transverse motion: betatron oscillation

- The general case of equation of motion in an accelerator

$$x''+kx=0$$ Where *k* can also be negative

- For k > 0

$$x(s)=A\cos(\sqrt{k}s+\chi) \qquad x'(s)=-A\sqrt{k}\sin(\sqrt{k}s+\chi)$$

- For k < 0

$$x(s)=A\cosh(\sqrt{k}s+\chi) \qquad x'(s)=-A\sqrt{k}\sinh(\sqrt{k}s+\chi)$$

Hill's equation

- In an accelerator which consists individual magnets, the equation of motion can be expressed as,

$$x''+k(s)x=0 \qquad k(s+L_p)=k(s)$$

- Here, k(s) is an periodic function of L_p, which is the length of the periodicity of the lattice, i.e. the magnet arrangement. It can be the circumference of machine or part of it.
- Similar to harmonic oscillator, expect solution as

$$x(s)=A(s)\cos(\psi(s)+\chi)$$

- or:

$$x(s)=A\sqrt{\beta_x(s)}\cos(\psi(s)+\chi) \qquad \beta_x(s+L_p)=\beta_x(s)$$

Hill's equation: cont'd

$$x'(s) = -A\sqrt{\beta_x(s)}\psi'(s)\sin(\psi(s)+\chi) + \frac{\beta'_x(s)}{2}A\sqrt{1/\beta_x(s)}\cos(\psi(s)+\chi)$$

- with

$$\psi'(s) = \frac{1}{\beta_x(s)} \qquad \frac{\beta_x''}{2}\beta_x - \frac{\beta_x'^2}{4} + k\beta_x^2 = 1$$

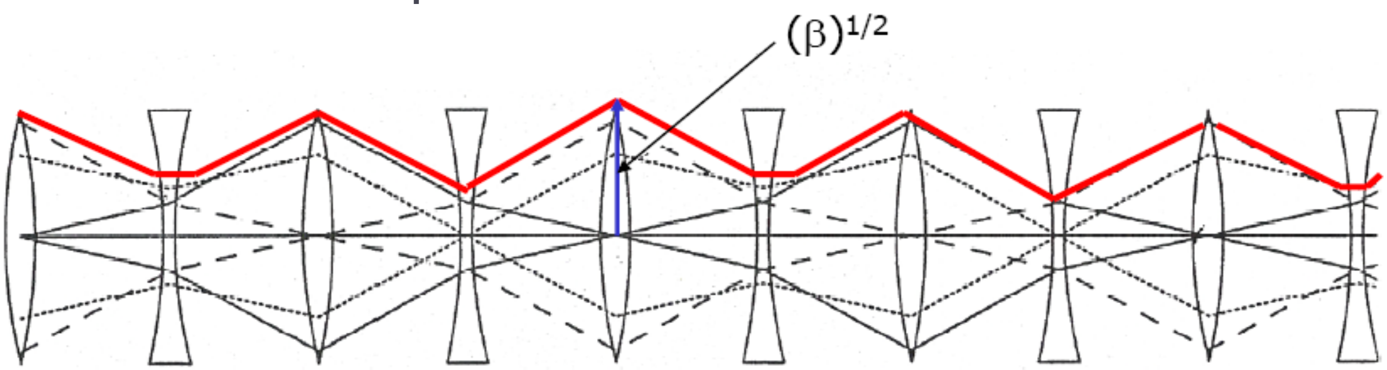
- Hill's equation $x''+k(s)x=0$ is satisfied

$$x(s) = A\sqrt{\beta_x(s)}\cos(\psi(s)+\chi)$$

$$x'(s) = -A\sqrt{1/\beta_x(s)}\sin(\psi(s)+\chi) + \frac{\beta'_x(s)}{2}A\sqrt{1/\beta_x(s)}\cos(\psi(s)+\chi)$$

Betatron oscillation

- Beta function $\beta_x(s)$:
  - Describes the envelope of the betatron oscillation in an accelerator



- Phase advance: $$\psi(s) = \int_0^s \frac{1}{\beta_x(s)} ds$$
- Betatron tune: number of betatron oscillations in one orbital turn

$$Q_x = \frac{\psi(0 \mid C)}{2\pi} = \oint \frac{ds}{\beta_x(s)} / 2\pi = \frac{R}{\langle \beta_x \rangle}$$

