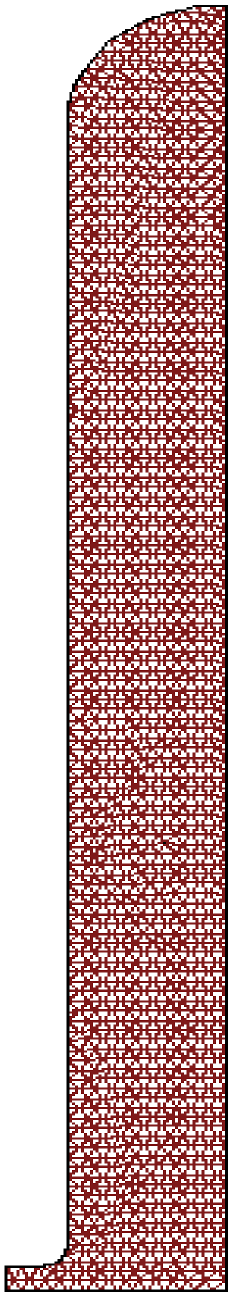
The absence of a sharp inside corner inhibits multipactoring in that region.

```
title
buncher 3
run 1
fish
rmax 60.
zmax 8.

nseg 6

rseg 0. 1. 1. 52.4 57.4 0.
zseg 0. 0. 2. 2.   7.  7.
cseg 0. 0. 0. 0.  -5.   0.

dz 0.5
rdrive 20.
zdrive 5.
freq 200.
power 2 3 4 5
boundary 1 0 1 1
begin
end
```



The peak field around the beam pipe is 1.169 MV/m.

Iteration 4 and 5: Round the Beam Hole



The sharp transition from the cavity to the beam pipe results in a high electric field at that point. Rounding the geometry will reduce the field and tendency for sparking.

Here, the mesh is shown. The values, scaled to 100 kV field integral are:

Frequency = 203.770 MHz
Power = 7964 watts
Epeak = 1.106 MV/m, down from 1.169 MV/m

Iteration 5: increase the outer radius proportionally to bring the frequency back to 200 MHz.   The new outer radius is 58.48 cm, and the resultant frequency is 199.935 MHz.

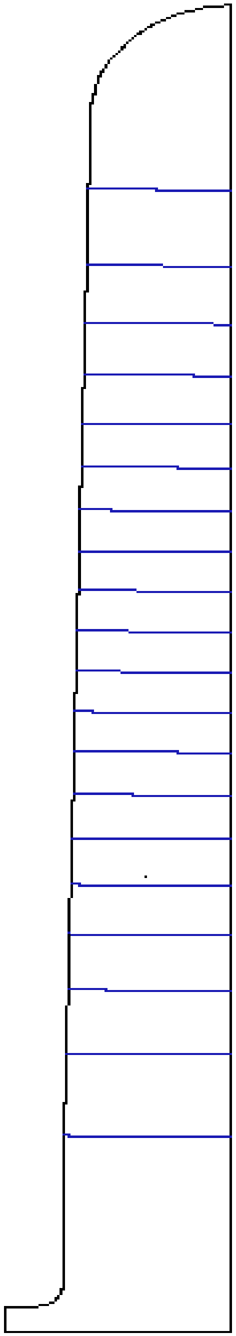
Iteration 6: Tilt the Sidewall

Parallel sidewalls tend to promote multipactoring. Also, parallel sidewalls require thick walls to counteract the effect of atmospheric pressure. A tilted sidewall allows the wall thickness to be reduced.

A 2 cm tilt from beam hole to outer curved section is introduced. The half-cavity length is 8 cm, including beam pipe. Normalized to 100 kV field integral for a full cavity, the power requirement is 6941 watts, the lowest yet. The peak electric field is 0.901 MV/m.

The *shunt impedance* is defined as $Z_{shunt} = \frac{1}{2P_{rms}} V^2$

The shunt impedance of this cavity is 0.72 Megohms, or 4.5 Megohms/meter, lower than 5.9 Megohms/meter for the pillbox example. Lowering the peak electric field with the rounded beam pipe has the consequence of lowering the shunt impedance.



Iteration 7: Increase Shunt Impedance with Nose Cones

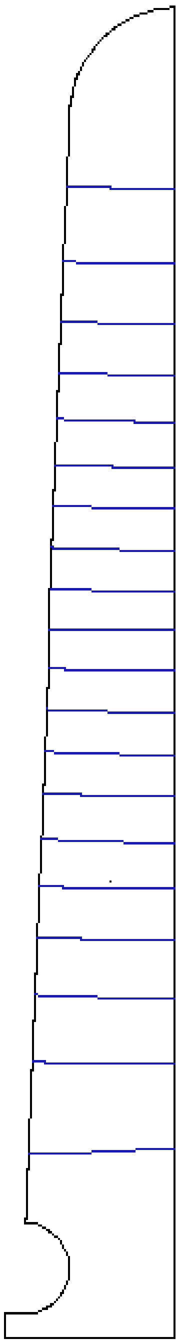
The nose cones concentrate the field toward the center of the cavity. The power for the full cavity is 7344 watts, and the shunt impedance is 0.87 Megohms, or 5.44 Megohms/meter for the 16 cm long cavity.

