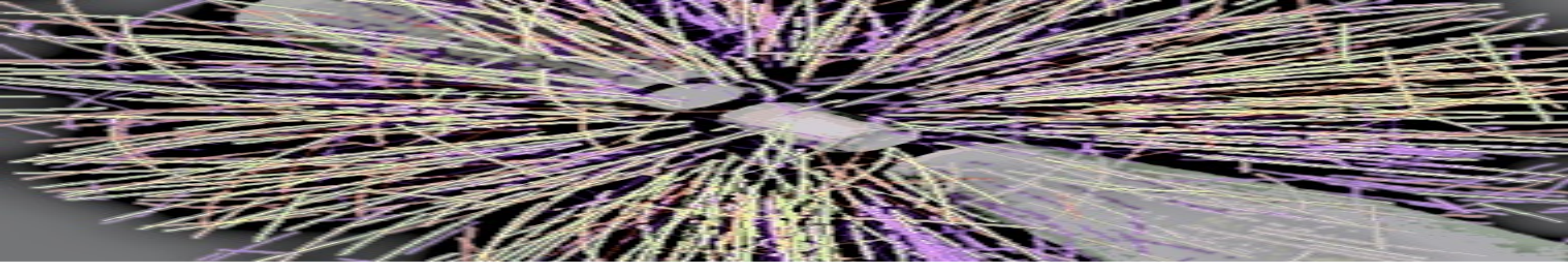
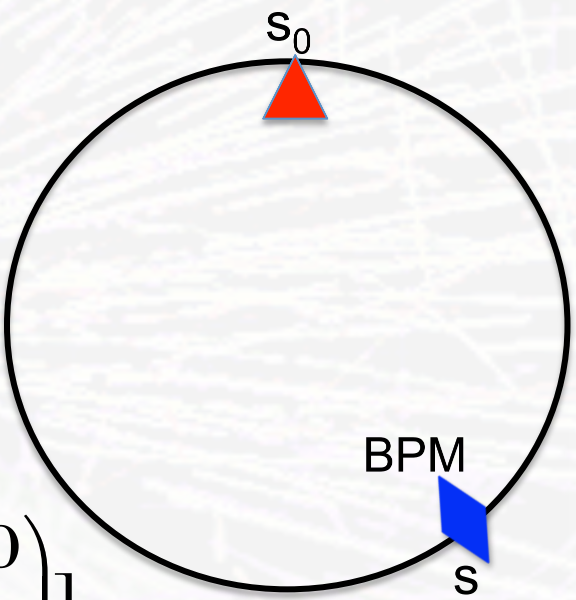


Effects of Errors

- dipole errors
- quadrupole errors
- resonance

Closed orbit distortion

- Dipole kicks can cause particle's trajectory deviate away from the designed orbit
  - Dipole error
  - Quadrupole misalignment



- Assuming a circular ring with a single dipole error, closed orbit then becomes:

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = M(s,s_0)[M(s_0,s)\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} + \begin{pmatrix} 0 \\ \theta \end{pmatrix}]$$

Closed orbit: single dipole error

- Let's first solve the closed orbit at the location where the dipole error is

$$\begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix} = M(s_0 + C, s_0)\begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix} + \begin{pmatrix} 0 \\ \theta \end{pmatrix}$$

$$x(s_0) = \beta_x(s_0)\frac{\theta}{2\sin\pi Q_x}\cos\pi Q_x$$

$$x(s) = \sqrt{\beta_x(s_0)\beta_x(s)}\frac{\theta}{2\sin\pi Q_x}\cos[\psi(s,s_0) - \pi Q_x]$$

- The closed orbit distortion reaches its maximum at the opposite side of the dipole error location

