1 Dipole errors and correction

1.1 Closed orbit change from magnetic dipole errors

In order for a particle to repeat the same orbit around an accelerator on every revolution, its position and angle must return to the same value when it returns to the same longitudinal location in the ring. A machine is designed so that a particle initially on the reference orbit (with position x_0 and angle $x_0^{'}$) will stay on that orbit. Denoting the transfer matrix of the the ring as M, this means that

$$\begin{pmatrix} x_0 \\ x_0^{'} \end{pmatrix}_{c.o.} = M \begin{pmatrix} x_0 \\ x_0^{'} \end{pmatrix}_{c.o.}$$

The subscript c.o. indicates that the particle must be on the closed orbit for the relation to hold. A particle injected with a position or angle error will not return to its initial coordinates after a single revolution. It oscillates about the design trajectory, and does not undergo an integer number of oscillations per revolution. The phase space ellipse for a particular location in the machine is defined by the machine parameters β and α and doesn't change. The errant particle will hit a different point on its phase space ellipse every time it goes another turn around, but it is constrained to be somewhere on its particular ellipse. On the other hand, the ideal particle is in the center of the phase space ellipse on every revolution, returning to the same location in phase space every time.

There are sources of error in an accelerator that cause the closed orbit to deviate from the original designed trajectory. The entire beam envelope oscillates around the original design orbit; there is a new closed orbit. On this new orbit there is still an ideal closed trajectory; a particle on this particular trajectory will retrace its path turn after turn. Note that when a beam is injected into a machine properly, the beam centroid is injected onto the closed orbit with no error in position or angle. Then, the center of mass of the beam follows the actual closed orbit (as opposed to the designed closed orbit). This orbit can then be seen by a decent beam position monitoring system.

The orbit is determined by the magnetic fields through which the beam travels. So, there are various types of errors that can be in a machine, dipole field errors, quadrupole field errors, and so on. Possible sources of dipole error include dipole magnets that have been 'rolled' in their placement, or quadrupoles with their centers misaligned to the reference orbit. A dipole can also have a weaker or stronger $\vec{B}$ field due to errors in its construction. These errors cause an angular deflection of the beam, causing the closed orbit to oscillate around the design orbit. A single steering error in some locations can cause a visible kink in the closed orbit. It is desirable to analyze the effect of different types of errors on the beam. A single dipole error gives an unwanted angular kick to the beam. So, as the closed orbit passes through the field error, the ideal particle's position does not change, but its angle changes according to the angular kick. The new closed orbit incorporates this kick. If the new closed orbit coordinates are x, x'; then for each revolution, the effect of the fields experienced by design plus the kick must result in the same coordinates x, x'.

$$I\begin{pmatrix} x \\ x^{'} \end{pmatrix}_{c.o.} = M\begin{pmatrix} x \\ x^{'} \end{pmatrix}_{c.o.} + \begin{pmatrix} 0 \\ \theta \end{pmatrix}$$

where I is the identity matrix,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then,

$$(M - I)\begin{pmatrix} x \\ x^{'} \end{pmatrix}_{c.o.} = \begin{pmatrix} 0 \\ -\theta \end{pmatrix}$$

Now we have a relation between the strength of the error and the change in position and angle of the closed orbit trajectory at the location of the error. Make this more explicit by writing M in terms of machine parameters;

$$M - I = \begin{pmatrix} \cos(2\pi\nu) + \alpha\sin(2\pi\nu) - 1 & \beta\sin(2\pi\nu) \\ -\gamma\sin(2\pi\nu) & \cos(2\pi\nu) - \alpha\sin(2\pi\nu) - 1 \end{pmatrix}$$

The two simultaneous equations with two unknowns given by the matrix equation are,

$$[\cos(2\pi\nu) + \alpha\sin(2\pi\nu) - 1]x + [\beta\sin(2\pi\nu)]x^{'} = 0$$

$$[-\gamma\sin(2\pi\nu)]\,x + [\cos(2\pi\nu) - \alpha\sin(2\pi\nu) - 1]x^{'} = -\theta$$