Hill's equation: cont'd

$$x_0 = -A\sqrt{\beta_0}\cos\chi \qquad x_0' = -\frac{A}{\sqrt{\beta_0}}\sin\chi + \frac{\beta_0'}{2}\frac{A}{\sqrt{\beta_0}}\cos\chi$$

$$\cos\chi = -\frac{x_0}{A\sqrt{\beta_0}} \qquad \sin\chi = \frac{\beta_0'}{2A\sqrt{\beta_0}}x_0 - \frac{\sqrt{\beta_0}}{A}x_0'$$

$$x(s) = -\sqrt{\frac{\beta(s)}{\beta_0}}(\cos\Delta\psi + \alpha_0\sin\Delta\psi)\,x_0 - \sqrt{\beta(s)\beta_0}\sin\Delta\psi\, x_0'$$

- With:

$$\alpha(s) = -\frac{\beta'(s)}{2}$$

Transfer Matrix of beam transport

- Proof the transport matrix from point 0 to point s is

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \sqrt{\dfrac{\beta(s)}{\beta_0}}(\cos\Delta\psi + \alpha_0 \sin\Delta\psi) & \sqrt{\beta_0\beta(s)}\sin\Delta\psi \\ -\dfrac{1+\alpha_0\alpha(s)}{\sqrt{\beta_0\beta(s)}}\sin\Delta\psi + \dfrac{\alpha_0 - \alpha(s)}{\sqrt{\beta_0\beta(s)}}\cos\Delta\psi & \sqrt{\dfrac{\beta_0}{\beta(s)}}(\cos\Delta\psi - \alpha(s)\sin\Delta\psi) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

- with:

$$x(s) = A\sqrt{\beta_x(s)}\cos(\psi(s) + \chi)$$

$$x'(s) = -A\sqrt{1/\beta_x(s)}\sin(\psi(s) + \chi) + \frac{\beta'_x(s)}{2} A\sqrt{1/\beta_x(s)}\cos(\psi(s) + \chi)$$

One Turn Map

- Transfer matrix of one orbital turn

$$\begin{pmatrix} x(s_0+C) \\ x'(s_0+C) \end{pmatrix} = \begin{pmatrix} (\cos 2\pi Q_x + \alpha_{x,s_0} \sin 2\pi Q_x) & \beta_{x,s_0} \sin 2\pi Q_x \\ -\frac{1+\alpha_{x,s_0}^2}{\beta_{x,s_0}} \sin 2\pi Q_x & (\cos 2\pi Q_x - \alpha_{x,s_0} \sin 2\pi Q_x) \end{pmatrix} \begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix}$$

- With Qx is the betatron tune, # of betatron oscillations in one orbital revolution

$$2\pi Q_x = \int \frac{1}{\beta(s)} ds$$

$$Tr(M_{s,s+C}) = 2\cos 2\pi Q_x$$

Stable condition

$$\left| \frac{1}{2} Tr(M_{s,s+C}) \right| \leq 1.0$$

Stability of transverse motion

- Matrix from point 1 to point 2

$$M_{s_2|s_1} = M_n \cdots M_2 M_1$$

- Stable motion requires each transfer matrix to be stable, i.e. its eigen values are in form of oscillation

$$|M - \lambda I| = 0 \qquad \text{With } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \det(M) = 1$$

$$\lambda^2 - Tr(M)\lambda + \det(M) = 0$$

$$\lambda = \frac{1}{2} Tr(M) \pm \sqrt{\frac{1}{4}[Tr(M)]^2 - 1} \Rightarrow \left|\frac{1}{2} Tr(M)\right| \leq 1.0$$

Closed Orbit

- Closed orbit:

$$\begin{pmatrix} x(s+C) \\ x'(s+C) \end{pmatrix} = \begin{pmatrix} x(s) \\ x'(s) \end{pmatrix}$$

$$\begin{pmatrix} x(s+C) \\ x'(s+C) \end{pmatrix} = M(s+C,s)\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix}$$

