Transverse Resonances

- resonances mechanisms
- Linear coupling
- Resonance conditions
- 3rd order resonances

Resonance mechanism

- Errors in the accelerators perturbs beam motions
- Coherent buildup of perturbations

Driven harmonic oscillator

- Equation of motion

$$\frac{d^2x(t)}{dt^2} + \omega^2 x(t) = f(t) = \sum_{m=0} C_m e^{i\omega_m t}$$

- for $f(t) = C_m e^{i\omega_m t}$

$$\frac{d^2x(t)}{dt^2} + \omega^2 x(t) = C_m e^{i\omega_m t}$$

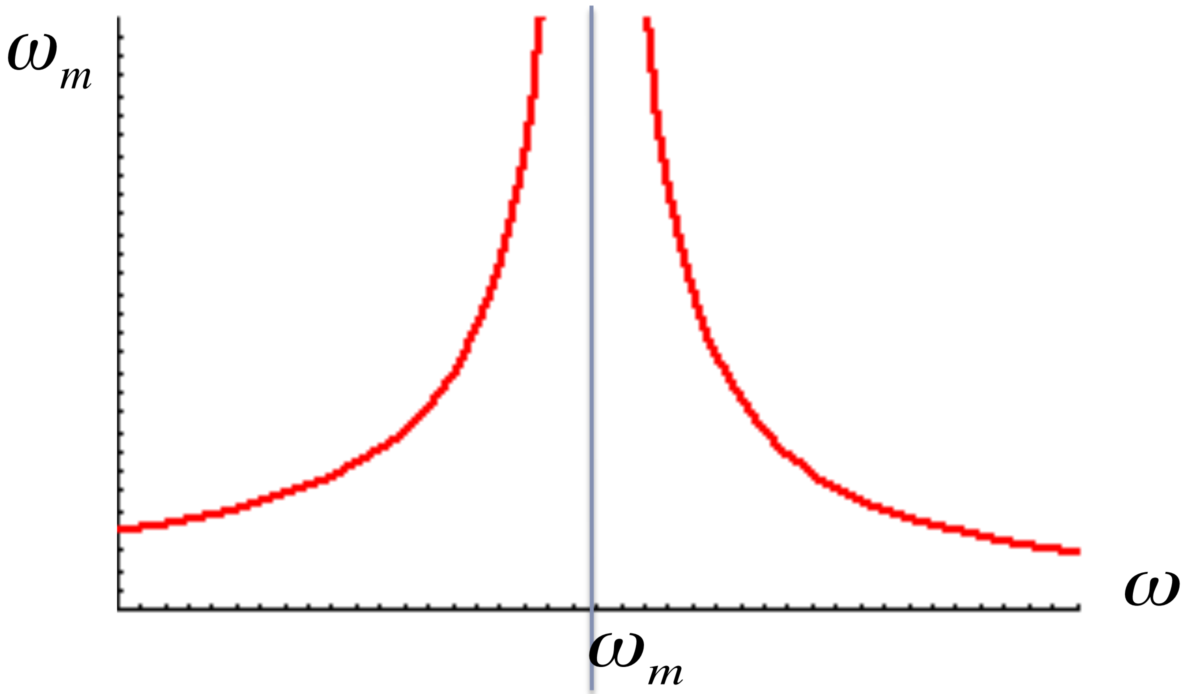
- Assume solution is like $x(t) = Ae^{i\omega t} + A_m e^{i\omega_m t}$

$$\boxed{A_m = \frac{C_m}{\omega^2 - \omega_m^2}}$$

Resonance response

- Response of the harmonic oscillator to a periodic force is

$$x(t) = Ae^{i\omega t} + \frac{C_m}{\omega^2 - \omega_m^2}$$



Betatron oscillation

- Equation of motion

$$x'' + K(s)x = 0 \qquad K(s + L_p) = K(s)$$

$$x = A\sqrt{\beta_x}\cos(\psi + \chi)$$

- In the presence of field errors including mis-aglinments, the equation of motion then becomes

$$x'' + K(s)x = -\frac{\Delta B_y}{B\rho}$$

where

$$\Delta B_y = B_0(b_0 + b_1 x + b_2 x^2 + ...)$$

Dipole error quadrupole error sextupole error

Floquet Transformation

- Re-define () as:

$$x'' + K(s)x = 0 \qquad K(s + L_p) = K(s)$$

$$\zeta(s) = x(s)/\sqrt{\beta_x(s)} \qquad \phi(s) = \psi(s)/Q_x \qquad or\ \phi' = 1/(Q_x\beta_x)$$