The solutions are,

$$x = \frac{\beta\theta}{2}\left(\frac{\cos(\pi\nu)}{\sin(\pi\nu)}\right)$$

$$x^{'} = \frac{\theta}{2}\left(\frac{\sin(\pi\nu) - \alpha\cos(\pi\nu)}{\sin(\pi\nu)}\right)$$

What about the new closed orbit positions at some other location in the ring?

$$\begin{pmatrix} x_2 \\ x_2^{'} \end{pmatrix} = M_{12}\begin{pmatrix} x_1 \\ x_1^{'} \end{pmatrix}$$

where

$$M_{12} = \begin{pmatrix} \frac{\beta_2}{\beta_1}[\cos(\psi_{12}) + \alpha_1\sin(\psi_{12})] & (\beta_1\beta_2)^{\frac{1}{2}}\sin(\psi_{12}) \\ -\frac{1+\alpha_1\alpha 2}{(\beta_1\beta_2)^{\frac{1}{2}}}\sin(\psi_{12}) + \frac{\alpha_1-\alpha_2}{(\beta_1\beta_2)^{\frac{1}{2}}}\cos(\psi_{12}) & \left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{2}}\cos(\psi_{12}) - \alpha_2\sin(\psi_{12}) \end{pmatrix}$$

so that

$$x_2 = \frac{\theta\sqrt{\beta_1\beta_2}}{2\sin(\pi\nu)}\cos(\psi_{12} - \pi\nu) \tag{1}$$

Notice that Eq. 1 goes to infinity if the number of betatron oscillations per turn, ν, is an integer value. Dipole errors drive what is called an 'integer resonance'. Should the tune (ν) of a circular machine be set to an integer value, the presence of a dipole error (and there is always *some* dipole error present) will cause the betatron amplitude to grow without bound.

When an accelerator is built, there are multiple errors causing a closed orbit distortion. The locations and number of errors is unknown! In the process of designing an accelerator, it is important to make an estimation of what these random errors do to the orbit, so that an appropriate correction system can be put into place. The errors may be summed, assuming a random distribution.

$$x = \frac{\sqrt{\beta_s}}{2\sin(\pi\nu)}\sum_i^{N_{error}}\sqrt{\beta_i}\theta_i\cos[(\psi_s - \psi_i) - \pi\nu]$$

Some of the errors will cancel each other since they are of opposite sign. Due to the variation in the signs of the errors, it is more useful to estimate the mean square error of the beam centroid.

$$\left\langle x_{c.o.}^2\right\rangle = \beta_s\frac{1}{4\sin^2(\pi\nu)}\left\langle\sum_i^{N_{error}}\beta_i\theta_i^2\cos^2[(\psi_s - \psi_i) - \pi\nu]\right\rangle \tag{2}$$

The cross terms have been thrown out, because they go away when the averaging is done (since some θ_i are negative and some are positive). We know the average value of cosine squared is one half. Then Eq. 2 becomes,

$$\left\langle \frac{x_{c.o.}^2}{\beta_s} \right\rangle = \frac{1}{8\sin^2(\pi\nu)} \left\langle \sum_i^{N_{error}} \beta_i \theta_i^2 \right\rangle$$

The average of N_{error} numbers is the same as N_{error} times the average of the numbers. Furthermore, β_i and θ_i are uncorrelated (the placement of errors is random with respect to the beta function). This allows the average of the product to be written as a product of the averages. Then,

$$\left\langle \frac{x_{c.o.}^2}{\beta_s} \right\rangle = \frac{1}{8\sin^2(\pi\nu)} N_{error} \left\langle \beta_i \right\rangle \left\langle \theta_i^2 \right\rangle$$

The angular deflections θ_i in Eq. 2 may be due to rotated dipoles, or dipoles with the wrong field strength. However, they can actually also arise when the centroid of the beam passes through a quadrupole magnet off-axis. The quadrupole will 'focus' the entire beam in this case, rather than acting only on individual particles. The beam will get an angular kick, $\frac{x}{f}$, where x is the position of the beam centroid at the quadrupole, and f is the focal length of the quadrupole. Considering the case of a distribution of quadrupole errors (N_Q quadrupoles in the machine), the deviation of the closed orbit due to those errors can be estimated,

$$\frac{\sigma_{x_{c.o.}}}{\sqrt{\beta_s}} = \frac{\sqrt{N_Q}}{\sqrt{8}\sin(\pi\nu)} \frac{\left\langle \sqrt{\beta_i} \right\rangle}{f} \sigma_x$$

If the errors were due to rotated dipoles rather than off-axis quadrupoles, the formula would be the same except that $N_Q \to N_D$, and $\frac{\sigma_x}{f} \to \sigma_\theta$.

