1 Electromagnets

A beam passes through the aperture of a magnet, a region under vacuum containing no material. The magnetic field that results from powering the coils is present whether the beam is there or not. We are interested in finding the field in this aperture, and also in knowing how to build a magnet to get the desired field. Typically accelerators and beamlines have magnets that serve individual functions. Dipole magnets bend the entire beam, quadrupole magnets focus a beam, and sextupole magnets are used to control chromaticity. A picture of these three types of magnets is shown in Fig. 1.

Figure 1: The electromagnets from left to right are a quadrupole magnet, a dipole magnet, and a sextupole magnet. Courtesy Fermilab visual media services.

The following table gives the magnetic fields of some of typically used magnets.

Magnet type	field $\vec{B}$
Horizontally bending dipole	$B_0\,\hat{y}$
Focusing quadrupole	$B' y\,\hat{x} + B' x\,\hat{y}$
Focusing sextupole	$B'' xy\,\hat{x} + \frac{B''}{2}(x^2 - y^2)\,\hat{y}$
Skew quadrupole	$B' x\,\hat{x} + B' y\,\hat{y}$
Skew sextupole	$\frac{B''}{2}(x^2 - y^2)\,\hat{x} - B'' xy\,\hat{y}$

It is desirable to calculate how the strength of the magnetic field depends on the current in the coils. The shape of the magnetic field depends on the number and location and of the coils, and the shape of the pole tips around which they are wound.

Dipole electromagnets are constructed with a pair of current loops. There is a reasonably uniform magnetic field in the region between the loops. The field can be made stronger in this region if the current loops are wound around blocks of iron or steel. It can be shown that the field is related to the current in the coils as $B_0 = \frac{2\mu_0 NI}{h}$, where N is the number of turns per coil, I is the current in each turn, and h is the gap height of the magnet.

Quadrupole magnets have four coils. The pole tips around which the coils are wrapped around are alternating north and south poles. Figure 2 shows a quadrupole magnet.

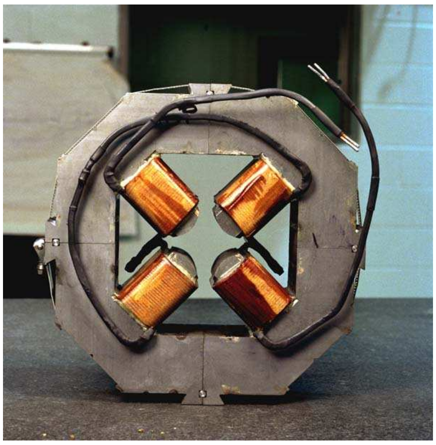

Figure 2: Quadrupole electromagnet. Courtesy Fermilab visual media services.

Ampere's law can be used to calculate the magnetic field, $\vec{B}$, in the gap of an electromagnetic quadrupole constructed in this fashion. Ampere's law relates the magnetic field integrated around a closed loop to the current passing through the plane enclosed by the loop;

$$\oint \vec{H} \cdot \vec{dl} = \int \vec{j} \cdot \vec{da}$$

where j is the current density (current/area), integrated over the area of the loop, and H is related to the magnetic field, B, by the permeability of the material, $\vec{H} = \frac{\vec{B}}{\mu}$

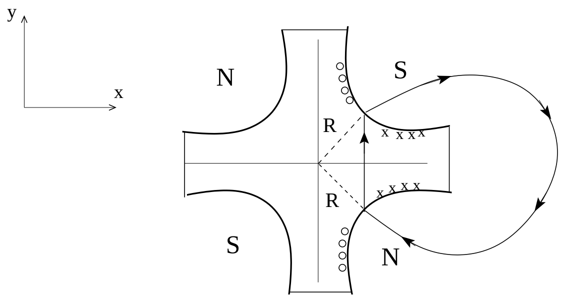

Figure 3: Sketch of an Amperian loop for a quadrupole magnet.

