Classical Mechanics and Electromagnetism

- MKS vs CGS
- Maxwell equations
- Electromagnetic waves
- special relativity
- coordinate system
- harmonic oscillators

MKS vs CGS unit conversion

	MKS	CGS
Speed of the light	$1/\sqrt{\varepsilon_0\mu_0}$	c
Electric field, potential	$E \quad \phi$	$E/\sqrt{4\pi\varepsilon_0} \quad \phi/\sqrt{4\pi\varepsilon_0}$
Charge density, current	$q \quad \rho \quad j$	$q\sqrt{4\pi\varepsilon_0} \quad \rho\sqrt{4\pi\varepsilon_0} \quad j\sqrt{4\pi\varepsilon_0}$
Magnetic induction	B	$B\sqrt{\mu_0/4\pi}$
Vacuum impedance	$\sqrt{\mu_0/\varepsilon_0} \approx 377\Omega$	$4\pi/c$

Maxwell's equation

- James Clerk Maxwell, "On Physical Lines of Force", 1861-1862

	Differential form	Integral form
Gauss's Law	$\nabla \bullet E = \rho / \varepsilon_0$	$\oiint E \bullet dA = Q / \varepsilon_0$
	$\nabla \bullet B = 0$	$\oiint B \bullet dA = 0$
Farady's Law	$\nabla \times E = -\frac{\partial B}{\partial t}$	$\oint E \bullet dl = -\frac{\partial \phi_B}{\partial t}$
Ampere's Law	$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$	$\oint B \bullet dl = -\frac{\partial \phi_E}{\partial t}$

Continuity: $\nabla \bullet \vec{J} + \frac{\partial \rho}{\partial t} = 0$

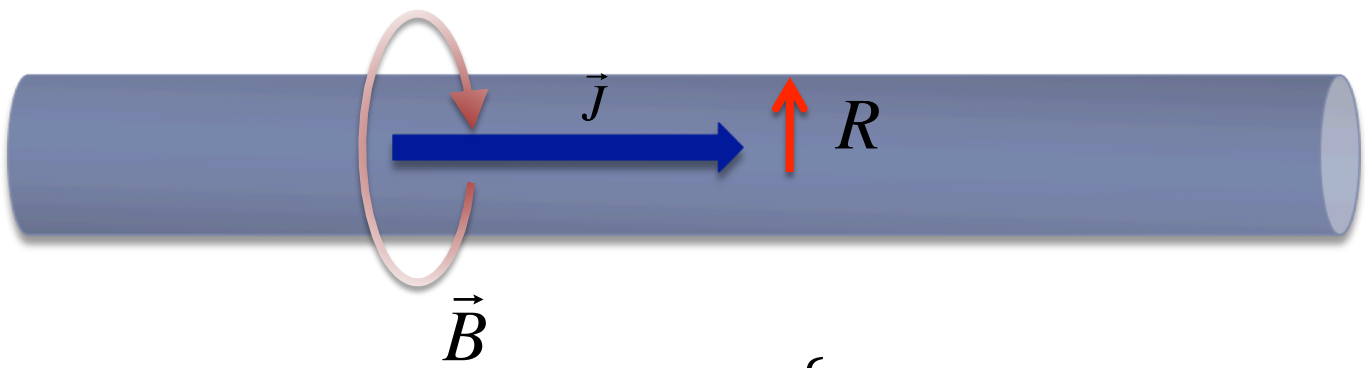
Electric field of a beam of charged particles

- Uniformly distributed beam of charged particles with charge intensity ρ

$$\nabla \bullet E = \rho / \varepsilon_0 \quad \Rightarrow \quad E_r = \begin{cases} \dfrac{\rho}{2\varepsilon_0} r & for\ r < R \\ \dfrac{\rho}{2\varepsilon_0} \dfrac{R^2}{r} & for\ r > R \end{cases}$$

Magnetic field of a uniform current

- Uniformly distributed beam of charged particles with current intensity of $\vec{J}$



$$\nabla \times \vec{B} = \mu \vec{J} \quad \Rightarrow \quad B_\phi = \begin{cases} \dfrac{J}{2\mu} r & \text{for } r < R \\ \dfrac{J}{2\mu} \dfrac{R^2}{r} & \text{for } r > R \end{cases}$$

Electromagnetic wave propagation

- propagation of an electromagnetic wave in vacuum

$$\begin{cases} (\nabla^2 - \mu\varepsilon \dfrac{\partial^2}{\partial t^2})\vec{E} = 0 \\ (\nabla^2 - \mu\varepsilon \dfrac{\partial^2}{\partial t^2})\vec{B} = 0 \end{cases} \Rightarrow \begin{cases} \vec{E}(\vec{r},t) = \vec{E}_0 e^{i\omega t - i\vec{k}\cdot\vec{r}} \\ \vec{B}(\vec{r},t) = \vec{B}_0 e^{i\omega t - i\vec{k}\cdot\vec{r}} \end{cases}$$