This is a lower shunt impedance, but the **Transit Time Factor** for this cavity is improved.

$$TTF = \frac{\int E(z)\cos(kz)\,dz}{\int E(z)\,dz}$$

The TTF is a measure of the energy gain of an ion as it crosses the gap as the field is varying in time.

Superfish requires the symmetry plane to start at the left to calculate the correct numerical value.

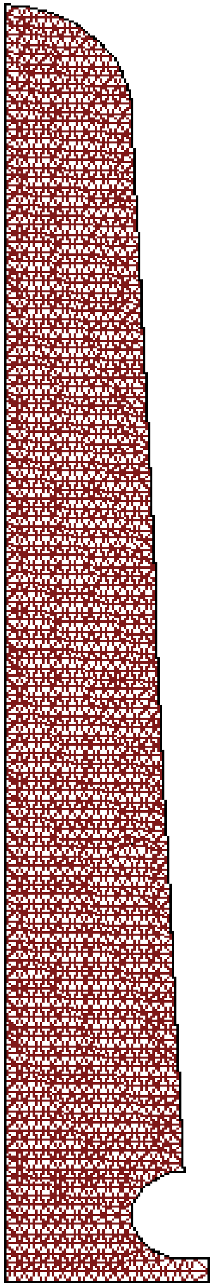


Reverse the Symmetry

```
1SUPERFISH DTL summary
 Problem name =buncher 6
 SUPERFISH calculates the frequency [f] to at most six place accuracy
 depending on the input mesh spacing.


 Full cavity length [2L]  =   16.0000 cm         Diameter =  116.9600 cm
 Mesh problem length [L]  =    8.0000 cm
 Full drift-tube gap [2g] =   10.0000 cm
 Beta                     =  .2251015       Proton energy =    24.714 MeV
 Frequency [f] (starting value = 200.000) =               210.886703 MHz
 Eo normalization factor (CON(74)=ASCALE) for  1.000 MV/m =    4896.1
 Stored energy [U] for mesh problem only =                 118.51226 mJ
 Power dissipation [P] for mesh problem only =              14105.21 W
 Q (2.0*pi*f(Hz)*U(J)/P(W)) =                                  11133
 Transit time factor [T] =                                    .80439
 Shunt impedance [Z] mesh problem only, ((Eo*L)**2/P) =       .45373 Mohm
 Shunt impedance per unit length [Z/L] =                       5.672 Mohm/m
 Effective shunt impedance per unit length [Z/L*T*T] =         3.670 Mohm/m
 Magnetic field on outer wall =                                 1838 A/m
 Hmax for wall and stem segments at z=   .00,r= 45.05 cm =      2055 A/m
 Emax for wall and stem segments at z=  5.12,r=  3.66 cm =     2.409 MV/m
```

Beta	T	Tp	S	Sp	g/L	Z/L
.22510152	.80439	.11587	.49431	.12204	.625000	5.671661

ISEG	zbeg (cm)	rbeg (cm)	zend (cm)	rend (cm)	Emax*epsrel (MV/m)	Power (W)	df/dz (MHz/mm)	df/dr (MHz/mm)
Wall								Wall
2	.0000	.0000	.0000	58.4800	1.4512	6741.2168	-.2675	.0000
3	.0000	58.4800	5.0000	53.4800	.1081	1852.0092	-.3504	-.3470
4	5.0000	53.4800	7.0000	5.0000	1.0830	5507.0117	-.0291	-.0012
5	7.0000	5.0000	5.0000	3.0000	2.4094	4.8204	.0794	.0455
6	5.0000	3.0000	7.0000	1.0000	2.2559	.1566	.0309	.0096
7	7.0000	1.0000	8.0000	1.0000	.0301	.0000	.0000	.0000
Wall					Total =	14105.2139		Wall



An apparent bug in the code requires the symmetry plane to be on the left for correct calculation of the transit time factor.

Summary of Buncher Calculations

Iteration	Type	Radius (cm)	Symmetry	Frequency (MHz)	Voltage Integral (kV)	Power (Watts)	Peak E-Field (MV/m)	Zsh MegOhm	Zsh/L MOhm/m
0	Pillbox	57.40	full	199.9	100	8455	1.000	0.591	5.91
1	+Beam Tube	57.40	full	199.9	140	8480	1.172	0.590	4.21
2		57.40	half	199.9	70	8441	1.172	0.592	4.23
3	Round Outer	57.40	half	203.8	70	7957	1.169	0.628	4.49
4	Round Inner	57.40	half	203.8	70	7964	1.106	0.628	4.48
5	Adj. radius	58.49	half	199.9	70	8175	1.106	0.611	4.37
6	Tilt Wall	58.49	half	206.1	80	6941	0.901	0.720	4.50
7	Nose cone	58.49	half	210.9	80	5737	1.500	0.871	5.45

The final iteration has the best shunt impedance, but also the highest peak electric field.