Closed orbit distortion

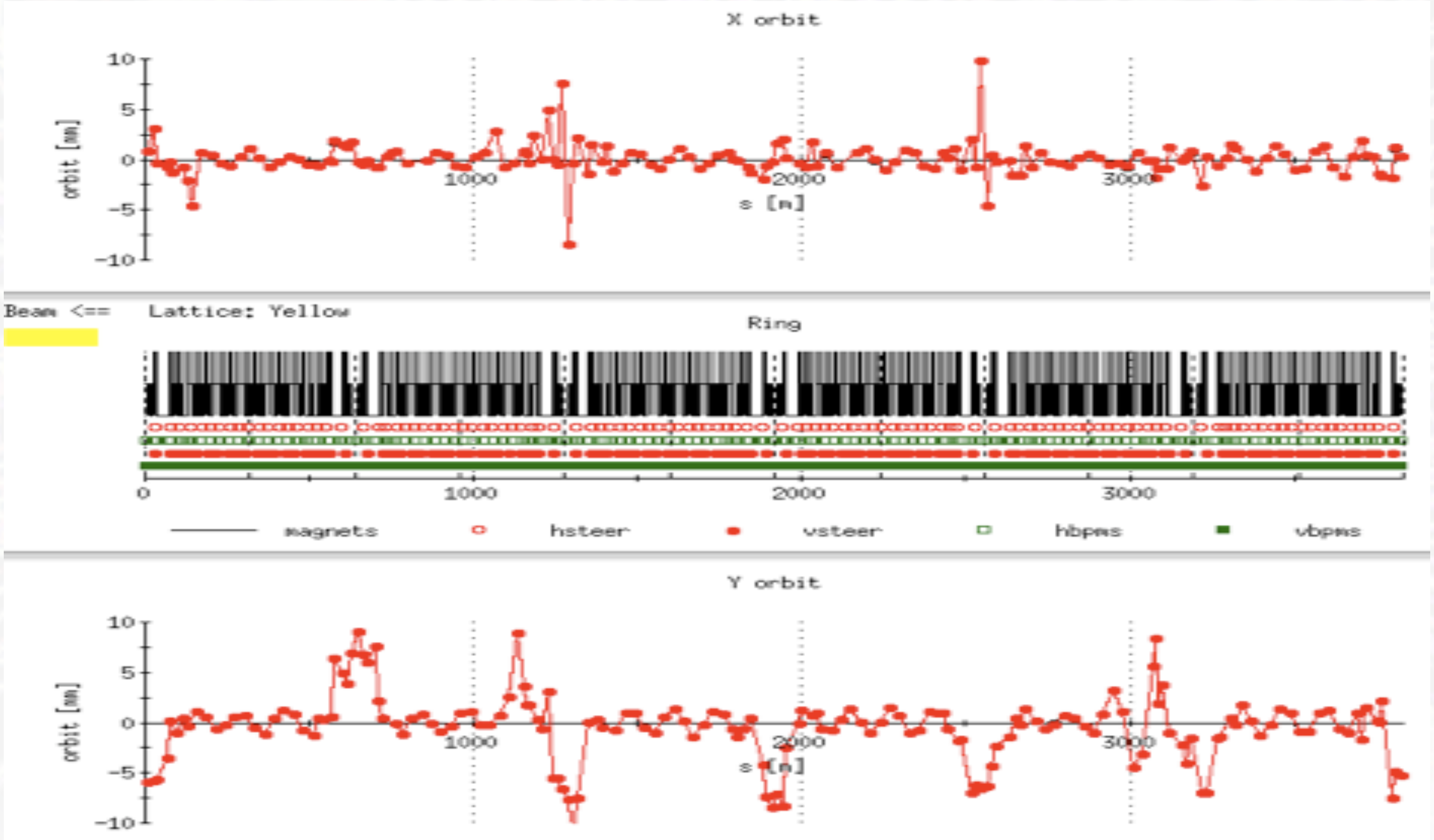
- In the case of multiple dipole errors distributed around the ring. The closed orbit is

$$x(s) = \sqrt{\beta_x(s)} \sum_i \sqrt{\beta_x(s_i)} \frac{\theta_i}{2\sin \pi Q_x} \cos[\psi(s_i, s_0) - \pi Q_x]$$

- Amplitude of the closed orbit distortion is inversely proportion to $\sin\pi Q_{x,y}$
  - **No stable orbit if tune is integer!**

Measure closed orbit

- Distribute beam position monitors around ring.



Control closed orbit

- minimized the closed orbit distortion.
  - Large closed orbit distortions cause limitation on the physical aperture
  - Need dipole correctors and beam position monitors distributed around the ring
    - Assuming we have m beam position monitors and n dipole correctors, the response at each beam position monitor from the n correctors is:

$$x_k = \sqrt{\beta_{x,k}} \sum_{k=1}^{n} \sqrt{\beta_{x,i}} \frac{\theta_i}{2\sin \pi Q_x} \cos\left[\psi(s_i, s_0) - \pi Q_x\right]$$

Control closed orbit

- Or,

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = (M) \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{pmatrix}$$

- To cancel the closed orbit measured at all the bpms, the correctors are then

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{pmatrix} = (M^{-1}) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

Quadrupole errors

- Misalignment of quadrupoles
  - dipole-like error: kx
  - results in closed orbit distortion
- Gradient error:
  - Cause betatron tune shift
  - induce beta function deviation: beta beat

Tune change due to a single gradient error

- Suppose a quadrupole has an error in its gradient, i.e.

$$M = \begin{pmatrix} 1 & 0 \\ -kl & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & 0 \\ -(kl+\Delta kl) & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -kl & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\Delta kl & 1 \end{pmatrix}$$

$$M(s+C,s) = \begin{pmatrix} (\cos 2\pi Q_{x0} + \alpha_{x,s_0} \sin 2\pi Q_{x0}) & \beta_{x,s0} \sin 2\pi Q_{x0} \\ -\frac{1+\alpha_{x,s_0}^2}{\beta_{x,s_0}} \sin 2\pi Q_{x0} & (\cos 2\pi Q_{x0} - \alpha_{x,s_0} \sin 2\pi Q_{x0}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\Delta kl & 1 \end{pmatrix}$$

$$\cos 2\pi (Q_{x0} + \delta Q_x) = \frac{1}{2} Tr(M(s+C,s)) \qquad \delta Q_x = \frac{1}{4\pi} \beta_{x,s_0} \Delta kl$$

Tune shift due to multiple gradient errors

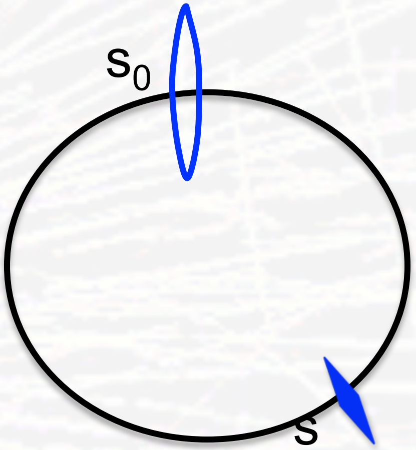
- In a circular ring with a multipole gradient errors, the tune shift is

$$\delta Q_x = \frac{1}{4\pi}\sum_i \beta_{x,s_i}\Delta k_i l$$

Beta beat

- In a circular ring with a gradient error at s0, the tune shift is

$$M(s+C,s) = M(s,s_0)\begin{pmatrix} 1 & 0 \\ -\Delta kl & 1 \end{pmatrix} M(s_0,s)$$



$$\beta_x(s)\sin 2\pi Q_x = \beta_{x0}(s)\sin 2\pi Q_{x0} +$$

$$\Delta kl \frac{\beta_{x0}(s)\beta_{x0}(s_0)}{2}[\cos(2\pi Q_{x0} + 2|\Delta\psi_{s,s0}|)]$$

$$\frac{\Delta\beta}{\beta} = \Delta kl \frac{\beta_{x0}(s_0)}{2\sin 2\pi Q_{x0}}\cos(2\pi Q_{x0} + 2|\Delta\psi_{s,s0}|)$$

Unstable betatron motion if tune is half integer!