ω: angular frequency,

$k = |\vec{k}| = \dfrac{2\pi}{\lambda}$: wave number, λ is the wavelength

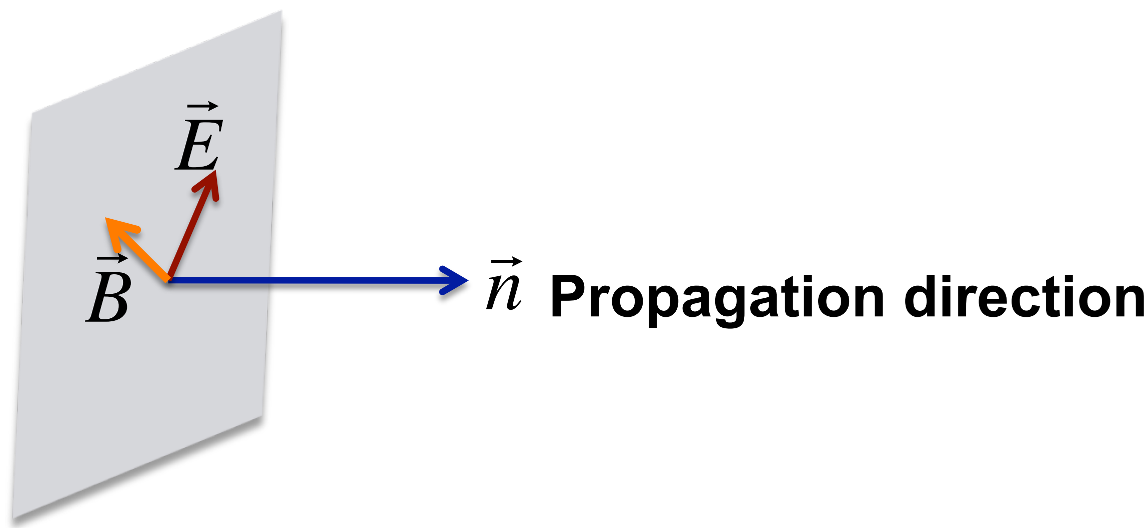
$k = \dfrac{\omega}{c}$: dispersion relation

Traveling wave in vacuum

- transverse wave:

$$\vec{E}\cdot\vec{n} = 0$$

$$\vec{B}\cdot\vec{n} = 0$$



$$\vec{B}_0 = \sqrt{\mu\varepsilon}\,\vec{n}\times\vec{E}_0$$

Special Relativity

- The laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to the one or the other of the two systems in uniform translational motion relative to each other.

- Invariant of the speed of the light, i.e. the speed of the light in vacuum is the same for all reference systems

- Nothing can be faster than the speed of the light

Lorentz Transformation

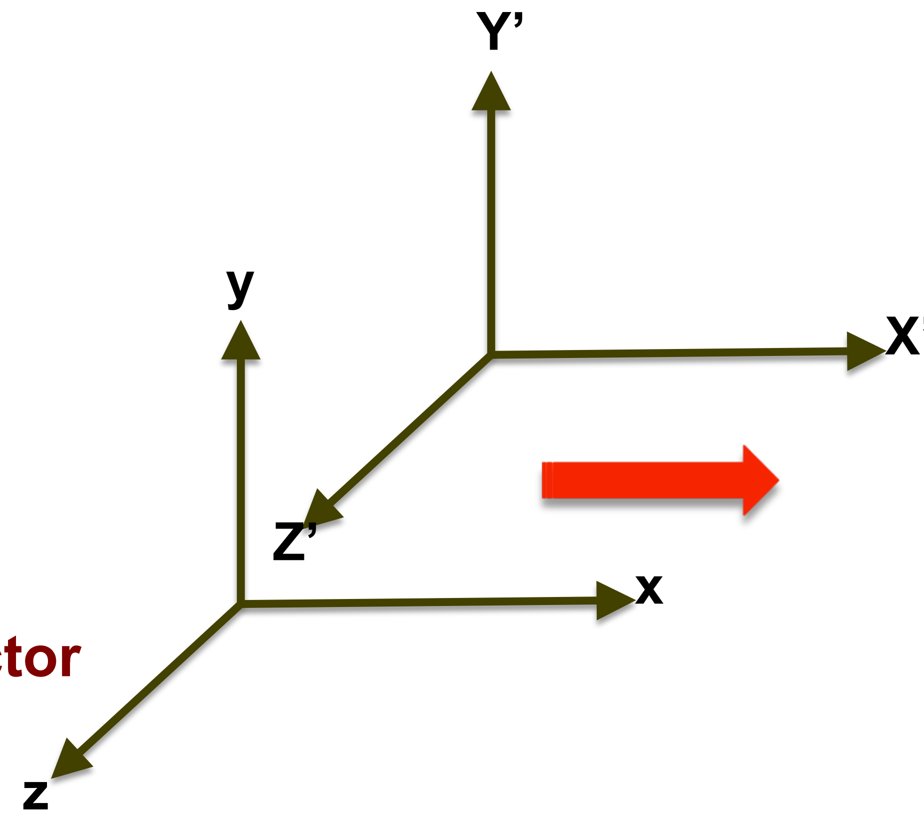
- two space-time reference systems, (t, x, y, z) and (t', x', y', z') moving w.r.t. at a velocity of v along the axis of x

