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Transverse Kinetic Stability

Outline:

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- Limits of model.

§2 Linearized Vlasov Equation

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Time Permiting we will add:

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§1 Overview

Collective Modes and Transverse Kinetic Stability

In discussions of transverse beam physics to this point we focused on:

- Equilibrium
  - Current limits (Q_{max}) resulting by balancing space-charge and focusing forces.
  - Self-consistent Vlasov equilibrium distributions with space-charge forces including KV with s-varying focusing forces and a general analysis for continuous focusing.
- Envelope Modes
  - Collective oscillations in the beam-edge resulting from a "mismatch" between the focusing structure and the beam.
  - Only consistent for a KV distribution with exactly conserved emittance, but approximately true in many situations with small emittance growth.

RMS Envelope Equations

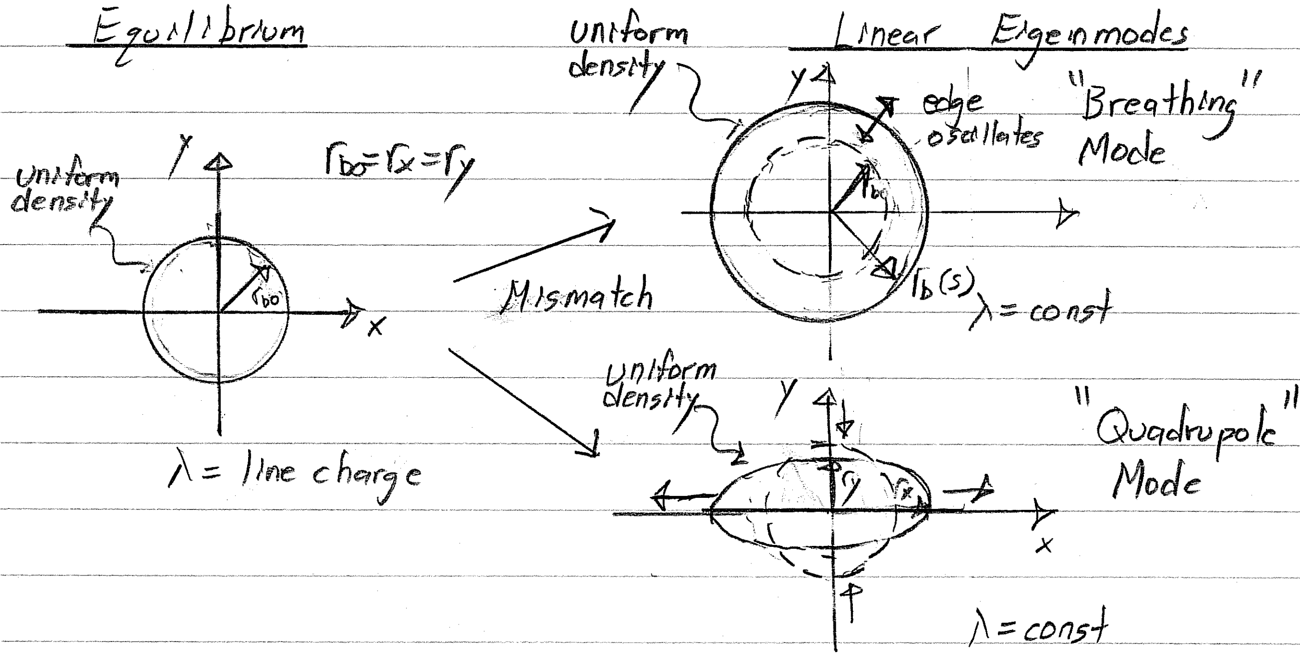
$$r_x'' + \kappa_x(s) r_x - \frac{Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} = 0$$

$$r_y'' + \kappa_y(s) r_y - \frac{Q}{r_x + r_y} - \frac{\varepsilon_y^2}{r_y^3} = 0$$

$\varepsilon_x = \text{const}$
$\varepsilon_y = \text{const}$ $\Longrightarrow$ Analyze for envelope modes

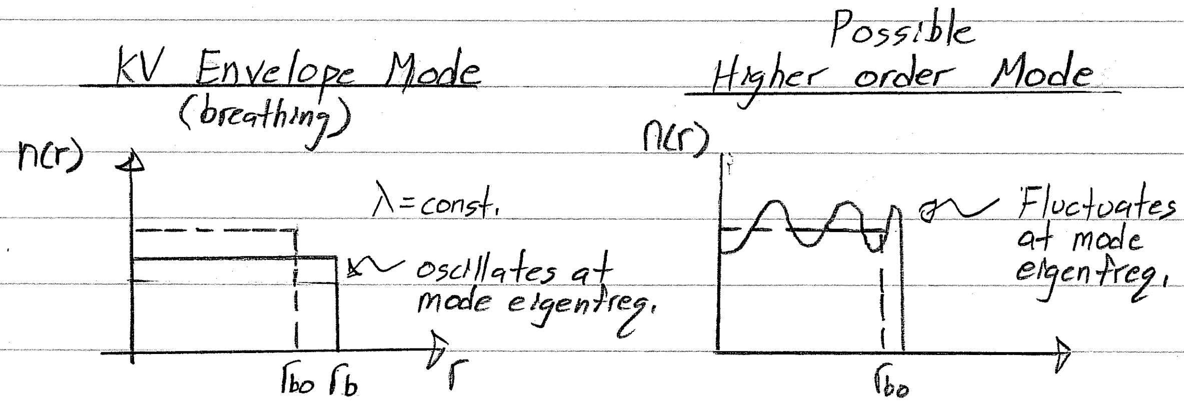
The envelope modes derived can be interpreted as oscillations in the core KV distribution where the <u>form</u> of the distribution does not change.

<u>Illustration</u> – <u>Envelope modes on a round beam equilibrium:</u>



Such low-order oscillations are highly important and may be the <u>most important</u> to control in a linear focusing channel in practical situations. However, at significant space-charge intensities we have shown that the beam behaves as a plasma with screened interactions etc. Thus we expect in analogy with plasma physics:

- Higher order collective modes exist internal to the beam and at the edge of the beam and may become unstable.
  - Perturbations will more generally have nonlinear space charge forces
  - The evolution of such perturbations may change the beam rms emittance etc.



- Expect many possible modes of oscillation, some of which may become unstable or resonate with various machine structures, etc.
  - Frequencies of higher order modes may differ significantly from the envelope modes.

Therefore, it is desireable to understand collective modes possible <u>beyond</u> those predicted by the simple envelope model.

- Expect intuitively that higher order collective modes become more important at higher space-charge intensities.

To begin characterizing the many possible modes of oscillation and the collective stability properties of intense beams, we take:

- Vlasov model of the beam
- Electrostatic self-interactions satisfying Poisson's equation.

and analyze the perturbations as oscillations about a Vlasov equilibrium

$$\boxed{f_\perp = f_0 + \delta f}$$

f_0 = Equilibrium distribution
δf = distribution perturbation associated with the mode.

We will also require (for simplicity) that:

$$\boxed{f_0 = \text{matched equilibrium}}$$

If f_0 is mismatched, more collective modes can result due to the "sloshing" motion of the equilibrium providing more "free energy" to drive perturbations.

Also, to keep the analysis reasonably simple, and allow analysis of equilibria beyond KV, we restrict our treatment to a <u>continuous focusing channel</u>

- Only KV equilibrium is known for s-varying focusing.

Continuous Focusing Model

Vlasov Poisson System:

$\gamma_b\beta_b = \text{const.}$ No momentum spread $\kappa_x = \kappa_y = k_{\beta 0}^2 = \text{const.}$

$$\frac{df_\perp}{ds} = \left\{ \frac{\partial}{\partial s} + \frac{\partial H}{\partial \vec{x}_\perp'} \cdot \frac{\partial}{\partial \vec{x}_\perp} - \frac{\partial H}{\partial \vec{x}_\perp} \cdot \frac{\partial}{\partial \vec{x}_\perp'} \right\} f_\perp(\vec{x}_\perp, \vec{x}_\perp', s) = 0$$

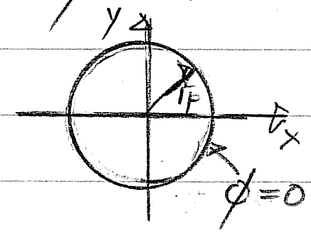
$$\frac{d\vec{x}_\perp}{ds} = \frac{\partial H}{\partial \vec{x}_\perp} = \vec{x}_\perp'$$

$$\frac{d\vec{x}_\perp'}{ds} = -\frac{\partial H}{\partial \vec{x}_\perp'} = -k_{\beta 0}^2 \vec{x}_\perp - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi}{\partial \vec{x}_\perp}$$

$$H = \frac{1}{2}\vec{x}_\perp'^2 + \frac{k_{\beta 0}^2}{2}\vec{x}_\perp^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2}$$

$$\nabla_\perp^2 \phi = \frac{-q}{\epsilon_0}\int d^2x' \, f_\perp \qquad \phi(r = r_p) = 0$$

Round, perfectly conducting beam-pipe.



