Fundamentals of Accelerators

Day 1 – Motivations and Preliminaries:

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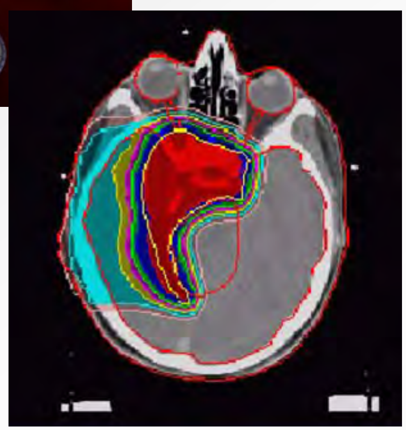
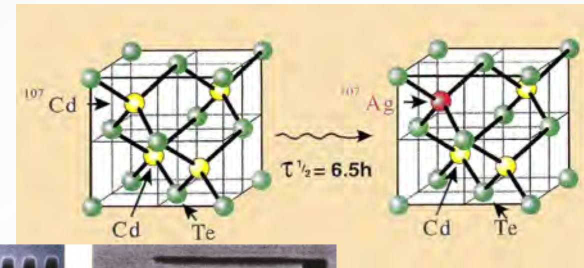
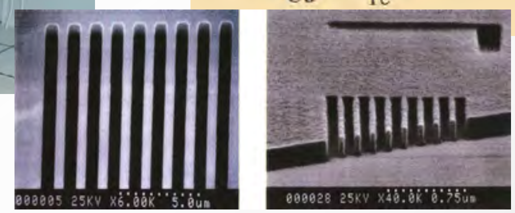
Motivations:
Why does anyone care about accelerators?



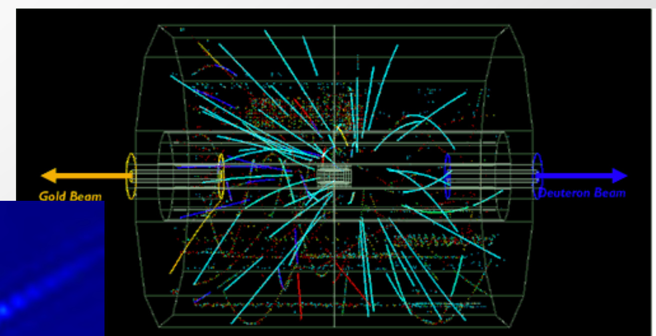
Medicine



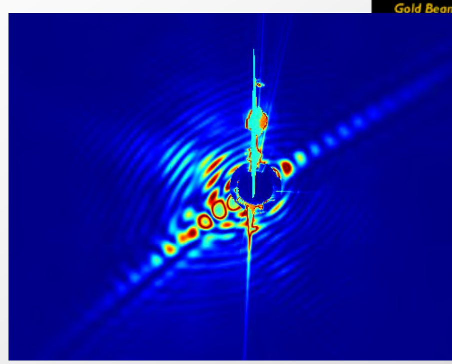


Materials

Basic Research



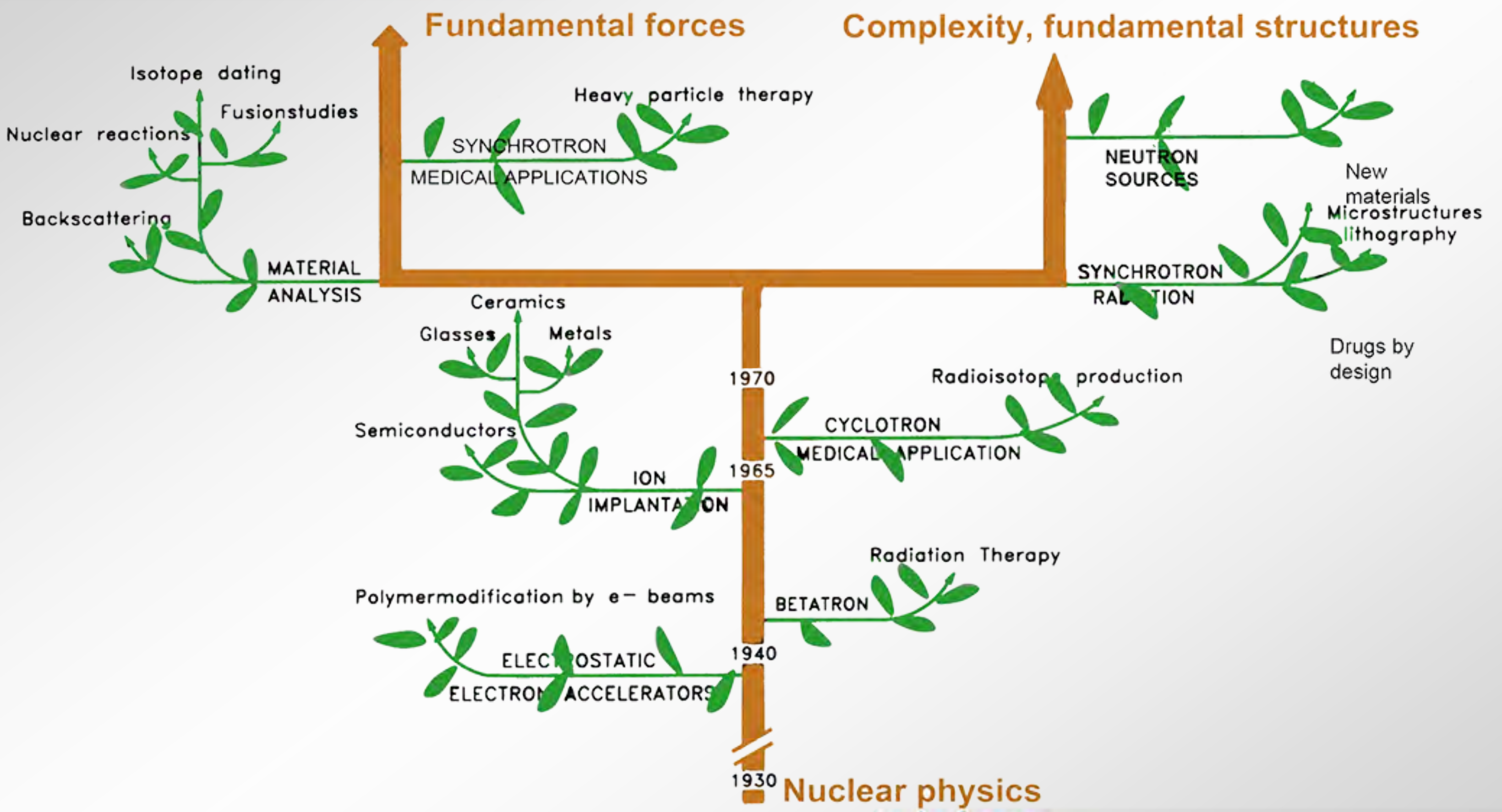
Brookhaven National Lab / RHIC-STAR Collaboration