A convenient loop of integration crosses the gap from pole tip to pole tip in the $\hat{y}$ direction, the remainder of the loop going through the iron of the magnet, as shown in Fig. 3. The distance from the center of the magnet to any pole tip is R. The current carrying coils shown in the figure are penetrating perpendicular to the plane of the loop. The loop shown has current from two coils penetrating the loop, each coil has N turns carrying current I. The total enclosed current is then,

$$\int \vec{j} \cdot d\vec{a} = \int j da = 2NI$$

The length of the path through the gap is $\sqrt{2}R$. In the gap

$$\vec{H} = \frac{1}{\mu_0}\vec{B}$$

where μ_0 is the permeability of free space. The permeability of the iron is much larger than the permeability of vacuum ($\mu_0 << \mu_{iron}$), which results in the contribution of the portion of the loop in the iron to be negligible compared to the contribution from the portion of the loop in the gap.

$$\begin{aligned} \oint \vec{H} \cdot d\vec{l} &= \int \vec{H}_{gap} \cdot d\vec{l}_{gap} + \int \vec{H}_{iron} \cdot d\vec{l}_{iron} \\ &= \frac{1}{\mu_0}\int_0^{\sqrt{2}R} \vec{B} \cdot \hat{y} dy + \frac{1}{\mu_{iron}} \int \vec{B}_{iron} \cdot d\vec{l}_{iron} \\ &= \frac{1}{\mu_0}\int_0^{\sqrt{2}R} (B^{'} y\hat{x} + B^{'} x\hat{y}) \cdot \hat{y} dy \end{aligned}$$

The magnetic field in the quadrupole is $B^{'} y\hat{x} + B^{'} x\hat{y}$, where $B^{'}$ is a constant characteristic of the particular quadrupole magnet. Derivation of this field will follow later in the notes. The x position is constant along the path of integration in the magnet gap, $x = \frac{R}{\sqrt{2}}$ by geometry. Then,

$$B^{'}\left(\frac{R}{\sqrt{2}}\right)(\sqrt{2}R) = B^{'} R^2 = 2\mu_0 NI$$

This can be solved for $B^{'}$, the constant magnetic field gradient of the quadrupole.

$$B' = \frac{2\mu_0 NI}{R^2} \qquad \text{T/m}$$

2 Calculating magnetic fields

The two Maxwell's equations for the magnetic field are

$$\nabla \cdot \vec{B} = 0 \tag{1}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{2}$$

Electromagnets are made to produce a field inside the beam chamber, but the coils of electromagnets are outside the beam chamber region. If the coils carry a steady current that doesn't change with time, the problem of finding the field in the chamber reduces to a magnetostatics problem. So, both of the source terms on the right side of Eq. 2 for this problem are zero. Since the curl of a gradient is zero, then $\vec{B}$ can be written as the gradient of a scalar function (say, Φ_m). Putting $\vec{B} = -\vec{\nabla}\Phi_m$ into Eq. 1 results in Laplace's equation,

$$\nabla^2 \Phi_m = 0$$

Laplace's equation may be solved using the separation of variables technique, resulting in solutions that are sums of harmonics appropriate to the geometry of the problem. In the lecture on accelerating structures, it was mentioned that expansions in a Cartesian coordinate system were sine and cosine functions, while in cylindrical coordinates the radial harmonics were Bessel functions. Here, it is assumed that the magnetic fields in the dipoles, quadrupoles, etc. are uniform in the longitudinal direction. Then, the field varies only in the transverse cross-section, and Laplace's equation is two dimensional. Circular coordinates are appropriate, so the general solution to Laplace's equation is given as

$$\Phi_m = A_0 + B_0 \ln(r) + \sum_{k=1}^{\infty} \left(A_k r^k + B_k r^{-k} \right) \left(C_k \cos(k\theta) + D_k \sin(k\theta) \right) \tag{3}$$

Since there is no current inside the vacuum chamber, B_0 and B_k in Eq. 3 must be zero, or else Φ_m would become infinite as $r \to 0$, which is not physical. Further, A_0 may also be set to zero since the derivative of a constant is zero and will not change $\vec{B}$.

Then,

$$\Phi_m = \sum_{k=1}^{\infty} r^k \left(a_k \cos(k\theta) + b_k \sin(k\theta) \right) \tag{4}$$

where the as yet undetermined coefficient A_k has been absorbed into the new (still undetermined) coefficients a_k and b_k. In addition, only one of the two terms in the expansion ($a_k \cos(k\theta)$ or $b_k \sin(k\theta)$) should be non-zero, the choice depending on whether even or odd symmetry is required. Normally oriented magnets are associated with the sine terms, while skew elements (rotated 90° with respect to a normal element) are associated with the cosine terms. Separated function magnets ideally are described by only one term in the harmonic expansion for the magnetic potential.