Beta beat

- In a circular ring with multiple gradient errors,

$$\frac{\Delta\beta}{\beta}(s) = \frac{\sqrt{\beta_{x0}(s)}}{2\sin 2\pi Q_{x0}} \sum_i \sqrt{\beta_{x0}(s_i)}\Delta k_i l \cos(2\pi Q_{x0} + 2|\Delta\psi_{s,si}|)$$

Unstable betatron motion if tune is half integer!

Beta beat wave varies twice of betatron tune around the ring

Resonance condition

- Tune change due to a single quadrupole error

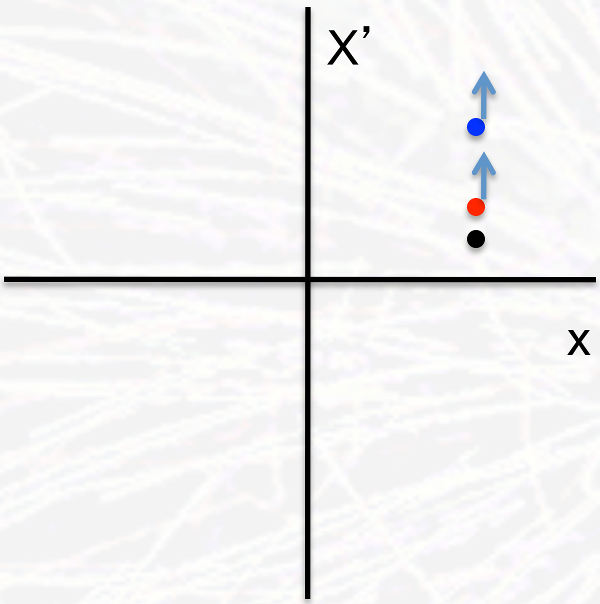
$$\cos[2\pi(Q_{x0}+\delta Q_x)] = \cos 2\pi Q_{x0} - \frac{1}{2}\beta_{x,s_0}\Delta kl \sin 2\pi Q_{x0}$$

- If $Q_{x0} = (2k+1)\frac{1}{2} + \varepsilon$, the above equation becomes

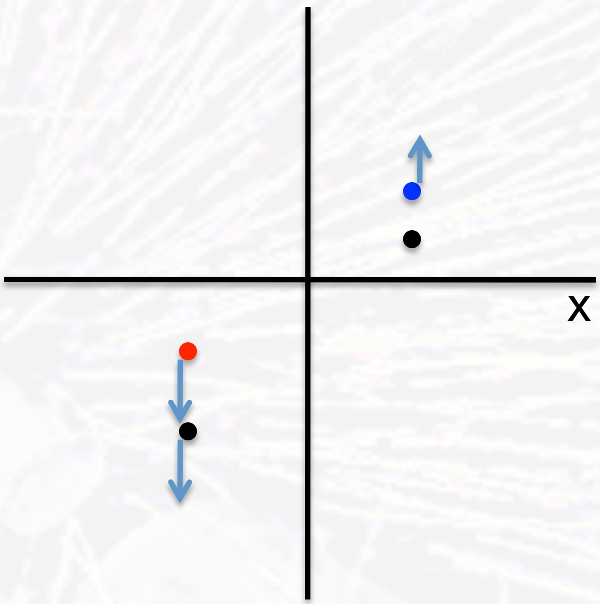
$$\cos[2\pi(Q_{x0}+\delta Q_x)] \approx 1 + \frac{1}{2}\beta_{x,s_0}\Delta kl\varepsilon$$

and Qx can become a complex number which means the betatron motion can become unstable

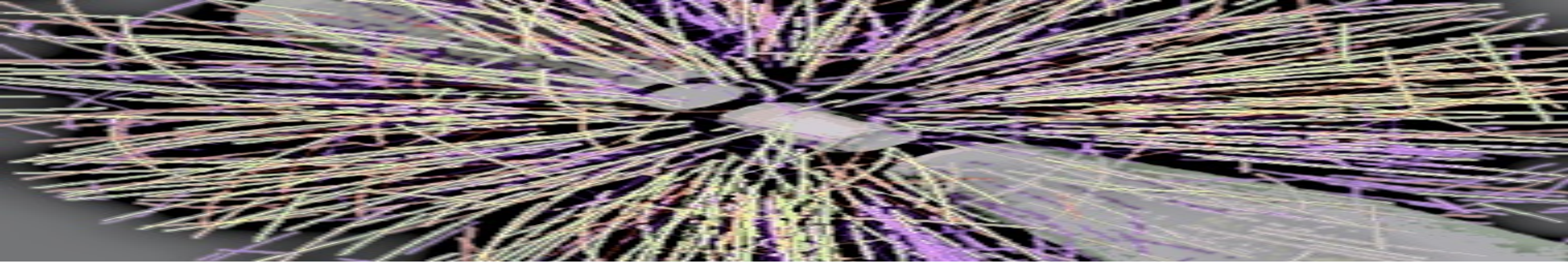
Resonance



Integer resonance



Half Integer resonance



Transverse Resonances

- Linear coupling
- resonances mechanisms
- Resonance conditions
- 3rd order resonances