$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

$$x' = \gamma(x - vt)$$

$$y' = y \qquad z' = z$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$: **Lorentz factor**



Special Relativity

- Time dilution: a clock in the moving reference frame(primed frame) runs slower than in the rest frame of the clock

$$\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) \xrightarrow{\Delta x = 0} \Delta t' = \gamma \Delta t$$

- Length contraction: the length of a project in the direction of primed frame relative to the rest frame is shorter in the moving frame

$$\Delta x = \gamma\left(\Delta x' + v\Delta t'\right) \xrightarrow{\Delta t' = 0} \Delta x' = \frac{\Delta x}{\gamma}$$

Space-time Four vector

- definition

$$S = (ct, x, y, z)$$

- Length of four vector: Lorentz invariant

$$|S| = \sqrt{(ct)^2 - x^2 - y^2 - z^2}$$

- Lorentz invariant: length of four vector

$$S'^2 = (ct')^2 - x'^2 - y'^2 - z'^2 = c^2t^2 - x^2 - y^2 - z^2 = S^2$$

Four momentum

$$\vec{P} = \left(\gamma mc, \vec{p}\right)$$

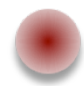
- Conservation of Four momentum:
  - Energy conservation
  - Momentum conservation

Fixed target experiment

- In the lab frame

$\vec{v}_1$

$\vec{v}_2 = 0$



$$\vec{P}_1 = \left(E_1/c, \vec{p}_1 \right) \qquad \vec{P}_2 = \left(m_0 c, 0 \right)$$

- In the center of mass frame:

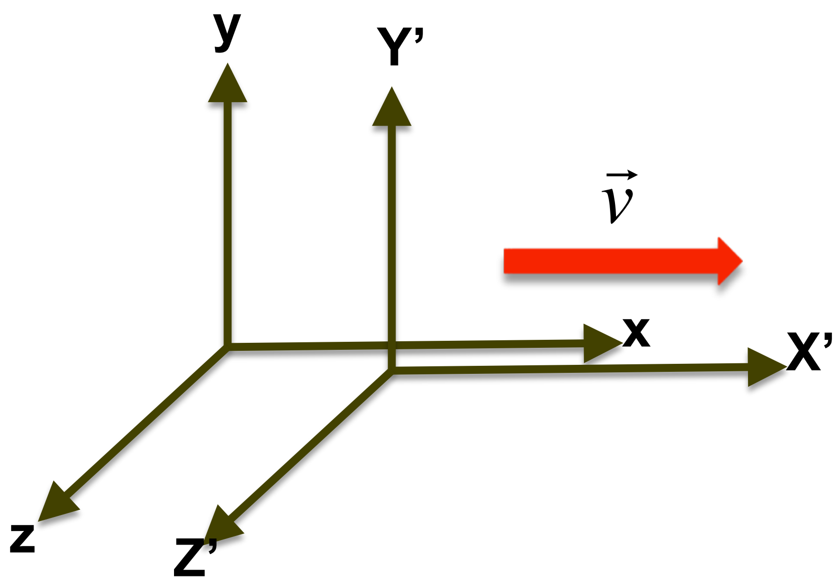
$$\left(E_{cm}/2c, \vec{p} \right) \qquad \left(E_{cm}/2c, -\vec{p} \right)$$

$$\left| \vec{P}_1 + \vec{P}_2 \right|^2_{LF} = \left| \vec{P}_1 + \vec{P}_2 \right|^2_{CM}$$