Exciting products…

exciting opportunities

Accelerators are the hallmark of highly technological societies

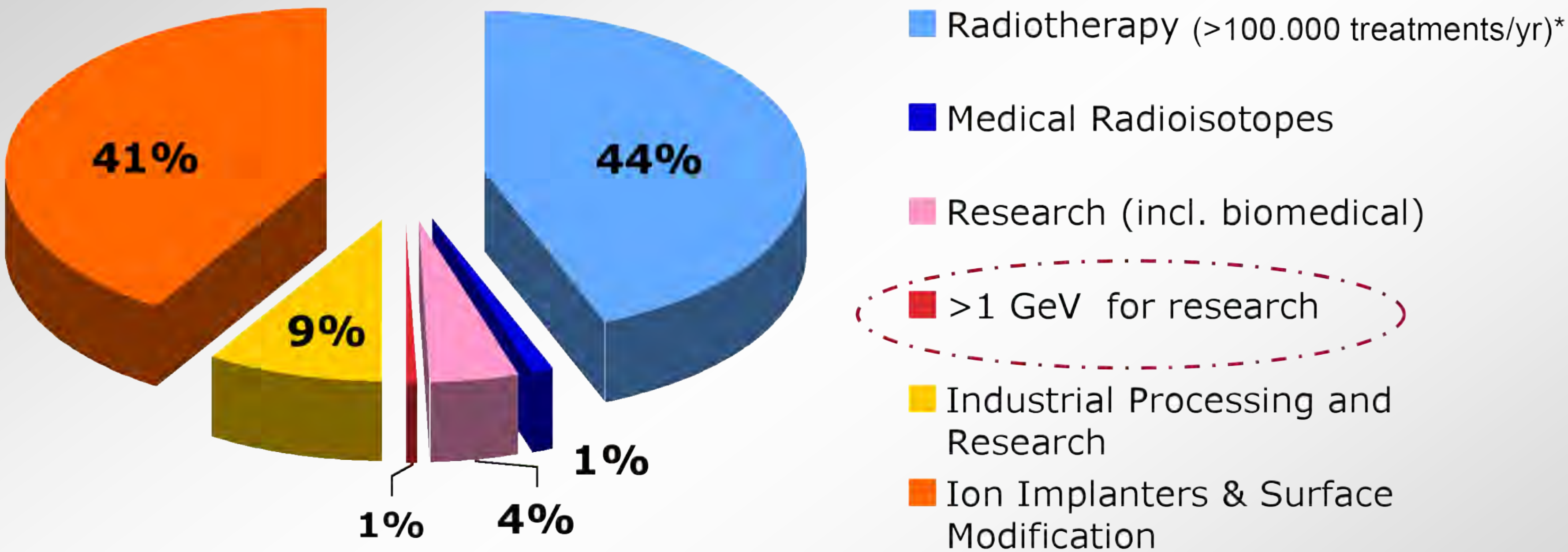


Societal applications & their technology develop from basic research

Source: U. Amaldi

Accelerators are big business

Number of accelerators worldwide
~ 26,000



Annual growth is several percent

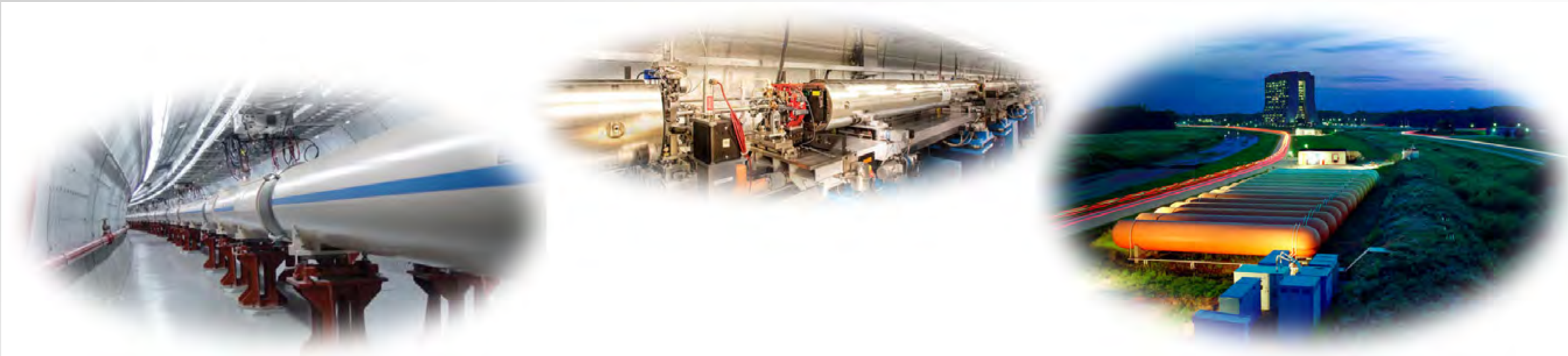
Sales >3.5 B$/yr

Value of treated good > 50 B$/yr **

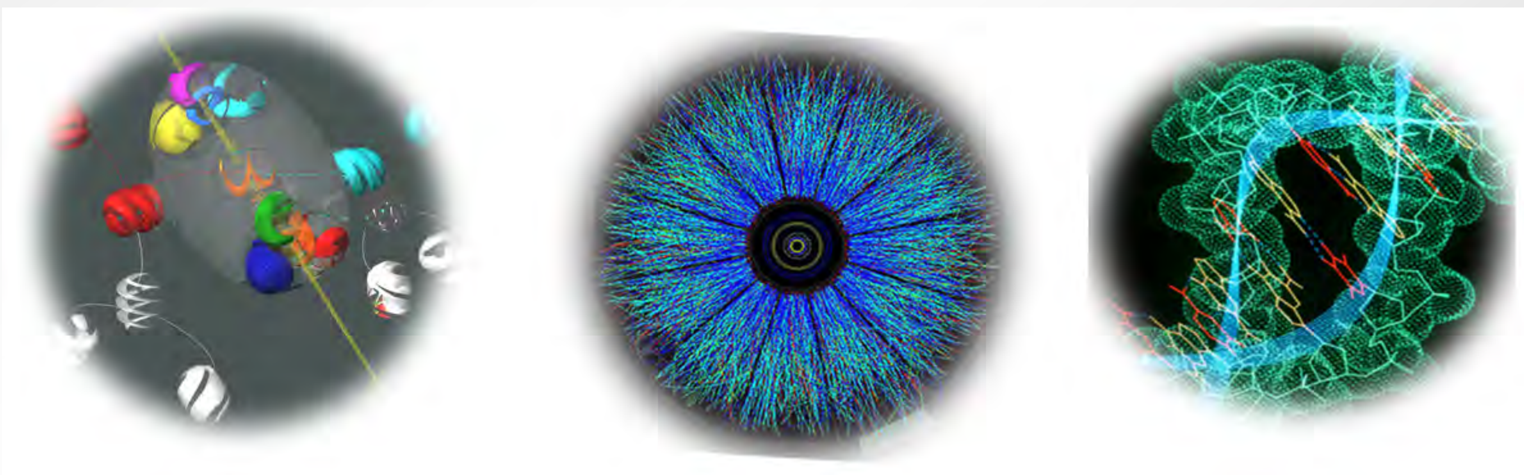
Major research machines are a tiny fraction of the total, but…

Sources: W. Maciszewski & W. Scharf, L. Rivkin, * EPP2010, ** R. Hamm

World-leading discovery science is the key
to competitive economic advantage



Accelerators are essential tools for discovery in physics, chemistry & biology



5 Minute Exercise

Why are you here?

What do you hope to learn

Discuss with your group of 5

The Basics: Special Relativity

Energy & Mass units

- To describe the energy of individual particles, we use the eV, the energy that a unit charge

$$e = 1.6 \times 10^{-19} \text{ Coulomb}$$

gains when it falls through a potential, $\Delta\Phi = 1$ volt.

$$1\ eV = 1.6 \times 10^{-19} \text{ Joule}$$

- We can use Einstein's relation to convert rest mass to energy units

$$E_o = mc^2$$

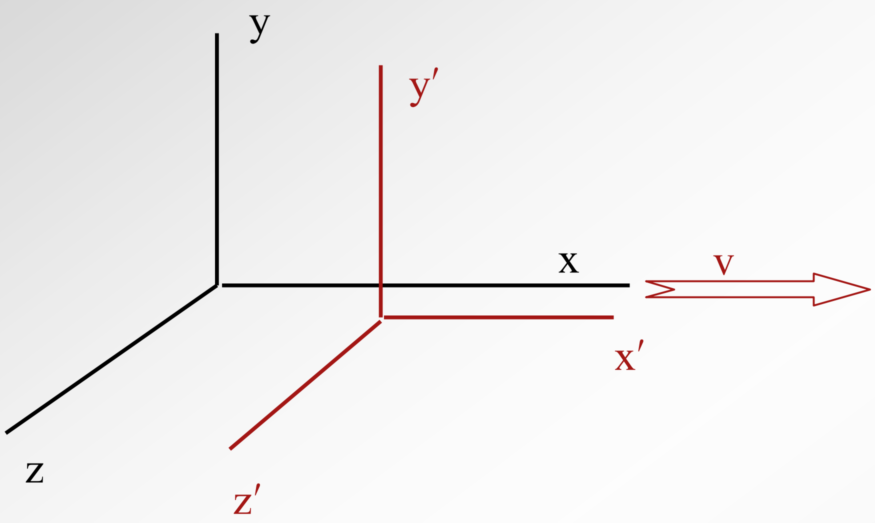
- For electrons,

$$E_{o,e} = 9.1\text{x}10^{-31} \text{ kg x } (3\text{x}10^{8} \text{ m/sec})^2 = 81.9\text{x}10^{-15} \text{ J } = 0.512 \text{ MeV}$$

- For protons,

$$E_{o,p} = 938 \text{ MeV}$$

Relativity: transformation of physical laws between inertial frames



What is an inertial frame?

How can you tell?

In an inertial frame free bodies have no acceleration

Postulate of Galilean relativity

Under the Galilean transformation

$$x' = x - V_x t$$
$$y' = y \qquad\qquad \Rightarrow \quad v'_x = v_x - V_x$$
$$z' = z$$
$$t' = t$$

the laws of physics remain invariant in all inertial frames.

Not true for electrodynamics !

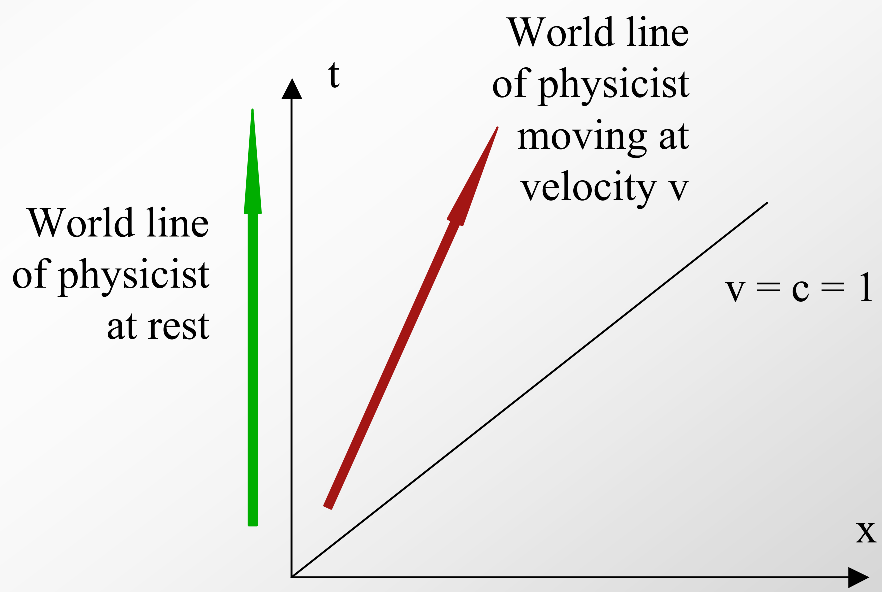
For example, the propagation of light

Observational basis of special relativity

*Observation 1: Light **never** overtakes light in empty space*
==> Velocity of light is the same for all observers

For this discussion let c = 1

Space-time diagrams



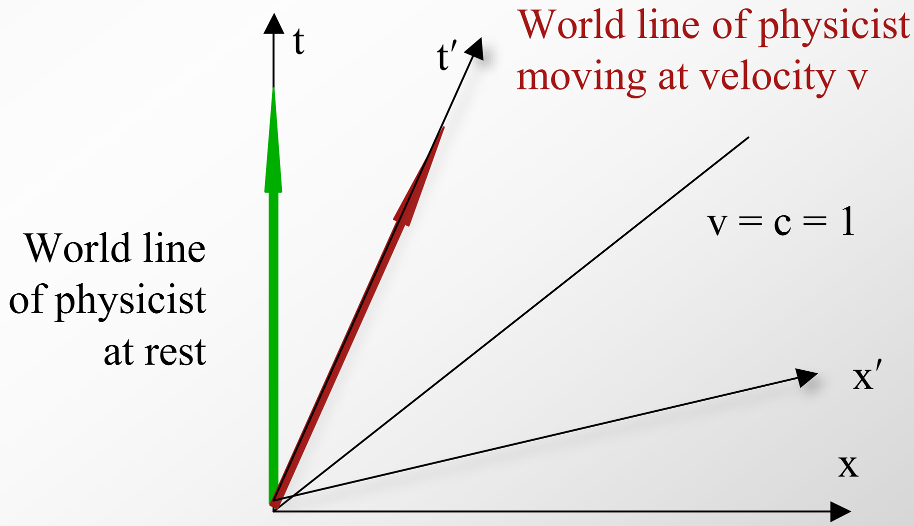
Relativistic invariance

Observation 2:

All the laws of physics are the same in all inertial frames

❖ This requires the invariance of the space-time interval

$$(ct')^2 - x'^2 - y'^2 - z'^2 = (ct)^2 - x^2 - y^2 - z^2$$



Lorentz boost replaces Galilean transformation

$$ct' = \gamma(ct - \beta z)$$

$$x' = x$$

$$y' = y$$

$$z' = \gamma(z - \beta ct)$$

where Einstein's relativistic factors are

$$\beta = \frac{|\mathbf{v}|}{c} \quad \text{and} \quad \gamma = 1 \Big/ \sqrt{1 - \beta^2}$$

Thus we have the Lorentz transformation

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \quad , \quad t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}$$

$$y' = y \quad , \quad z' = z$$

Or in matrix form

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z' \end{pmatrix}$$

Proper time & proper length

- Define proper time, τ, as duration *measured in the rest frame*
- The length of an object in its rest frame is L_o
- As seen by an observer moving at v, the duration, $\mathscr{T}$, is

$$\mathscr{T} = \frac{\tau}{\sqrt{1 - v^2/c^2}} \equiv \gamma\tau > \tau$$

And the length, L, is

$$L = L_o/\gamma$$

Four-vectors & scalar invariants

- Introduce 4-vectors, w^α, with 1 time-like & 3 space-like components ($\alpha = 0, 1, 2, 3$)
  - $x^\alpha = (ct, x, y, z) = (ct, \mathbf{x})$ [Also, $x_\alpha = (ct, -x, -y, -z)$
  - Note Latin indices i =1, 2, 3

- Norm of w^α is a *Lorentz scalar* ***(invariant in all frames)***

$$|w| = (w^\alpha w_\alpha)^{1/2} = (w_o^2 - w_1^2 - w_2^2 - w_3^2)^{1/2}$$

$$|w|^2 = g_{\mu\nu} w^\mu w^\nu$$ where the metric tensor is

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Velocity, energy and momentum

- For a particle with 3-velocity $\boldsymbol{v}$, the 4-velocity is

$$u^\alpha = (\gamma c, \gamma \mathbf{v}) = \frac{dx^\alpha}{d\tau}$$

- Total energy, E, of a particle equals its rest mass, m_o, plus kinetic energy, T

$$\boxed{E = m_o c^2 + T = \gamma m_o c^2}$$

Tutorial exercise: 10 minutes

- The Fermilab Alvarez Linac accelerates protons to a *kinetic energy* of 400 MeV

  - a) Calculate the total energy of the protons in units of MeV
  - b) Calculate the momentum of the protons in units of MeV/c
  - c) Calculate the relativistic gamma factor
  - d) Calculate the proton velocity in units of the velocity of light.