Source of linear coupling

- Skew quadrupole

$$B_x = -qx; \quad B_y = qy$$

$$x'' + K_x(s)^2 x = -\frac{B_y l}{B\rho} = -qy$$

$$y'' + K_y(s)^2 y = \frac{B_x l}{B\rho} = -qx$$

Coupled harmonic oscillator

- Equation of motion

$$x'' + \omega_x{}^2 x = q^2 y \qquad y'' + \omega_y{}^2 y = q^2 x$$

- Assume solutions are:

$$x = Ae^{i\omega t} \qquad y = Be^{i\omega t}$$

$$-\omega^2 A + \omega_x{}^2 A = q^2 B \qquad -\omega^2 B + \omega_y{}^2 B = q^2 A$$

$$(\omega_x{}^2 - \omega^2)(\omega_y{}^2 - \omega^2) = q^4$$

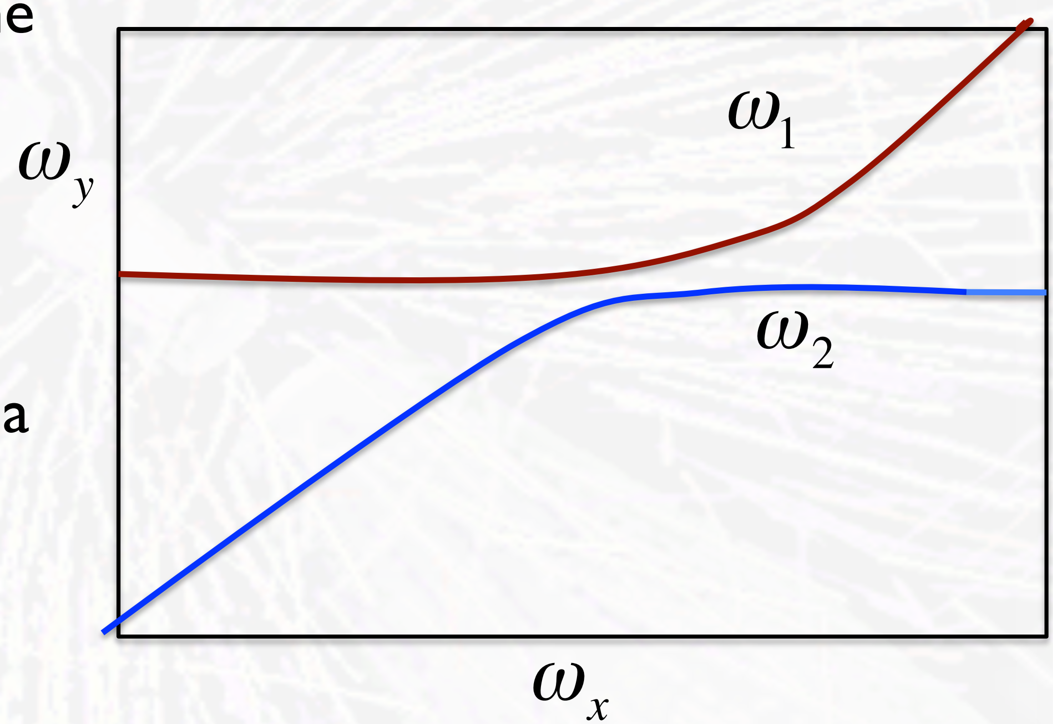
$$\omega^2 = \frac{\omega_x^2 + \omega_y^2 \pm \sqrt{(\omega_x^2 - \omega_y^2)^2 + 4q^4}}{2}$$