Take

$$\boxed{f_\perp = f_0(H_0) + \delta f}$$

$f_0(H_0)$ = equilibrium
δf = perturbation

where $f_0(H_0)$ satisfies the equilibrium Vlasov-Poisson system:

$$\frac{df_0}{ds} = \left\{ \frac{\partial}{\partial s} + \vec{x}_\perp' \cdot \frac{\partial}{\partial \vec{x}_\perp} - \left(k_{\beta 0}^2 \vec{x}_\perp + \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi_0}{\partial \vec{x}_\perp}\right) \cdot \frac{\partial}{\partial \vec{x}_\perp'} \right\} f_0 = 0$$

$f_0 = f_0(H_0)$ ~ automatically satisfies Vlasov's equation above with

$$H_0 = \frac{1}{2}\vec{x}_\perp'^2 + \frac{k_{\beta 0}^2}{2}\vec{x}_\perp^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi_0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi_0}{\partial r}\right) = \frac{-q}{\epsilon_0}\int d^2x' f_0(H_0) \quad ; \quad \phi_0(r = r_p) = 0$$

Some insight can be gained on what possible unstable modes that might be lost can be attained by revisiting the simple envelope model of a KV breathing mode (constant emittance).

$$r_b'' + k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon_x^2}{r_b^3} = 0 \qquad r_b = r_x = r_y$$

$$r_b = r_{bo} + \delta r_b \quad ; \quad \frac{|\delta r_b|}{r_{bo}} \ll 1$$

r_{bo} = equilibrium envelope radius
δr_b = perturbed envelope radius

$$k_{\beta 0}^2 r_{bo} - \frac{Q}{r_{bo}} - \frac{\varepsilon_x^2}{r_{bo}} = 0 \qquad \text{eq. constraint}$$

Expand the envelope eqn. to linear order and employ the equilibrium constraint:

$$\Rightarrow \quad \delta r_b'' = -\left(4k_{\beta 0}^2 - \frac{2Q}{r_{bo}^2}\right)\delta r_b \equiv -k_{env}^2\,\delta r_b$$

$$k_{env} = 2\left(k_{\beta 0}^2 - \frac{Q}{2r_{bo}^2}\right)^{1/2} \qquad \text{envelope mode breathing "freq."}$$

$$= \left(2k_{\beta 0}^2 + 2k_\beta^2\right)^{1/2}$$

Thus:
The breathing mode envelope modes are always stable in a continuous focusing channel. But we know that they can become unstable for s-varying focusing from previous lectures.

$\Rightarrow$ Shows that important physics can be lost from the model – expect it to underestimate instability.

Nevertheless, in spite of thes limitations of the continuous focusing model the results derived from it provide valuable insight on the more complicated (realistic) situation with s-varying focusing and/or possible equilibrium mismatch. Moreover, kinetic stability analysis can be notoriously complicated

- Difficult algebra
- Methods of plasma physics are complicated in beams due to the finite geometry and intense self-fields. in the equilibrium

Solution of more complicated situations requires a through understanding of methods and ideas that are more simply expressed in the continuous focusing model. So it is a useful model in spite of the limitations!

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§2 Linearized Vlasov Equation and the Method of Characteristics

Vlasov's equation for the continuous focusing model can be expressed as:

$$\boxed{\begin{gathered}\frac{df_\perp}{ds} = \left\{ \frac{\partial}{\partial s} + \vec{x}_\perp' \cdot \frac{\partial}{\partial \vec{x}_\perp} - \left(k_{\beta 0}^2 \vec{x}_\perp + \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi}{\partial \vec{x}_\perp}\right) \cdot \frac{\partial}{\partial \vec{x}_\perp'} \right\} f_\perp(\vec{x}_\perp, \vec{x}_\perp', s) = 0 \\ \nabla_\perp^2 \phi = -\frac{q}{\epsilon_0}\int d^2x' \; f_\perp(\vec{x}_\perp, \vec{x}_\perp', s) \\ \phi(r=r_p) = 0\end{gathered}}$$

Then we expand perturbations as:

$$\boxed{\begin{aligned} f_\perp &= \underbrace{f_0(H_0)}_{\text{Equilibrium}} + \underbrace{\delta f}_{\text{Perturbation}} \\ \phi &= \phi_0 \quad\;\; + \delta\phi \end{aligned}}$$

δf is the distribution perturbation and $\delta\phi$ the consistent perturbed potential.

where:

$$\boxed{\begin{gathered} H_0 = \frac{1}{2}\vec{x}_\perp'^2 + \frac{k_{\beta 0}^2}{2}\vec{x}_\perp^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi_0 \\ f_0(H_0) \sim \text{any differentiable function} \\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi_0}{\partial r}\right) = -\frac{q}{\epsilon_0}\int d^2x' \; f_0(H_0) \\ \phi_0(r=r_p) = 0 \end{gathered}}$$

Since Poisson's equation is linear in ϕ, note that $\delta\phi$ must satisfy:

$$\boxed{\begin{gathered} \nabla_\perp^2 \delta\phi = -\frac{q}{\epsilon_0}\int d^2x' \; \delta f(\vec{x}_\perp, \vec{x}_\perp', s) \\ \delta\phi(r=r_p) = 0 \end{gathered}}$$

We now insert the perturbations in Vlasov's equation:

$$\underbrace{\left\{ \frac{\partial}{\partial s} + \vec{x}_\perp' \cdot \frac{\partial}{\partial \vec{x}_\perp} - \left(k_{\beta 0}^2 \vec{x}_\perp + \frac{q}{m\gamma_b^3\beta_b^2c^2} \frac{\partial \phi_0}{\partial \vec{x}_\perp} \right) \cdot \frac{\partial}{\partial \vec{x}_\perp'} \right\} f_0(H_0)}_{\to 0 \text{ by equilibrium Vlasov Equation}}$$

$$+ \left\{ \frac{\partial}{\partial s} + \vec{x}_\perp' \cdot \frac{\partial}{\partial \vec{x}_\perp} - \left(k_{\beta 0}^2 \vec{x}_\perp + \frac{q}{m\gamma_b^3\beta_b^2c^2} \frac{\partial \phi_0}{\partial \vec{x}_\perp} \right) \cdot \frac{\partial}{\partial \vec{x}_\perp'} \right\} \delta f$$

$$= \frac{q}{m\gamma_b^3\beta_b^2c^2} \frac{\partial \delta\phi}{\partial \vec{x}_\perp} \cdot \frac{\partial}{\partial \vec{x}_\perp'} f_0(H_0) + \frac{q}{m\gamma_b^3\beta_b^2c^2} \frac{\partial \delta\phi}{\partial \vec{x}_\perp} \cdot \frac{\partial}{\partial \vec{x}_\perp'} \delta f$$

Generally, to investigate stability, one need only analyze small amplitude perturbations with:

$$\boxed{\begin{array}{l} f_0 \gg |\delta f| \\ \phi_0 \gg |\delta\phi| \end{array}}$$

and terms of $\mathcal{O}(\delta^2)$ may be dropped to obtain the linearized Vlasov equation.

$$\boxed{\begin{gathered} \left\{ \frac{\partial}{\partial s} + \vec{x}_\perp' \cdot \frac{\partial}{\partial \vec{x}_\perp} - \left(k_{\beta 0}^2 \vec{x}_\perp + \frac{q}{m\gamma_b^3\beta_b^2c^2} \frac{\partial \phi_0}{\partial \vec{x}_\perp} \right) \cdot \frac{\partial}{\partial \vec{x}_\perp'} \right\} \delta f(\vec{x}_\perp, \vec{x}_\perp', s) \\ = \frac{q}{m\gamma_b^3\beta_b^2c^2} \frac{\partial \delta\phi}{\partial \vec{x}_\perp} \cdot \frac{\partial}{\partial \vec{x}_\perp'} f_0(H_0) \end{gathered}}$$

To analyze Vlasov stability we must solve this equation for a particular equilibrium $f_0(H_0)$ subject to:

$$\boxed{\nabla_\perp^2 \delta\phi = -\frac{q}{\epsilon_0} \int d^2x' \, \delta f \qquad \delta\phi(r = r_p) = 0}$$

How can we do this? The equation is complicated!

Note that the equilibrium Vlasov equation is:

$$\left\{ \frac{\partial}{\partial s} + \vec{x}_\perp' \cdot \frac{\partial}{\partial \vec{x}_\perp} - \left(k_{\beta 0}^2 \vec{x}_\perp + \frac{q}{m\gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi_0}{\partial \vec{x}_\perp} \right) \cdot \frac{\partial}{\partial \vec{x}_\perp'} \right\} f_0 = 0$$

$$\Rightarrow \left. \frac{d}{ds} \right|_{\substack{\text{equilibrium} \\ \text{orbit}}} f_0 = 0$$

Thus we may interpret

$$\boxed{\left\{ \frac{\partial}{\partial s} + \vec{x}_\perp' \cdot \frac{\partial}{\partial \vec{x}_\perp} - \left(k_{\beta 0}^2 \vec{x}_\perp + \frac{q}{m\gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi_0}{\partial \vec{x}_\perp} \right) \cdot \frac{\partial}{\partial \vec{x}_\perp'} \right\} = \left. \frac{d}{ds} \right|_{\substack{\text{equilibrium} \\ \text{orbit}}}}$$

as the total derivative evaluated <u>along an equilibrium particle orbit.</u> This suggests using the <u>method of characteristics</u> to solve the linearized Vlasov equation.

<u>Method of Characteristics:</u>

Denote:

$$\boxed{\begin{aligned} \frac{d}{d\tilde{s}} \vec{x}_\perp(\tilde{s}) &= \vec{x}_\perp'(\tilde{s}) \\ \frac{d}{d\tilde{s}} \vec{x}_\perp'(\tilde{s}) &= -k_{\beta 0}^2 \vec{x}_\perp(\tilde{s}) - \frac{q}{m\gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi_0(\vec{x}_\perp(\tilde{s}))}{\partial \vec{x}_\perp(\tilde{s})} \end{aligned}}$$

- Equilibrium trajectories as a function of the variable $\tilde{s}$.

Require that the trajectory pass through the phase space point $\vec{x}_\perp, \vec{x}_\perp'$ at $\tilde{s} = s$

$$\boxed{\begin{aligned} \vec{x}_\perp(\tilde{s}=s) &= \vec{x}_\perp \\ \vec{x}_\perp'(\tilde{s}=s) &= \vec{x}_\perp' \end{aligned}}$$

Then the linearized Vlasov equation can be expressed as:

$$\boxed{\frac{d}{d\tilde{s}}\delta f(\vec{x}_\perp(\tilde{s}),\vec{x}_\perp'(\tilde{s}),\tilde{s}) = \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\delta\phi(\vec{x}_\perp(\tilde{s}))}{\partial\vec{x}_\perp(\tilde{s})}\cdot\frac{\partial}{\partial\vec{x}_\perp'(\tilde{s})}f_0(H_0(\vec{x}_\perp(\tilde{s}),\vec{x}_\perp'(\tilde{s})))}$$

For amplifying perturbations that grow in s, we may integrate this equation from $\tilde{s}\rightarrow-\infty$, to $\tilde{s}=s$ and neglect the "initial" conditions at $\tilde{s}=-\infty$ to obtain:

$$\boxed{\delta f(\vec{x}_\perp,\vec{x}_\perp',s) = \frac{q}{m\gamma_b^3\beta_b^2c^2}\int_{-\infty}^{s}d\tilde{s}\;\frac{\partial\delta\phi(\vec{x}_\perp(\tilde{s}))}{\partial\vec{x}_\perp(\tilde{s})}\cdot\frac{\partial}{\partial\vec{x}_\perp'(\tilde{s})}f_0(H_0(\vec{x}_\perp(\tilde{s}),\vec{x}_\perp'(\tilde{s})))}$$

This solution is then inserted into the Poisson equation

$$\boxed{\begin{gathered}\nabla_\perp^2\delta\phi = -\frac{q}{\epsilon_0}\int d^2x'\,\delta f(\vec{x}_\perp,\vec{x}_\perp',s)\\ \delta\phi(r=r_p)=0\end{gathered}}$$

to determine the <u>self-consistent</u> evolution of the <u>field perturbations</u> $\delta\phi$ in the small-amplitude regime.