Motivations: How it all began
Paradigm 1: Fixed target experiments

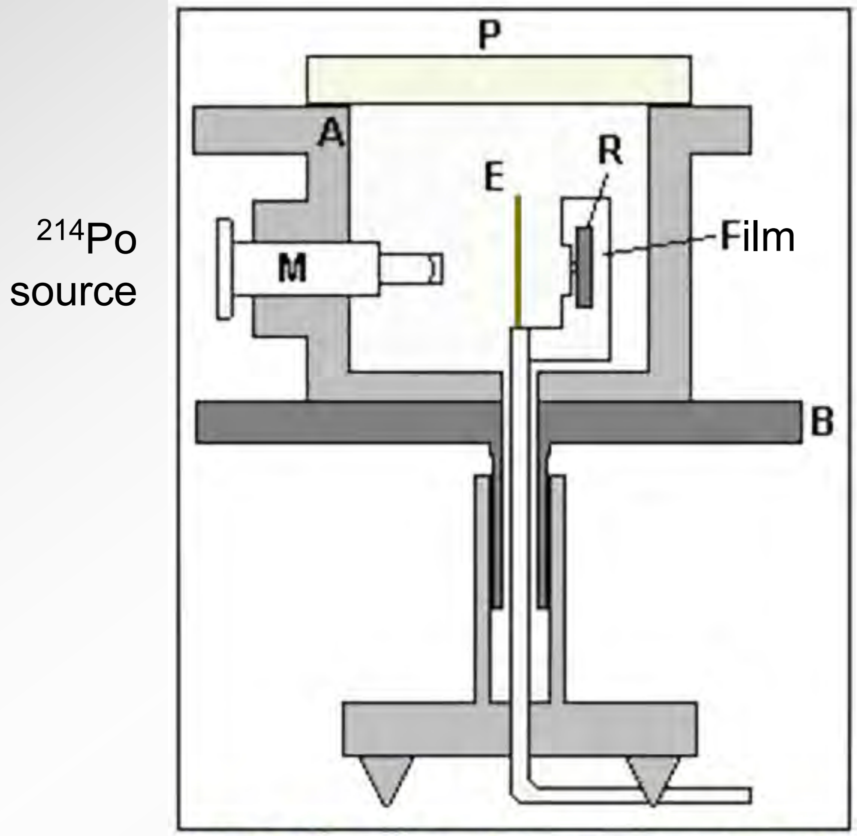


Fig1. Marsden-Geiger experiment.

Rutherford explains scattering of *alpha particles* on *gold* discovering the nucleus & urges … *On to higher energy probes!*

Rutherford articulated Figure of Merit 1

Particle energy on target

Why we use energetic beams for research?

- Resolution of "Matter" Microscopes
  - Wavelength of Particles (γ, e, p, ...) (de Broglie, 1923)

$$\boxed{\lambda = h/p} = 1.2\ \mathrm{fm}/p\ [\ \mathrm{GeV/c}]$$

  - Higher momentum => shorter wavelength => better resolution

- Energy to Matter
  - Higher energy produces heavier particles

The advantage of the fixed target physics:
Figure of Merit 2

$$\frac{Events}{second} = \sigma_{process} \circ \underbrace{Flux \circ Target\ Number\ Density \circ Path\ Length}_{Luminosity}$$

Typical values:

Flux $\sim 10^{12} - 10^{14}\ s^{-1}$
Number density $\sim \rho N_A Z/A \sim 5 \times 6 \times 10^{23} / 2$
Path length $\sim 10\ cm$

Luminosity $\sim 15 \times 10^{23} \times 10^{14} \sim 10^{36} - 10^{38}\ cm^{-2}s^{-1}$

Ideal for precision & rare process physics,
BUT how much energy is available for new physics

Momentum & available energy

- The 4-momentum, p^μ, is

$$p^\mu = m_o u^\mu = (c\gamma m_0, \gamma m_0 \mathbf{v})$$

- Recalling that

$$\boxed{E = m_o c^2 + T = \gamma m_o c^2}$$

we have

$$p^\mu = m_o u^\mu = (c\gamma m_0, \gamma m_0 \mathbf{v}) = (\frac{E}{c}, \gamma m_0 \mathbf{v})$$

$$p^2 = (m^2 c^2 \gamma^2 - \gamma^2 m^{\ 2} v^2) = \left[m^2 c^2 \gamma^2 - \gamma^2 m^{\ 2} c^2 (1 - \frac{1}{\gamma^2}) \right]$$

$$= (m^2 c^2 \gamma^2 - \gamma^2 m^{\ 2} c^2 + m^2 c^2) = m^{\ 2} c^2$$

Particle collisions

❖ Two particles have equal rest mass m_0.

Laboratory Frame (LF): one particle at rest, total energy is E.

projectile

target

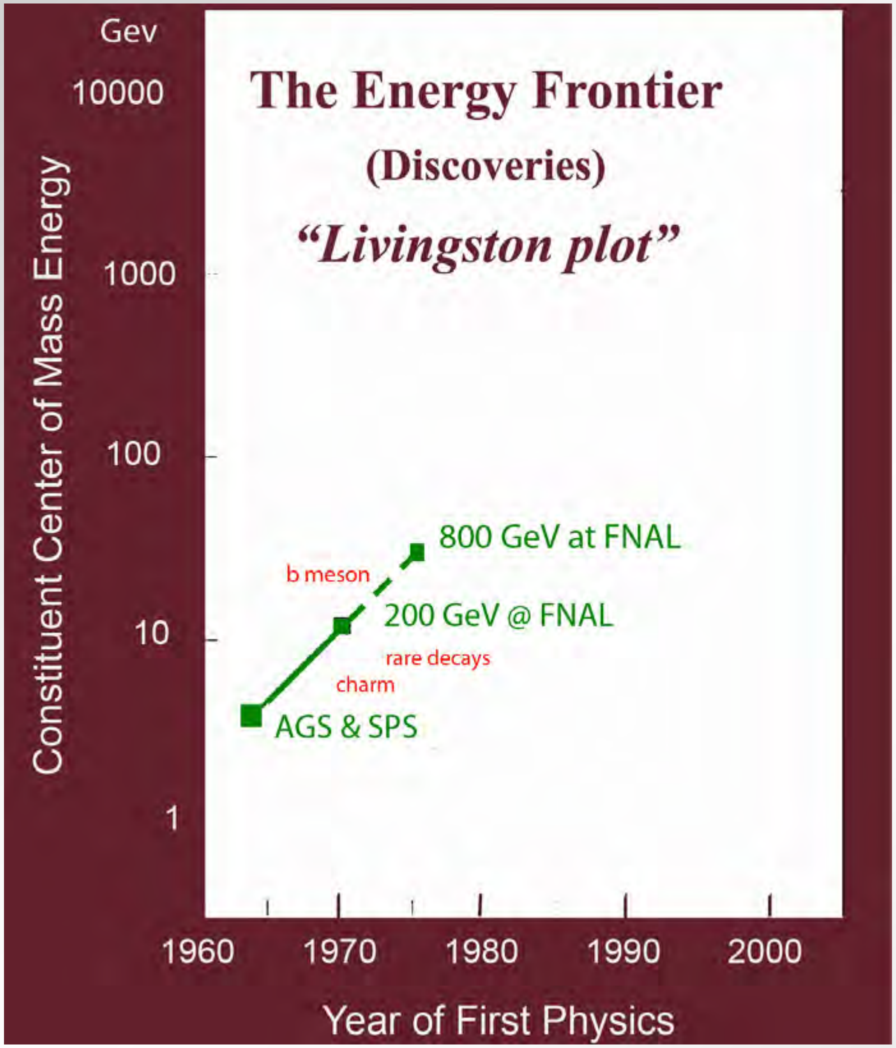
$$\mathbf{P_1} = (E_1/c, \mathbf{p_1}) \qquad \mathbf{P_2} = (m_0 c, \mathbf{0})$$

Centre of Momentum Frame (CMF): Velocities are equal & opposite, total energy is E_{cm}.

$$\mathbf{P_1} = (E_{\rm cm}/(2c), \mathbf{p}) \qquad \mathbf{P_2} = (E_{\rm cm}/(2c), -\mathbf{p})$$

Exercise: Relate E to E_{cm}

The fixed target paradigm has its limits



Invariance of

$$(p_1 + p_2)^\mu \bullet (p_1 + p_2)_\mu$$

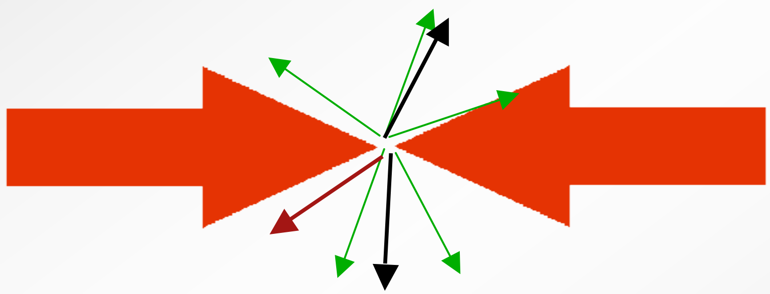
in Lorentz frames implies that

$$E_{cm} = \sqrt{m_1^2 + m_2^2 + 2m_2c^2E_{beam}}$$

$$\approx \sqrt{2mc^2E_{beam}} \quad \textit{for equal masses}$$

A great invention comes to the rescue

Collide beams !