Phase space

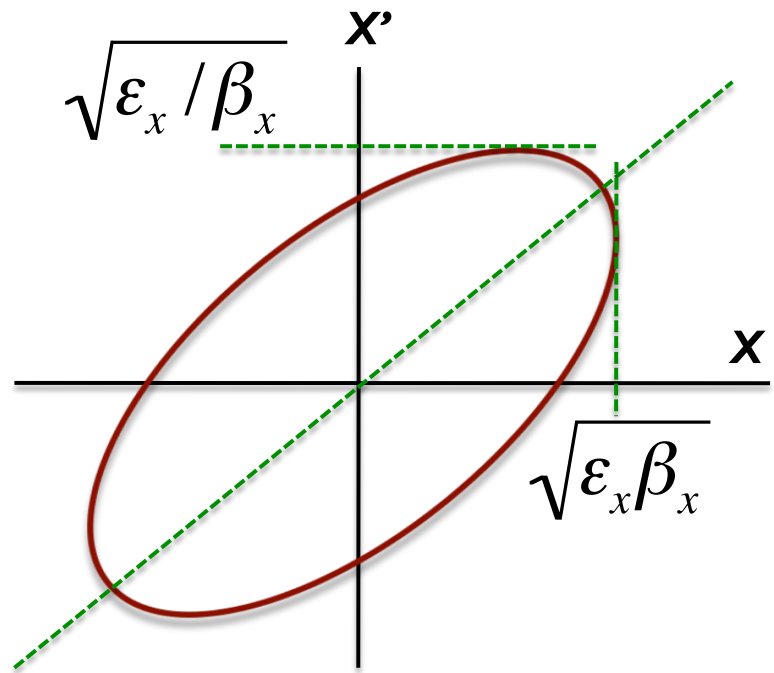
- In a space of x-x', the betatron oscillation projects an ellipse

$$\beta_x x'^2 + \gamma_x x^2 + 2\alpha_x x x' = \varepsilon$$

where

$$\alpha_x = -\frac{1}{2}\beta_x'$$

$$\beta_x \gamma_x = 1 + \alpha_x^2$$



- The are of the ellipse is $\pi\varepsilon$

Courant-Snyder parameters

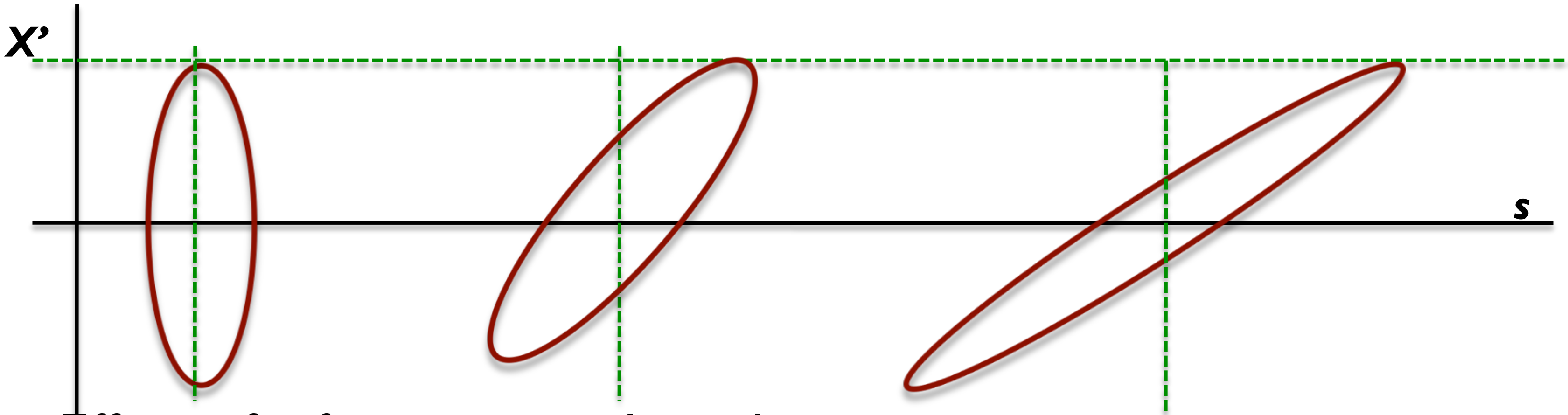
- The set of parameter ($\beta_{x,}$ α_x and γ_x) which describe the phase space ellipse