Beam Capture: Bunching

The acceleration process requires the beam to be formed into bunches with the correct separation in {time, phase, distance}. The beam from the ion source is continuous (direct current) during the length of the *macropulse*.

Several techniques may be used to form the d.c. beam into bunches.

Chopping. Throw away beam between bunches. This is very inefficient as most of the beam from the ion source is discarded. This is sometimes used in electron accelerators.

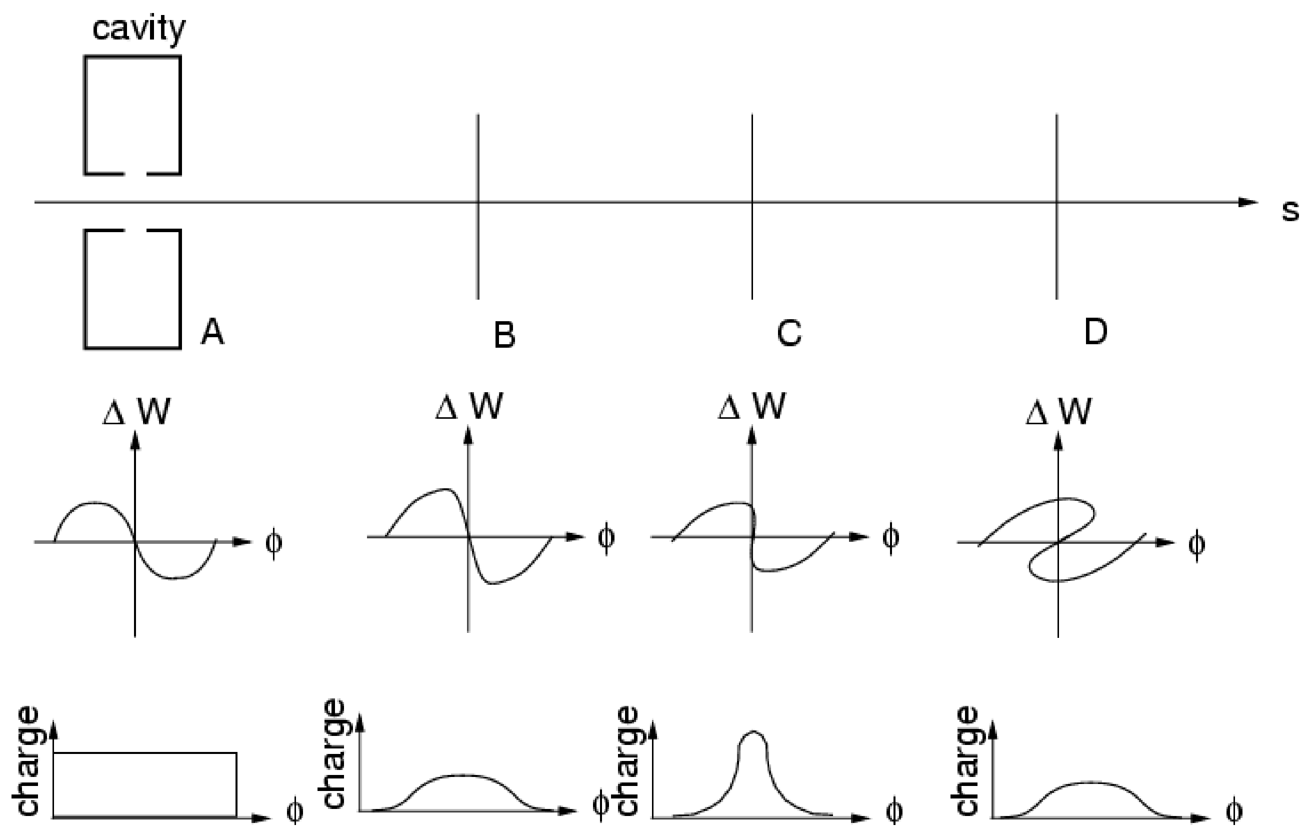
Kick Bunching. Introduce a coherent energy spread in the d.c. beam with a single cavity and then allow the off-energy components to catch up with or lag to points of concentration of particles, which will form the centroids of the bunch.

Adiabatic Bunching. Almost the same as kick bunching, but accomplished over a large number of kicks to allow more precise and efficient gathering of the all the ions into a bunch. Used in RFQs.

Kick Bunching

This method was used on almost all linacs before the invention of RFQs. A high-voltage injector (Cockcroft-Walton) provided beam to an Alvarez linac with a kick buncher in the LEBT (Low Energy Beam Transport).

A cavity at point A induces an energy variation in the input beam. As the beam drifts through points B, C and D, the charge density increases and maximizes at point C, and then begins to diffuse again.



What is the best waveform to produce a tight bunch?

The kick buncher works by introducing a coherent energy variation, which leads to density variations in the beam by a coherent velocity variation.