$$E^2_{cm} = (E_1 + m_0 c^2) - c^2 p_1^2$$

Lorentz transformation of E&M fields

- Prime reference system boost w.r.t. the unprime system along x direction



$$E'_x = E_x$$

$$E'_y = \gamma(E_y - \beta B_z)$$

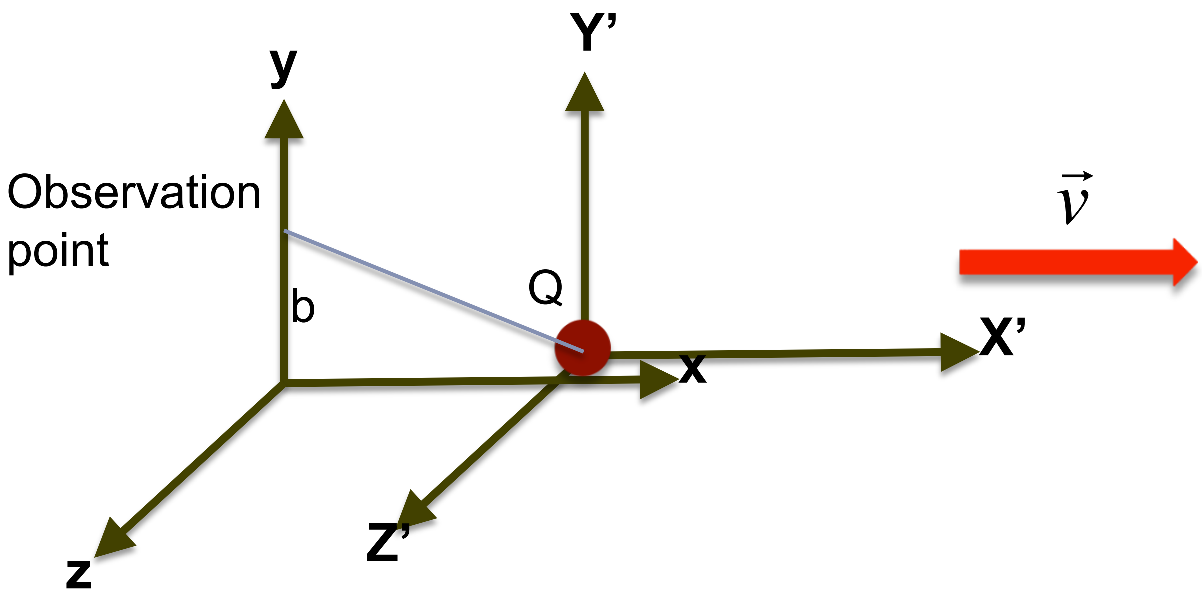
$$E'_z = \gamma(E_z + \beta B_y)$$

$$B'_x = B_x$$

$$B'_y = \gamma(B_y + \beta E_z)$$

$$B'_z = \gamma(B_z - \beta E_y)$$

Electromagnetic field of a moving charge



- Ratio of transverse electric field and longitudinal electric field is ~vt/b,
- Generate transverse magnetic field
- Electric field lines becomes whiskbroom shape instead of isotropically distributed

Lorentz force

- A moving charged particle with velocity V in a electric-magnetic field

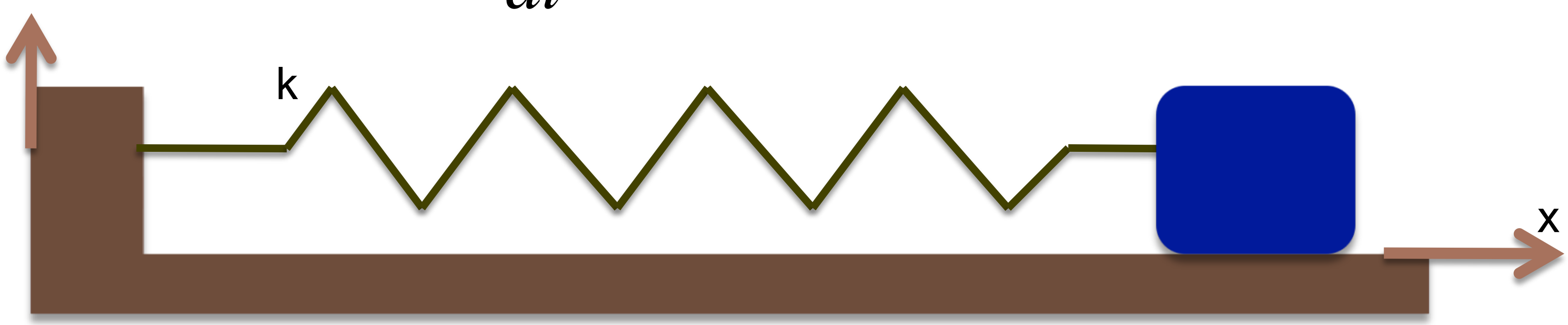
$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

- Home work: Design a Wien filter, a device of static electric and magnetic field arranged in a way that only allow charged particle q with a specific velocity v to pass. In other words, select electric field and magnetic field which is transparent to a 35keV electron beam. Please also describe the trajectory of electron in side the wien filter.

Harmonic oscillator

- Equation of motion
  - In absence of any frictions or other external force

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$



$$x(t) = x_0 \cos(\omega t + \chi) \qquad \frac{dx(t)}{dt} = -x_0 \omega \sin(\omega t + \chi)$$

Driven Harmonic oscillator

- In the presence of an external driving force

$$\frac{d^2x}{dt^2} + \omega^2 x = f(t)$$

$$x(t) = x_0 \cos(\omega t + \chi) + \frac{1}{\omega}\int_0^t \sin\omega(t - t')f(t')dt'$$