- Courant-Snyder invariant: the area of the ellipse

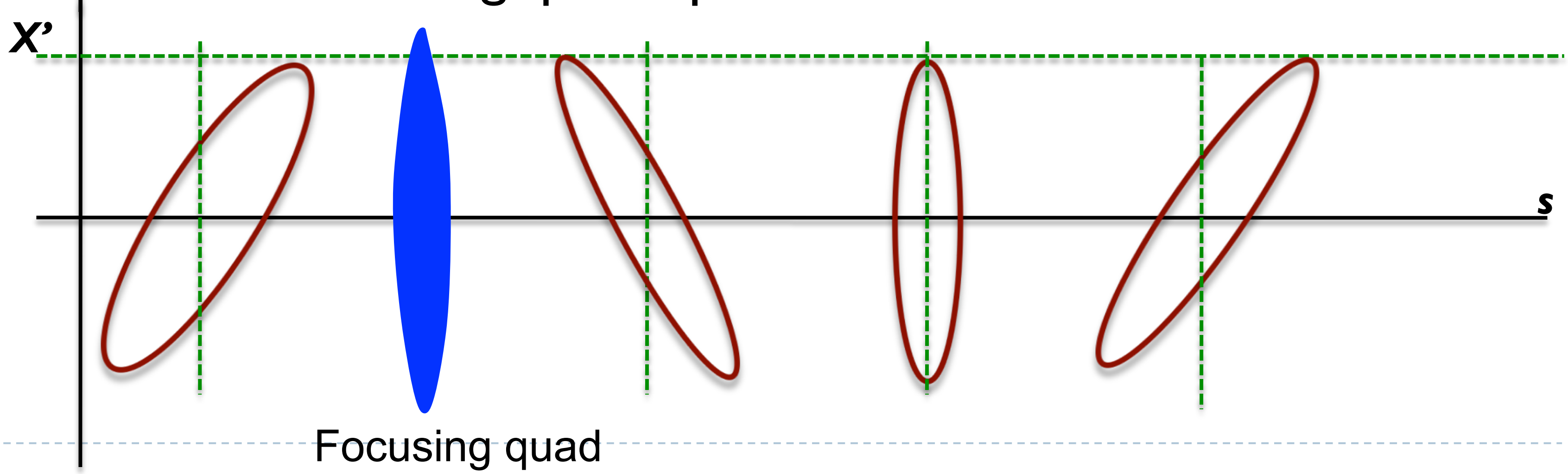
$$\varepsilon = \beta_x x'^2 + \gamma_x x^2 + 2\alpha_x x x'$$

Phase space transformation

- In a drift space from point 1 to point 2



- Effect of a focusing quadrupole



How to measure betatron oscillation

- How to measure betatron tune?
- How to measure beta function?
- How to measure beam emittance?

Dispersion function

- Transverse trajectory is function of particle momentum.

$$x'' - \frac{\rho + x}{\rho^2} = -\frac{qB_y}{\gamma m}\left(1 + \frac{x}{\rho}\right)^2 \qquad B_y = B_0 + B' x$$

$$x'' + \left[\frac{1}{\rho^2}\frac{2p_0 - p}{p} + \frac{B'}{B\rho_0}\frac{p_0}{p}\right]x = \frac{1}{\rho}\frac{\Delta p}{p}$$

$$x = D(s)\frac{\Delta p}{p} \qquad D(s + C) = D(s)$$

$$D'' + \left[\frac{1}{\rho^2}\frac{2p_0 - p}{p} + \frac{B'}{B\rho_0}\frac{p_0}{p}\right]D = \frac{1}{\rho}$$

Dispersion function: cont'd

- In drift space

$$\frac{1}{\rho} = 0 \quad \text{and} \quad B' = 0 \;\Rightarrow\; D'' = 0$$

dispersion function has a constant slope

- In dipoles,

$$\frac{1}{\rho} \neq 0 \quad \text{and} \quad B' = 0$$

$$D'' + [\frac{1}{\rho^2} \frac{2p_0 - p}{p}]D = \frac{1}{\rho}$$

Dispersion function: cont'd

- For a focusing quad,

$$\frac{1}{\rho} = 0 \quad \text{and} \quad B' > 0 \qquad \Rightarrow D'' + B' \frac{p_0}{p} D = 0$$

dispersion function oscillates sinusoidally

- For a defocusing quad,

$$\frac{1}{\rho} = 0 \quad \text{and} \quad B' < 0 \qquad \Rightarrow D'' - B' \frac{p_0}{p} D = 0$$