- In the presence of field errors including mis-aglinments, the equation of motion then becomes

$$\frac{d^2\zeta}{d\phi^2} + Q_x^2\zeta = -Q_x^2\beta_x^{3/2}\frac{\Delta B_y}{B\rho}$$

where

$$\frac{d^2\zeta}{d\phi^2} + Q_x^2\zeta = -\frac{Q_x^2 B_0}{B\rho}[b_0 + \beta_x b_1\zeta + \beta_x^2 b_2\zeta^2 + \cdots]$$

Resonance contd

- For each n:

$$\frac{d^2\zeta}{d\phi^2} + Q_x^2\zeta = -\frac{Q_x^2\beta_x^{3/2}}{B\rho}\beta_x^n b_n \zeta^n$$

  - When the term on the right side of the equation contain same frequency as Qx, a resonance occurs. And the solution has a form of

$$\zeta = A_k e^{-iQ_x\phi}$$

- Express the perturbation term as:

$$\beta_x^{(n+3)/2} b_n = \sum_k c_k e^{ik\phi}$$

$$k - nQ_x = Q_x \qquad \boxed{k = (n+1)Q_x}$$

Resonance condition

- In the absence of coupling between horizontal and vertical

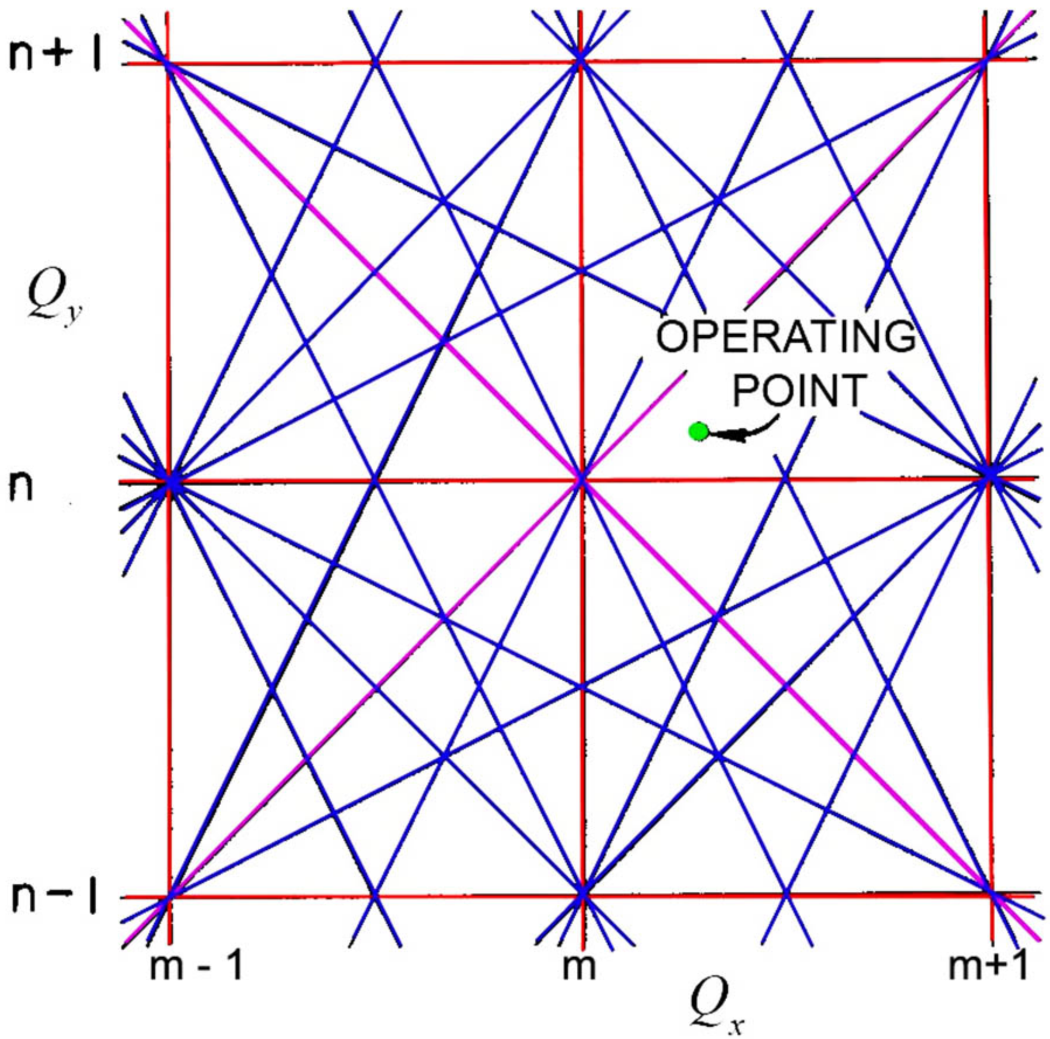
$$k = (n+1)Q_{x,y}$$

error	n	
dipole	0	Qx,y=integer
quadrupole	1	2Qx,y=integer
Sextupole	2	3Qx,y=integer
Octupole	3	4Qx,y=integer

- In the presence of coupling between horizontal and vertical

$$MQ_x + NQ_y = k$$

Tune diagram



- the resonance strength decreases as the order goes higher

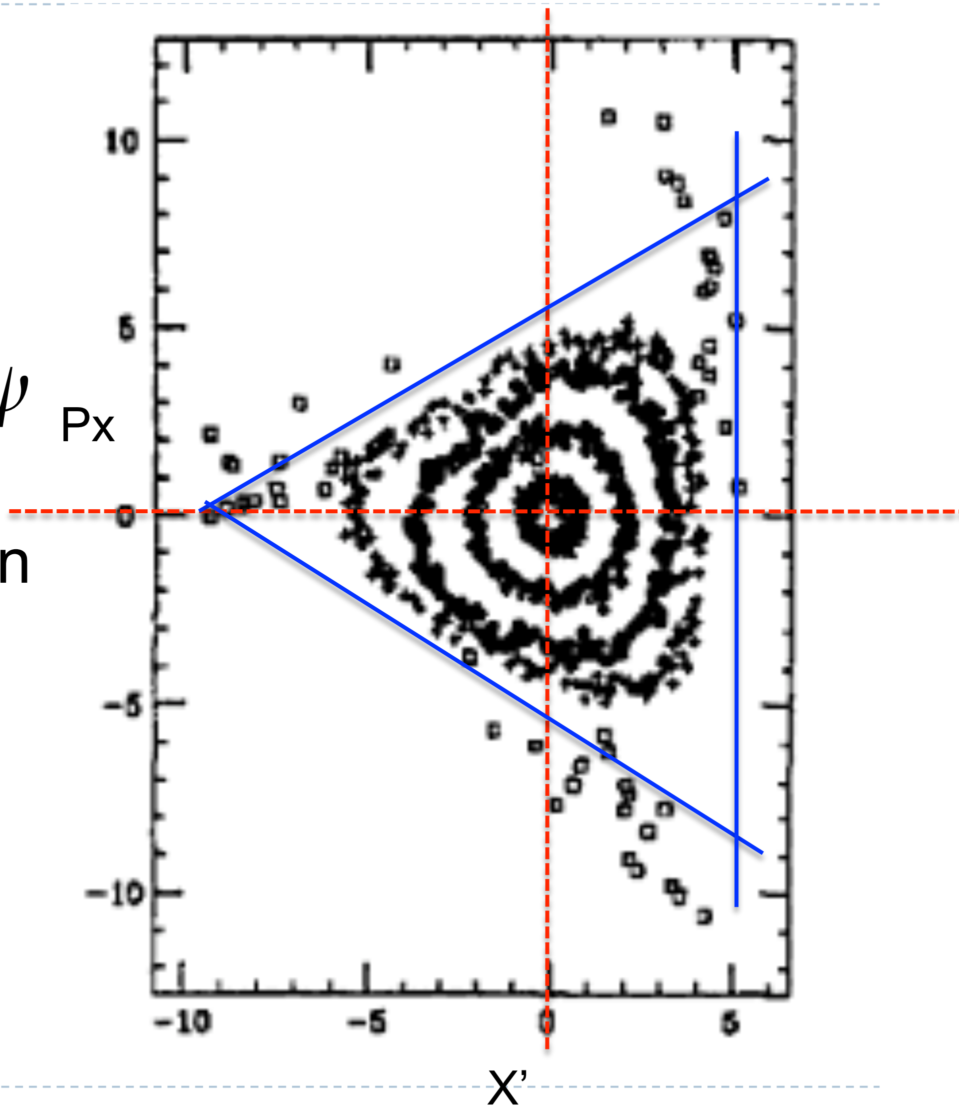
- the working point should be located in an area between resonances there are enough tune space to accommodate tune spread of the beam