- A similar procedure can be applied for the nonlinear perturbations too (homework problem).

<u>The Poisson equation is now cast as a complicated differential-integral equation that must be solved to understand the evolution of the perturbations.</u>

- Still complicated, but easier to analyze than other forms <u>if the equilibrium characteristics can be calculated.</u>

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§3 Collective Modes on a KV Equilibrium Beam

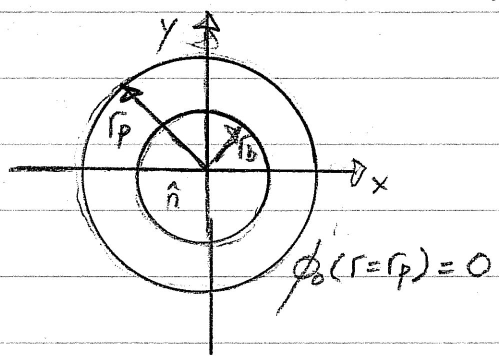
Here we take a KV equilibrium distribution with

$$\boxed{f_0(H_0) = \frac{\hat{n}}{2\pi}\,\delta\left[H_0 - \frac{\varepsilon_x^2}{2r_b^2}\right]}$$

$\hat{n}$ = constant density of KV equilibrium
ε_x^2 = x-emittance.
r_b = equilibrium beam radius.

$$k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon_x^2}{r_b^3} = 0$$

$$H_0 = \frac{1}{2}\vec{x}_\perp'^2 + \frac{k_{\beta 0}^2}{2}\vec{x}_\perp^2 + \frac{q\,\phi_0}{m\gamma_b^3\beta_b^2c^2}$$



and assume small-amplitude axisymmetric ($\partial/\partial\theta = 0$) perturbations with normal mode form:

$$\boxed{\begin{aligned}\delta f(\vec{x}_\perp, \vec{x}_\perp', s) &= \delta f(r, \vec{x}_\perp', k)\, e^{-iks} \\ \delta\phi(\vec{x}_\perp, s) &= \delta\phi(r, k)\, e^{-iks}\end{aligned}}$$

k = const (mode eigenfrequency)

The equilibrium characteristics in the core of the KV beam can be expressed as:

$$\boxed{\begin{aligned} r^2(\tilde{s}) &= r^2\cos^2[k_\beta(\tilde{s}-s)] + \frac{rr'}{k_\beta}\cos\psi\,\sin[2k_\beta(\tilde{s}-s)] \\ &\quad + \frac{r'^2}{k_\beta^2}\sin^2[k_\beta(\tilde{s}-s)] \\[1em] x(\tilde{s}=s) &= r\cos\theta \quad ; \quad x'(\tilde{s}=s) = r'\cos\theta_p \\ y(\tilde{s}=s) &= r\sin\theta \quad ; \quad y'(\tilde{s}=s) = r'\sin\theta_p \\[1em] \psi &\equiv \theta - \theta_p \\[1em] k_\beta &= \left(k_{\beta 0}^2 - \frac{Q}{r_b^2}\right)^{1/2} = \frac{\varepsilon_x}{r_b^2} \quad \text{Depressed } \beta\text{-tron wavenumber of particle oscillations} \end{aligned}}$$

These results can be inserted into the characteristic equation

$$\delta f(\vec{x}_\perp, \vec{x}_\perp', s) = \frac{q}{m\gamma_b^3\beta_b^2c^2}\int_{-\infty}^{s} d\tilde{s}\; \frac{\partial\,\delta\phi(\vec{x}_\perp(\tilde{s}))}{\partial\vec{x}_\perp(\tilde{s})}\cdot\frac{\partial}{\partial\vec{x}_\perp'(\tilde{s})} f_0\left(H_0\left(\vec{x}_\perp(\tilde{s}), \vec{x}_\perp'(\tilde{s})\right)\right)$$

to derive an expression for $\delta f(r, \vec{x}_\perp')$. This expression can then be inserted into the Poisson equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\,\delta\phi(r)}{\partial r}\right) = -\frac{q}{\epsilon_0}\int d^2x'\,\delta f(r, \vec{x}_\perp')$$

to derive a linear eigenvalue equation for $\delta\phi(r)$:

A significant amount of manipulation obtains the following form for the eigenvalue equation:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\delta\phi(r) = \frac{\hat{\omega}_p^2}{\gamma_b \beta_b^2 c^2}\,\Theta(r_b - r)\,\frac{1}{r_\perp'}\frac{\partial}{\partial r_\perp'} I_{orb}(r, r_\perp', k)\Big|_{r_\perp'^2 = \frac{\varepsilon_x^2}{r_b^2}\left(1-\frac{r^2}{r_b^2}\right)} \quad (1)$$

$$+ \frac{\hat{\omega}_p^2/(\gamma_b \beta_b^2 c^2)}{\varepsilon_x^2/r_b^2}\,\delta(r - r_b)\left[\delta\phi + I_{orb}(r, r_\perp', k)\right]_{r_\perp'=0} \quad (2)$$

Subject to: $\delta\phi(r = r_b) = 0$, $\hat{\omega}_p^2 = \dfrac{q^2 \hat{n}}{\varepsilon_0 m}$ = Plasma Freq. Squared.

where:

$$\Theta(r_b - r) \equiv \begin{cases} 1 & r_b > r \\ 0 & r_b < r \end{cases} \qquad \text{Heaviside step function}$$

$$I_{orb}(r, r_\perp', k) = ik\int_{-\pi}^{\pi}\frac{d\psi}{2\pi}\int_{-\infty}^{s} d\tilde{s}\;\delta\phi(r(\tilde{s}), k)\, e^{-ik(\tilde{s}-s)}$$

Orbit integral.

Note:

- Term (1) $\propto \Theta(r_b - r)$ is a body-wave perturbation existing only in the core ($r < r_b$) of the equilibrium beam.
- Term (2) $\propto \delta(r - r_b)$ is a surface-wave perturbation existing only at the edge ($r = r_b$) of the equilibrium beam.
- The orbit integral $I_{orb}(r, r', k)$ depends on both $\delta\phi$ and the eigenfrequency k.

$$\boxed{\text{The Poisson equation has become a linear integro-differential eigenvalue equation fixing the mode perturbed potential } \delta\phi \text{ and eigenfrequency } k.}$$

Gluckstern Mode Solution

This eigenvalue equation is difficult, but it has been solved analytically

- A finite polynomial in r^2 expansion of $\delta\phi$ for $r<r_b$ can satisfy the equation (terms truncate)
- Expansions are inserted into the characteristic integrals and coefficients are identified power-by-power in r^2, and assembled.

Solution (after much analysis)

Eigenfunction:

$$\boxed{\delta\phi_n(r) = \begin{cases} \frac{A_n}{2}\left[P_{n-1}(1-2r^2/r_b^2) + P_n(1-2r^2/r_b^2)\right]; & 0 \le r < r_b \\ 0 & ; r_b < r \le r_p \end{cases}}$$

$n = 1, 2, 3, \ldots$	radial mode index
$A_n = \text{const.}$	linear mode amplitude.
$P_n(x)$	n'th order Legendre Polynomial

Dispersion Relation:

Each n-labeled eigenfunction has $2n$ (degenerate) "eigenfrequencies" k satisfying an nth degree polynomial in k^2 dispersion relation.

$$\boxed{2n + \frac{1-(\sigma/\sigma_0)^2}{(\sigma/\sigma_0)^2}\left[B_{n-1}\left(\frac{k/k_{\beta 0}}{\sigma/\sigma_0}\right) - B_n\left(\frac{k/k_{\beta 0}}{\sigma/\sigma_0}\right)\right] = 0}$$

where:
$$\frac{\sigma}{\sigma_0} \equiv \frac{k_\beta}{k_{\beta 0}} = \frac{(k_{\beta 0}^2 - Q/r_b^2)^{1/2}}{k_{\beta 0}}$$

and

$$\boxed{B_n(\alpha) \equiv \begin{cases} 1 & n=0 \\ \frac{[(\alpha/2)^2-0^2]}{[(\alpha/2)^2-1^2]}\cdot\frac{[(\alpha/2)^2-2^2]}{[(\alpha/2)^2-3^2]}\cdots\cdots\frac{[(\alpha/2)^2-(n-1)^2]}{[(\alpha/2)^2-n^2]} & n=1,3,5\ldots \\ \frac{[(\alpha/2)^2-1^2]}{[(\alpha/2)^2-2^2]}\cdot\frac{[(\alpha/2)^2-3^2]}{[(\alpha/2)^2-4^2]}\cdots\cdots\frac{[(\alpha/2)^2-(n-1)^2]}{[(\alpha/2)^2-n^2]} & n=2,4,6\ldots \end{cases}}$$

Properties:

Radial Eigenfunction:

- Vanishes outside the equilibrium beam edge ($r > r_b$).
- Has $n-1$ nodes with $\delta\phi = 0$ within the equilibrium beam ($r < r_b$).
- Each n labeled eigenfunction has $2n$ distinct frequencies

Corresponding perturbed density can be calculated from Poisson's equation:

$$\boxed{\begin{aligned}\delta n_n &= \delta n_n(r) e^{-iks} \\ \delta n_n(r) &= -\frac{\epsilon_0}{q}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \delta\phi_n}{\partial r}\right)\end{aligned}}$$

- Find that the perturbed density of the mode is more larger near the outer ($r \lesssim r_b$) edge of the beam for larger n.