If the ion that receive the largest energy variation are assumed to get a velocity kick necessary to advance their phase by 90 degrees ($\pi/4$ radians),

$$\beta^2 = \frac{2W}{mc^2} \quad \rightarrow \quad 2\beta\, d\beta = \frac{2\, dW}{mc^2}$$

For a buncher cavity with a frequency *f* with free space wavelength λ, the phase slip is

$$\frac{\Delta\phi}{2\pi} = \frac{ds}{\beta\lambda} \qquad \text{so} \qquad \frac{\Delta\phi}{2\pi} = \frac{L_{drift}}{\beta^3\lambda}\frac{dW}{mc^2}$$

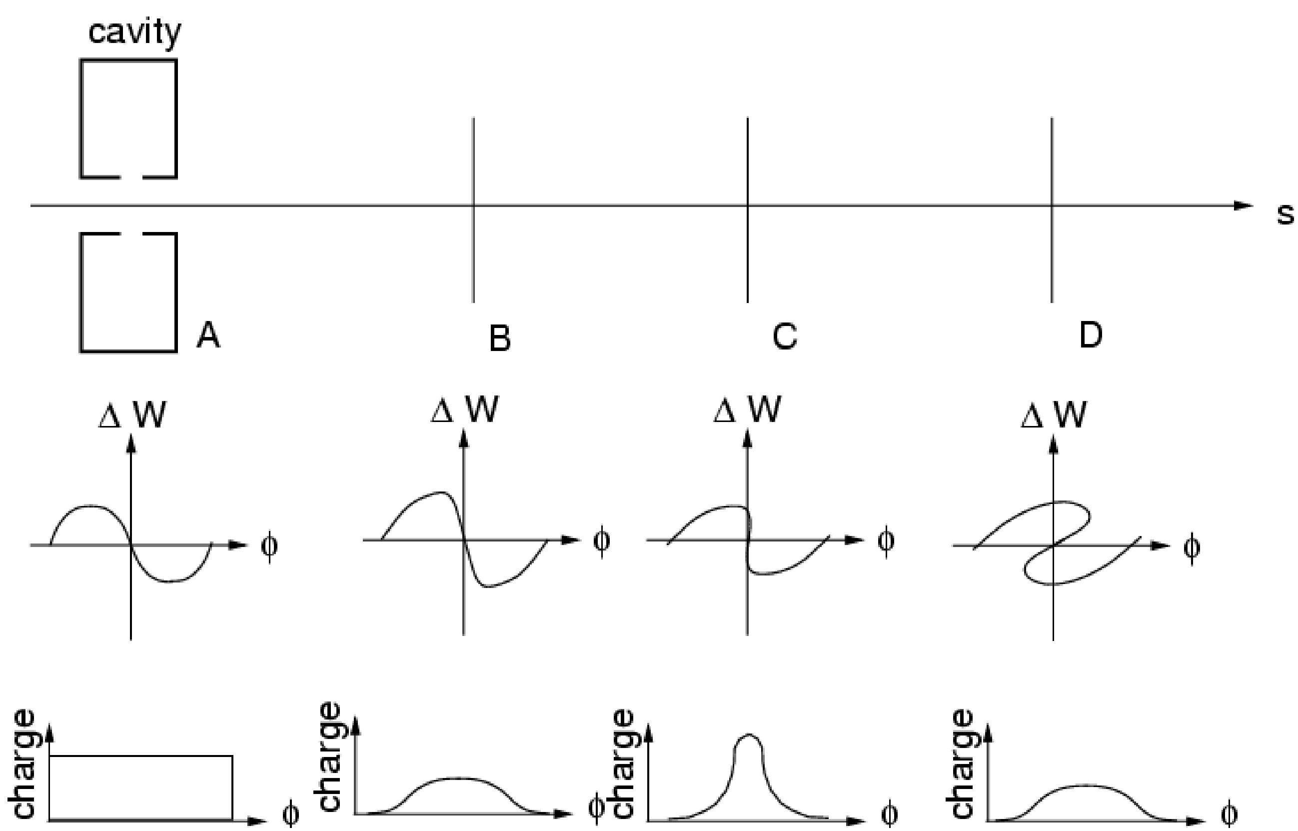
Requiring ions at $\pm\pi/2$ to drift by $\pm\pi/2$ over drift length L_{drift},

$$\frac{\Delta W}{mc^2} = \frac{1}{4}\frac{\beta^3\lambda}{L_{drift}} \qquad \text{thus} \qquad V_{gap} = \frac{1}{4}\left(\frac{mc^2}{e}\right)\frac{\beta^3\lambda}{L_{drift}}$$

Note that this process produces an energy spread in the beam, which must be small enough to fit into the accelerating bucket.

The energy spread from V_{gap} may be made smaller by increasing L_{drift}, but this imposes a tighter requirement on the energy of the beam from the ion source. Why?

This bunching method usually results in only 50-60% of the beam being captured, and space charge forces reduces even this.