Driven Harmonic oscillator

- Equation of motion

$$\frac{d^2x(t)}{dt^2} + \omega^2 x(t) = f(t) = \sum_{m=0} C_m e^{i\omega_m t}$$

- for $f(t) = C_m e^{i\omega_m t}$

$$\frac{d^2x(t)}{dt^2} + \omega^2 x(t) = C_m e^{i\omega_m t}$$

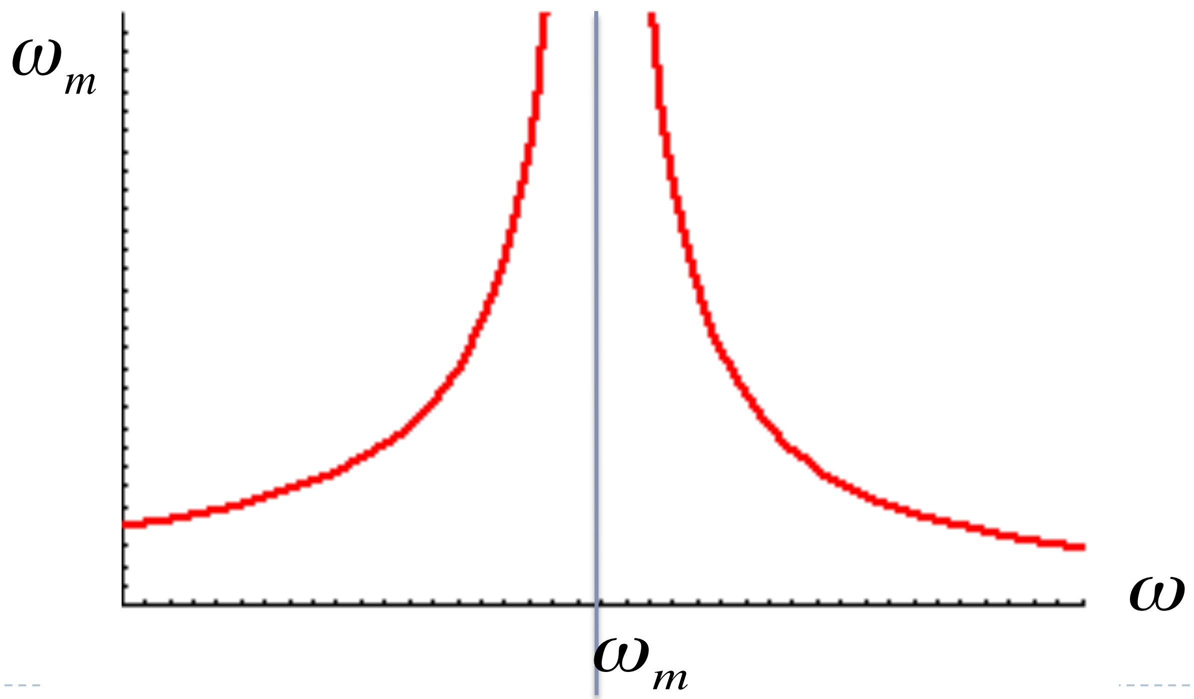
- Assume solution is like $x(t) = Ae^{i\omega t} + A_m e^{i\omega_m t}$

$$\boxed{A_m = \frac{C_m}{\omega^2 - \omega_m^2}}$$

Resonance Response

- Response of the harmonic oscillator to a periodic force is

$$x(t) = Ae^{i\omega t} + \frac{C_m}{\omega^2 - \omega_m^2}$$



Coupled Harmonic Oscillator

- Equation of motion

$$x'' + \omega_x^{\,2} x = q^2 y \qquad y'' + \omega_y^{\,2} y = q^2 x$$

- Assume solutions are:

$$x = Ae^{i\omega t} \qquad y = Be^{i\omega t}$$

$$-\omega^2 A + \omega_x^{\,2} A = q^2 B \qquad -\omega^2 B + \omega_y^{\,2} B = q^2 A$$