2 Gradient errors and correction

A gradient error (focusing strength error in a quadrupole) will shift the tune, ν, by

$$\Delta\nu = \frac{\beta}{4\pi f} = \frac{1}{4\pi} \beta \frac{B' L}{B\rho}$$

where β is the beta function at the location of the gradient error, f is the error in focal strength, B' is the gradient strength, L is the length of the magnet, and $B\rho = p/e$ is the magnetic rigidity. So, if the current is increased, the gradient B' goes up, and the focal length gets shorter. If the focal length in the horizontal plane were systematically shortened around an accelerator ring, the horizontal tune (number of horizontal betatron oscillations per turn) would increase.

Since a quadrupole defocuses in one plane and focuses in the other, changing a quadrupole gradient will change the tune in both planes. The tune changes for a change in gradient, B', from a single quadrupole located at s_0 would be:

$$\Delta\nu_x = \frac{\beta_x(s_0)}{4\pi f} = \frac{1}{4\pi}\,\beta_x(s_0)\,\frac{B'L}{B\rho}$$

$$\Delta\nu_y = -\frac{\beta_y(s_0)}{4\pi f} = -\frac{1}{4\pi}\,\beta_y(s_0)\,\frac{B'L}{B\rho}$$

If it were a horizontally focusing quadrupole, an increase in gradient, B', would increase the horizontal tune, but decrease the vertical tune (hence the negative sign).

Often the focusing quadrupoles in a ring are on the same circuit, and have the same strength; while the same is true for the defocusing magnets. Changes in the gradients of both the focusing circuit and the defocusing circuit must be made in order to have independent control of the horizontal and vertical tunes. The tune changes in a FODO lattice with N cells would be:

$$\Delta\nu_x = \frac{N}{4\pi}\left[\beta_{Hmax}\left(\frac{B'L}{B\rho}\right)_{FQ} + \beta_{Hmin}\left(\frac{B'L}{B\rho}\right)_{DQ}\right]$$

$$\Delta\nu_y = -\frac{N}{4\pi}\left[\beta_{Vmin}\left(\frac{B'L}{B\rho}\right)_{FQ} + \beta_{Vmax}\left(\frac{B'L}{B\rho}\right)_{DQ}\right]$$

3 Momentum dependent effects

The bend angle given to a particle going through a dipole magnet depends inversely on the particle momentum, as per the relation

$$\theta = \frac{eB_0L}{p}$$

where B_0 is the field in the dipole magnet, L is the length of the magnet, and e/p is the charge to momentum ratio of the particle. Particles with momenta greater than the momentum of an ideal particle will travel radially outside of the design trajectory, while those of lower momenta will travel to the inside. The dependence of particle position on its momentum as it travels through a machine is described by the dispersion function, $D(s)$.

$$\Delta x = D(s)\frac{\Delta p}{p_0}$$

Quadrupole magnets also bend higher momenta particles less, resulting in a shorter focal length for lower momenta particles, and longer focus for higher momenta particles. So, in an accelerator ring, the number of oscillations per turn (the tune) will be shifted for off-momentum particles. The proportionality between the momentum spread in a beam and the resulting tune spread is called the chromaticity, ξ, of the machine.

$$\Delta \nu = \xi \frac{\Delta p}{p_0}$$

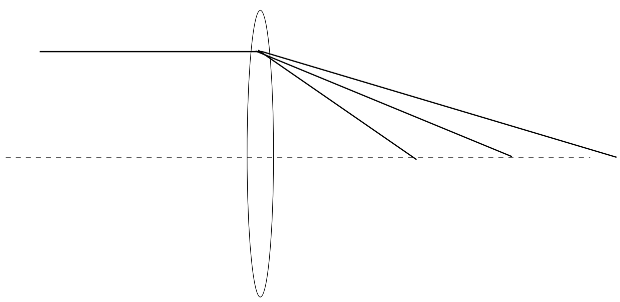

Figure 1: Particles of differing momenta entering a quadrupole lens on the same trajectory will be bent different amounts. The lower momenta particles are bent more than higher momenta particles.