Coupled harmonic oscillator

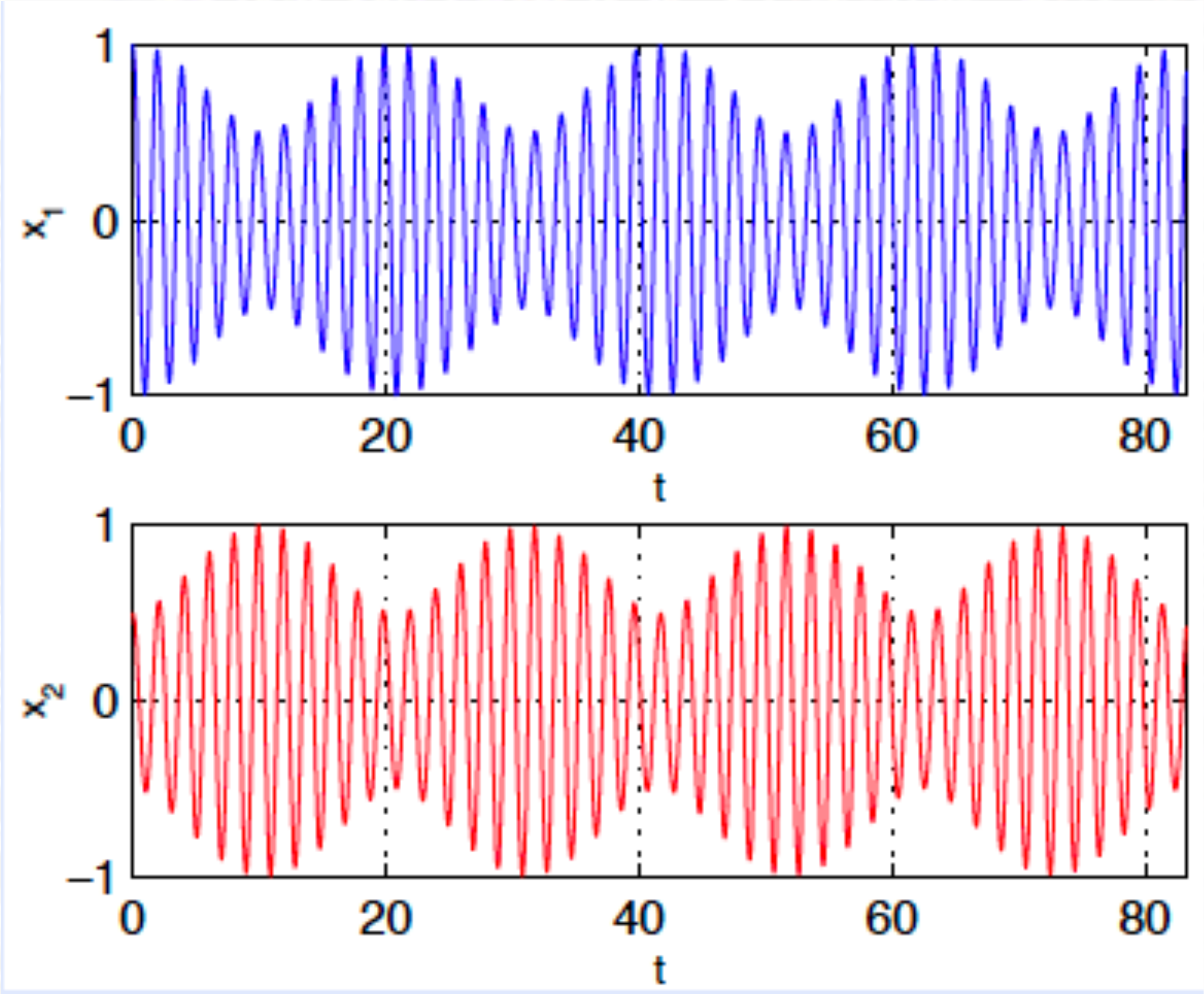
$$\omega^2 = \frac{\omega_x^2 + \omega_y^2 \pm \sqrt{(\omega_x^2 - \omega_y^2)^2 + 4q^4}}{2}$$

- The two frequencies of the harmonic oscillator are functions of the two unperturbed frequencies
- When the unperturbed frequencies are the same, a minimum frequency difference

$$\Delta\omega \approx \frac{q^2}{\omega}$$



Example of a Coupled harmonic oscillator



Resonance mechanism

- Errors in the accelerators perturbs beam motions
- Coherent buildup of perturbations

Driven harmonic oscillator

- Equation of motion

$$\frac{d^2x(t)}{dt^2} + \omega^2 x(t) = f(t) = \sum_{m=0} C_m e^{i\omega_m t}$$

- for $f(t) = C_m e^{i\omega_m t}$

$$\frac{d^2x(t)}{dt^2} + \omega^2 x(t) = C_m e^{i\omega_m t}$$

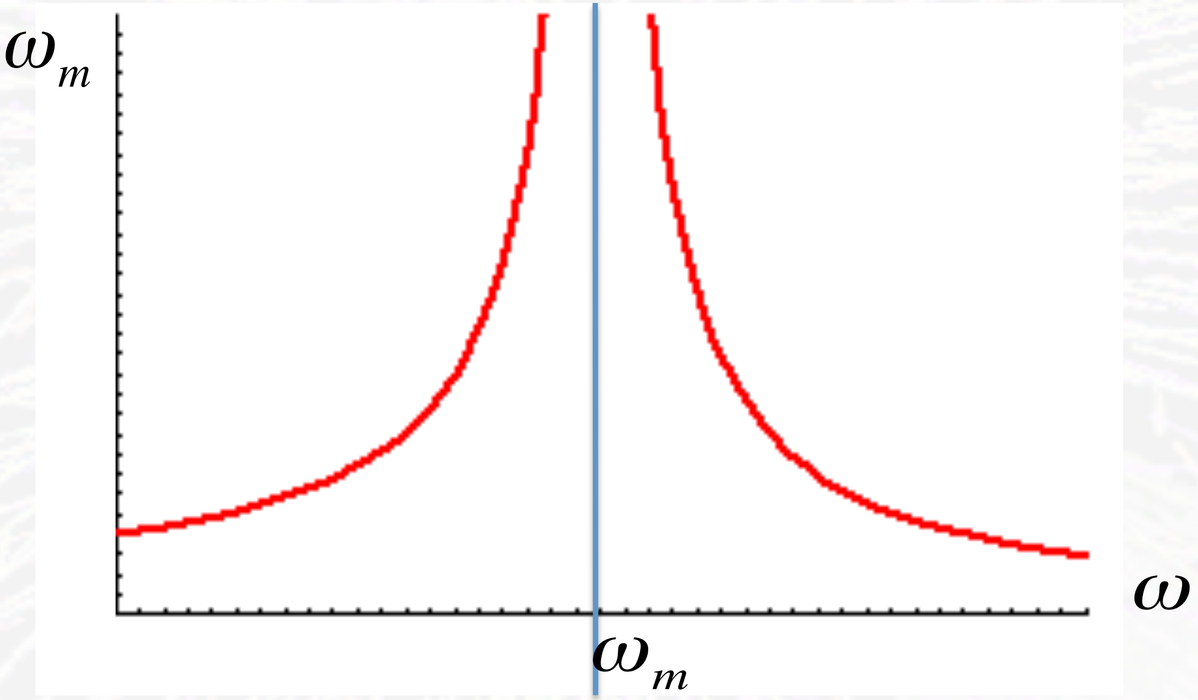
- Assume solution is like $x(t) = Ae^{i\omega t} + A_m e^{i\omega_m t}$