$$(\omega_x^{\,2} - \omega^2)(\omega_y^{\,2} - \omega^2) = q^4$$

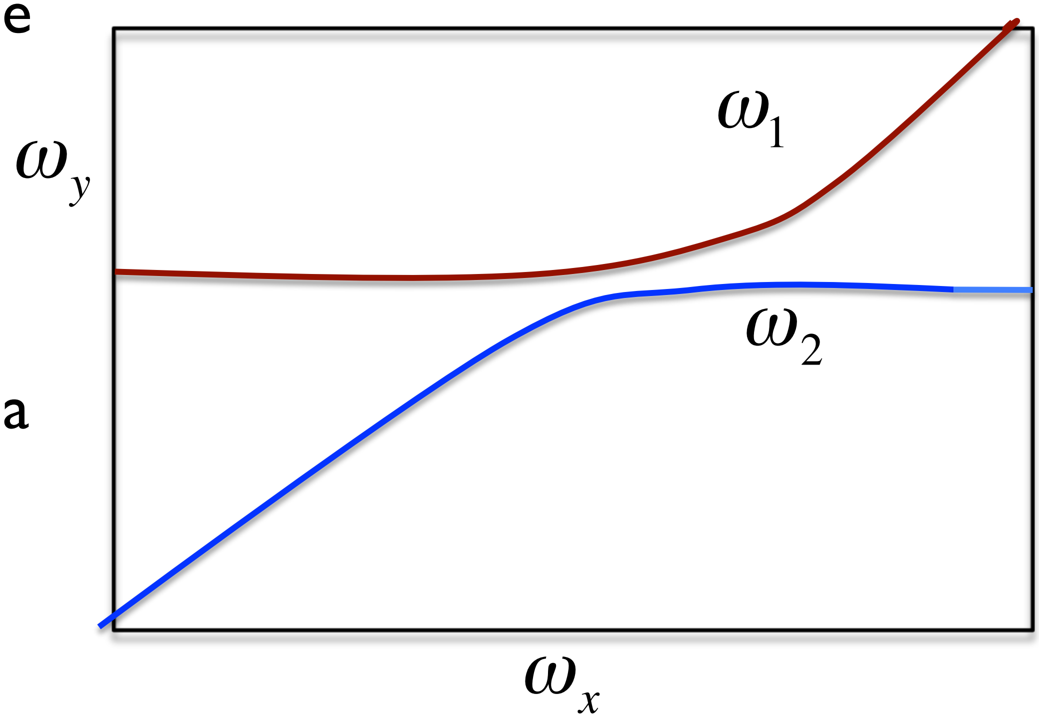
$$\omega^2 = \frac{\omega_x^2 + \omega_y^2 \pm \sqrt{(\omega_x^2 - \omega_y^2)^2 + 4q^4}}{2}$$

Coupled Harmonic Oscillator

$$\omega^2 = \frac{\omega_x^2 + \omega_y^2 \pm \sqrt{(\omega_x^2 - \omega_y^2)^2 + 4q^4}}{2}$$

- The two frequencies of the harmonic oscillator are functions of the two unperturbed frequencies
- When the unperturbed frequencies are the same, a minimum frequency difference

$$\Delta\omega \approx \frac{q^2}{\omega}$$



Nonlinear Harmonic oscillator

- Harmonic oscillator with high order terms

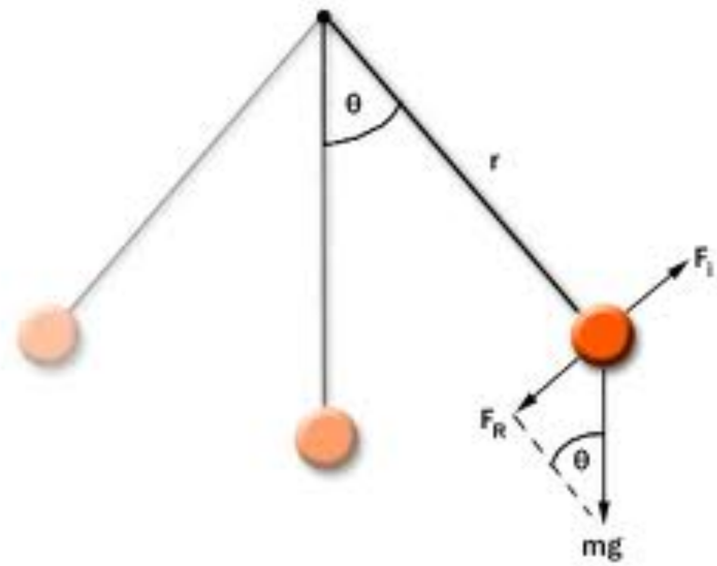
$$\frac{d^2x}{dt^2} + \omega^2 x = \sum_{i=2}^{n} a_i x^i$$

- Pendulum

$$\frac{d^2\theta}{dt^2} + \omega^2 \sin\theta = 0 \qquad \omega = \sqrt{\frac{g}{l}}$$

for $\theta << l$

$$\frac{d^2\theta}{dt^2} + \omega^2\theta \approx \omega^2 \frac{\theta^3}{6}$$



Exact Solution of Pendulum Equation

$$\frac{d^2\theta}{dt^2} + \omega^2 \sin\theta = 0 \qquad \frac{1}{2}\frac{d}{dt}\left(\frac{d\theta}{dt}\right)^2 - \omega^2 \frac{d}{dt}\cos\theta = 0$$

$$\frac{1}{2\omega^2}\left(\frac{d\theta}{dt}\right)^2 = \cos\theta + C$$

for C is a constant

- Unlike linear linear oscillator, pendulum has oscillation period as function of its amplitude.