Eigenfunction Form

Mode number n	$\delta\phi_n / A_n$ (potential)	δn_n (density, scaled units)
1	$1-\tilde{r}^2$	1
2	$1-4\tilde{r}^2+3\tilde{r}^4$	$4(1-3\tilde{r}^2)$
3	$1-9\tilde{r}^2+18\tilde{r}^4-10\tilde{r}^6$	$9(1-8\tilde{r}^2+10\tilde{r}^4)$
4	$1-16\tilde{r}^2+60\tilde{r}^4-80\tilde{r}^6+35\tilde{r}^8$	$16(1-15\tilde{r}^2+45\tilde{r}^4-35\tilde{r}^6)$
⋮	⋮	⋮

$$\tilde{r} \equiv r/r_b$$

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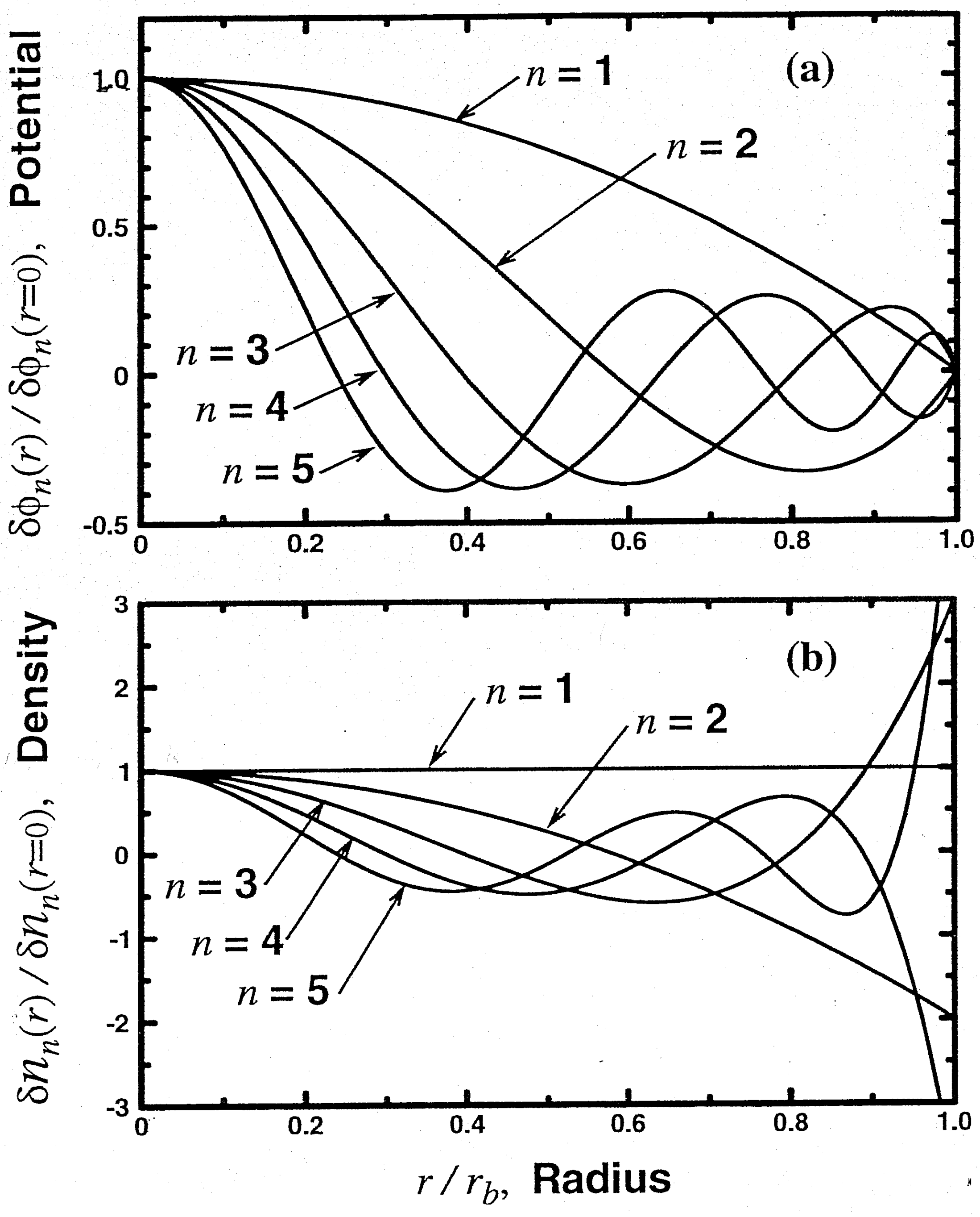
Radial Eigenfunction



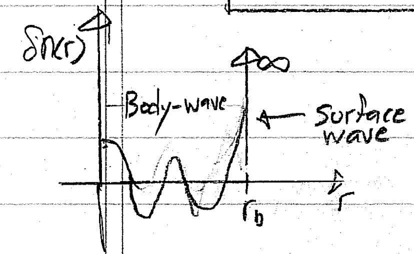
S. M. Lund

Perturbations should introduce no net charge into the system:

$$2\pi \int_0^{r_p} dr\, r\, \delta n(r) = 0$$

<u>The $r < r_b$ component of the perturbations are not the only terms present.</u> For the $r < r_b$ eigenfunctions calculated: $\int_0^{r_d} dr\, r\, \delta n_n(r) \neq 0$. A more detailed analysis shows that:

$$\boxed{\delta n_n(r) = \underbrace{\delta n_n(r)\Big|_{\text{body}} \Theta(r_b - r)}_{\substack{\text{body-wave perturbation}\\ \text{in core of equilibrium}}} + \underbrace{\delta n_n\Big|_{\text{surface}} \frac{r_b^2}{r}\delta(r - r_b)}_{\substack{\text{singular charge conserving}\\ \text{beam edge adjustment at } r = r_b}}}$$



where:

$$\delta n_n\Big|_{\text{body}} = -\frac{\varepsilon_0}{q}\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \delta\phi_n}{\partial r}\right)$$

$$\delta n_n\Big|_{\text{surface}} = \text{const} \times (-1)^n n A_n$$

To linear order this is equivalent to:

$$\boxed{n(r) = \left[\hat{n} + \delta n_n(r)\Big|_{\text{body}}\right] \Theta\left[r_b + \delta r_b - r\right]}$$

$$\delta r_b = \text{const} \times (-1)^n n A_n$$

readjustment of beam edge radius.

Dispersion Relation

- Polynomial in k^2 $\Rightarrow$ $\pm k$ solutions and therefore there will be unstable growing perturbations if k is complex:

$$\delta\phi \sim \delta\phi_n(r)\, e^{-iks}$$

$$k = k_r \pm i k_I \qquad \begin{array}{l} k_r = \text{real part} \\ k_I = \text{imaginary part} \end{array}$$

For the unstable branch:

$$\delta\phi \sim \delta\phi_n(r)\, e^{-ik_r s} \cdot e^{|k_I| s} \Rightarrow \text{exponential growth.}$$

- k is a function of n and σ/σ_0 <u>only</u>.
  - $0 \leq \sigma/\sigma_0 \leq 1$

    0 ↑ strongest possible space charge.

    1 ↑ zero space-charge.

  - <u>Instabilities will occur over a range of σ/σ_0 and will turn off for σ/σ_0 large enough</u> (weak space-charge).
    KV beam is always stable for zero space-charge since orbits are stable.

<u>Dispersion Relations:</u>

Mode number n	Dispersion relation
1	$(k/k_{\beta 0})^2 - 2(1 + \sigma^2/\sigma_0^2) = 0$
2	$(k/k_{\beta 0})^4 - 2(1 + 9\sigma^2/\sigma_0^2)(k/k_{\beta 0})^2 - 4(\sigma^2/\sigma_0^2)(1 - 17\sigma^2/\sigma_0^2) = 0$
⋮	⋮ Rapidly more complicated!