Phase space: 3rd order resonance

In the phase space of x, Px

$$x = A\sqrt{\beta_x}\cos\psi$$

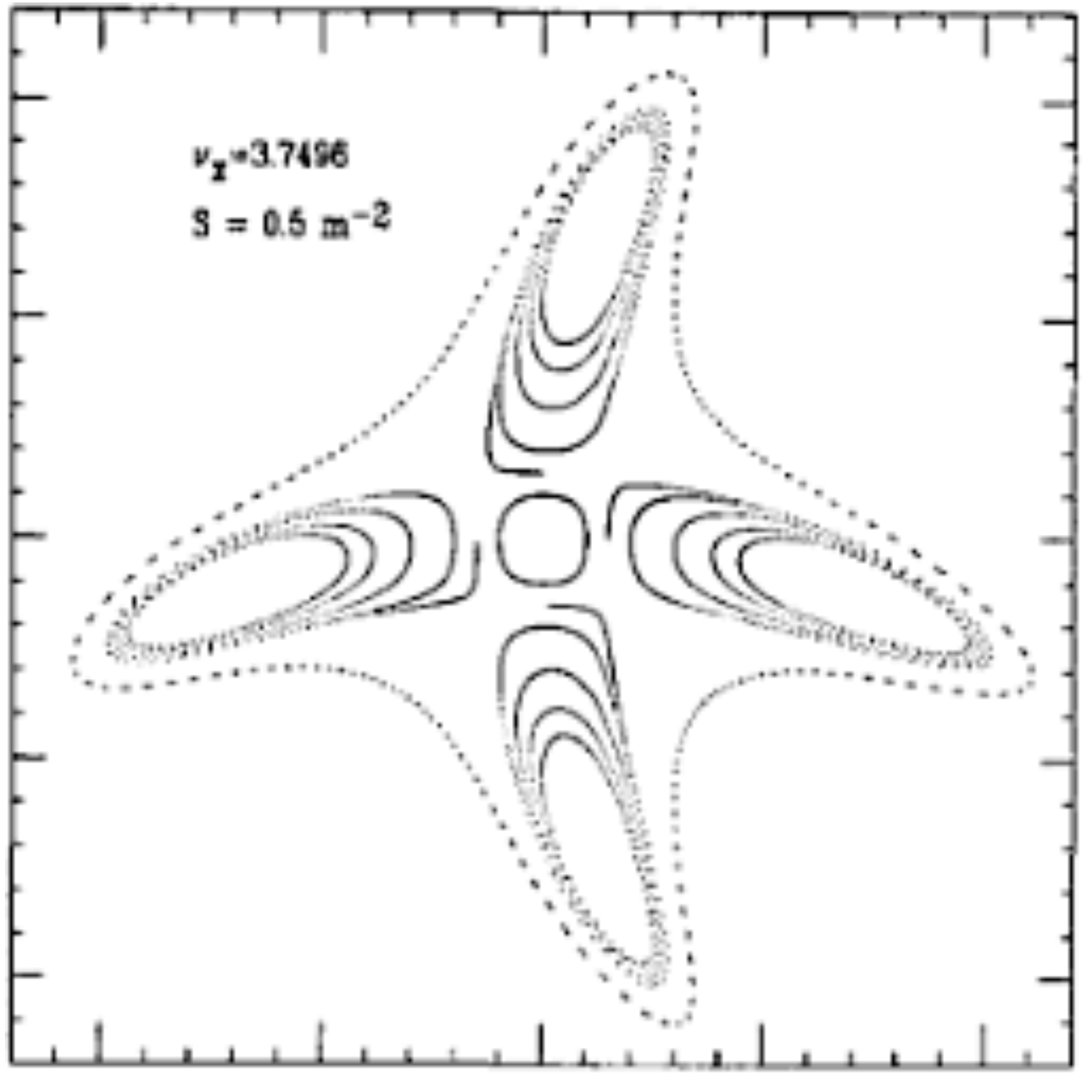
$$P_x = \beta_x x' + \alpha_x x = -A\sqrt{\beta_x}\sin\psi$$