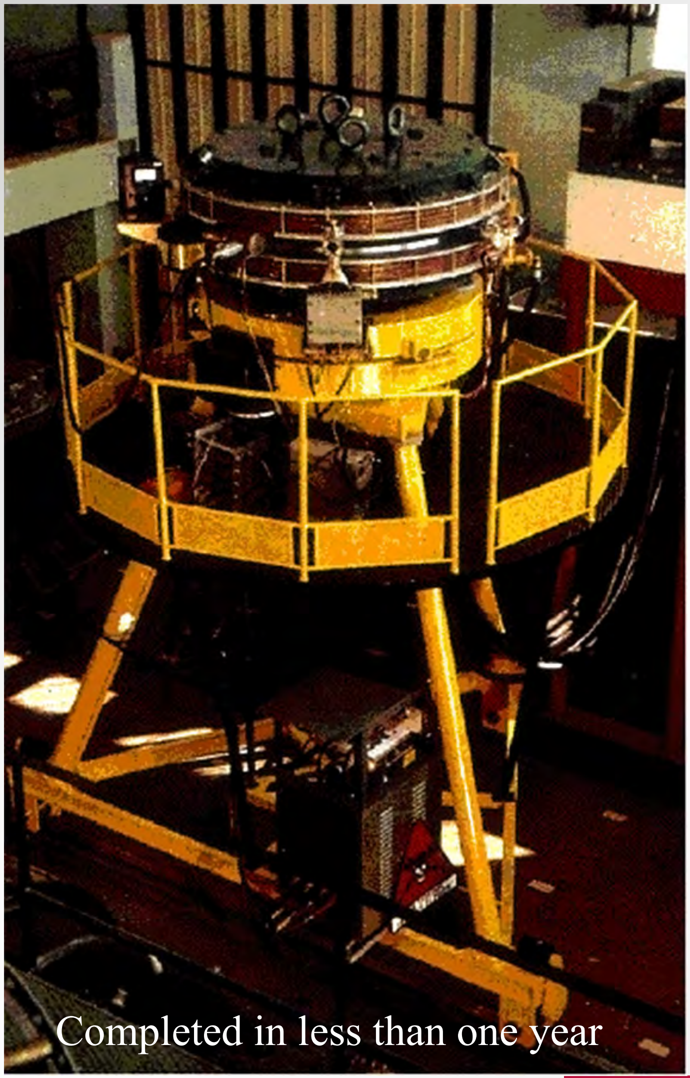


If $m_1 = m_2$ and if $E_1 = E_2 = E$

$$E_{cm} = 2\,E$$

The full kinetic energy of both particles is now available to physical processes

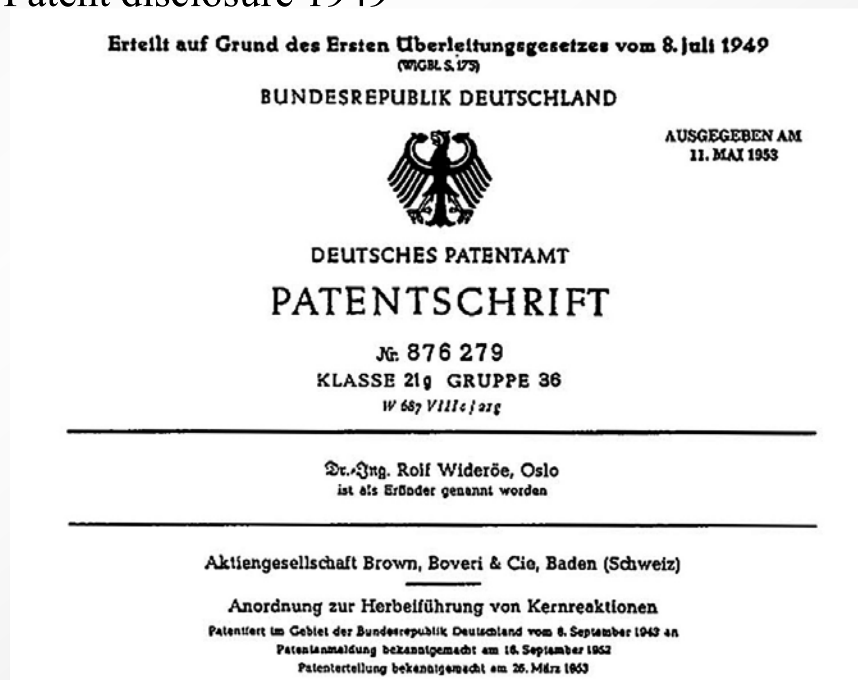
ADA - The first storage ring collider (e^+e^-) by B. Touschek at Frascati (1960)



Completed in less than one year

The storage ring collider idea was invented by Rolf Wiederoe in 1943!

- Collaboration with Bruno Touschek
- Patent disclosure 1949



Erteilt auf Grund des Ersten Überleitungsgesetzes vom 8. Juli 1949
(WiGBl. S. 175)

BUNDESREPUBLIK DEUTSCHLAND

AUSGEGEBEN AM
11. MAI 1953

DEUTSCHES PATENTAMT

PATENTSCHRIFT

Nr. 876 279

KLASSE 21g GRUPPE 36

W 687 VIIIc / 21g

Dr.-Ing. Rolf Wideröe, Oslo
ist als Erfinder genannt worden

Aktiengesellschaft Brown, Boveri & Cie, Baden (Schweiz)

Anordnung zur Herbeiführung von Kernreaktionen

Patentiert im Gebiet der Bundesrepublik Deutschland vom 8. September 1943 an
Patentanmeldung bekanntgemacht am 18. September 1952
Patenterteilung bekanntgemacht am 26. März 1953

$$\boxed{E_{cm} = 2E_{beam}}$$

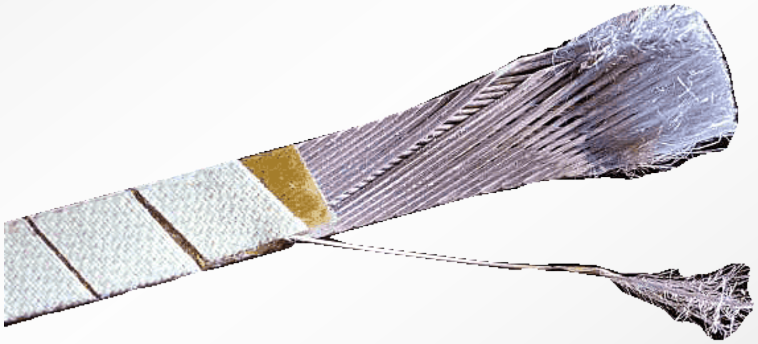
The next big step was the ISR at CERN

- 30 GeV per beam with > 60 A circulating current
  - Required extraordinary vacuum (10^{-11} Torr)
  - Great beam dynamics challenge - more stable than the solar system
- Then on to the 200 GeV collider at Fermilab (1972) and …
- The $Sp\bar{p}S$ at CERN
  - *Nobel invention:*
    Stochastic cooling
- And finally the Tevatron
  - Requires a *major technological invention*

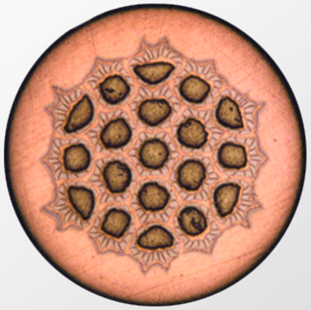
First machine to exploit superconducting magnet technology

Small things make a difference:
SC wire & cable ==> TeV colliders

64-strand cabling machine at Berkeley



NiTi superconducting cable showing stands & filaments



Sub-elements of a NiTi superconducting wire strand
The dark outlines are a resistive coating

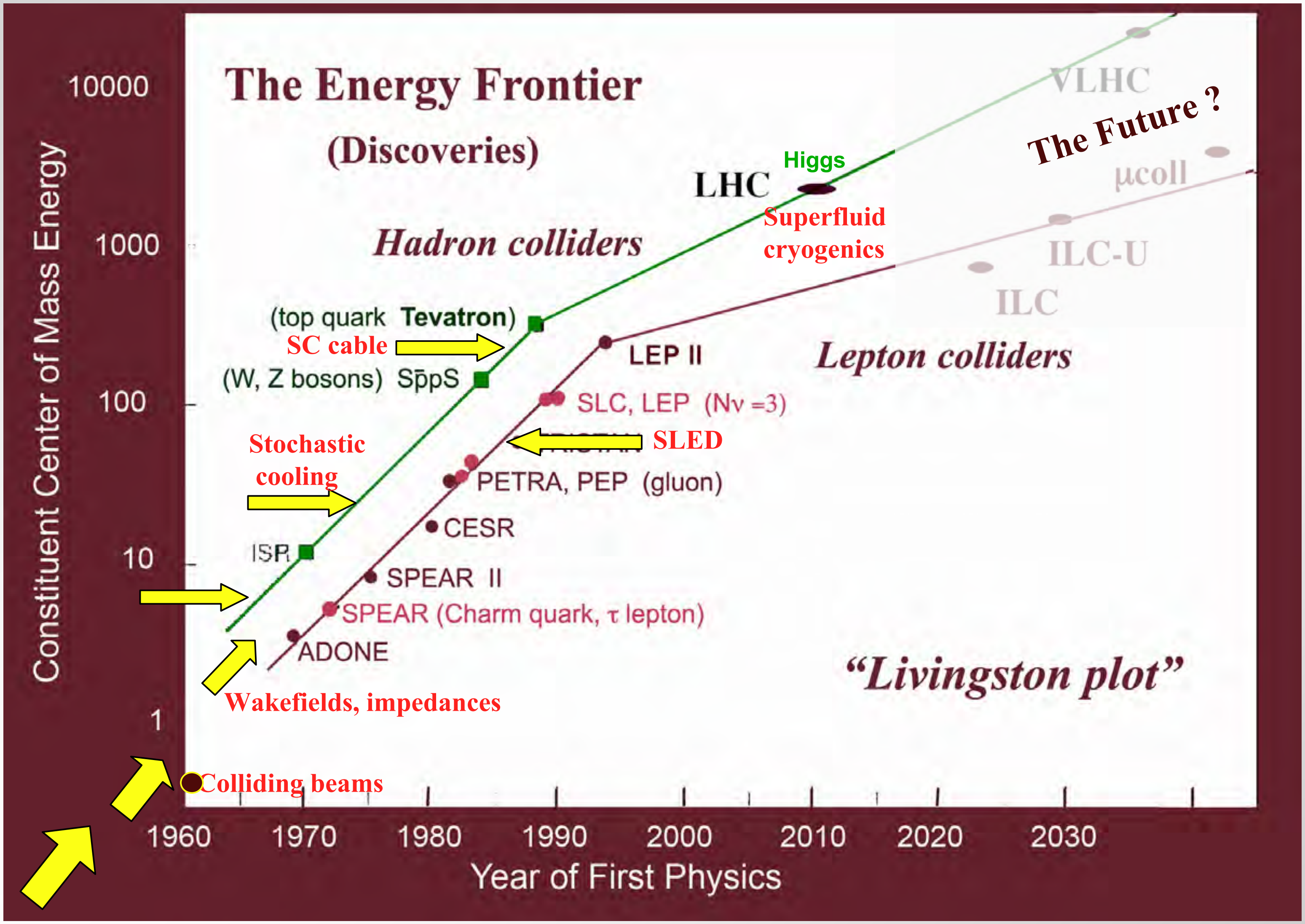
Cable design suppresses eddy currents as the current is increased

Cable concept invented by Martin Wilson

How big can this get? The LHC tunnel

This goes on for 28 km!