9/

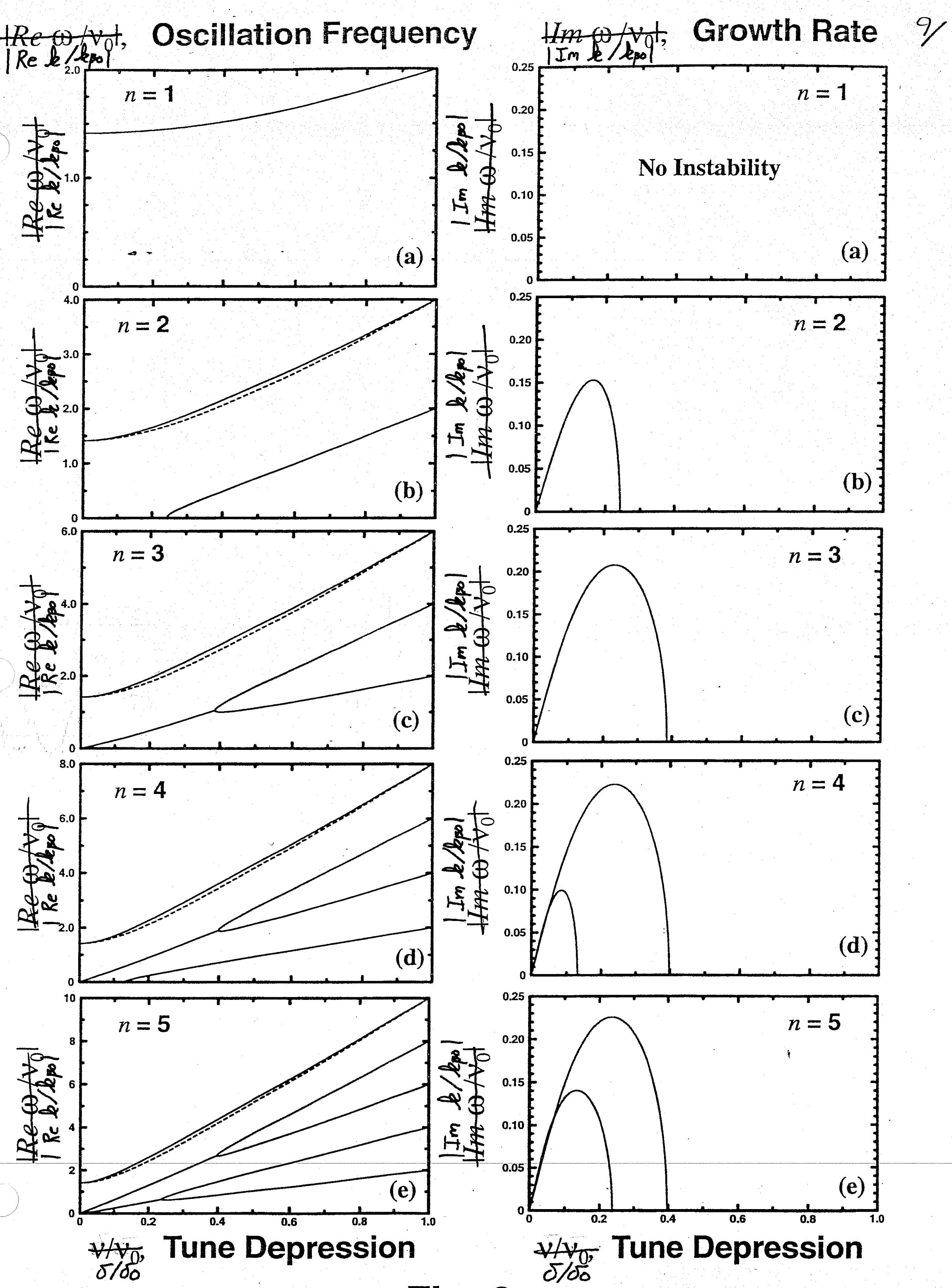
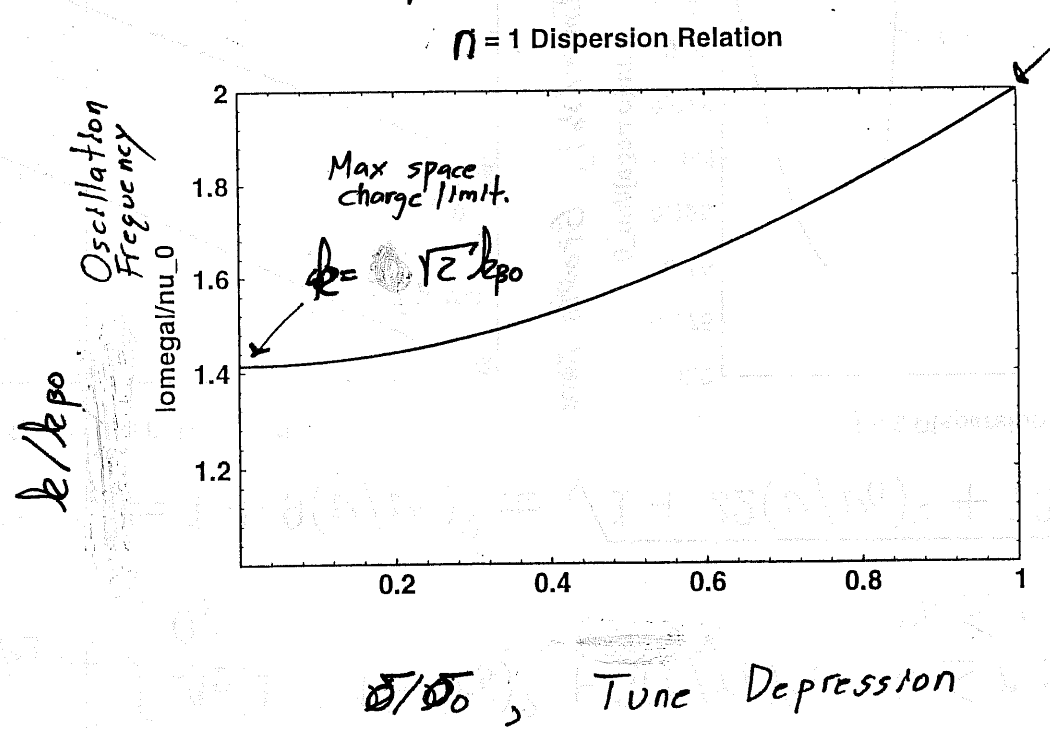

~~|Re ω/ν0|~~, |Re k/kpo| Oscillation Frequency
~~|Im ω/ν0|~~, |Im k/kpo| Growth Rate
n = 1
n = 2
n = 3
n = 4
n = 5
No Instability
(a)
(b)
(c)
(d)
(e)
~~ν/ν0~~, δ/δ0 Tune Depression

Fig. 8

Example: $n = 1$, Envelope Mode

$$\delta\phi_1 = \begin{cases} A_1[1-(r/r_b)^2], & 0 \le r \le r_b, \\ 0, & r_b \le r \le r_p, \end{cases}$$

$$(k/k_{\beta 0})^2 = 2 + 2(\sigma/\sigma_0)^2$$



$k = 2k_{\beta 0}$
Zero space charge limit

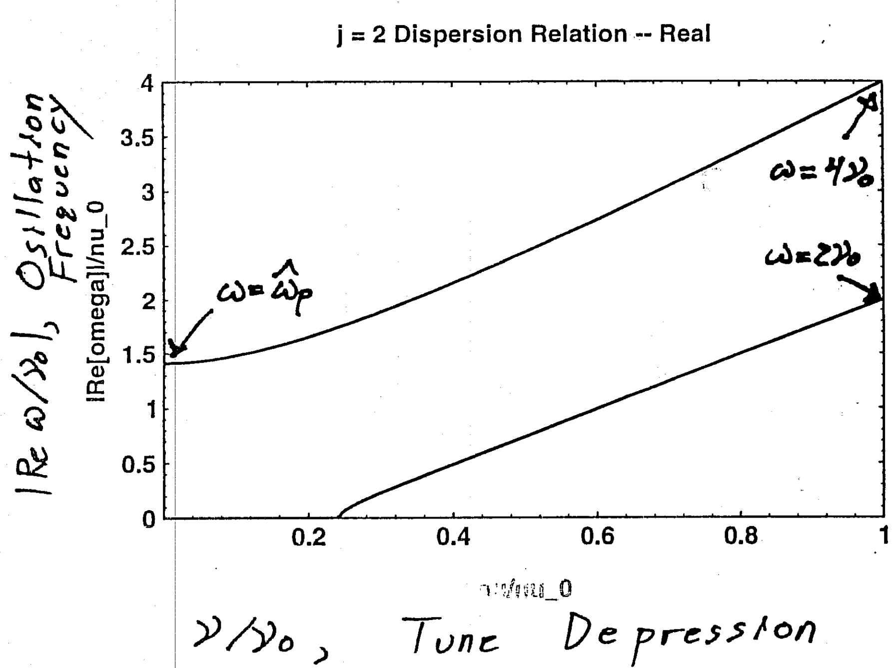
The $n=1$ mode will be shown (homework) to be the usual envelope mode

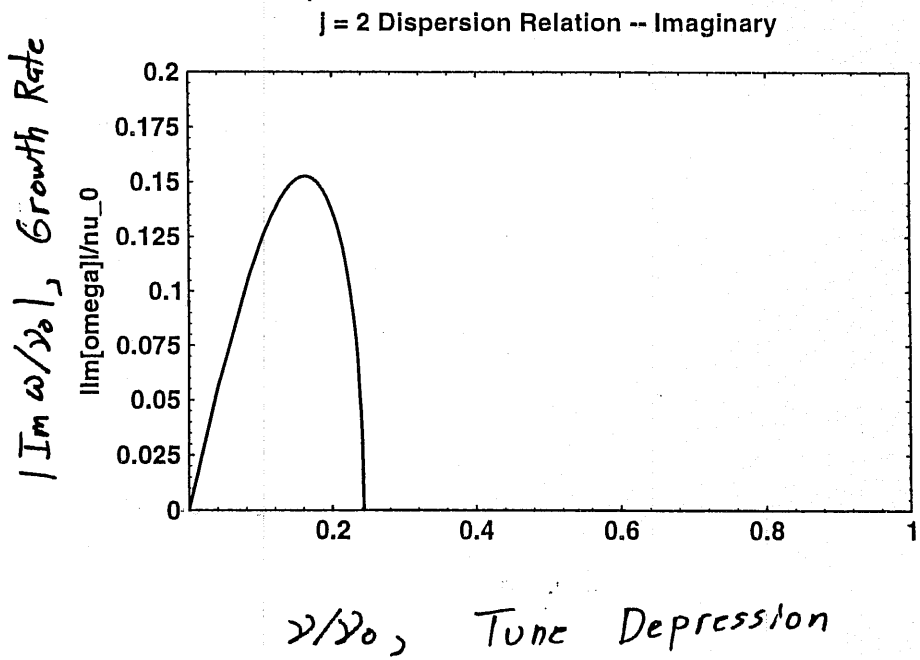
Oscillation Frequency

Example: $n = 2$ Mode

$$\delta\phi_2 = \begin{cases} A_2[1 - 4(r/r_b)^2 + 3(r/r_b)^4], & 0 \le r \le r_b, \\ 0, & r_b \le r \le r_p, \end{cases}$$

$$(\omega/\nu_0)^2 = 1 + 9(\nu/\nu_0)^2 \pm \sqrt{1 + 22(\nu/\nu_0)^2 + 13(\nu/\nu_0)^4}$$





Oscillation Frequency

Growth Rate

As might be expected on physical grounds, the singular KV distribution drives numerous, strong, collective instabilities. This implies that the KV model is suspect since real beams are often transported where the KV model would predict strong instability. However:

- Low-order KV features (envelope modes) are correct and well verified.
- Higher order collective modes observed on intense beam cores often look similar to the KV model predictions in density/potential etc, but are not unstable.

How is this situation resolved? A partial answer was suggested by a fluid model developed by Lund and Davidson. In this model:

- Density and tempeture profiles (i.e., low order features) of the KV model were preserved.
- The singular phase-space structures were eliminated.

