

Review of formulas for relativistic motion

parameter	value
Speed of light	$c = 3.0 \times 10^8$ m/s
Rest energy of a proton	938.26 MeV
Rest energy of an electron	.511 MeV
Rest energy of a muon	105.659 MeV
Charge of an electron	-1.6×10^{-19} C

A relativistic particle moving with velocity v is often characterized by β , the fraction of lightspeed at which it moves:

$$\beta = \frac{v}{c}$$

where c is the speed of light. The energy and momentum of the particle are more conveniently scaled with γ :

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}$$

Since nothing can go faster than the speed of light, the particle velocity in an accelerator increases significantly at lower energies, but doesn't change much at higher energies. The dependence of $\beta = \frac{v}{c}$ on the total energy is shown in figure 1 for both an electron and a proton.

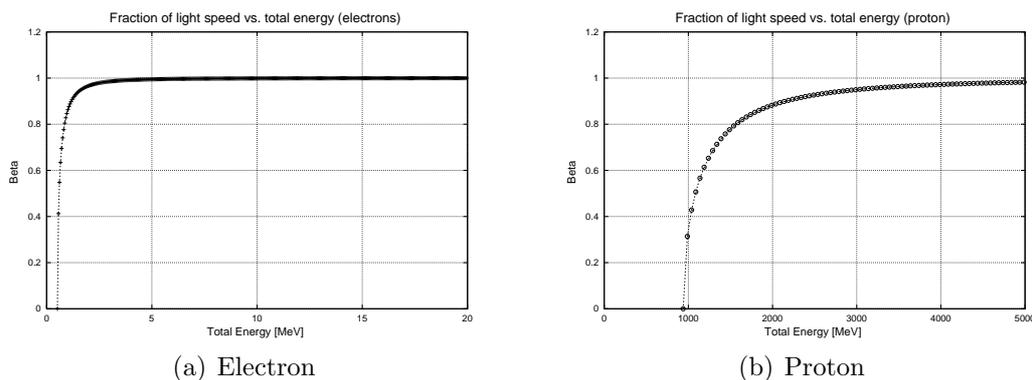


Figure 1: Dependence of β_{rel} on total energy.

The energy scaling (horizontal axis) in figure 1 has to be different for the electron and proton plots to clearly see the dependence. When plotted on the same scale, the result is shown in figure 2.

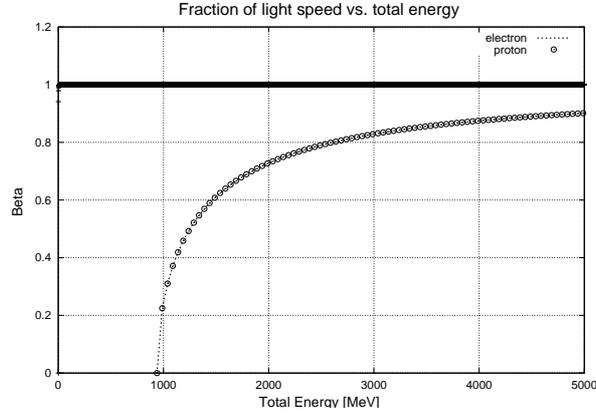


Figure 2: The dependence of β_{rel} on total energy, both for an electron and a proton.

Given the energy range of a particular accelerator, the associated change in particle velocity impacts the design of the accelerating structures. Electrons quickly reach lightspeed, while the heavier protons need to be at significantly higher energies before their velocity stops dramatically changing. The first accelerating stages for protons must handle this velocity swing.

Then total energy of a particle is the sum of its rest energy and its kinetic energy:

$$E_{total} = E_{rest} + T$$

where $E_{rest} = m_0c^2$ is the rest energy, the energy of a particle due to its mass, and T the kinetic energy of the particle. The total energy can also be expressed in terms of the gamma factor:

$$E_{total} = \gamma m_0 c^2$$

The particle momentum in terms of the γ factor is given by the following:

$$p = \gamma m_0 v = \gamma m_0 \beta c$$

The Lorentz invariant combination of E and \vec{p} is given by the following:

$$\left(\frac{E}{c}\right)^2 - p^2 = \left(\frac{E'}{c}\right)^2 - (p')^2 \quad (1)$$

where \vec{p} (and \vec{p}') is the total vector sum of the momenta of particles in the system. The expression with primed variables represents values of energy and momentum in one

frame of reference, while the expression with the unprimed variables represents values of energy and momentum in another frame of reference. The combination of energy and momentum in equation 1 has the same value regardless of the frame of reference.