Energy frontier of discovery is extended by inventions in accelerator technology (in red)

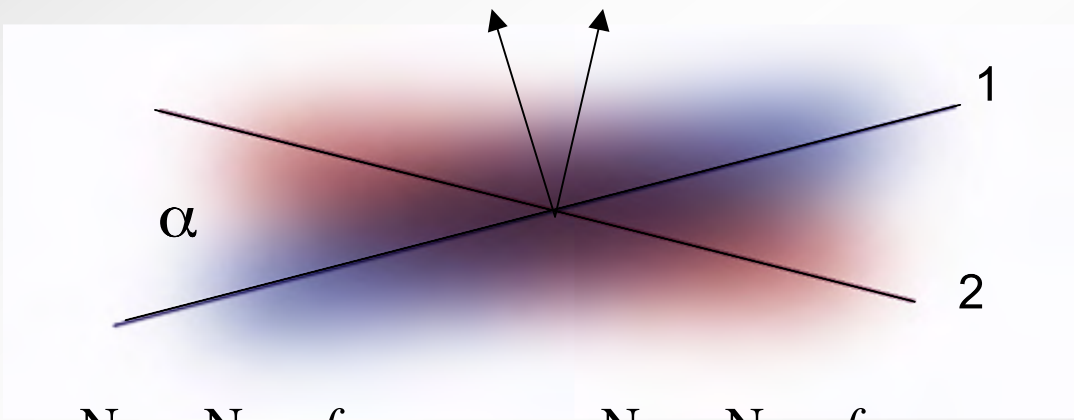


Strong focusing

This looks too good to be true!
What about the luminosity?

$$Events = \; Cross\text{-}section \; \times \; \langle Collision\ Rate \rangle \times Time$$

Beam energy: sets scale of physics accessible



$$\text{Luminosity} \; = \; \frac{\mathrm{N}_1 \times \mathrm{N}_2 \times frequency}{Overlap\ Area} = \frac{\mathrm{N}_1 \times \mathrm{N}_2 \times f}{4\pi\sigma_x\sigma_y} \times Correction\ factors$$

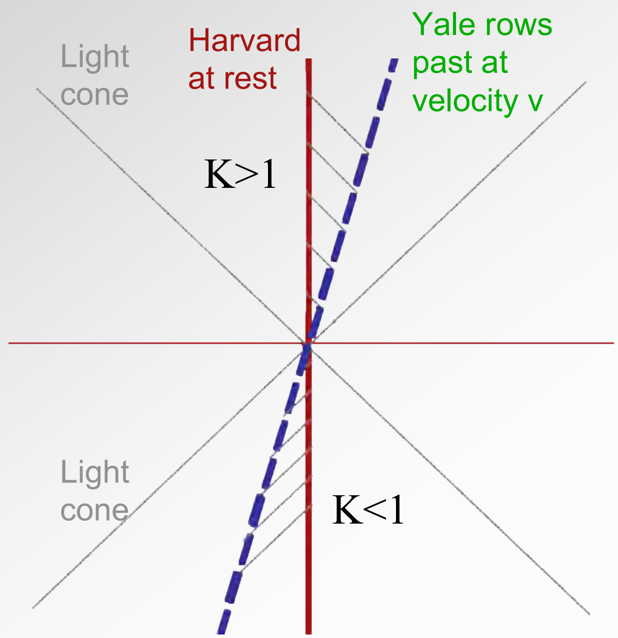
We want large charge/bunch, high collision frequency & small spot size

$$Luminosity \sim 10^{31} - 10^{34}\ cm^{-2}s^{-1}$$

10 minute exercise:

Show explicitly that the expression for collider luminosity is equivalent to the expression for fixed target luminosity.

Doppler shift of frequency
Harvard v. Yale crew teams



Distinguish between coordinate transformations & observations

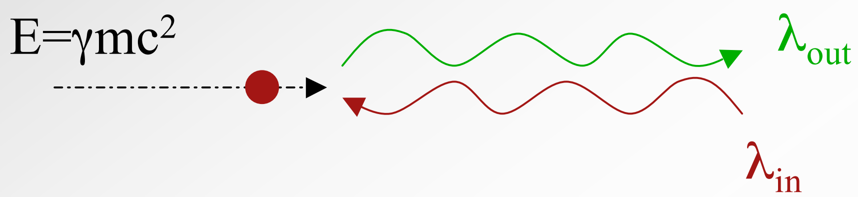
- Yale sets his signal to flash at a constant interval, $\Delta t'$
- Harvard sees the interval foreshortened by K(v) as Yale approaches
- Harvard see the interval stretched by K(-v) as Yale moves away

Homework: Using the world line diagram

Show $K(v) = K^{-1}(-v)$

For γ large find $K(\gamma)$

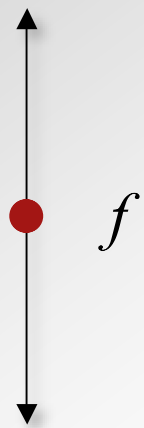
Head-on Compton scattering
by an ultra-relativistic electron

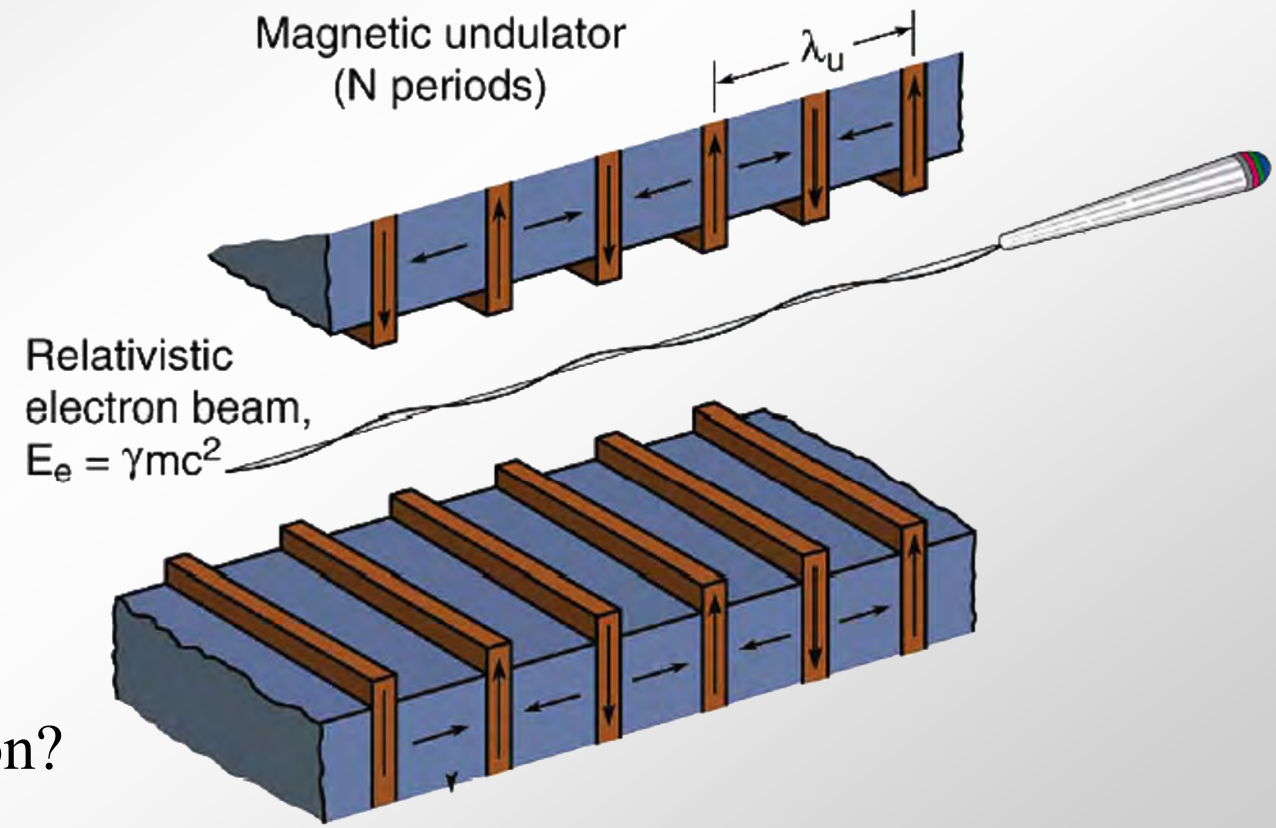


- What wavelength is the photon that is scattered by 180°?
  Write your answer in terms of K(γ)

Undulator radiation: What is λ_{rad}?

An electron in the lab oscillating at frequency, f, emits dipole radiation of frequency f





What about the relativistic electron?

Mechanics,

Maxwell's Equations

&

Special Relativity

The Basics - Mechanics

Newton's law

- We all know

$$\mathbf{F} = \frac{d}{dt}\mathbf{p}$$

- The 4-vector form is

$$F^{\mu} = \left(\gamma c \frac{dm}{dt}, \gamma \frac{d\mathbf{p}}{dt}\right) = \frac{dp^{\mu}}{d\tau}$$

- Differentiate $p^2 = m_o^{\,2} c^2$ with respect to τ

$$p_{\mu}\frac{dp^{\mu}}{d\tau} = p_{\mu}F^{\mu} = \frac{d(mc^2)}{dt} - \mathbf{F}\circ\mathbf{v} = 0$$

- The work is the rate of changing mc^2

Harmonic oscillators & pendula

- Motion in the presence of a linear restoring force

$$F = -kx$$

$$\ddot{x} + \frac{k}{m} x = 0$$

$$x = A \, \sin \omega_o t \quad \text{where} \quad \omega_o = \sqrt{k/m}$$

- It is worth noting that the simple harmonic oscillator is a linearized example of the pendulum equation

$$\ddot{x} + \omega_o^2 \sin(x) \approx \ddot{x} + \omega_o^2 (x - x^3/6) = 0$$

that governs the free electron laser instability

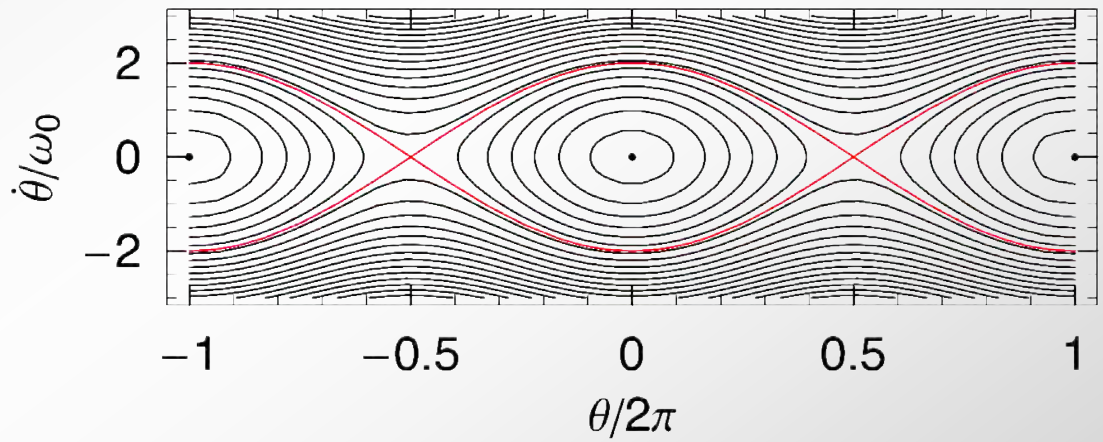
Solution to the pendulum equation

- Use energy conservation to solve the equation exactly
- Multiply $\ddot{x} + \omega_o^2 \sin(x) = 0$ by $\dot{x}$ to get

$$\frac{1}{2}\frac{d}{dt}\dot{x}^2 - \omega_o^2 \frac{d}{dt}\cos x = 0$$

- Integrating we find that the energy is conserved

$$\frac{1}{2\omega_o^2}\dot{x}^2 - \cos x = \text{constant} = \text{ energy of the system} = E$$