A stability analysis obtained:

Mode Eigenfunction: $(r<r_b)$

$$\delta\phi_n = \frac{A_n}{2}\left[P_{n-1}\left(1-2\frac{r^2}{r_b^2}\right) + P_n\left(1-2\frac{r^2}{r_b^2}\right)\right]$$

Mode Dispersion Relation:

$$\left(\frac{k}{k_{\beta 0}}\right)^2 = 2 + 2\left(\frac{\sigma}{\sigma_0}\right)^2(2n^2-1)$$

$$n = 1, 2, 3, \ldots\ldots$$

Features of fluid model:

- Identical radial eigenfunction to the full kinetic theory
- Fluid mode dispersion relation predicts stability for all modes and closely tracks the (stable) high frequency branch of the KV dispersion relation for the full range of space charge strength $0 \leq \sigma/\sigma_0 \leq 1$
  - Fluid mode dispersion relation plotted dashed on KV mode plots.
  - The $n=1$ fluid envelope mode is identical to the KV envelope mode.

Since the fluid model reproduces the coarse macroscopic features of the KV model – which can be a good approximation at high space-charge intensities, this implies:

- KV model mode eigenfunctions should roughly model those of intense beams with smooth distributions.
- Oscillation frequencies may be close to the (stable) high frequency KV mode branch
  - May be other lower frequency branches that are also physical.
- Many high-order KV instabilities may be of little relevance to real beams.
  - Low order (envelope and maybe others) can be relevant.

The real issue for high intensity collective modes may not be higher order KV instabilities but if low-order collective modes can:

- Be driven unstable by periodic (s-varying) focusing structures in machine lattices, errors in rings, etc.
- Drive the production of beam halo, etc.

References:

Material on the kinetic stability of KV beams is found mostly in journals.

Original references

Gluckstern, Proc. 1970 Proton Linac Conference, Nat. Accel. Lab., pg. 811 – First KV mode analysis.

T.F. Wang and L. Smith, Part. Accel. 12, 247 (1982). – Simplified (closed form) mode eigenfunction and dispersion relation.

Interpretation of Branches, Mode Structure, KV Fluid Stability

S.M. Lund and R.C. Davidson, Physics of Plasmas 5, 3028 (1998). Detailed analysis of eigenfunctions, dispersion relations, etc. in appendices. Fluid mode analysis and interpretations of KV modes.

Other papers by Hoffmann, Gluckstern, and others. Hoffmann et. al. analyzed KV in periodic focusing lattices.

J.J. Barnard and S.M. Lund
USPAS, June 2001, Boulder CO

§4 Global Conservation Constraints

In a continuous focusing channel, the Vlasov-Poisson system has several global invariants valid for *any* initial distribution function.

- Distribution need not be an equilibrium – any initial state
- Valid provided needed symmetries are respected and that no particles are lost in the evolution.

Generalized Entropy (All Vlasov evolutions)

$$\boxed{\begin{aligned} U_G &= \int d^2x \int d^2x' \, G(f_\perp) = \text{const.} \\ G(f_\perp) &= \text{Any differentiable function satisfying } G(f_\perp \to 0) = 0 \end{aligned}}$$

Special cases:

1) Line charge conservation; $G(f_\perp) = q f_\perp$

$$\lambda = q \int d^2x \int d^2x' \, f_\perp = \text{const.}$$

2) System entropy conservation; $G(f_\perp) = -\dfrac{f_\perp}{A} \ln\left(\dfrac{f_\perp}{f_0}\right)$

f_0, A constants defining measure.

$$S = -\int \frac{d^2x}{A} \int d^2x' \, f_\perp \ln\left(\frac{f_\perp}{f_0}\right)$$

Note that Vlasov's equation says that phase space density can be mixed but the local measures are preserved. The generalized entropy constraints simply express this using differing functions to weight differing values of the distribution.

Angular Momentum* (Rotationally invariant focusing)

$$\boxed{U_\theta = \int d^2x \int d^2x' \, (y'x - x'y) f_\perp = \text{const.}}$$

* Units chosen for convenience

- Often useful for analysis of transport in a focusing channel with magnetic solenoids.

Axial Momentum* (No acceleration)

$$\boxed{U_z = \int d^2x \int d^2x' \, m\gamma_b \beta_b c \, f_\perp = \text{const.}}$$

For our model: $= m\gamma_b\beta_b c \dfrac{\lambda}{q}$

* Units chosen for convenience

- Trivial in the simple theory employed, but useful for coasting beams with axial momentum spread

Transverse Energy*

$$\boxed{U_\varepsilon = \int d^2x \int d^2x' \left\{ \frac{1}{2}\vec{x}_\perp'^2 + \frac{k_{\beta 0}^2}{2}\vec{x}_\perp^2 \right\} f_\perp + \int d^2x \, \frac{\varepsilon_0 |\nabla_\perp \phi|^2}{2m\gamma_b^3\beta_b^2c^2} = \text{const}}$$

*Expressed in per unit axial length units.

Note that the terms can be interpreted as:

in U_ε:

$\int d^2x \int d^2x' \, \frac{1}{2}\vec{x}_\perp'^2 f_\perp \sim$ Kinetic Energy

$\int d^2x \int d^2x' \, \frac{k_{\beta 0}^2}{2}\vec{x}_\perp^2 f_\perp \sim$ Potential Energy due to applied focusing

$\int d^2x \int d^2x' \, \frac{|\nabla_\perp \phi|^2}{2m\gamma_b^3\beta_b^2c^2} \sim$ Self field Energy

The energy constraint can be related to terms in the single-particle Hamiltonian integrated over the beam:

$$U_{\mathcal{E}} = \int d^2x \int d^2x' \left\{ \frac{1}{2}\vec{x}_\perp'^2 + \frac{k_{\beta 0}^2}{2}\vec{x}_\perp^2 \right\} f_\perp + \int d^2x \frac{\epsilon_0 |\nabla\phi|^2}{2m\gamma_b^3\beta_b^2c^2} = \text{const.}$$

$$H = \frac{1}{2}\vec{x}_\perp'^2 + \frac{k_{\beta 0}^2}{2}\vec{x}_\perp^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2}$$

Note that:

$$\int d^2x \frac{|\nabla_\perp\phi|^2}{2} = \underbrace{\int d^2x \cdot \frac{\nabla_\perp\cdot(\phi\nabla_\perp\phi)}{2}}_{\to 0} - \int d^2x \cdot \frac{\phi\nabla_\perp^2\phi}{2}$$

Div. theorem – perfectly conducting boundary, take $\phi = 0$ on

$$\int d^2x \frac{|\nabla_\perp\phi|^2}{2} = -\int d^2x \frac{\phi\nabla_\perp^2\phi}{2} = \int d^2x\, \phi \frac{q}{2}\int d^2x' \frac{f_\perp}{\epsilon_0}$$

$$= \frac{q}{2}\int d^2x \int d^2x' \frac{\phi f_\perp}{\epsilon_0} = \frac{1}{2}\int d^2x \frac{\lambda\phi}{\epsilon_0}$$

Thus:

$$\boxed{U_{\mathcal{E}} = \int d^2x \int d^2x' \left\{ \frac{1}{2}\vec{x}_\perp'^2 + \frac{k_{\beta 0}^2}{2}\vec{x}_\perp^2 + \frac{1}{2}\frac{q\phi}{m\gamma_b^3\beta_b^2c^2} \right\} f_\perp = \text{const.}}$$

Symmetry factor in calculating self-field interactions.

// Analogous illustration of symmetry factors:

Energy of point charges in 3D free space:

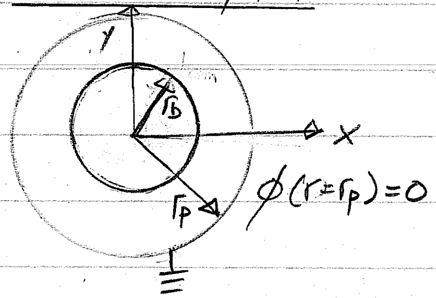
$$\phi(\vec{x}) = \sum_i \frac{q}{4\pi\epsilon_0}\frac{1}{|\vec{x}-\vec{x}_i|} \qquad \mathcal{E} = \sum_i q\phi(\vec{x}_i) = \sum_{\substack{i,j \\ \text{distinct pairs}}} \frac{q^2}{4\pi\epsilon_0}\frac{1}{|\vec{x}_i-\vec{x}_j|}$$

$$= \frac{1}{2}\sum_{\substack{i,j \\ \text{all pairs}}} \frac{q^2}{4\pi\epsilon_0}\frac{1}{|\vec{x}_i-\vec{x}_j|} \quad //$$

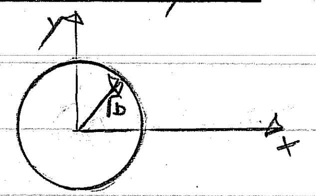
Aside

In certain cases (e.g. thermal equilibrium) it is desired to apply the conservation constraints to radially unbounded systems in two-dimensions:

bounded system



unbounded system



$$\lim_{r\to\infty} -\frac{\partial\phi}{\partial r} = \frac{\lambda}{2\pi\epsilon_0 r}$$

In both situations the energy constraint

$$U_\mathcal{E} = \int d^2x \int d^2x' \left\{ \frac{1}{2}\vec{x}'^2_\perp + \frac{k_{\beta 0}^2}{2}\vec{x}_\perp^2 \right\} f_\perp + \frac{\epsilon_0}{m\gamma_b^3\beta_b^2c^2}\int d^2x \frac{|\nabla_\perp\phi|^2}{2} = \text{const}$$

is valid. However in the unbounded system the field energy:

$$\epsilon_0 \int d^2x \frac{|\nabla_\perp\phi|^2}{2} \simeq \epsilon_0\int_0^{r_\infty} dr\, r \int d\theta \frac{|\nabla_\perp\phi|^2}{2} + \frac{\lambda^2}{4\pi\epsilon_0}\int_{r_\infty}^{\infty} dr \frac{1}{r}$$

Logrithmically diverges.

diverges. However, the rate of divergance is $\propto \lambda^2$ and this can be used to <u>subtract out an infinite constant</u> and thereby <u>regularize</u> the system energy. This can also be done in a manner independent of the detailed structure of the distribution, to render the energy constraint useful in practical applications.