$$A_m = \frac{C_m}{\omega^2 - \omega_m^2}$$

Resonance response

- Response of the harmonic oscillator to a periodic force is

$$x(t) = Ae^{i\omega t} + \frac{C_m}{\omega^2 - \omega_m^2}$$



Betatron oscillation

- Equation of motion

$$x'' + K(s)x = 0 \qquad K(s + L_p) = K(s)$$

$$x = A\sqrt{\beta_x}\cos(\psi + \chi)$$

- In the presence of field errors including mis-aglinments, the equation of motion then becomes

$$x'' + K(s)x = -\frac{\Delta B_y}{B\rho}$$

where

$$\Delta B_y = B_0(b_0 + b_1 x + b_2 x^2 + ...)$$

Dipole error quadrupole error sextupole error

Floquet Transformation

- Re-define () as:

$$x'' + K(s)x = 0 \qquad K(s + L_p) = K(s)$$

$$\zeta(s) = x(s)/\sqrt{\beta_x(s)} \qquad \phi(s) = \psi(s)/Q_x \qquad or\ \phi' = 1/(Q_x\beta_x)$$

- In the presence of field errors including mis-aglinments, the equation of motion then becomes

$$\frac{d^2\zeta}{d\phi^2} + Q_x^2\zeta = -Q_x^2\beta_x^{3/2}\frac{\Delta B_y}{B\rho}$$

where

$$\frac{d^2\zeta}{d\phi^2} + Q_x^2\zeta = -\frac{Q_x^2 B_0}{B\rho}\left[b_0 + \beta_x b_1\zeta + \beta_x^2 b_2\zeta^2 + \ldots\right]$$

Resonance contd

- For each n:

$$\frac{d^2\zeta}{d\phi^2} + Q_x^2\zeta = -\frac{Q_x^2\beta_x^{3/2}}{B\rho}\beta_x^n b_n \zeta^n$$

The solution of the homogenous differential equation

$$\frac{d^2\zeta}{d\phi^2} + Q_x^2\zeta = 0$$

is $\zeta = e^{-iQ_x\phi}$. Let's put this back to the right side of the inhomogeneous equation of motion, one then gets

$$\frac{d^2\zeta}{d\phi^2} + Q_x^2\zeta = -\frac{Q_x^2\beta_x^{3/2}}{B\rho}\beta_x^n b_n e^{-inQ_x\phi}$$

Resonance contd

- for a circular machine, beta functions and lattices are periodic. One can then expand

$$\beta_x^{(n+3)/2} b_n = \sum_k c_k e^{ik\phi}$$

- Now, the inhomogeneous equation of motion then becomes

$$\frac{d^2\zeta}{d\phi^2} + Q_x^2\zeta = -\frac{Q_x^2}{B\rho}\sum_k e^{i(k-nQ_x)\phi}$$

- Compare this with the driven harmonic oscillator

$$\frac{d^2x(t)}{dt^2} + \omega^2 x(t) = f(t) = \sum_{m=0} C_m e^{i\omega_m t}$$

Resonance contd

$$\frac{d^2\zeta}{d\phi^2} + Q_x^2\zeta = -\frac{Q_x^2}{B\rho}\sum_k e^{i(k-nQ_x)\phi}$$

- This means the motion becomes unstable(on resonance) when

$$Q_x = \pm(k - nQ_x)$$

  - i.e. resonance location at, here k and n both are integers

$$(n+1)Q_x = k \quad \text{and} \quad (n-1)Q_x = k$$

  - If *k*=0, no resonance condition
  - Any error of x^n can drive a (n+1)th order of resonance
- Driving term is then given

$$C_{k,n} = \frac{1}{(B\rho)}\frac{1}{2\pi Q_X}\oint \beta^{(n+1)/2}(s)b_n(s)e^{-ik\phi}\,ds$$