Appendix - The Pillbox Cavity

Detailed Analysis

- Resonant Frequency Spectrum
- Field Distribution
- Power Dissipation
- Stored Energy

The Pillbox Cavity Field Configuration

The fields in the pillbox cavity are (Wangler, page 28)

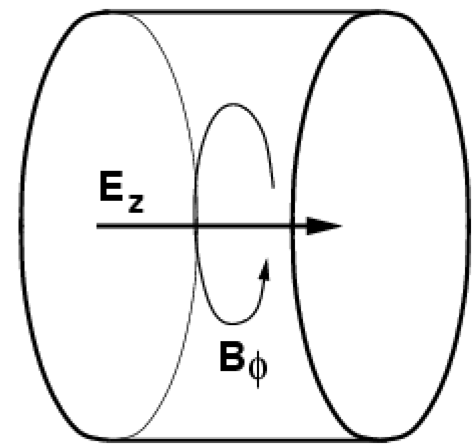
$$E_z = E_0 J_0(k_r r) \cos\omega t$$

$$B_\theta = -\frac{E_0}{c} J_1(k_r r) \sin\omega t$$

$$\omega_c = k_r c = \frac{2.405\, c}{R_c}$$

Length = L

Radius = R_c



The stored energy, power and unloaded quality factor are

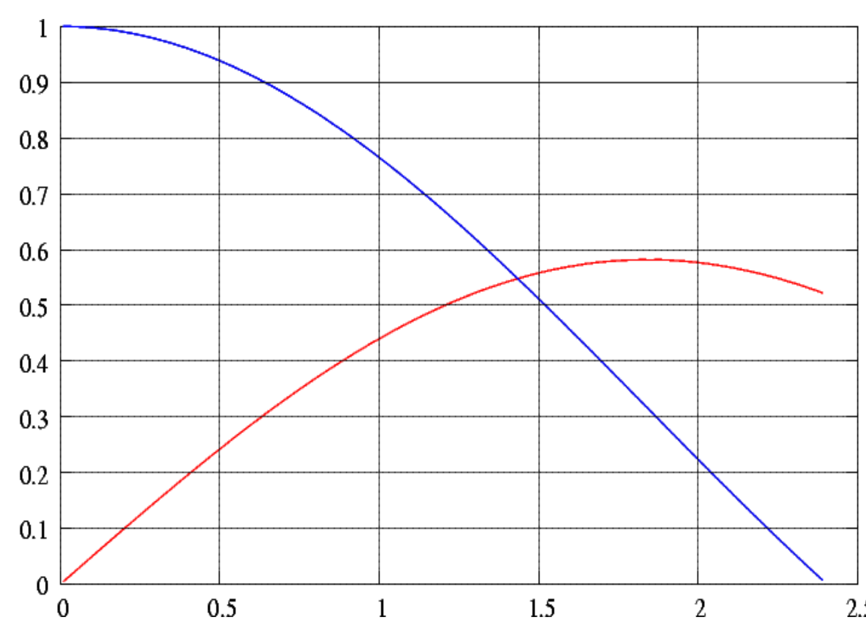
$$U = \frac{\pi \epsilon_0 L R_c^2}{2} E_0^2 J_1^2(2.405)$$

$$P = \pi R_c R_s E_0^2 \frac{\epsilon_0}{\mu_0} J_1^2(2.405)\,(L+R_c)$$

$$Q = \frac{\omega_c U}{P} = \frac{2.405\sqrt{\mu_0/\epsilon_0}}{2 R_s} \frac{1}{1+R_c/L}$$

$$R_s = \frac{1}{\delta\sigma}, \qquad \delta = \sqrt{\frac{\lambda}{\pi\sigma\mu_0 c}}$$

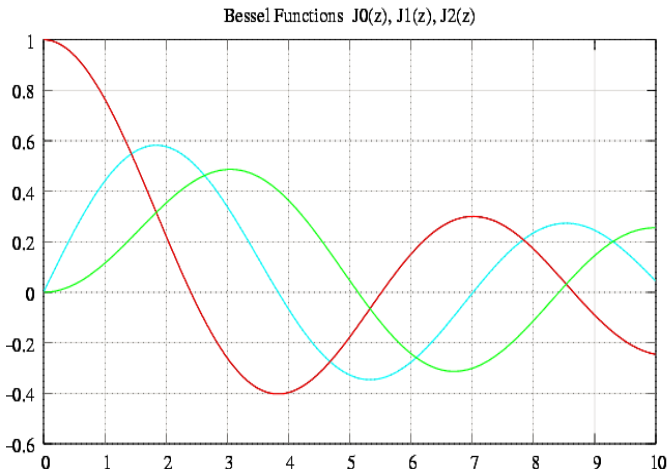
$$\frac{1}{\sigma_{copper}} = \rho_{copper} = 1.724\times 10^{-8}\,\Omega - m$$



The TM_{010} Pillbox Mode

The radius of the cavity is *a*.
$m = 0$, $n = 1$, $p = 0$.

$$E_z(r) = A J_0(k_{01} r)$$



The boundary condition that $E_z(a) = 0$ is satisfied if $k_{01}\, a$ = the first zero of J_0. $(k_{01}\, a) = 2.405$.

$$k_z = \frac{\pi p}{L_{cav}} = 0, \qquad \frac{\omega^2}{c^2} = k_{mn}^2 + k_z^2 = k_{01}^2$$

We can solve this for the resonant frequency of the TM_{010} mode

$$k_{01} = \frac{2.405}{a}, \qquad \omega = k_{01} c, \qquad f = \frac{\omega}{2\pi} = \frac{2.405}{2\pi} \frac{c}{a}$$

For a pillbox cavity with a radius $a = 1$ meter, the TM_{010} mode frequency is

$f = 114.9$ MHz

and is independent of the length of the cavity.