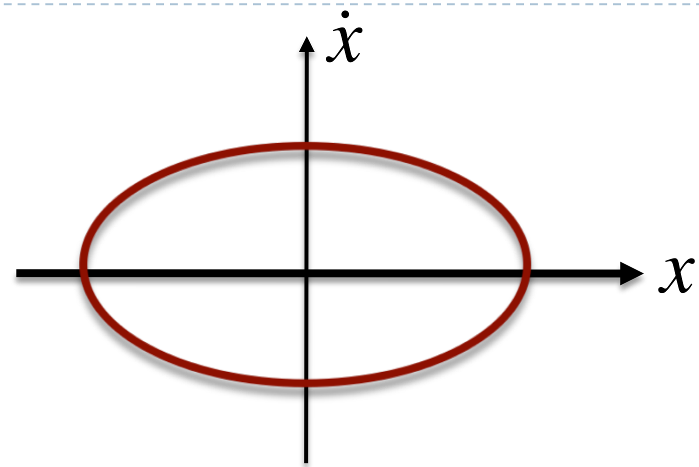
- Stable condition: $-1 < C < 1$

- Separatrix: $C = 0 \quad \frac{d\theta}{dt} = 0$

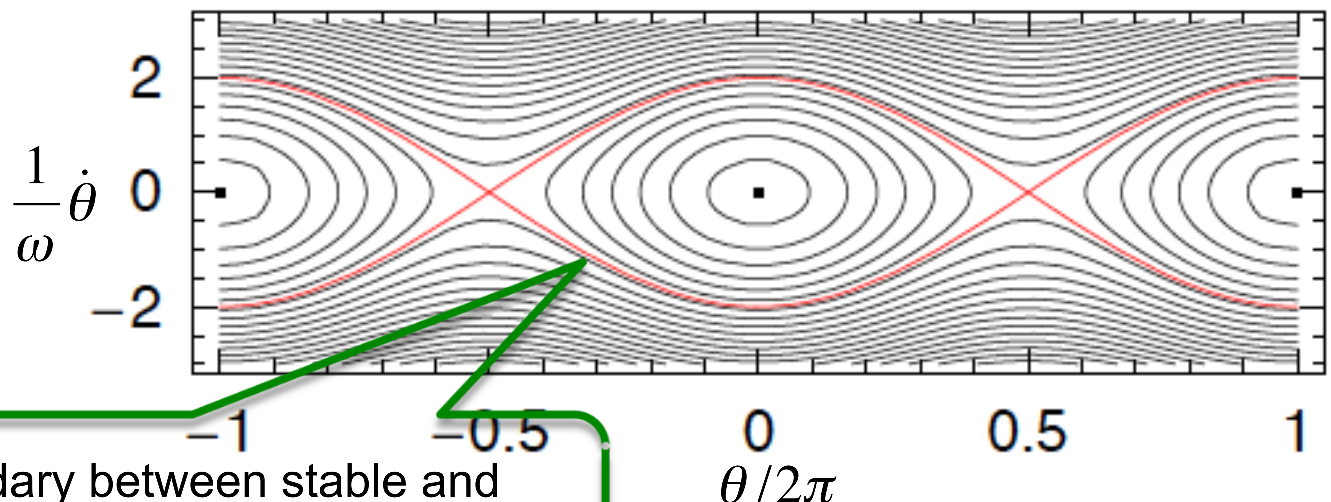
Phase Space

- Linear harmonic

$$\left(\frac{x}{x_0}\right)^2 + \left(\frac{\dot{x}}{x_0\omega}\right)^2 = 1$$



- Pendulum

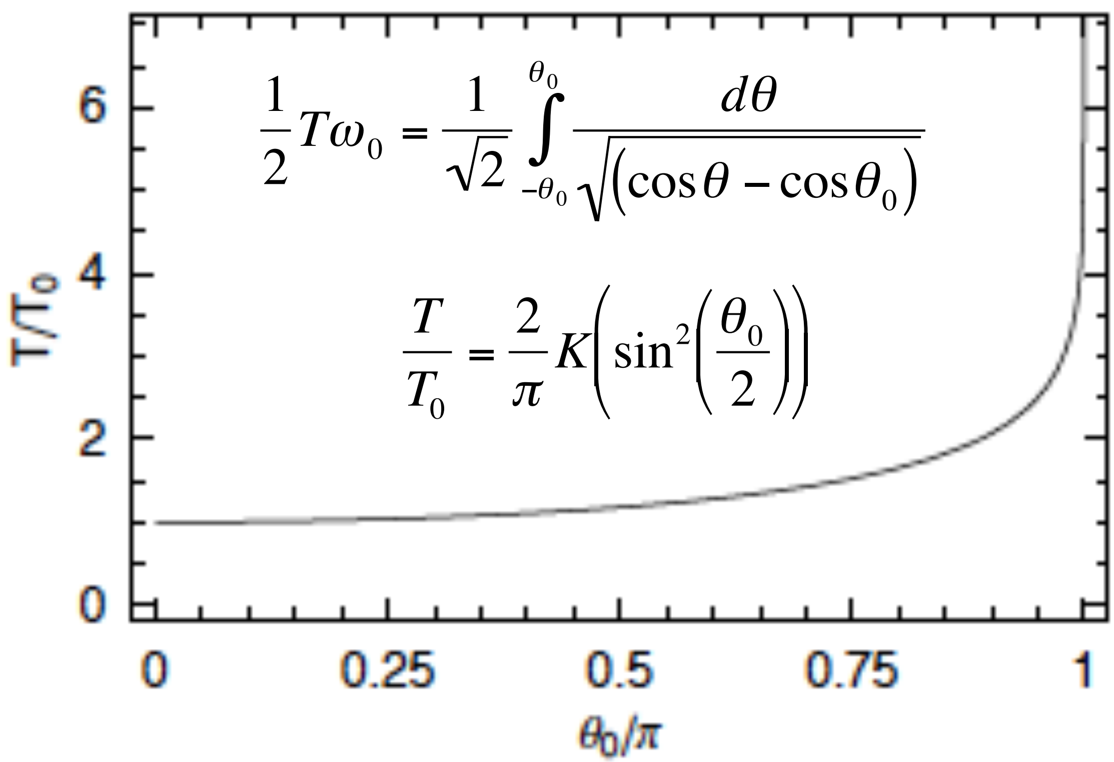


Separatrix, boundary between stable and unstable region on which motion is frozen

Nonlinear Oscillator Frequency

- Pendulum period expression

$$\frac{1}{2\omega^2}\left(\frac{d\theta}{dt}\right)^2 = \cos\theta + C \quad \Rightarrow \quad \omega_0 dt = \frac{1}{\sqrt{2}}\sqrt{\frac{d\theta}{|\cos\theta - \cos\theta_0|}}$$

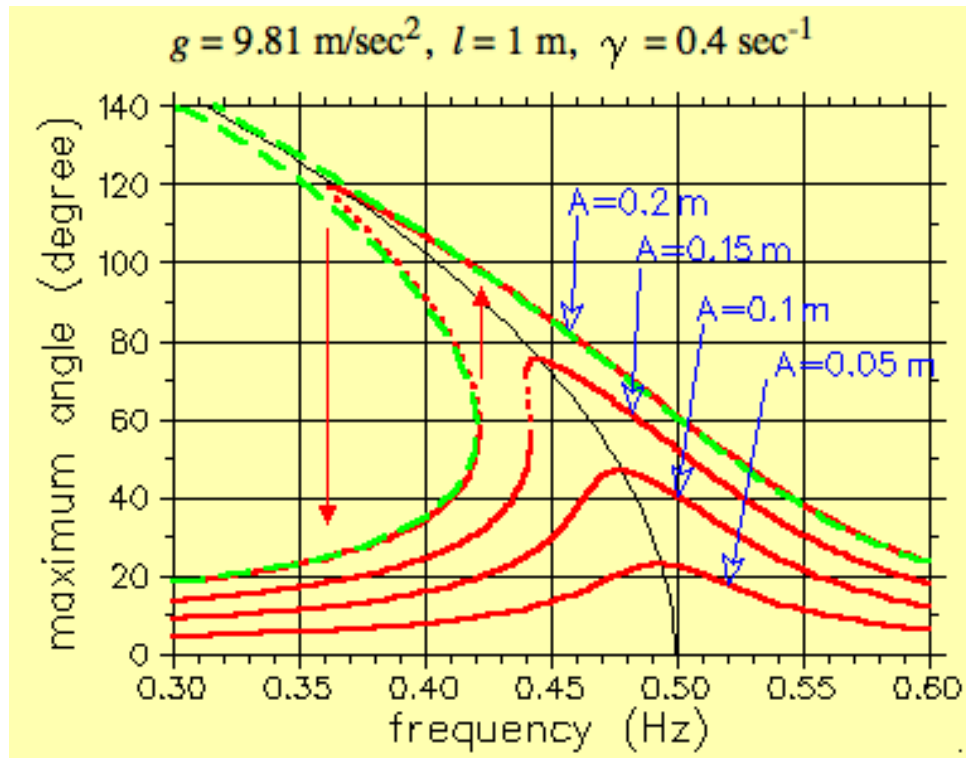


Nonlinear Resonance

- Unlike linear harmonic resonance, the frequency of a nonlinear oscillator is a function of amplitude
  - Small angle pendulum, i.e. small x $(2/\pi)K(x) \approx 1 + x/4$

$$\omega \approx \omega_0 \left(1 - \frac{\theta_0^2}{16}\right)$$

- Nonlinear resonance amplitude doesn't grow unlimited because of the detuning, i.e. frequency moves away from resonant condition as amplitude grows



Home works

- The Stanford LINAC accelerates electrons to 50 GeV in a distance of 2 miles at a constant rate of energy gain. For an observer who rides precariously on an electron, how long does this journey last?

- A positron beam accelerated to 50 GeV in the linac hits a fixed hydrogen target. What is the available energy from a collision with a target electron assumed to be at rest? Compare this with that obtained in a linear collider where electrons and positrons from two linacs collide head on with the same energy.