Resonance condition

- In the absence of coupling between horizontal and vertical

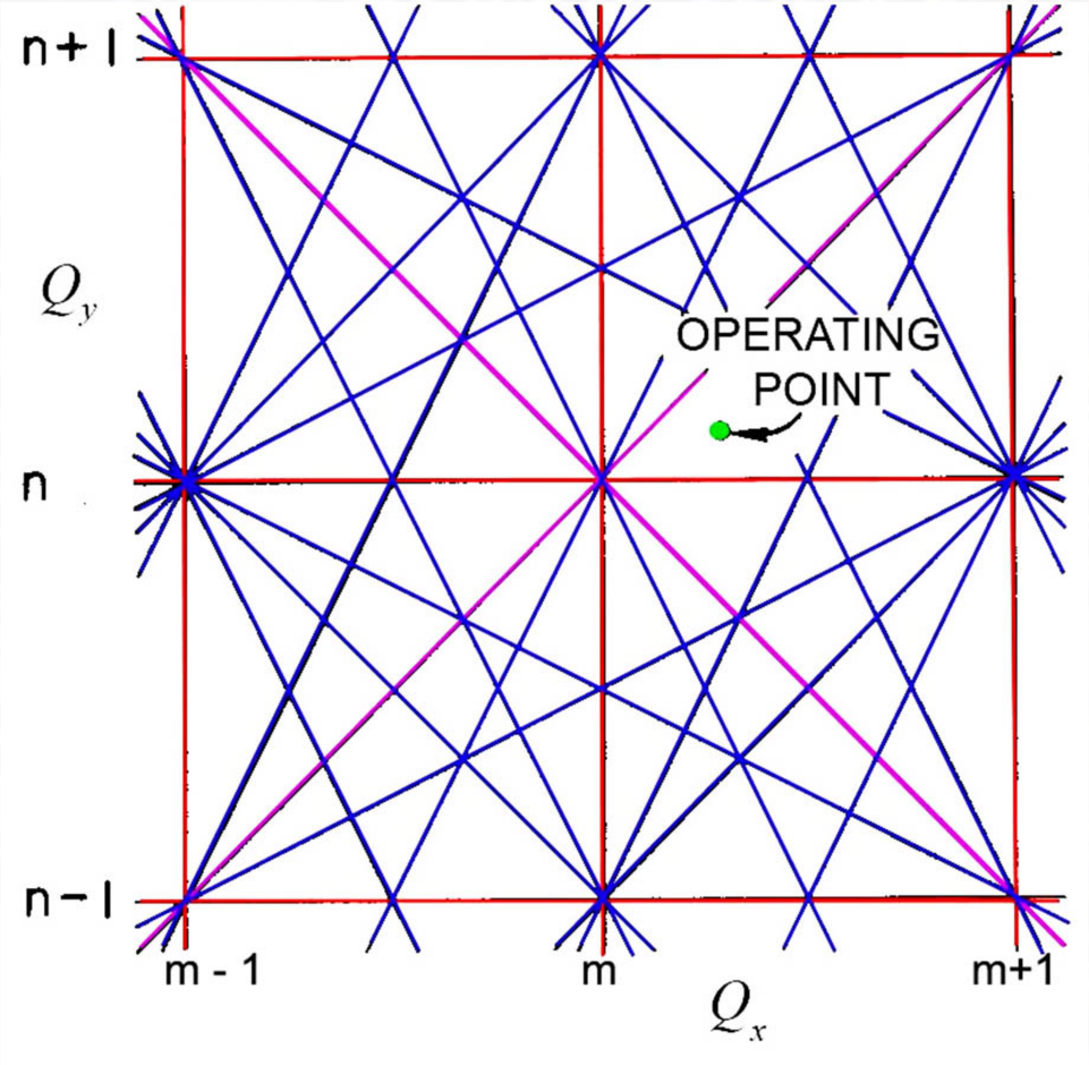
$$k = (n \pm 1) Q_{x,y}$$

error	n	
dipole	0	Qx,y=integer
quadrupole	1	2Qx,y=integer
Sextupole	2	3Qx,y=integer
Octupole	3	4Qx,y=integer

- In the presence of coupling between horizontal and vertical

$$MQ_x + NQ_y = k$$

Tune diagram



- the resonance strength decreases as the order goes higher
- the working point should be located in an area between resonances there are enough tune space to accommodate tune spread of the beam