dispersion function evolves exponentially

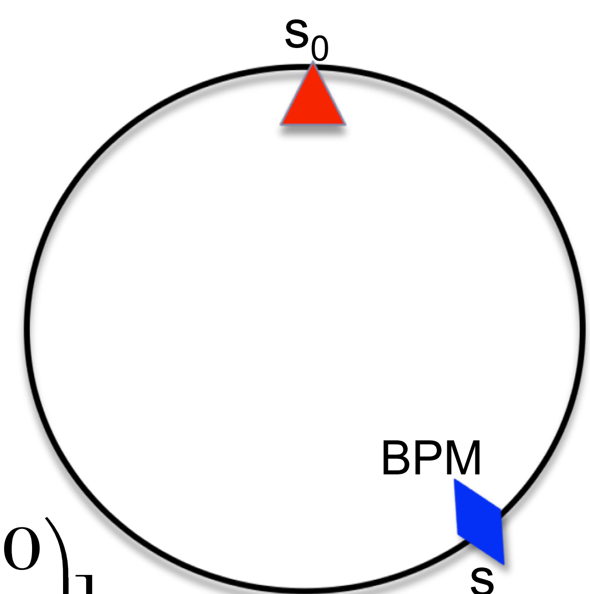
Effects of Errors

- dipole errors
- quadrupole errors
- resonance

Closed orbit distortion

- Dipole kicks can cause particle's trajectory deviate away from the designed orbit
  - Dipole error
  - Quadrupole misalignment

- Assuming a circular ring with a single dipole error, closed orbit then becomes:



$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = M(s,s_0)[M(s_0,s)\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} + \begin{pmatrix} 0 \\ \theta \end{pmatrix}]$$

Closed orbit: single dipole error

- Let's first solve the closed orbit at the location where the dipole error is

$$\begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix} = M(s_0 + C, s_0) \begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix} + \begin{pmatrix} 0 \\ \theta \end{pmatrix}$$

$$x(s_0) = \beta_x(s_0) \frac{\theta}{2\sin \pi Q_x} \cos \pi Q_x$$

$$\boxed{x(s) = \sqrt{\beta_x(s_0)\beta_x(s)} \frac{\theta}{2\sin \pi Q_x} \cos[\psi(s,s_0) - \pi Q_x]}$$

- The closed orbit distortion reaches its maximum at the opposite side of the dipole error location

Closed orbit distortion

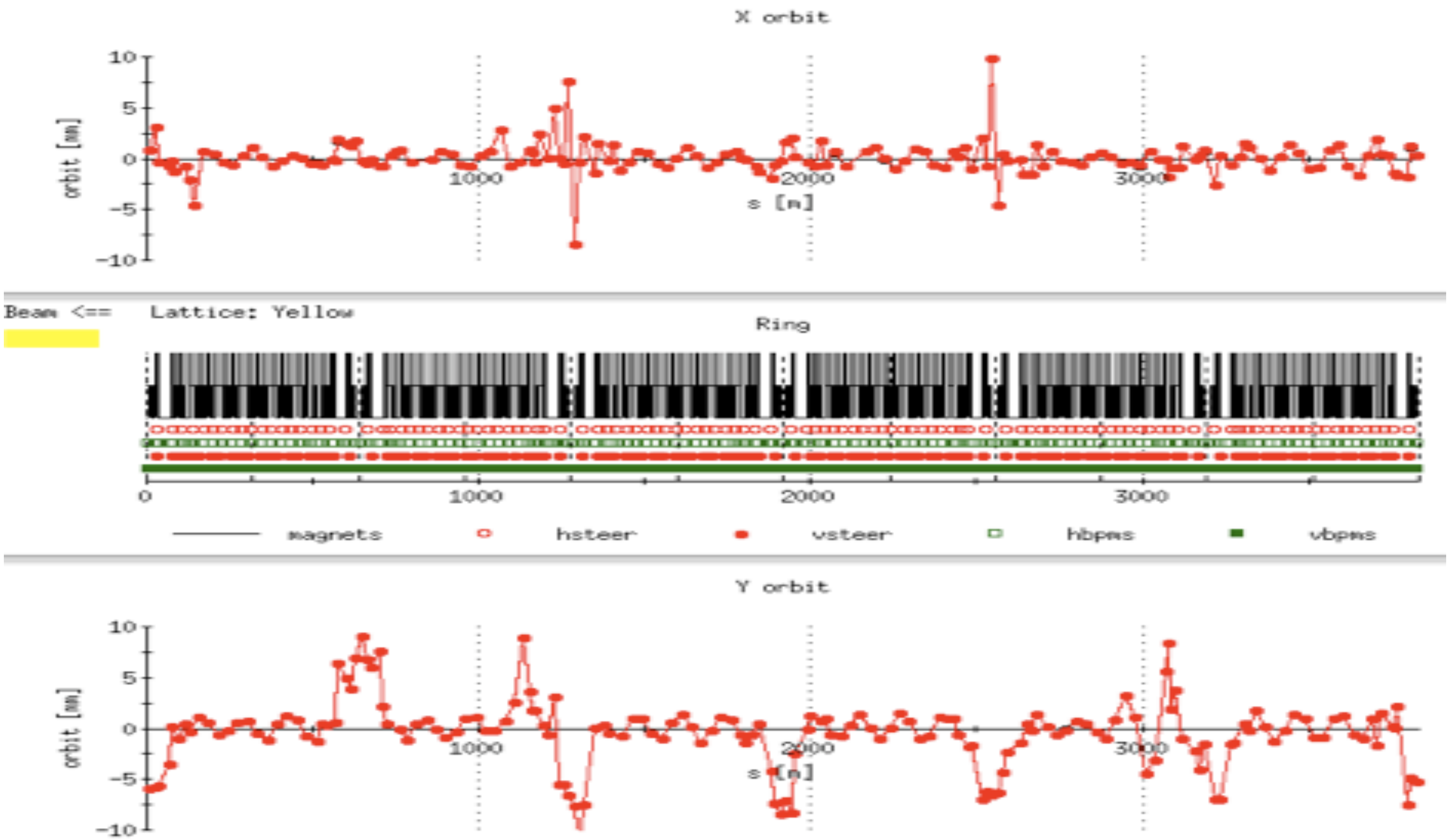
- In the case of multiple dipole errors distributed around the ring. The closed orbit is