A useful formula can be obtained by using the Lorentz invariant combination of E and \vec{p} (the scalar product $p^\mu p_\mu$). Equate $p^\mu p_\mu$ as written in the center of mass frame (net momentum is zero), to the expression written for a general frame of reference.

$$\begin{aligned}
 -\left(\frac{E_{cm}}{c}\right)^2 &= -\left(\frac{E}{c}\right)^2 + p^2 \\
 \left(\frac{E}{c}\right)^2 &= \left(\frac{m_0 c^2}{c}\right)^2 + p^2 = m_0^2 c^2 + p^2 \\
 E^2 &= (pc)^2 + m_0^2 c^4 \tag{2}
 \end{aligned}$$

Example 1 - The γ and β of a beam

If a proton has a total energy of 1 TeV, what is its value of β ? The highest energy proton ring at Fermilab was run at close to 1 TeV.

The proton rest energy is $m_0 c^2 = 938$ MeV. The ratio of the total energy to the rest energy is the gamma factor:

$$\begin{aligned}
 \gamma &= \frac{E_{total}}{E_{rest}} = \frac{\gamma m_0 c^2}{m_0 c^2} \\
 &= \frac{1 \times 10^6 \text{ MeV}}{938 \text{ MeV}} = 1066
 \end{aligned}$$

Now, β can be found:

$$\beta = \sqrt{1 - \left(\frac{1}{\gamma}\right)^2} = .99999956$$

For comparison, let's check the value of γ for an electron in the 7 GeV APS ring at Argonne.

$$\gamma = \frac{E_{total}}{E_{rest}} \approx \frac{7\text{GeV}}{.511 \times 10^{-3}\text{GeV}} = 13,700$$

Example 2 - Collider versus fixed target energies

There is an advantage to colliders, machines where two beams collide head-on, compared to fixed target arrangements, where a beam hits a fixed target. In a collider all the available energy goes into the collision, whereas in a fixed target experiment some energy goes into motion after the collision (target recoil, for example). Let's compare the the center-of-mass energy for these cases. In the collider, two protons with kinetic energy 900 GeV hit head-on coming from opposite directions. The net momentum is zero, since the momenta of the protons have opposite signs.

$$E_{cm} = E_{lab} = 900.938 + 900.938 \approx 1802 \text{ GeV}$$

In the fixed target case, a proton with kinetic energy 900 GeV hits a stationary proton. The momentum of the first (moving) proton is the total momentum, and may be found with the relation,

$$E_1^2 = p_1^2 c^2 + m_0^2 c^4 \rightarrow p_1 = \left[\left(\frac{E_1}{c} \right)^2 - m_0^2 c^2 \right]^{\frac{1}{2}} \quad (3)$$

The center of mass energy may be found using the momentum-energy invariant,

$$\left(\frac{E_{cm}}{c} \right)^2 = \left[\left(\frac{E_{tot}}{c} \right)^2 - p_{tot}^2 \right] \quad (4)$$

The energy of the moving proton is the kinetic energy plus the rest energy.

$$E_1 = 900 \text{ GeV} + 0.938 \text{ GeV}$$

So, the total energy is E_1 plus the rest energy of the target proton, $E_{total} = E_1 + m_0 c^2$ GeV. Combining Eq.s 3 and 4,

$$\begin{aligned} E_{cm}^2 &= E_{tot}^2 - p_1^2 c^2 \\ &= (E_1 + m_0 c^2)^2 - E_1^2 + m_0^2 c^4 \\ &= 2E_1 m_0 c^2 + 2m_0^2 c^4 \\ &= 2(900.938)(0.938) + 2(0.938)^2 \end{aligned}$$

Taking the square root to get E_{cm} ,

$$E_{cm} \approx 41 \text{ GeV}$$

Example 3 - Relating momentum change to energy change

Find the relation between the fractional change in total energy of a particle, and the fractional change in the particle momentum. This can be done by taking the derivative of the energy with respect to the momentum, and re-arranging the resulting expression.

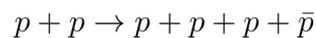
$$\begin{aligned} \frac{dE}{dp} &= \frac{d}{dp} \left([p^2 c^2 + m_0^2 c^4]^{1/2} \right) \\ &= \frac{1}{2} [p^2 c^2 + m_0^2 c^4]^{-1/2} 2pc^2 \\ &= \frac{p}{E} c^2 = \frac{E}{p} \left(\frac{p}{E} \right)^2 c^2 \\ &= \frac{E}{p} \left(\frac{\gamma m_0 \beta c}{\gamma m_0 c^2} \right)^2 c^2 = \frac{E}{p} \beta^2 \end{aligned}$$

Then,

$$\frac{dp}{p} = \frac{1}{\beta^2} \frac{dE}{E}$$

Example 4 - Antiproton production

What is the threshold energy for the following reaction?



A proton in a beam hits a proton in a target, and protons and an antiproton come off the target. Note that at least three protons are needed to have charge conservation.

The threshold energy is the minimum energy for the reaction; at this energy all of the particles end up at rest in the center of mass frame.