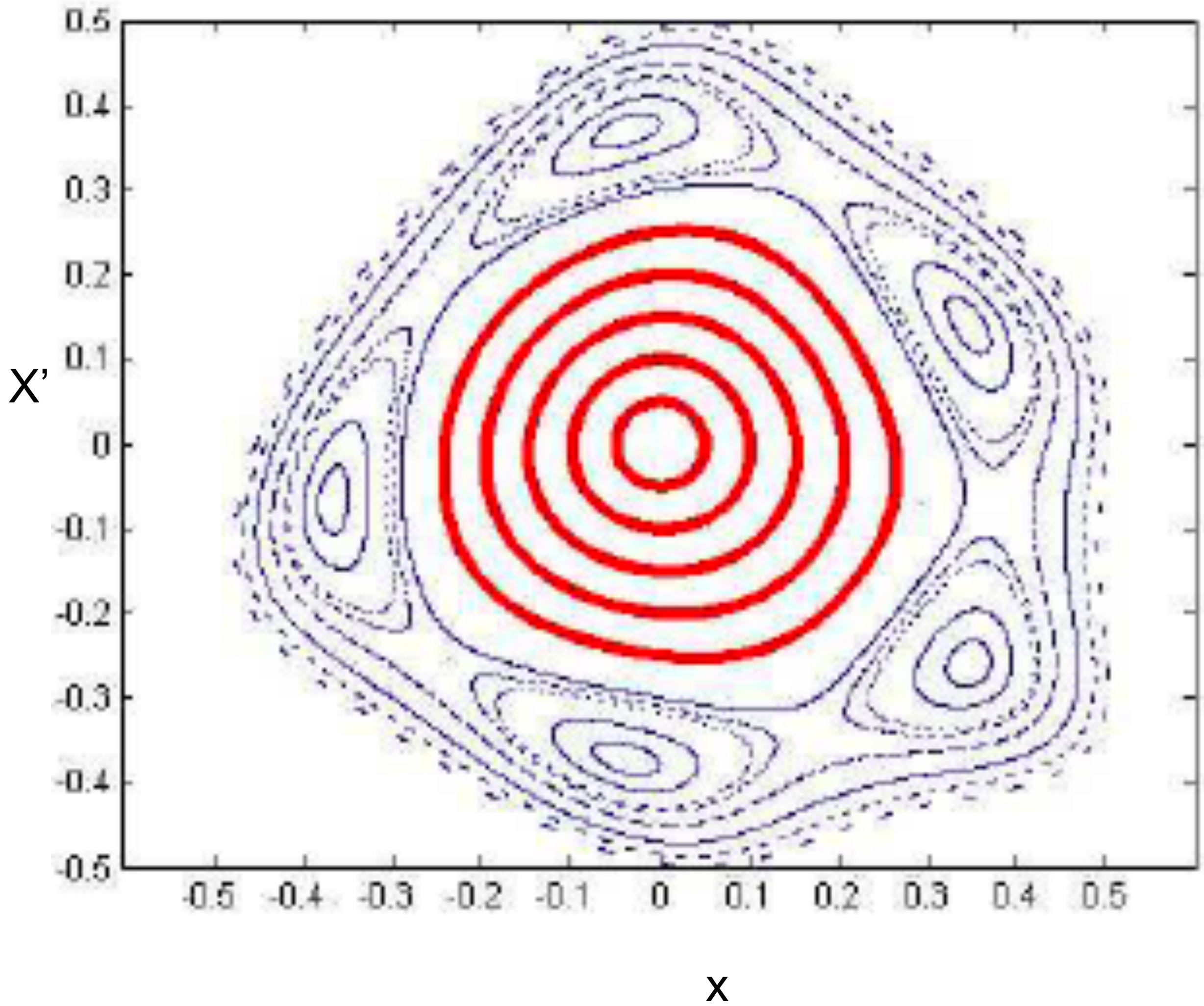
- separatrix: boundery between stable region and unstable region
- Fixed points: where