With $x = \theta$

Beams subject to non-linear forces
are commonplace in accelerators

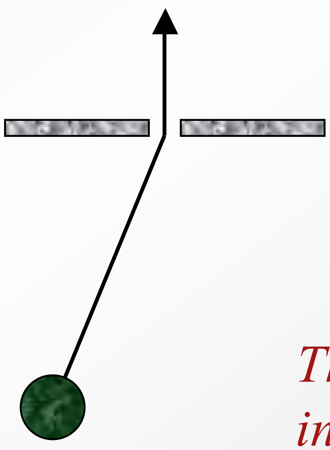
- Examples include
  - Space charge forces in beams with non-uniform charge distributions
  - Forces from magnets higher order than quadrupoles
  - Electromagnetic interactions of beams with *external* structures
    - Free Electron Lasers
    - Wakefields

Properties of harmonic oscillators

- Total energy is conserved

$$U = \frac{p^2}{2m} + \frac{m\omega_o^2 x^2}{2}$$

- If there are *slow* changes in m or ω, then $I = U/\omega_o$ *remains invariant*



$$\frac{\Delta\omega_o}{\omega_o} = \frac{\Delta U}{U}$$

This effect is important as a diagnostic in measuring resonant properties of structures

Hamiltonian systems

- In a Hamiltonian system, there exists generalized positions q_i, generalized momenta p_i, & a function $H(q, p, t)$ describing the system evolution by

$$\boxed{\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}} \qquad \boxed{\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}} \qquad \begin{aligned} \bar{q} &\equiv \{ q_1, q_2, \ldots, q_N \} \\ \bar{p} &\equiv \{ p_1, p_2, \ldots, p_N \} \end{aligned}$$

- H is called the Hamiltonian and q & p are canonical conjugate variables

- For q = usual spatial coordinates $\{x, y, z\}$ & p their conjugate momentum components $\{p_x, p_y, p_z\}$
  - H coincides with the total energy of the system

$$H = U + T = \text{Potential Energy} + \text{Kinetic Energy}$$

Dissipative, inelastic, & stochastic processes are non-Hamiltonian

Lorentz force on a charged particle

- Force, **F**, on a charged particle of charge q in an electric field **E** and a magnetic field, **B**

$$\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right)$$

- $E =$ electric field with units of force per unit charge, newtons/coulomb = volts/m.
- B = magnetic flux density or magnetic induction, with units of newtons/ampere-m = Tesla = Weber/m^2.

A simple problem - bending radius

- Compute the bending radius, R, of a non-relativistic particle particle in a uniform magnetic field, B.
  - Charge = q
  - Energy = $mv^2/2$

$$F_{Lorentz} = q\frac{v}{c}B = F_{centripital} = \frac{mv^2}{\rho}$$

$$\Rightarrow \rho = \frac{mvc}{qB} = \frac{pc}{qB}$$

$$\boxed{\rho(\mathrm{m}) = 3.34\left(\frac{p}{1\ \mathrm{GeV/c}}\right)\left(\frac{1}{q}\right)\left(\frac{1\ \mathrm{T}}{B}\right)}$$

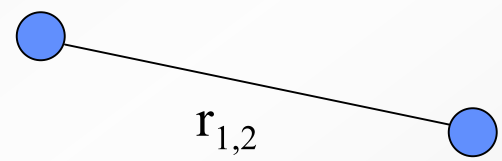
10 minute exercise from Whittum

- **Exercise:** A charged particle has a kinetic energy of 50 keV. You wish to apply as large a force as possible. You may choose either an electric field of 500 kV/m or a magnetic induction of 0.1 T. Which should you choose
  - (a) for an electron,
  - (b) for a proton?

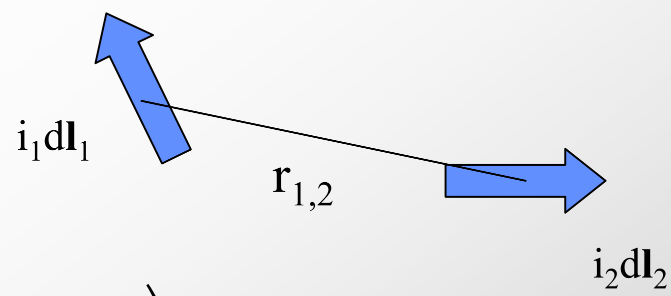
The fields come from charges & currents

- Coulomb's Law

$$\mathbf{F}_{1\to 2} = q_2\left(\frac{1}{4\pi\varepsilon_o}\frac{q_1}{r_{1,2}^2}\hat{\mathbf{r}}_{1\to 2}\right) = q_2\mathbf{E}_1$$



- Biot-Savart Law

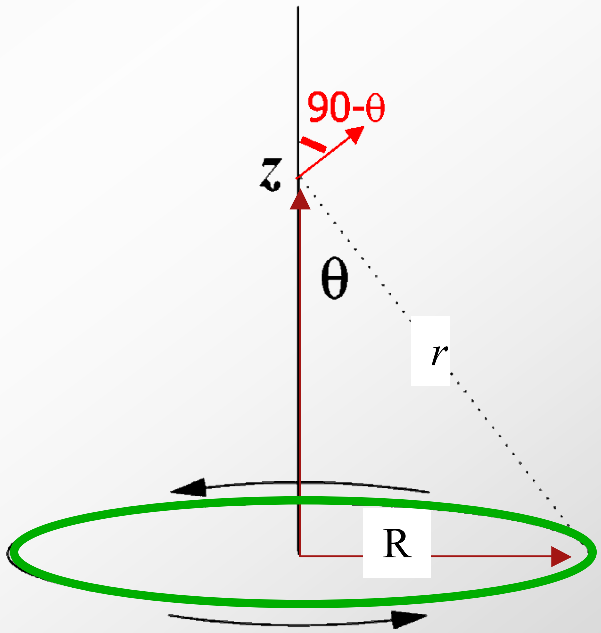


$$d\mathbf{F}_{1\to 2} = i_2 d\mathbf{l}_2 \times \left(\frac{\mu_0}{4\pi}\frac{(i_1 d\mathbf{l}_1 \times \hat{\mathbf{r}}_{12})}{r_{1,2}^2}\right) = i_2 d\mathbf{l}_2 \times \mathbf{B}_2$$

Compute the B-field from current loop

- On axis there is only B_z by symmetry
- The Biot-Savart law says

$$B = \int_{wire} \left(d\vec{B}\right)_z = \int_{wire} \frac{I}{cr^2} \left| d\vec{l} \times \hat{r} \right| \sin\theta$$

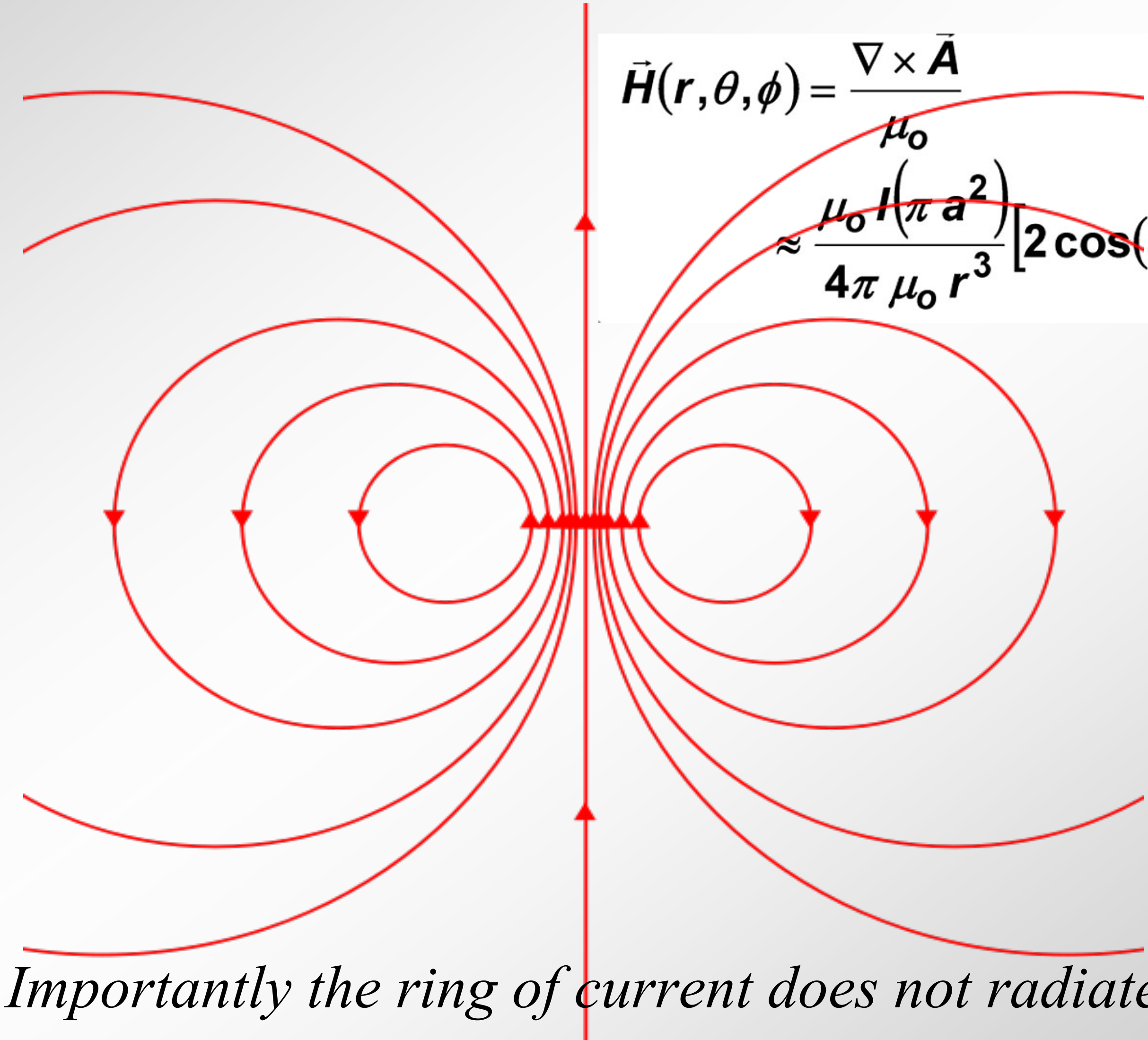


$$\left| d\mathbf{l} \times \hat{\mathbf{r}} \right| = \left| d\mathbf{l} \right| = R d\varphi$$

$$\sin\theta = R/r \quad \text{and} \quad r = \sqrt{R^2 + z^2}$$

$$\mathbf{B} = \frac{I}{cr^2} R \sin\theta \int_0^{2\pi} d\varphi\, \hat{\mathbf{z}} = \frac{2\pi I R^2}{c\left(R^2 + z^2\right)^{3/2}} \hat{\mathbf{z}}$$