Phase space: 3rd order resonance

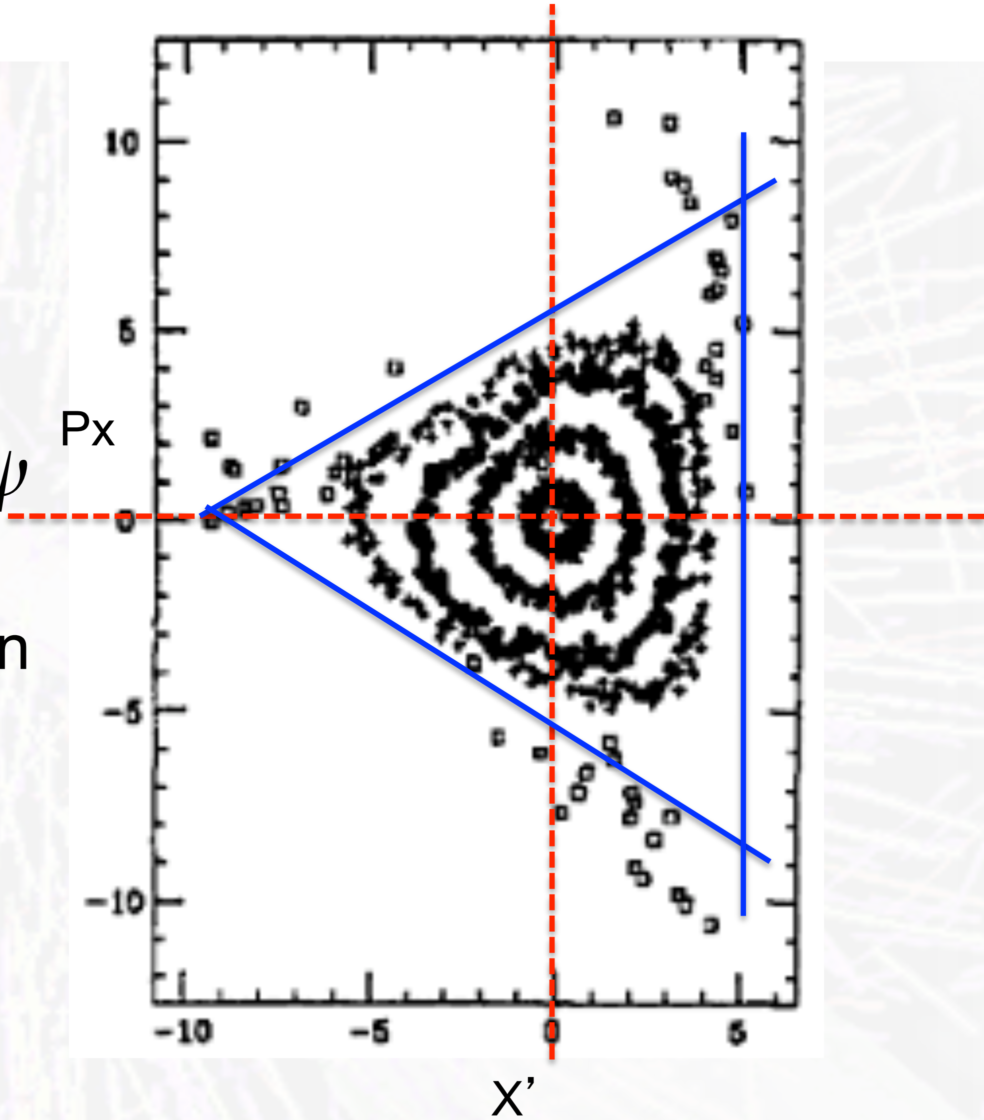
In the phase space of x, Px

$$x = A\sqrt{\beta_x}\cos\psi$$

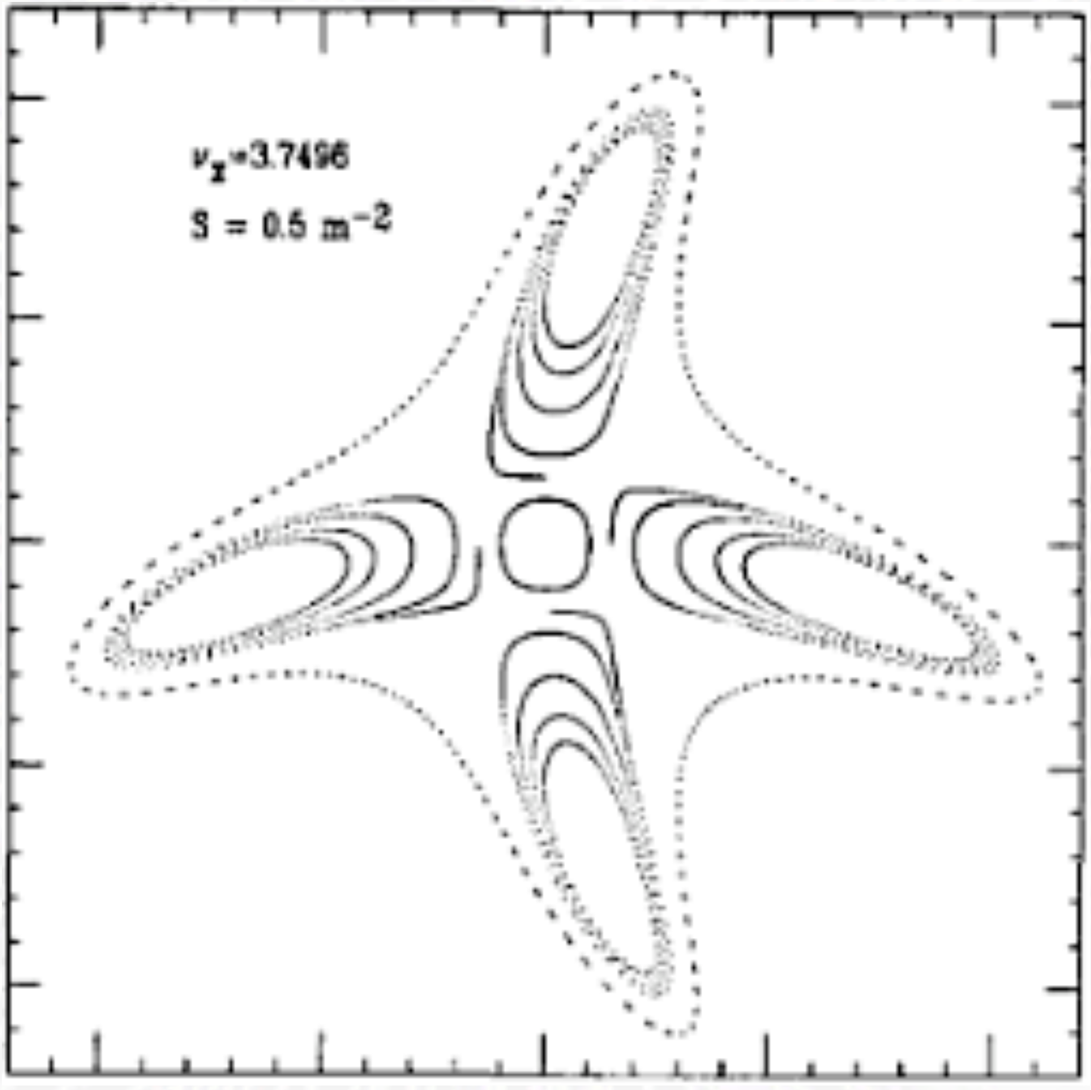
$$P_x = \beta_x x' + \alpha_x x = -A\sqrt{\beta_x}\sin\psi$$

- separatrix: boundery between stable region and unstable region
- Fixed points: where

$$\frac{dx}{dn} = \frac{dP_x}{dn} = 0$$



Phase space: 4th order resonance



Phase space: 5th order resonance