$$\frac{dx}{dn} = \frac{dP_x}{dn} = 0$$

Phase space: 4^{th} order resonane



Phase space: 5resonane



Source of linear coupling

- Skew quadrupole

$$B_x = -qx; \quad B_y = qy$$

$$x'' + K_x(s)^2 x = -\frac{B_y l}{B\rho} = -qy$$

$$y'' + K_y(s)^2 y = \frac{B_x l}{B\rho} = -qx$$

Coupled harmonic oscillator

- Equation of motion

$$x'' + \omega_x{}^2 x = q^2 y \qquad y'' + \omega_y{}^2 y = q^2 x$$

- Assume solutions are:

$$x = Ae^{i\omega t} \qquad y = Be^{i\omega t}$$

$$-\omega^2 A + \omega_x{}^2 A = q^2 B \qquad -\omega^2 B + \omega_y{}^2 B = q^2 A$$

$$(\omega_x{}^2 - \omega^2)(\omega_y{}^2 - \omega^2) = q^4$$

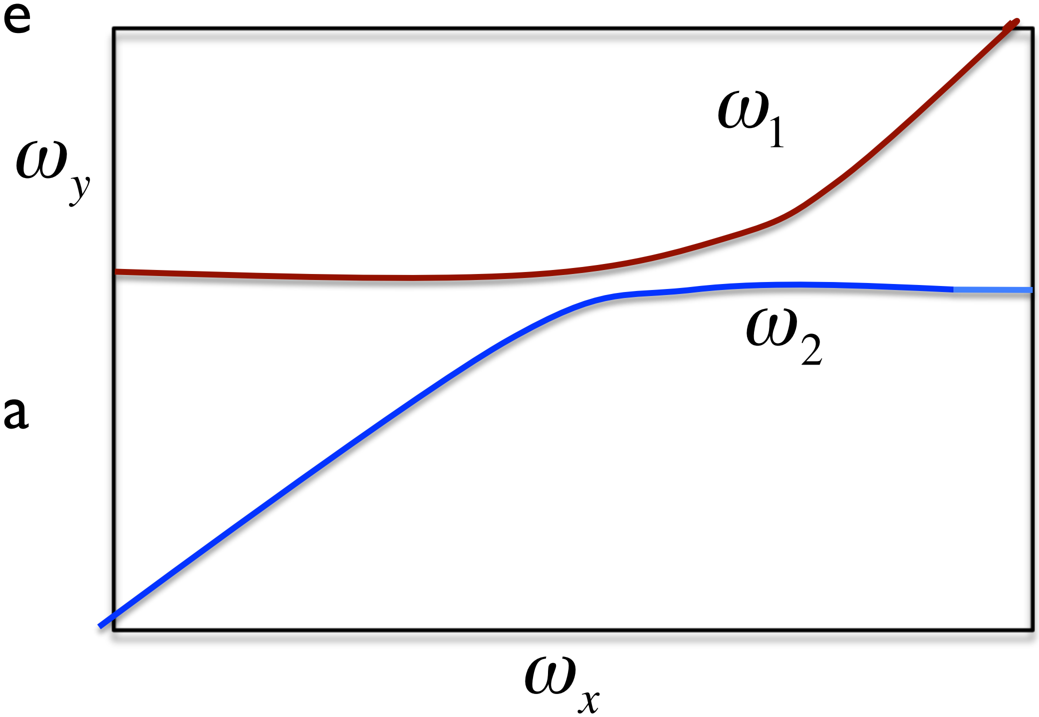
$$\omega^2 = \frac{\omega_x^2 + \omega_y^2 \pm \sqrt{(\omega_x^2 - \omega_y^2)^2 + 4q^4}}{2}$$

Coupled harmonic oscillator

$$\omega^2 = \frac{\omega_x^2 + \omega_y^2 \pm \sqrt{(\omega_x^2 - \omega_y^2)^2 + 4q^4}}{2}$$

- The two frequencies of the harmonic oscillator are functions of the two unperturbed frequencies
- When the unperturbed frequencies are the same, a minimum frequency difference

$$\Delta\omega \approx \frac{q^2}{\omega}$$



Example of a Coupled harmonic oscillator