The TM_{010} Fields

$m = 0, n = 1, p = 0$

$$
\begin{aligned}
E_z &\propto J_m(k_{mn} r) \quad \cos m\theta \quad \cos k_z z \\
E_r &\propto p J'_m(k_{mn} r) \quad \cos m\theta \quad \sin k_z z \\
E_\theta &\propto \frac{p}{r} J_m(k_{mn} r) \quad \sin m\theta \quad \sin k_z z
\end{aligned}
$$

$p = 0$ so E_r and $E_\theta = 0$

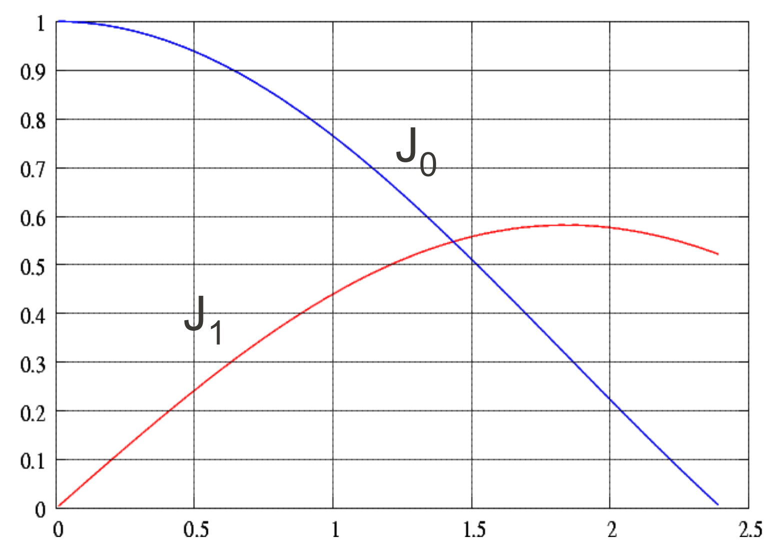
$$E_z(r) = E_0 J_0\left(2.405 \frac{r}{a}\right)$$

$$
\begin{aligned}
B_z &= 0 \\
B_r &\propto -\frac{i}{r} J_m(k_{mn} r) \quad \sin m\theta \quad \cos k_z z \\
B_\theta &\propto -i J'_m(k_{mn} r) \quad \cos m\theta \quad \cos k_z z
\end{aligned}
$$

$m = 0$ so $\sin m\theta = 0$ so $B_r = 0$

B_z is always zero

$$B_\theta(r) = i B_0 J_1\left(2.405 \frac{r}{a}\right)$$



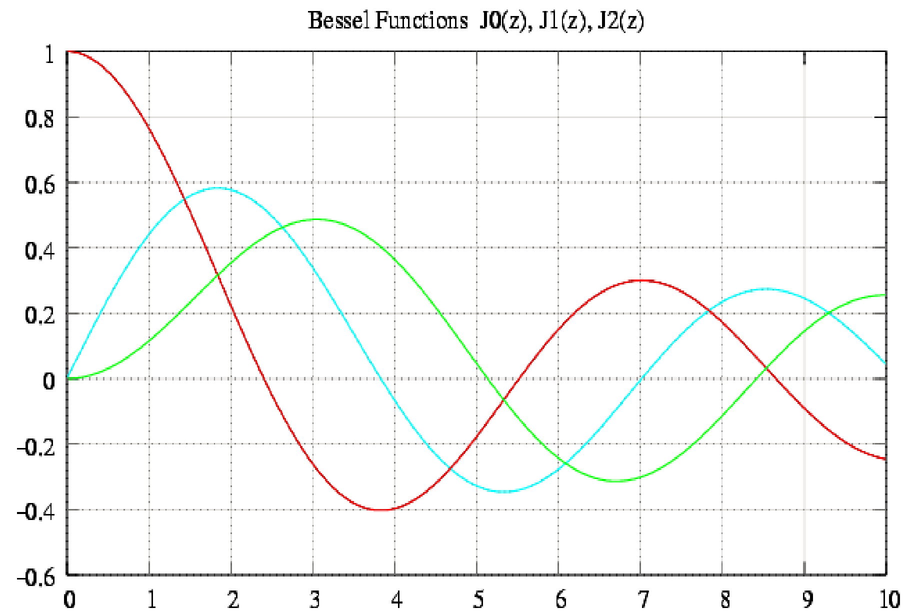
Note that E_z is at a maximum on the axis and zero at $r = a$, and that B_θ is maximum about ¾ of the way out.

E_0 and B_0 are constants.