//

Proofs of conservation relations:

Vlasov Equation:

$$\frac{df_\perp}{ds} = \left\{ \frac{\partial}{\partial s} + \frac{\partial H}{\partial \vec{x}_\perp'} \cdot \frac{\partial}{\partial \vec{x}_\perp} - \frac{\partial H}{\partial \vec{x}_\perp} \cdot \frac{\partial}{\partial \vec{x}_\perp'} \right\} f_\perp = 0$$

$$\left\{ \frac{\partial}{\partial s} + \vec{x}_\perp' \cdot \frac{\partial}{\partial \vec{x}_\perp} - \left(k_{\beta 0}^2 \vec{x}_\perp + \frac{q}{m\gamma_b^3\beta_b^2c^2} \frac{\partial \phi}{\partial \vec{x}_\perp} \right) \cdot \frac{\partial}{\partial \vec{x}_\perp'} \right\} f_\perp = 0$$

Example Proof:

Generalized Entropy:

$$\frac{d}{ds} G(f_\perp) = G'(f_\perp) \frac{df_\perp}{ds} = 0 \qquad \text{since} \qquad \frac{df_\perp}{ds} = 0$$

operate with $\int d^2x'$

$$\int d^2x' \frac{\partial G(f_\perp)}{\partial s} + \int d^2x'\, \vec{x}_\perp' \cdot \frac{\partial}{\partial \vec{x}_\perp} G(f_\perp)$$

$$- \underbrace{\int d^2x' \left(k_{\beta 0}^2 \vec{x}_\perp + \frac{q}{m\gamma_b^3\beta_b^2c^2} \frac{\partial \phi}{\partial \vec{x}_\perp} \right) \cdot \frac{\partial G(f_\perp)}{\partial \vec{x}_\perp'}}_{\to 0} = 0$$

integrate by parts

provided $G(f_\perp \to 0) = 0$

operate with $\int d^2x \ldots$

$$\Rightarrow \frac{\partial}{\partial s} \int d^2x \int d^2x'\, G(f_\perp) + \int d^2x \int d^2x' \frac{\partial}{\partial \vec{x}_\perp} \cdot \left(\vec{x}_\perp' G(f_\perp) \right) = 0$$

$$\Rightarrow \frac{\partial}{\partial s} \int d^2x \int d^2x'\, G(f_\perp) = 0$$

$$\Rightarrow \int d^2x \int d^2x'\, G(f_\perp) = \text{const.}$$

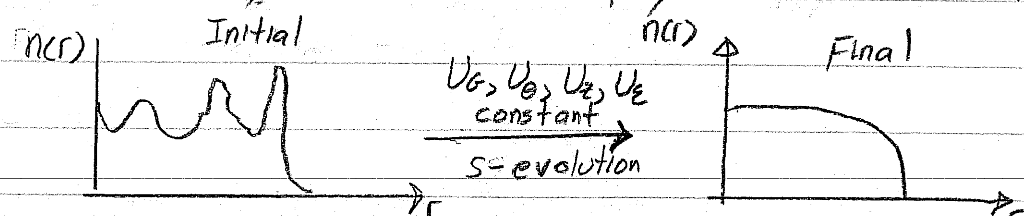
Other proofs are similar. Multiply Vlasov's equation by appropriate factors and integrate. Several examples will be left as problems.

The global invariants strongly constrain the nonlinear evolution of the system from the initial distribution function. The use of such constants is appealing since in reality, the initial distribution in a machine may be far from an equilibrium state. Thus the use of general constraints to bound kinematically accessible states can be interpreted more generally than the evolution of particular modes on a particular class of Vlasov equilibrium.

The global constraints have been analyzed to:

1) Obtain sufficient conditions for beam stability using functional bounds on perturbations. These methods are analogous to those used by Newcomb and Gardner in neutral plasma physics. Davidson has employed such analyses in nonneutral plasma and beam physics.

2) Calculate changes in beam moments (e.g. emittance) when one (initial) distribution relaxes to another. For example, an initial nonequilibrium distribution to thermal equilibrium. Reiser, Davidson, Barnard and Lund and many others have employed such methods.



3) Bound possible particle losses: O'Neil, Lee, others.

We will cover 1) and 2) in some detail in these lectures.

Although the conservation constraints are powerful and apply to general evolutions in the model - regardless of how complex, there are practical limits to the use of such results.

- Does not provide information on the time-scales of evolutions and dynamical processes.
  - Knowledge of characteristic mode frequencies is highly valuable in searching for determental resonances etc.

- The energy constraint is one of the more useful conservation law but becomes invalid in the case of (realistic) s-varying focusing. This follows because s-varying applied fields can add and remove beam energy.
  - In practice the constraint can still be a useful (average) guide if the beam is matched, etc.

- The angular momentum constraint does not apply to alternating gradient focusing.
  - Mostly useful in the analysis of solenoidal focusing channels.

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USPAS, Jan. 2004

§5: Kinetic Stability Theorem

In a continuous focusing channel, the global conservation constraints can be employed to derive a <u>sufficient condition for beam stability</u>. These techniques were developed in neutral plasma physics by Newcomb, Gardner, and Fowler. Davidson also has employed the methodology extensively to analyze nonneutral plasmas and beams.

Let $f_\perp$ be a distribution undergoing Vlasov evolution

Vlasov-Poisson System:

$$\left\{ \frac{\partial}{\partial s} + \frac{\partial H}{\partial \vec{x}_\perp'} \cdot \frac{\partial}{\partial \vec{x}_\perp} - \frac{\partial H}{\partial \vec{x}_\perp} \cdot \frac{\partial}{\partial \vec{x}_\perp'} \right\} f_\perp(\vec{x}_\perp, \vec{x}_\perp', s) = 0$$

$$\frac{d\vec{x}_\perp}{dt} = \frac{\partial H}{\partial \vec{x}_\perp'} = \vec{x}_\perp'$$

$$\frac{d\vec{x}_\perp'}{dt} = -\frac{\partial H}{\partial \vec{x}_\perp} = -k_{\beta 0}\vec{x}_\perp - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi}{\partial \vec{x}_\perp}$$

$$H = \frac{1}{2}\vec{x}_\perp'^2 + \frac{k_{\beta 0}^2}{2}\vec{x}_\perp^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2}$$

$$\nabla_\perp^2 \phi = -\frac{q}{\epsilon_0}\int d^2x' f_\perp \qquad \phi(r = r_p) = 0$$

Resolve $f_\perp$ into an equilibrium plus perturbation:

$$\boxed{f_\perp = f_0(H_0) + \delta f}$$

$f_0(H_0)$ = Equilibrium (subscript 0) distribution
δf = Perturbation about equilibrium.

The equilibrium by itself satisfies the Vlasov Poisson System:

$$\boxed{\begin{aligned} &\frac{df_0}{ds} = \left\{ \frac{\partial}{\partial s} + \vec{x}_\perp' \cdot \frac{\partial}{\partial \vec{x}_\perp} - \left(k_{\beta 0}\vec{x}_\perp + \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi_0}{\partial \vec{x}_\perp}\right)\cdot \frac{\partial}{\partial \vec{x}_\perp'} \right\} f_0 = 0 \\ &f_0 = f_0(H_0) \\ &H_0 = \frac{1}{2}\vec{x}_\perp'^2 + \frac{k_{\beta 0}^2}{2}\vec{x}_\perp^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi_0 \\ &\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi_0}{\partial r}\right) = \frac{-q}{\epsilon_0}\int d^2x'\, f_0(H_0), \qquad \phi_0(r=r_p)=0 \end{aligned}}$$

Since f_0 is an equilibrium, all integrals and moments of f_0 are also constants. We denote the "generalized entropy" and energy of the equilibrium by:

$$\boxed{\begin{aligned} &U_{G0} \equiv \int d^2x \int d^2x'\, G(f_0) = \text{const.} \\ &U_{\mathcal{E}0} \equiv \int d^2x \int d^2x' \left\{ \frac{1}{2}\vec{x}'^2 + \frac{k_{\beta 0}^2}{2}\vec{x}_\perp^2 + \frac{1}{2}\frac{q\phi_0}{m\gamma_b^3\beta_b^2c^2} \right\} f_0 = \text{const.} \end{aligned}}$$

Likewise, we apply the Global conservation constraints previously discussed for the generalized entropy and energy of the full system to obtain:

$$\boxed{\begin{aligned} &U_G = \int d^2x \int d^2x'\, G(f_\perp) = \text{const.} \\ &U_\mathcal{E} = \int d^2x \int d^2x' \left\{ \frac{1}{2}\vec{x}_\perp'^2 + \frac{k_{\beta 0}^2}{2}\vec{x}_\perp^2 \right\} f_0 + \frac{\epsilon_0}{m\gamma_b^3\beta_b^2c^2}\int d^2x\, \frac{|\nabla_\perp \phi|^2}{2} \end{aligned}}$$

It follows that:

$$\boxed{\begin{aligned}\Delta U_G &\equiv U_G - U_{G0} = \int d^2x \int d^2x' \left[G(f_\perp) - G(f_0) \right] = \text{const.} \\ \Delta U_\mathcal{E} &\equiv U_\mathcal{E} - U_{\mathcal{E}0} = \int d^2x \int d^2x' \left\{ \frac{1}{2} \vec{x}_\perp'^2 + \frac{k_{\beta 0}^2}{2} \vec{x}_\perp^2 \right\} (f_\perp - f_0) \\ &\quad + \frac{\epsilon_0}{m\gamma_b^3\beta_b^2c^2} \int d^2x \left\{ \frac{|\nabla_\perp \phi|^2}{2} - \frac{|\nabla_\perp \phi_0|^2}{2} \right\} \\ &= \text{const.}\end{aligned}}$$

From this we form an effective free-energy

- Analogous to a <u>Helmholtz free energy in Thermodynamics</u>

$$\boxed{\begin{aligned}F &\equiv \Delta U_\mathcal{E} + \Delta U_G \\ &= \int d^2x \int d^2x' \left\{ \frac{1}{2} \vec{x}_\perp'^2 + \frac{k_{\beta 0}^2}{2} \vec{x}_\perp^2 \right\} \delta f + \frac{\epsilon_0}{m\gamma_b^3\beta_b^2c^2} \int d^2x \left\{ \frac{|\nabla_\perp \phi|^2}{2} - \frac{|\nabla_\perp \phi_0|^2}{2} \right\} \\ &\quad + \int d^2x \int d^2x' \left[G(f_\perp) - G(f_0) \right]\end{aligned}}$$