The 'natural' chromaticity of a machine is due to the large focusing and defocusing quadrupoles in the ring (as opposed to the correction quadrupoles). An ideal particle would have the following inverse focal length,

$$\frac{1}{f_0} = \frac{eB'L}{p_0}$$

This would shift slightly for a particle with a small momentum error, Δp,

$$\begin{aligned} \frac{1}{f} = \frac{1}{f_0} + \Delta\left(\frac{1}{f}\right) &= \frac{eB'L}{p_0 + \Delta p} \\ &= \frac{eB'L}{p_0(1+\frac{\Delta p}{p_0})} = \frac{eB'L}{p_0}\left(1 + \frac{\Delta p}{p_0}\right)^{-1} \end{aligned}$$

$$\approx \quad \frac{eB'L}{p_0}\left(1-\frac{\Delta p}{p_0}\right)$$

where a binomial expansion was done since $\Delta p/p_0$ is small. The change in the inverse focal length must be:

$$\Delta\left(\frac{1}{f}\right) = -\left(\frac{eB'L}{p_0}\right)\frac{\Delta p}{p_0}$$

The off-momentum particle then has a tune shift,

$$\Delta\nu = -\frac{\beta_0}{4\pi}\left(\frac{eB'L}{p_0}\right)\frac{\Delta p}{p_0} = -\frac{\beta_0}{4\pi f_0}\frac{\Delta p}{p_0}$$

A positive momentum error decreases the focusing and decreases the tune.

The chromaticity for a FODO cell can be found by summing the tune change from the two quadrupoles, and identifying the coefficient to $\Delta p/p_0$.

$$\begin{aligned}
\Delta\nu_x &= -\frac{1}{4\pi}\left[\frac{\beta_{Hmax}}{f_{FQ}} - \frac{\beta_{Hmin}}{f_{DQ}}\right]\frac{\Delta p}{p_0} \\
\Delta\nu_y &= \frac{1}{4\pi}\left[\frac{\beta_{Vmin}}{f_{FQ}} - \frac{\beta_{Vmax}}{f_{DQ}}\right]\frac{\Delta p}{p_0}
\end{aligned}$$

The natural chromaticities for one FODO cell are then,

$$\begin{aligned}
\xi_x &= -\frac{1}{4\pi}\left[\frac{\beta_{Hmax}}{f_{FQ}} - \frac{\beta_{Hmin}}{f_{DQ}}\right] \\
\xi_y &= \frac{1}{4\pi}\left[\frac{\beta_{Vmin}}{f_{FQ}} - \frac{\beta_{Vmax}}{f_{DQ}}\right]
\end{aligned}$$

4 Beam motion

Previously only the effect of quadrupole magnets was considered, giving the simplest form of Hill's equations for horizontal and vertical motion. These were obtained using the equations of motion;

$$
\begin{aligned}
\frac{dx^2}{d^2s} + \frac{e}{p}B_y &= 0 \\
\frac{dy^2}{d^2s} - \frac{e}{p}B_x &= 0
\end{aligned}
$$

Typically in an accelerator ring there are major dipoles in the horizontal plane, and not so in the vertical. The effect of the dipoles on the motion can be found a few different ways. In the Edwards and Syphers text, cylindrical coordinates are used, so that $x \to \rho_0 + x$, where ρ_0 is the radius of curvature of the ideal trajectory. The equation of motion $\gamma m \ddot{\vec{R}} = e\vec{v} \times \vec{B}$ is solved in cylindrical coordinates, which results in an extra term in the x equation.

$$
\begin{aligned}
x^{''} + \frac{x^2}{\rho_0} &= -\frac{B_y}{B\rho} \\
y^{''} &= \frac{B_x}{B\rho}
\end{aligned}
$$

The extra term in the x equation represents the effect of the dipoles on particles which are not following the design trajectory. Due to their different path lengths through the dipole, they experience more or less bend than the ideal particle. In high energy machines with strong focusing this term is usually much smaller than the term due to strong focusing from the quadrupole fields. Note that also the magnetic fields have been left in general form, so they may be expanded into a series of terms; dipole, quadrupole, sextupole, etc. The effect of each order of magnetic field in the expansion may be treated as an error that perturbs the motion. In addition, since $B\rho = p/e$, the effect of momentum spread may be investigated.