The TE_{mnp} Modes

These are the transverse electric modes.

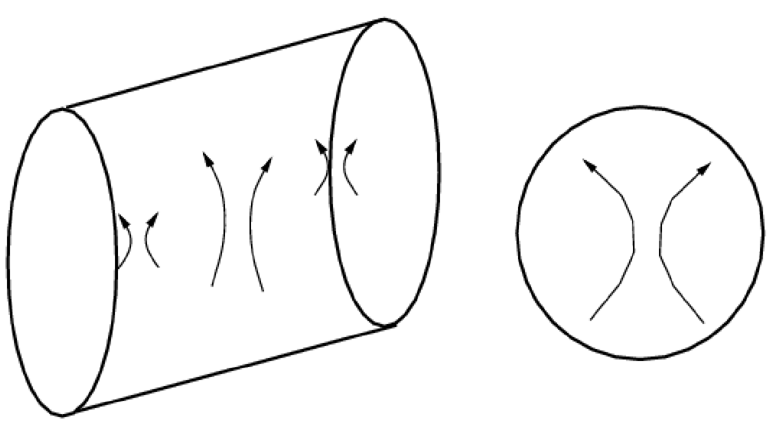
$$
\begin{aligned}
E_z &= 0 \\
E_r &\propto \frac{i}{r} J_m(k_{mn} r) \quad \sin m\theta \quad \sin k_z z \\
E_\theta &\propto i J'_m(k_{mn} r) \quad \cos m\theta \quad \sin k_z z \\
\\
B_z &\propto J_m(k_{mn} r) \quad \cos m\theta \quad \sin k_z z \\
B_r &\propto p J_m(k_{mn} r) \quad \cos m\theta \quad \cos k_z z \\
B_\theta &\propto -\frac{p}{r} J_m(k_{mn} r) \quad \sin m\theta \quad \cos k_z z
\end{aligned}
$$



$$k_z = \frac{\pi p}{L_{cav}}, \qquad \frac{\omega^2}{c^2} = k_{mn}^2 + k_z^2$$

m = 0, 1, 2,... azimuthal
n = 1, 2, 3,... radial
p = 1, 2, 3,... longitudinal (why not 0?)

Here, $k_{mn} = x'_{mn}/R_{cavity}$ and the x'_{mn} are the zeros of J'_m.



$x'_{01} = 3.832$, $x'_{02} = 7.016$, $x'_{03} = 10.174$,...

The RFQ uses a TE_{210} mode of operation.

TM_{0np} Mode Spectrum

For m = 0, the modes are azimuthally symmetric (no ϕ dependence).

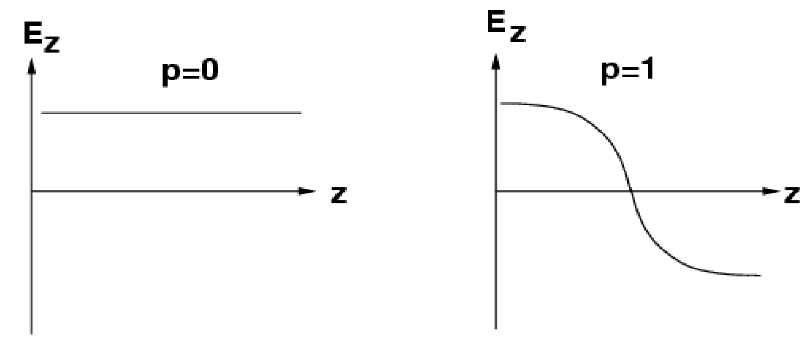
The TM_{0n0} modes show a radial dependence of $E_z(r)$ that has n zeros (including the zero at the outer radius). These modes are not harmonically related, but lie along the zeros of $J_0(k_{0n})$.
Those values of k_{0n} are 2.405, 5.520, 8.654...

For $p > 0$, $E_z(z)$ has p nodes (zeros) along the z-axis. The frequency of the TM_{0np} modes for $p > 0$ depend on the length of the cavity, and E_r and E_θ have components which are non-zero, except at the outer radius.

$$\frac{\omega^2}{c^2} = k_{mn}^2 + k_z^2, \qquad f = \frac{c}{2\pi}\sqrt{k_{mn}^2 + \left(\frac{\pi p}{L_{cav}}\right)^2}$$

$$E_z \propto J_m(k_{mn} r) \quad \cos m\theta \quad \cos k_z z$$
$$E_r \propto p\, J'_m(k_{mn} r) \quad \cos m\theta \quad \sin k_z z$$
$$E_\theta \propto \frac{p}{r} J_m(k_{mn} r) \quad \sin m\theta \quad \sin k_z z$$