We define a perturbed potential:

$$\boxed{\begin{aligned}\delta\phi &= \phi - \phi_0 \\ \nabla_\perp^2 \delta\phi &= \frac{-q}{\epsilon_0} \int d^2x' (f_\perp - f_0) = \frac{-q}{\epsilon_0} \int d^2x' \, \delta f \\ \delta\phi(r = r_p) &= 0\end{aligned}}$$

Then

$$\frac{1}{2}\int d^2x\left\{|\nabla_\perp\phi|^2 - |\nabla_\perp\phi_0|^2\right\} = \frac{1}{2}\int d^2x\left\{|\nabla_\perp\delta\phi|^2 + 2\nabla_\perp\phi_0\cdot\nabla_\perp\delta\phi\right\}$$

$$= \frac{1}{2}\int d^2x\,|\nabla_\perp\delta\phi|^2 + \int d^2x\left\{\underbrace{\nabla_\perp\cdot(\phi_0\nabla_\perp\delta\phi)}_{\to 0} - \phi_0\nabla_\perp^2\delta\phi\right\}$$

Divergance theorem and $\phi_0(r=r_p)=0$

$$= \frac{1}{2}\int d^2x\,|\nabla_\perp\delta\phi|^2 + \frac{q}{\epsilon_0}\int d^2x\int d^2x'\,\phi_0\,\delta f$$

Thus:

$$\boxed{\begin{aligned} F = &\int d^2x\,d^2x'\left\{\frac{1}{2}\vec{x}_\perp'^2 + \frac{k_{\beta 0}^2}{2}\vec{x}_\perp^2 + \frac{q\phi_0}{m\gamma_b^3\beta_b^2c^2}\right\}\delta f \\ &+ \frac{\epsilon_0}{m\gamma_b^3\beta_b^2c^2}\int d^2x\,\frac{|\nabla_\perp\delta\phi|^2}{2} + \int d^2x\int d^2x'\left[G(f_0+\delta f) - G(f_0)\right] \\ &= \text{const.}\end{aligned}}$$

Now assume that the perturbations are small with $|\delta f| \ll f_0$ and Taylor expand

$$\boxed{G(f_0+\delta f) = G(f_0) + G'(f_0)\,\delta f + G''(f_0)\frac{\delta f^2}{2} + \cdots}$$

Here $G'(f_0) = \frac{dG(f_0)}{df_0}$; $G''(f_0) = \frac{d^2G(f_0)}{df_0^2}$

Then

$$F = \int d^2x\int d^2x'\left\{\frac{1}{2}\vec{x}_\perp'^2 + \frac{k_{\beta 0}^2}{2}\vec{x}_\perp^2 + \frac{q\phi_0}{m\gamma_b^3\beta_b^2c^2} + G'(f_0)\right\}\delta f$$

$$+ \frac{\epsilon_0}{m\gamma_b^3\beta_b^2c^2}\int d^2x\,\frac{|\nabla_\perp\delta\phi|^2}{2} + \int d^2x\int d^2x'\,\frac{G''(f_0)}{2}\delta f^2$$

$$+ \mathcal{O}(\delta f^3) = \text{const.}$$

We are free to choose the form of G:

$$\boxed{\frac{dG(f_0)}{df_0} = G'(f_0) = -H_0 = -\left\{ \frac{1}{2}\vec{x}_\perp'^2 + \frac{k_{\beta 0}^2}{2}\vec{x}_\perp^2 + \frac{q\phi_0}{m\gamma_b^3\beta_b^2c^2} \right\}}$$

- This choice can be made without loss in generality since $f_0 = f_0(H_0)$.

Then:

$$\boxed{G''(f_0) = -\frac{\partial H_0}{\partial f_0} = \frac{-1}{\partial f_0(H_0)/\partial H_0}}$$

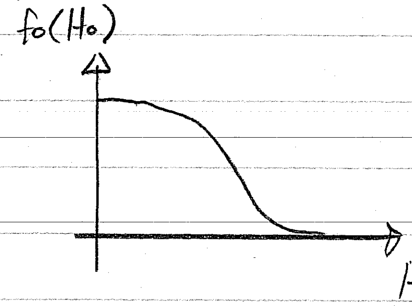
and the conserved free energy becomes:

$$\boxed{F \equiv \frac{\epsilon_0}{m\gamma_b^3\beta_b^2c^2}\int d^2x \left\{ \frac{|\nabla_\perp \delta\phi|^2}{2} - \frac{m\gamma_b^3\beta_b^2c^2}{\epsilon_0}\int d^2x' \frac{(\delta f)^2}{\partial f_0(H_0)/\partial H_0} \right\} + \mathcal{O}(\delta f^3) = \text{const.}}$$

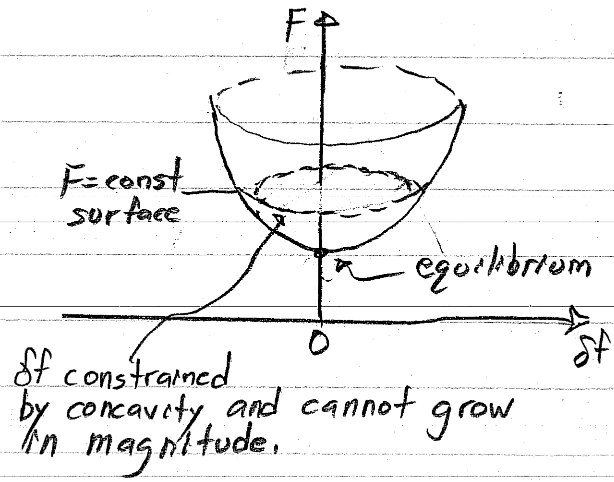
If $f_0(H_0)$ is a monotonic decreasing function of H_0, then

$$\frac{\partial f_0(H_0)}{\partial H_0} < 0$$

and then F is a sum of two positive definite terms that cannot grow without bound!



$\Longleftrightarrow$



Kinetic Stability Theorem

> If $f_0(H_0)$ is a monotonic decreasing function of $H_0 = \frac{1}{2}\vec{x}_\perp'^2 + \frac{k_{\beta 0}^2}{2}\vec{x}_\perp^2 + \frac{q\phi_0}{m\gamma_b^3\beta_b^2c^2}$, with $\partial f_0(H_0)/\partial H_0 < 0$, then the equilibrium defined by $f(H_0)$ is stable to arbitrary small amplitude perturbations.

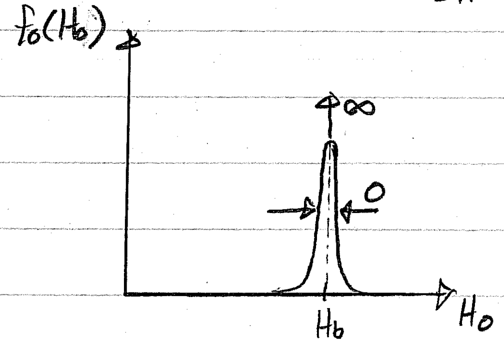
Note:

- This is a sufficient condition for stability
  - Equilibria not satisfying this theorem meet a necessary condition for instability but may or may not be unstable.
- The mean value theorem may be employed to generalize the kinetic stability theorem to show that $f_0(H_0)$ being a monotonic decreasing function of H_0 is also sufficient for nonlinear stability to arbitrary amplitude perturbations. (Davidson).
  - Must define notion of nonlinear stability.

Example Applications:

1) KV Equilibrium

$$f_0(H_0) = \frac{\hat{n}}{2\pi}\delta[H_0 - H_b] \qquad H_b = \text{const.}$$



The δ-function can be represented by the limit of a distribution that diverges at H_b while having vanishing width and constant area in the limit.

$$\Rightarrow \quad \frac{\partial f_0(H_0)}{\partial H_0} = \frac{\hat{n}}{2\pi}\delta'[H_0 - H_b]$$

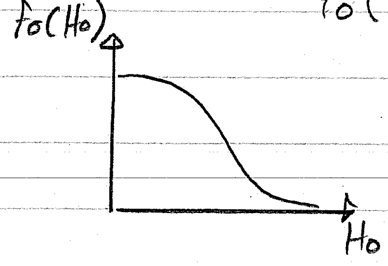
changes sign to the right and left of $H_0 = H_b$

The KV distribution does not satisfy the stability theorem and can be unstable.

- More detailed analysis earlier quantified the instabilities
- Intuitively, the highly inverted population of the distribution provides large free energy. It is not surprising that the KV distribution can be strongly unstable.

2) Thermal Equilibrium $(\gamma_b = 1, \ \beta_b c = v_b)$

$$f_0(H_0) = \frac{\gamma_b m \beta_b^2 c^2 \hat{n}}{2\pi T}\exp\left\{-\frac{\gamma_b m \beta_b^2 c^2}{T} H_0\right\}$$



$$\frac{\partial f_0(H_0)}{\partial H_0} = -\frac{(\gamma_b m \beta_b^2 c^2)^2 \hat{n}}{2\pi T} f_0(H_0) < 0$$

Thermal equilibrium is stable and has no free energy to amplify perturbations.

J.J. Barnard and S.M. Lund
USPAS, Jan 2004

1

References

The course textbook Reiser provides material on several aspects of transverse stability

- Envelope modes ..... which can be thought of as the lowest order collective modes. (covered by J.J. Barnard)
- The use of global conservation constraints in continuous focusing to bound emittance growth resulting from the relaxation of initial perturbations from mismatch etc. (covered in part in this lecture).

In general, the topic of transverse collective stability arising from space-charge effects is quite involved if information beyond envelope (lowest order) stability is desired. In such cases the current literature must typically be consulted. Some introductory (analytical) material applicable to beams can be found in the textbooks:

R.C. Davidson, "An Introduction to the Physics of Nonneutral Plasmas" (Addison-Wesley Publishing Company, New York, 1990.).

R.C. Davidson and H. Qin, "Physics of Intense Charged Particle Beams in High Energy Accelerators" (World Scientific, 2001)

Numerous properties of KV equilibria and stability properties can be found in

Lund and Davidson, "Warm-Fluid Description of Intense Beam Equilibrium and Electrostatic Stability Properties" Phys. of Plasmas, (1998)

along with Fluid model approximations of collective modes.