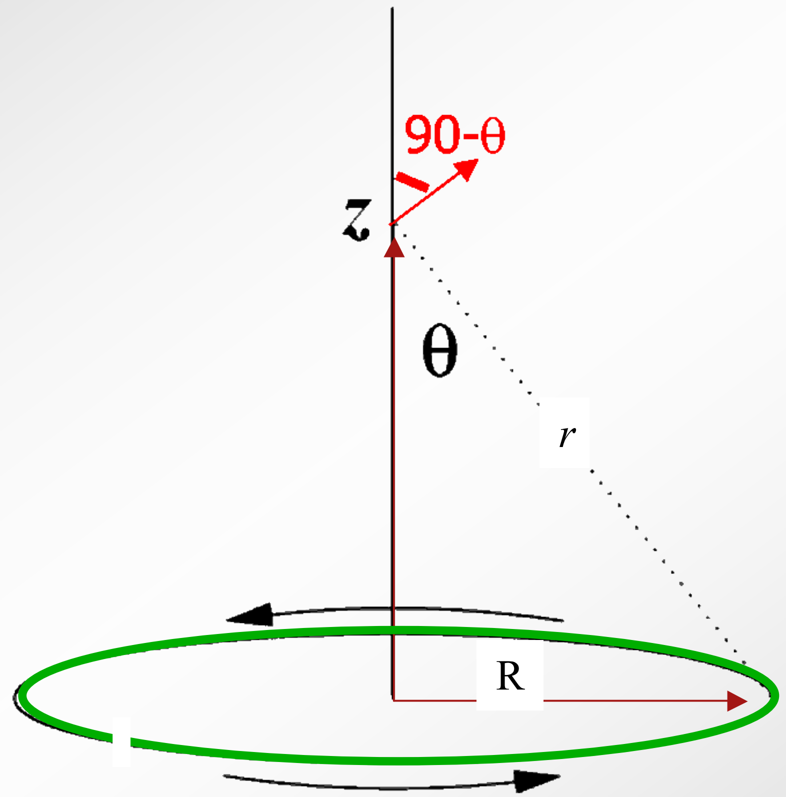
The far field B-field has a static dipole form



$$\vec{H}(r,\theta,\phi) = \frac{\nabla \times \vec{A}}{\mu_o}$$
$$\approx \frac{\mu_o I\left(\pi a^2\right)}{4\pi\, \mu_o\, r^3}\left[2\cos(\theta)\hat{r} + \sin(\theta)\hat{\theta}\right]$$

Importantly the ring of current does not radiate

Question to ponder:
What is the field from this situation?



We'll return to this question in the second half of the course

Electric displacement & magnetic field

In vacuum,

- The electric displacement is $\mathbf{D} = \varepsilon_o \mathbf{E}$,
- The magnetic field is $\mathbf{H} = \mathbf{B}/\mu_o$

Where

$$\varepsilon_o = 8.85\text{x}10^{-12} \text{ farad/m} \quad \& \quad \mu_o = 4\,\pi\,\text{x}10^{-7} \text{ henry/m.}$$

Maxwell's equations (1)

- Electric charge density ρ is source of the electric field, **E** (Gauss's law)

$$\nabla \bullet \mathbf{E} = \rho$$

- Electric current density **J** = ρ**u** is source of the magnetic induction field **B** (Ampere's law)

$$\nabla \times \mathbf{B} = \mu_o \mathbf{J} + \mu_0 \varepsilon_o \frac{\partial \mathbf{E}}{\partial t}$$

If we want big magnetic fields, we need large current supplies

Maxwell's equations (2)

- Field lines of **B** are closed; i.e., no magnetic monopoles.

$$\nabla \bullet \mathbf{B} = 0$$

- Electromotive force around a closed circuit is proportional to rate of change of **B** through the circuit (Faraday's law).

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Maxwell's equations: integral form

$$\vec{\nabla} \bullet \vec{E} = \frac{\rho}{\varepsilon_0} \quad => \quad \oint_S \vec{E} \bullet d\vec{a} = \frac{Q_{enclosed}}{\varepsilon_0} \quad \text{Gauss' Law}$$

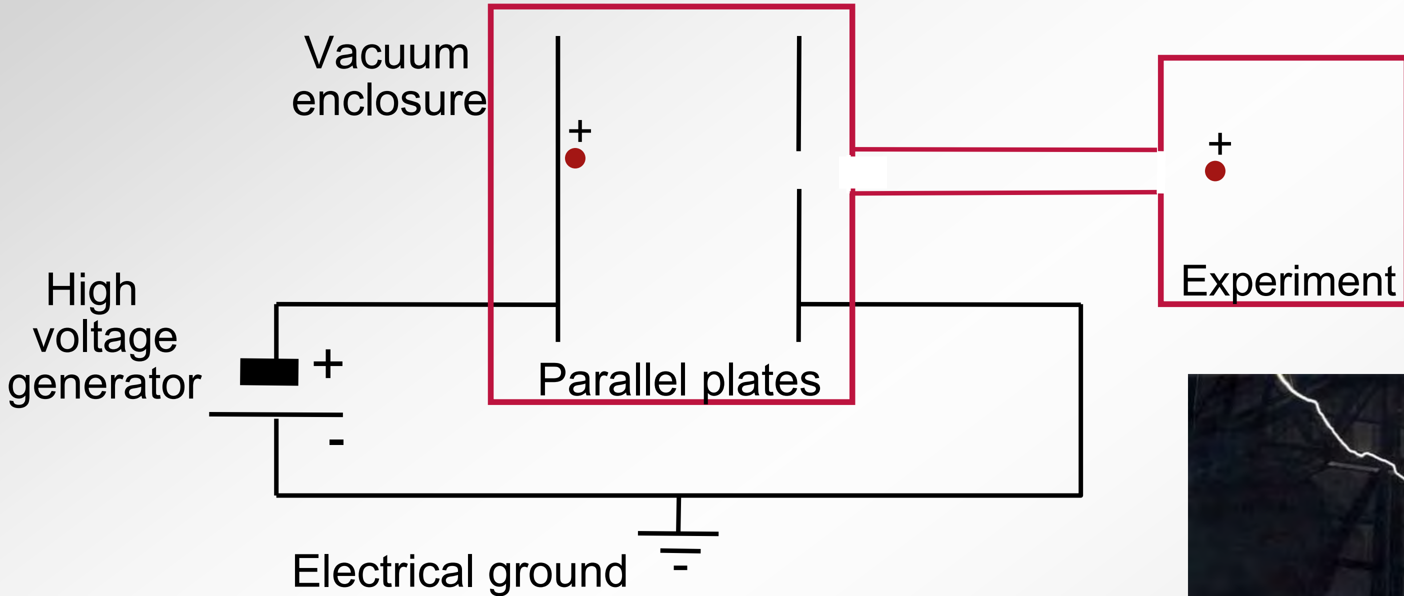
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad => \quad \oint_C \vec{E} \bullet d\vec{l} = -\oint_S \frac{\partial \vec{B}}{\partial t} \bullet d\vec{a} \quad \text{Faraday's Law}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \quad =>$$

Displacement current

$$\oint_C \vec{B} \bullet d\vec{l} = \mu_0 I_{enclosed} + \mu_0 \varepsilon_0 \oint_S \frac{\partial \vec{E}}{\partial t} \bullet d\vec{a} \quad \text{Ampere's Law}$$

The first accelerators:
DC (electrostatic) accelerators



Note the exposed high voltage hazard

The energy is limited

by high voltage break down

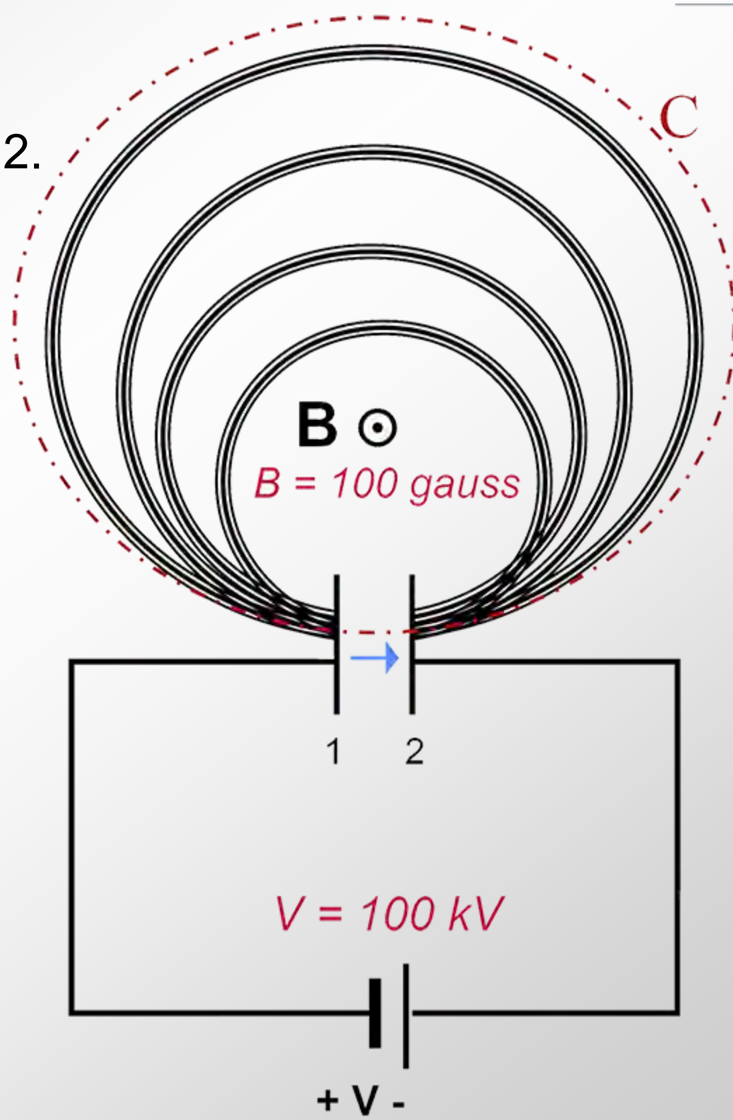
Possible high energy DC accelerator?

At t = 0 the ion source at 1 injects a proton of energy E_o in the gap pointed at a hole in plate 2. The entire device is imbedded in a constant magnetic (dipole) field, B, pointing out of the surface.

Exiting the plate 2, the proton enters the innermost virtual beam pipe.

If B = 100 Gauss and E_o = 100 keV, what is the radius of the first orbit?

After 10,000 revolutions, what is the energy of the proton as it leaves plate 2.