5 Transverse resonances

Resonances that drive the amplitude of transverse beam oscillations can be understood using the analogy of a driven harmonic oscillator. A driving force for a harmonic oscillator is a source of motion, and so is represented on the right side of the harmonic oscillator equation. The driving force may be made up of sinusoidal components, and the effect of each component on the motion may be analyzed one by one.

$$\ddot{x} + \omega_0^2 x = F/m$$
$$\ddot{x} + \omega_0^2 x = A \sin(\omega t + \phi) \tag{3}$$

The solution for this driven harmonic oscillator is the following,

$$x = A \frac{\sin(\omega t + \phi)}{\omega_0^2 - \omega^2}$$

This may be verified by substitution back into the original equation. When the drive frequency ω is equal to the natural frequency ω_0, the solution for x becomes infinite. As the drive frequency gets close to the natural frequency $\omega \to \omega_0$, or $\omega = \omega_0 + \delta$ where δ is small compared to ω_0, the oscillation of x becomes very large. Intuitively the same thing can be expected for a particle beam if there were a drive force in the accelerator with a harmonic relation to the betatron tune. Magnetic field errors can potentially supply this driving force. The particles traverse the ring many times, and in this case, they will sample the field error some significant fraction of the number of times they orbit.

Already equation 1 shows that in the presence of a dipole field error running the machine at integer tune causes particle positions to be infinite (i.e. larger than the beampipe radius). Quadrupole errors drive particle amplitudes for half integer tunes, and so on.

We'll now check out the specific case for a sextupole error, following a discussion by Steve Holmes. Circular phase space (see emittance discussion) is convenient for analysis of errors.

The angle ϕ gives the angular position of the particle in $(x, \beta x' + \alpha x)$ space, as shown in Fig. 2. In the absence of perturbations, $\phi \equiv 2\pi\nu_x$. Then, if the motion is stable, the amplitude A would not change from turn to turn ($\frac{dA}{dn} = 0$), and the phase evolution would be $\frac{d\phi}{dn} = 2\pi\nu_x$.

Consider a sextupole error that occurs as a kick in one location of the ring rather than a distributed error. Then at the location of the error, there is no position change ($\Delta x = 0$ going through a thin sextupole element representing the error), but there is an angle change due to the kick from the sextupole field. The analysis is for the x motion; so, B_y is the component of importance since it affects the horizontal motion. To further simplify the analysis, assume that the particle position is at $y = 0$ at the error location (and this won't change, because the perturbation is in $\hat{x}$). Then,

$$\theta = \Delta x' = \frac{e}{p} \int B_y \, ds = \frac{B'' L}{2B\rho} x^2 = s x^2$$

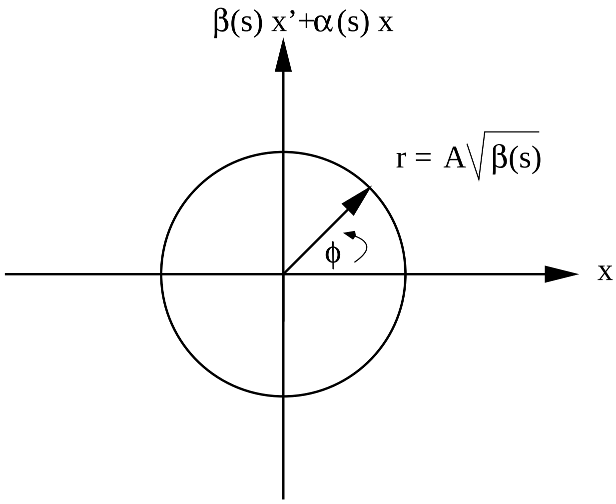

Figure 2: A circular trajectory in horizontal phase space of a particle at location s.

where s is a convenient symbol for the constant coefficient, and $x^2 = A^2\beta\cos^2\phi$.