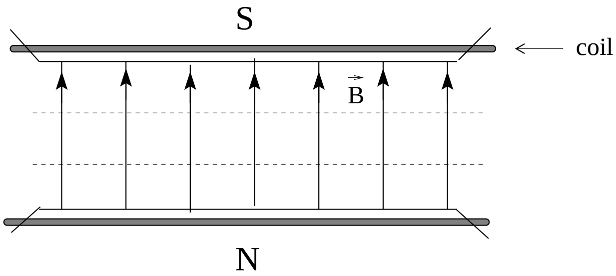

Figure 4: Cross-sectional sketch of a dipole magnet. A beam would go into or out of the plane of the paper. The force on the beam from the field would be $\vec{F} = q\vec{v} \times \vec{B}$, so for a positively charged particle going into the paper, the force would be to the right.

The only term desired for a horizontally bending dipole (with a vertically oriented magnetic field) is the $k = 1$ sine term, $b_1 r \sin(\theta)$. Note that in Cartesian coordinates $\Phi_m = b_1 r \sin(\theta) = b_1 y$. The magnetic field of the dipole is given by

$$\begin{aligned} \vec{B} &= -\vec{\nabla}\Phi_m = \left(-\hat{x}\frac{\partial}{\partial x} - \hat{y}\frac{\partial}{\partial y}\right) b_1 y \\ &= -b_1 \hat{y} \end{aligned}$$

The magnetic field is constant in the y direction. Also note that an equipotential will be a line at some constant value of y. Figure 4 shows a sketch of a dipole magnet cross-section. The magnetic field lines are in the $\hat{y}$ direction, and some equipotentials are shown as dashed lines at constant y.

Now consider a quadrupole magnet. The $k = 2$ term in the expansion for Φ_m is the quadrupole term,

$$\Phi_{quad} = b_2 r^2 \sin(2\theta)$$

Using the following trigonometric identity,

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

gives the following

$$\begin{aligned} \Phi_{quad} &= 2b_2(r\sin(\theta))(r\cos(\theta)) \\ &= 2b_2 xy \end{aligned}$$

The resulting magnetic field is then,

$$\begin{aligned} \vec{B} &= -\vec{\nabla}\Phi_{quad} = -\hat{x}\, 2b_2 \frac{\partial(xy)}{\partial x} - \hat{y}\, 2b_2 \frac{\partial(xy)}{\partial y} \\ &= -2b_2 y\, \hat{x} - 2b_2 x\, \hat{y} \\ &= B' y\, \hat{x} + B' x\, \hat{y} \end{aligned}$$

where the constant $-2b_2$ has been renamed B'. It must have units of T/m, to have the field end up in units of Tesla. Lets check out the force this field would exert on a particle of charge q traveling with speed v in the $\hat{z}$ direction. The force is given by $\vec{F} = q\vec{v} \times \vec{B}$.

$$\vec{F} = q \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & v \\ B'y & B'x & 0 \end{vmatrix} = q\hat{x}(-vB'x) + q\hat{y}(vB'y)$$

Notice that there is focusing in the $\hat{x}$ direction; the force is negative (restoring) and proportional to the offset x of the particle from the design orbit. In the $\hat{y}$ direction, there is a defocusing force, also linearly proportional to the particle offset y from the design orbit. A 'focusing' quadrupole focuses the beam horizontally but defocuses the beam in the vertical direction. A 'defocusing' quadrupole focuses vertically and defocuses horizontally. Net focusing in both planes in achievable, it requires the proper pattern of focusing and defocusing quadrupoles.

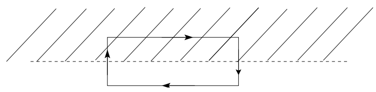

Figure 5: A loop across a boundary between vacuum and a ferromagnet.