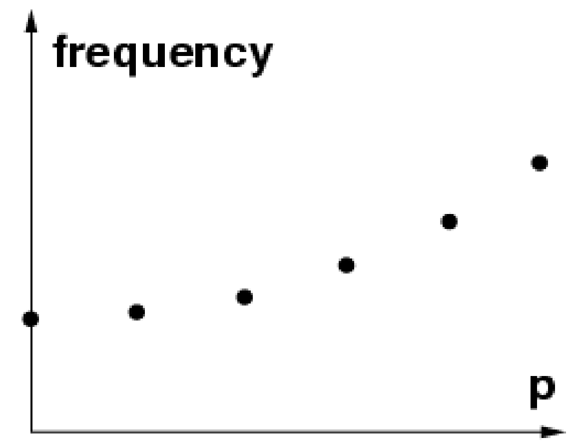
$$B_z = 0$$
$$B_r \propto -\frac{i}{r} J_m(k_{mn} r) \quad \sin m\theta \quad \cos k_z z$$
$$B_\theta \propto -i\, J'_m(k_{mn} r) \quad \cos m\theta \quad \cos k_z z$$



Higher Modes of a Single Pillbox Cavity

These structures can be characterized as having a spectrum of frequencies.

$$\frac{\omega^2}{c^2} = k_{mn}^2 + k_z^2, \qquad f = \frac{c}{2\pi}\sqrt{k_{mn}^2 + \left(\frac{\pi\, p}{L_{cav}}\right)^2}$$



This plot is a type of **dispersion plot**, which relates the resonant frequency to the phase shift along the axis of the field.

The lowest mode, p = 0, has a uniform $E_z(z)$ distribution along the axis of the accelerator. Notice that the next higher mode is only slightly removed from the fundamental mode, and has a $E_z(z)$ distrbution that has one node halfway down the linac.

Energy can be coupled into this and higher modes by several methods, such as construction errors or beam loading, and alter the desired field configuration of the linac. We will discuss this further.

Pillbox: Power on Walls

Power on the outer wall at $r = R_c$

$$P_{outer} = \frac{R_s}{2} H_{wall}^2 \times Area = \pi R_s R_c \, E_0^2 \, \frac{\epsilon_0}{\mu_0} \, J_1^2(2.405) \, L$$

Power on each endwall (this is a little more difficult)

$$P_{end} = \frac{1}{\mu_0^2} \frac{R_s}{2} \int B_\theta^2 \, 2\pi \, r \, dr = \pi \, R_s \, E_0^2 \frac{\epsilon_0}{\mu_0} \int_0^{R_c} J_1^2 (2.405 \frac{r}{R_c}) \, r \, dr$$

The identities that allow the integral to be evaluated are

$$\int_0^P [J_n(ax)]^2 x \, dx \equiv \frac{P^2}{2} \left([J_n'(aP)]^2 + (1 - \frac{n^2}{a^2 P^2}) [J_n(aP)]^2 \right)$$

and

$$J_1'(a) = J_0(a) - \frac{1}{a} J_1(a)$$

Some terms cancel and one goes to zero.

$$P_{end} = \pi R_s E_0^2 \frac{\epsilon_0}{\mu_0} \frac{R_c^2}{2} J_1^2(2.405)$$

The total power on the surfaces is then

$$P_{total} = P_{wall} + 2 P_{end} = \pi R_c R_s E_0^2 \frac{\epsilon_0}{\mu_0} J_1^2(2.405)\,(L + R_c)$$
$$= \pi R_s (Z_0 E_0)^2 R_c (L + R_c) J_1^2(2.405)$$

Note that this is the **rms** (thermal) power, and the fields are expressed as **peak** fields. We have converted from peak to rms power by the factor of ½ in the expression for power.

$$P_{rms} = \frac{R_s}{2} H^2 \times Area$$

Compare results with Superfish. A cavity with R_s = L = 0.1 meters is computed.

The default electric field of 1 MV/m on the axis is used.

```
title
.1 m radius, .1 m tall pillbox
run 1
fish
xmax 10.0
ymax 11.0
nseg 4
rseg 0 10 10 0
zseg 0 0 10 10
dz .1
boundary 1 0 1 1
freq 1150.0
power 2 3 4
beta 1.0
begin
end
```

Stored Energy in Pillbox Cavity

The peak stored energy (at the crest of the field) is

$$U_{peak} = \frac{1}{2}\epsilon_0 \int E_z^2(r)\,dV = \frac{1}{2}\mu_0 \int H_\phi^2(r)\,dV$$

$$U_{peak} = \frac{1}{2}\epsilon_0 E_0^2 2\pi L \int J_0^2(k_c r)\,r\,dr$$

Using the same integration identity, and $J_0'(z) = -J_1(z)$

$$U_{rms} = \frac{1}{2}U_{peak} = \frac{1}{2}\pi\epsilon_0 L R_c^2\, E_0^2\, J_1^2(2.405)$$

Evaluated for the pillbox cavity, U_{rms} = 3.746 mJoule (SF: 3.749 mJoule)

$$Q_0 = \frac{\omega_c U_{rms}}{P_{rms}} = \frac{2.405}{2}\frac{Z_0}{R_s}\frac{L}{L+R_c} \qquad Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \sim 377\,\Omega$$

The unloaded Q_0 for the cavity is 25667 (SF: 25809).

*Lesson learned: Superfish uses **peak** fields, but calculates **rms** power and stored energy.*