Setting $(p^\mu p_\mu)_{before} = (p^\mu p_\mu)_{after}$,

$$(\vec{p}_{beam} + \vec{p}_{target})^2 - \left(\frac{E_{beam} + E_{target}}{c}\right)^2 = -\left(\frac{E_{cm}}{c}\right)^2$$

The center of mass energy is the rest energy of the four particles, $E_{cm} = 4mc^2$. Multiplying out the terms on the left side of the equation,

$$p_{beam}^2 + p_{target}^2 + 2\vec{p}_{beam} \cdot \vec{p}_{target} - \left(\frac{E_{beam}}{c}\right)^2 - \left(\frac{E_{target}}{c}\right)^2 - 2\frac{E_{beam}E_{target}}{c^2} = -16m^2c^2$$

Grouping the terms,

$$-\left(p_{beam}^2 - \left(\frac{E_{beam}}{c}\right)^2\right) - \left(p_{target}^2 - \left(\frac{E_{target}}{c}\right)^2\right) - 2\left(\vec{p}_{beam} \cdot \vec{p}_{target} - \frac{E_{beam}E_{target}}{c^2}\right) = 16m^2c^2$$

The first term in parentheses on the left is the scalar product $p_{beam}^\mu p_{beam\mu}$, which is the same as the scalar product for that proton in the center of mass frame, $-m^2c^2$.

Similarly, the second term is also $-m^2c^2$. Examining the third term in parentheses on the left, $\vec{p}_{target} = 0$, and $E_{target} = mc^2$, since the target proton is stationary, so it becomes $2E_{beam}m$. Then,

$$E_{beam} = 7mc^2$$

The minimum beam energy to make antiprotons this way is seven times the rest energy.

Example 5 - Compton scattering

The inelastic scattering of a photon on an electron that results in a decrease in the photon energy is called Compton scattering. As the scattering angle increases, the photon loses more energy, and the photon wavelength gets longer.

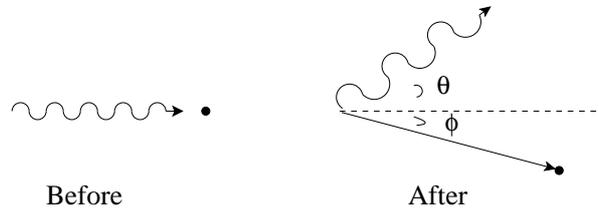


Figure 3: Compton scattering; a photon scatters off an electron, losing energy.

Conservation of momentum and energy are used to find the shift in wavelength.

Figure 3 shows the scattering angles of the photon and the electron. Take the incident

momentum of the photon, $p_\gamma = \frac{E_\gamma}{c}$, to be in the \hat{x} direction. The horizontal component of momentum is conserved;

$$\frac{E_\gamma}{c} = p'_\gamma \cos(\theta) + p'_e \cos(\phi)$$

The vertical component of momentum is conserved;

$$p'_\gamma \sin(\theta) = p'_e \sin(\phi)$$

The angle ϕ may be eliminated by squaring and adding,

$$\begin{aligned} (p'_e)^2 &= \left(\frac{E_\gamma}{c} - \frac{E'_\gamma}{c} \cos(\theta) \right)^2 + \left(\frac{E'_\gamma}{c} \right)^2 \sin^2(\theta) \\ (p'_e c)^2 &= E_\gamma^2 - 2E_\gamma E'_\gamma \cos(\theta) + (E'_\gamma)^2 \end{aligned} \quad (5)$$

There are two unknowns, p'_e and E'_γ . We are solving the system for the final energy of the photon. In order to eliminate the momentum of the electron after scattering, conservation of energy may be used, along with Eq. 5. The expression for energy conservation is the following,

$$\begin{aligned} E_\gamma + m_e c^2 &= E'_\gamma + E'_e \\ E_\gamma + m_e c^2 &= E'_\gamma + \left[(p'_e c)^2 + m_e^2 c^4 \right]^{\frac{1}{2}} \\ E_\gamma + m_e c^2 - E'_\gamma &= \left[E_\gamma^2 - 2E_\gamma E'_\gamma \cos(\theta) + (E'_\gamma)^2 + m_e^2 c^4 \right]^{\frac{1}{2}} \end{aligned}$$

Squaring both sides and solving for the final photon energy E'_γ ,

$$E'_\gamma = \frac{1}{\frac{(1-\cos(\theta))}{m_e c^2} + \frac{1}{E_\gamma}} \quad (6)$$

Given that for a photon, $E = h\nu = \frac{hc}{\lambda}$, the final photon wavelength may be obtained in terms of its initial wavelength, λ_0 , using Eq. 6,

$$\lambda = \lambda_0 + \frac{h}{m_e c} (1 - \cos(\theta)) \quad (7)$$

References

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