Using $\Delta x = 0$ (impulse error) the change in phase on a single turn in terms of the kick may now be found by differentiation of x.

$$x = A\sqrt{\beta}\cos(\phi)$$
$$\Delta x = 0 = \Delta A\sqrt{\beta}\cos(\phi) - A\sqrt{\beta}\sin(\phi)\Delta\phi$$

Solving for $\Delta\phi$,

$$\Delta\phi = -\frac{\sqrt{\beta}}{A}\cos(\phi)\Delta x'$$

In order to find ΔA, use Eq.4 in the emittance lecture:

$$A^2 = \frac{x^2}{\beta} + \frac{(\alpha x + \beta x')^2}{\beta}$$

the change in amplitude, ΔA, due to a kick $\Delta x'$ can be found by differentiation. ($\Delta x = 0$)

$$\Delta A = \frac{(\alpha x + \beta x')}{A}\Delta x' = -\sqrt{\beta}\sin(\phi)\Delta x'$$

where $x = A\sqrt{\beta}\cos(\phi)$, and $x' = \frac{dx}{ds}$ were used to get the expression in terms of $\sin(\phi)$. We have the change in amplitude and phase of the motion due to an impulse error:

$$\Delta A = -\sqrt{\beta}\sin(\phi)\Delta x'$$
$$\Delta\phi = -\frac{\sqrt{\beta}}{A}\cos(\phi)\Delta x'$$

To get a more specific result, use $\Delta x'$ for a sextupole error:

$$\begin{aligned}\frac{dA}{dn} &= -sA^2(\beta)^{3/2}\sin(\phi)\cos^2(\phi) \\ \frac{d\phi}{dn} &= -sA(\beta)^{3/2}\cos^3(\phi) + 2\pi\nu_x\end{aligned}$$

where $2\pi\nu_x$ is the normal contribution from the tune. Using trigonometric identities, these can be written,

$$\begin{aligned}\frac{dA}{dn} &= -sA^2(\beta)^{3/2}\frac{1}{4}(\sin(\phi) + \cos(3\phi)) \\ \frac{d\phi}{dn} &= -sA(\beta)^{3/2}\frac{1}{4}(3\cos(\phi) + \cos 3(\phi)) + 2\pi\nu_x\end{aligned}$$

Over many turns $\frac{dA}{dn}$ and $\frac{d\phi}{dn}$ average to zero, *unless* ν_x is close to integer or third integer. Sextupole errors can drive third integer resonances.

Extending the analysis to higher order errors and including skew elements, transverse resonances have the form,

$$m\nu_x + n\nu_y = p$$

where m, n, and p are integers of the same sign (including zero). These conditions for resonance can be plotted as lines in 'tune space', ν_x versus ν_y. A depiction of tune space is shown in Fig. 3.

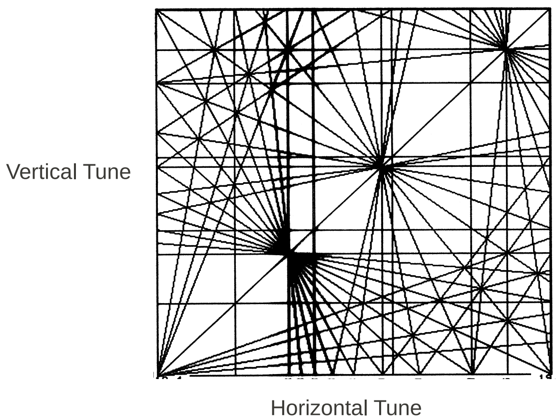

Figure 3: Plot of horizontal versus vertical tune. Solid lines are resonance lines. The tunes ν_x and nu_y should not lie on these lines, i.e. should not satisfy $m\nu + n\nu_y = p$, $m, n = 0, 1, 2, \ldots$ and $p = 1, 2, \ldots$.

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