How can a magnet be made to produce a field that is represented be a single term in the expansion of Φ_m? Shaping the pole faces correctly can achieve this goal. Ferromagnetic materials used for electromagnets have a very large permeability, μ. The curl equation $\vec{\nabla} \times \vec{H} = 0$ requires that the tangential component of $\vec{H}$ be continuous across a boundary. Place a rectangular loop straddling the boundary of two materials, in this case, the vacuum in the beam chamber and the ferromagnetic material the coil is wrapped around. This is depicted in Fig. 5. Since $\vec{\nabla} \times \vec{H} = 0$, the integral of the curl over the area of the loop must be zero;

$$\begin{aligned}
\int \vec{\nabla} \times \vec{H} \cdot d\vec{a} &= 0 && (5) \\
\oint \vec{H} \cdot d\vec{l} &= 0 && (6) \\
\int_{top} \vec{H} \cdot d\vec{l} + \int_{bottom} \vec{H} \cdot d\vec{l} &= 0 \\
H^{||}_{top} l + H^{||}_{bottom} l &= 0 \\
\frac{1}{\mu} B^{||}_{top} + \frac{1}{\mu_0} B^{||}_{bottom} &= 0 && (7)
\end{aligned}$$

Applying the curl theorem to Eq. 5 results in Eq. 6. The sides of the loop perpendicular to the interface can be made arbitrarily short and contribute nothing to the loop integral. Evaluating the contributions from the top and bottom sides of the loop results in Eq. 7. Since the top side of the boundary has a very large permeability, $\frac{1}{\mu}B^{||}_{top} \approx 0$. Then, Eq. 7, implies that there can be no component of $\vec{B}$ parallel to the magnetic surface. All magnetic field lines must be perpendicular to the ferromagnetic surface. Ferromagnetic surfaces are equipotential surfaces, just as the surfaces of perfect conductors are equipotential surfaces for electrostatic fields. (Perfect conductors cannot support a tangential component of the electrostatic field, or else current would flow along the surface.)

Then, to build a dipole, the pole faces should be flat at constant y with respect to the center plane of the magnet, just as depicted in Fig. 4. The expression for constant potential for a quadrupole is $Constant = \Phi_m = B^{'}xy$. This is the equation for a rotated hyperbola (such as the rough sketch of the pole faces in Fig. 3).

3 Magnetic field expansion

The magnetic fields present in particle accelerators, storage rings and transport lines can be represented with a multipole expansion, an expression where all the lower harmonics of Eq. 4 are kept, rather than a single term. In Cartesian coordinates the normal and skew components of the magnetic field takes the form:

$$B_y + iB_x = B_0 \sum_{n=0}^{\infty} (b_n + ia_n)(x + iy)^n \tag{8}$$

where n gives the order of the pole. For example, the $n = 0$ term corresponds to a dipole field, the $n = 1$ term to a quadrupole field, the $n = 2$ term to a sextupole field, and so on. The b_n coefficients go with the normal magnetic fields and the a_n coefficients with the skew magnetic fields.

Examining the dipole term, we have: $B_y + iB_x = B_0(b_0 + ia_0)$, or $B_y = B_0 b_0$ and $B_x = B_0 a_0$. If the dipole is an ideal horizontal dipole, with a constant field $B_y = B_0$, then $b_0 = 1$ and $a_0 = 0$.

Now check out the quadrupole term ($n = 1$) in Eq. 8. Equating the real part of the left side of the equation to the real part of the right side of the equation (and similarly for the imaginary part):

$$B_x = B_0(b_1 y + a_1 x) \tag{9}$$

$$B_y \quad = \quad B_0(b_1 x - a_1 y) \tag{10}$$

To solve Eq. 10 for b_1, take the derivative of both sides with respect to x:

$$b_1 = \frac{1}{B_0}\frac{\partial B_y}{\partial x}$$

This tells us that B_y in a quadrupole magnet is $B_y = \frac{\partial B_y}{\partial x} x = B' x$, and that the field gradient in a quadrupole is a constant (since b_1 is a constant).

The general magnetic field expansion may also be expressed in polar coordinates, the radial component of the field in polar coordinates has the form

$$B_r = B_0 \sum_{n=0}^{\infty} r^n [b_n \sin((n+1)\theta) + a_n \cos((n+1)\theta)]$$

The angular component of the field in polar coordinates has the form

$$B_\theta = B_0 \sum_{n=0}^{\infty} r^n [b_n \cos((n+1)\theta) - a_n \sin((n+1)\theta)]$$

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