$$x(s) = \sqrt{\beta_x(s)} \sum_i \sqrt{\beta_x(s_i)} \frac{\theta_i}{2 \sin \pi Q_x} \cos[\psi(s_i, s_0) - \pi Q_x]$$

- Amplitude of the closed orbit distortion is inversely proportion to $\sin \pi Q_{x,y}$
  - **No stable orbit if tune is integer!**

Measure closed orbit

- Distribute beam position monitors around ring.



Control closed orbit

- minimized the closed orbit distortion.
  - Large closed orbit distortions cause limitation on the physical aperture
  - Need dipole correctors and beam position monitors distributed around the ring
    - Assuming we have m beam position monitors and n dipole correctors, the response at each beam position monitor from the n correctors is:

$$x_k = \sqrt{\beta_{x,k}} \sum_{k=1}^{n} \sqrt{\beta_{x,i}} \frac{\theta_i}{2\sin \pi Q_x} \cos\left[\psi(s_i, s_0) - \pi Q_x\right]$$

Control closed orbit

- Or,

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} M \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{pmatrix}$$

- To cancel the closed orbit measured at all the bpms, the correctors are then

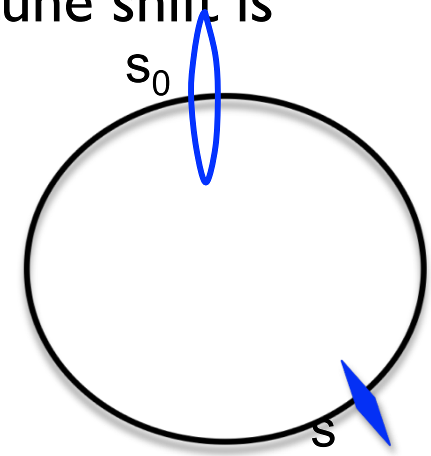
$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{pmatrix} = \begin{pmatrix} M^{-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

Quadrupole errors

- Misalignment of quadrupoles
  - dipole-like error: kx
  - results in closed orbit distortion
- Gradient error:
  - Cause betatron tune shift
  - induce beta function deviation: beta beat

Beta beat

- In a circular ring with a gradient error at s0, the tune shift is



$$M(s+C,s) = M(s,s_0)\begin{pmatrix} 1 & 0 \\ -\Delta k & 1 \end{pmatrix} M(s_0,s)$$

$$\beta_x(s)\sin 2\pi Q_x = \beta_{x0}(s)\sin 2\pi Q_{x0} + \Delta k \frac{\beta_{x0}(s)\beta_{x0}(s_0)}{2}[\cos(2\pi Q_{x0} + 2|\Delta\psi_{s,s0}|)]$$

$$\boxed{\frac{\Delta\beta}{\beta} = \Delta k \frac{\beta_{x0}(s_0)}{2\sin 2\pi Q_{x0}} \cos(2\pi Q_{x0} + 2|\Delta\psi_{s,s0}|)}$$

Unstable betatron motion if tune is half integer!