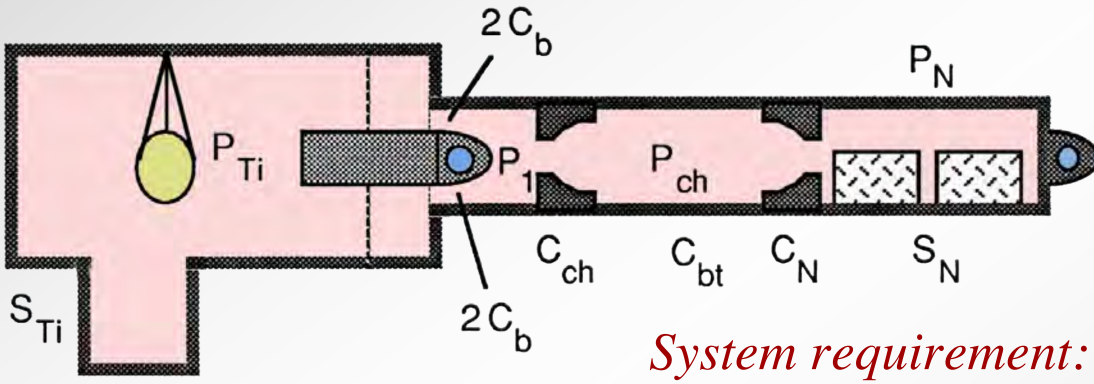
Circuit theory

Accelerator physicists often use
network (circuit) analogs of accelerator systems

1) RF systems
2) Vacuum systems
3) Control systems

Example: Vacuum design storage ring
Synchrotron radiation in hard bends of CESR-B

Estimate the pumping speed needed for Titanium pumps & NEG pumps



System requirement: $P_{ch} < 1.7$ nTorr

Figure 5. Schematic of the pumping scheme and beam chamber in the hard bend transition region of the high energy ring of CESR-B

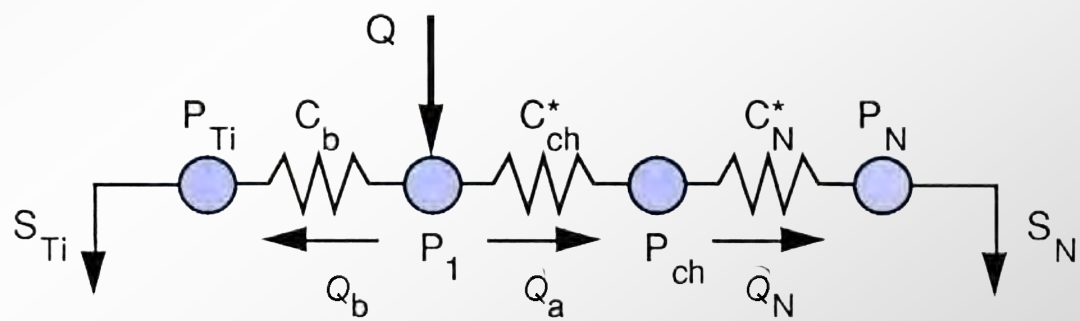
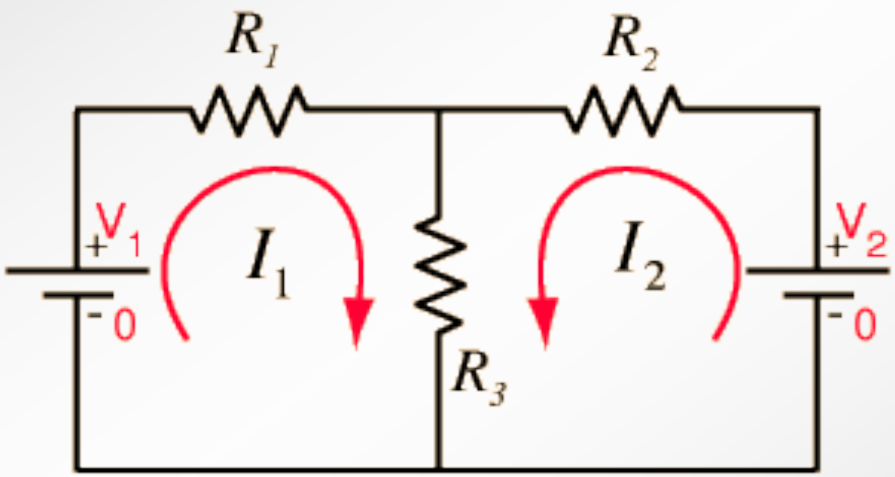


Figure 5. Circuit model of the pumping in the HER transition section

Basic concepts:
Start with dc circuits

- Kirchoff's law's
  - The sum of Voltage drops around any loop equals zero
  - The sum of the currents into any node equals zero



- Ohm's law:
  - The voltage drop across a resistance: V = I R

Ohm's Law Generalized

- Basic approach is the Fourier analysis of a circuit
- Start with

$$\tilde{V} = V e^{j(\omega t + \varphi)}$$

- Instead of V = IR where the quantities are real we write

$$\tilde{V}(\omega) = \tilde{I}(\omega)\overleftrightarrow{Z}(\omega)$$

- We know this works for resistors.

$V(t) = R\,I(t) \implies Z_R$ is real $= R$

- What about capacitors & inductors?

Impedance of Capacitors

- For a capacitor

$$I = C\left(\frac{dV}{dt}\right) \Rightarrow \tilde{I} = C\frac{d}{dt}Ve^{j(\omega t+\varphi)} = j\omega C\tilde{V}$$

- So our generalized Ohm's law is

$$\tilde{V} = \tilde{I}\overleftrightarrow{Z}_C$$

where

$$\tilde{Z}_C = \frac{1}{j\omega C}$$

Impedance of Inductors

❖ For a capacitor

$$V = L\left(\frac{dI}{dt}\right) \Rightarrow \tilde{V} = L\frac{d}{dt}Ie^{j(\omega t+\varphi)} = j\omega L\tilde{I}$$

❖ So our generalized Ohm's law is

$$\tilde{V} = \tilde{I}\overleftrightarrow{Z}_L$$

Where

$$\tilde{Z}_L = j\omega L$$

Combining impedances

- The algebraic form of Ohm's Law is preserved

==> impedances follow the same rules for combination in series and parallel as for resistors

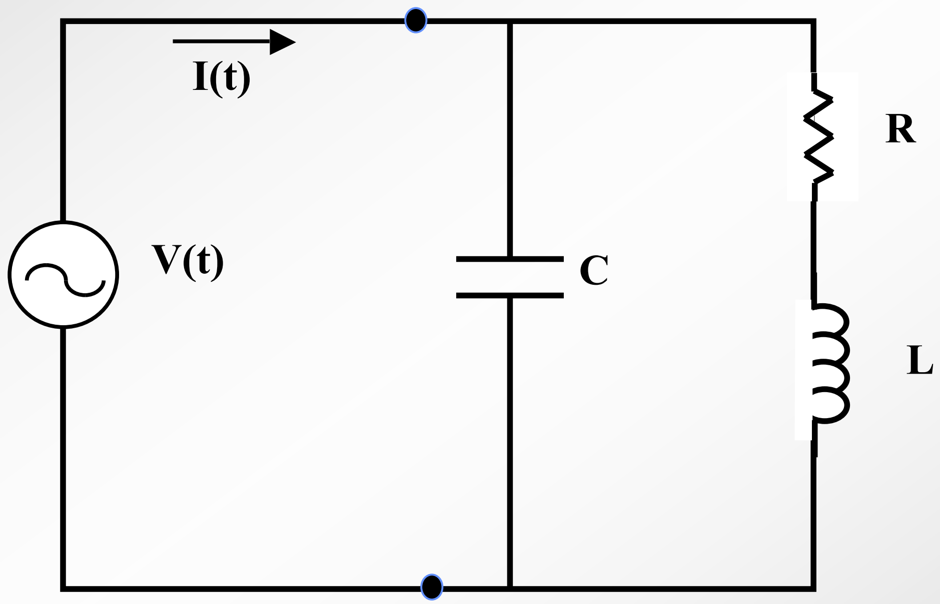
- For example

$$Z_{series} = Z_1 + Z_2$$

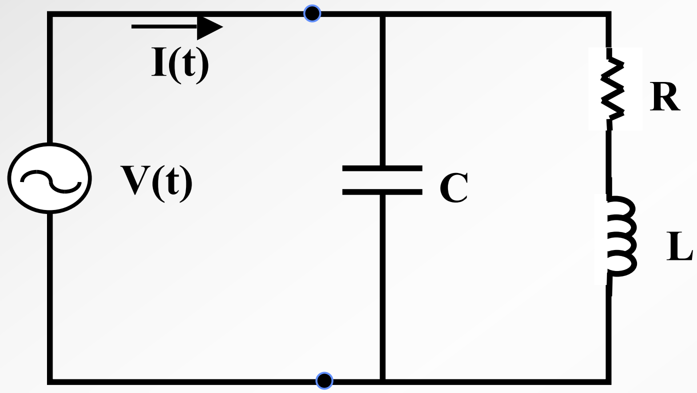
$$Z_{parallel} = \left[1/Z_1 + 1/Z_2 \right]^{-1} = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

- We can now solve circuits using Kirkhoff's laws, *but in the frequency domain*

Exercise: Compute the impedance Z
looking into the terminals (10 miutes)



Looking into the terminals, we have



$$Z(\omega) = \left[j\omega C + (j\omega L + R)^{-1} \right]^{-1}$$

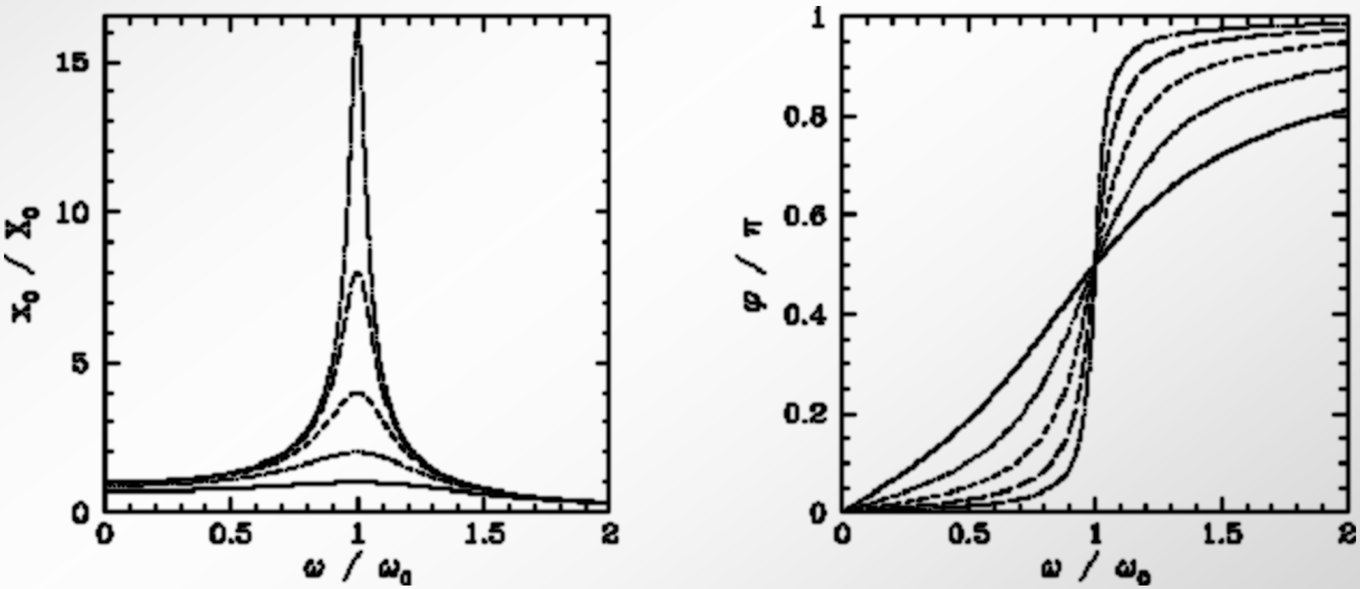
$$Z(\omega) = \frac{1}{j\omega C + (j\omega L + R)^{-1}} = \frac{(j\omega L + R)}{(j\omega L + R)\, j\omega C + 1} = \frac{(j\omega L + R)}{(1 - \omega^2 LC) + j\omega RC} = X + j\varphi$$

The resonant frequency is $\omega_o = 1/\sqrt{LC}$

Resonant behavior of the lumped circuit

Converting the denominator of Z to a real number we see that

$$|\, Z(\omega)\, | \sim \left[\left(1 - \frac{\omega^2}{\omega_o^2} \right)^2 + (\omega RC)^2 \right]^{-1}$$



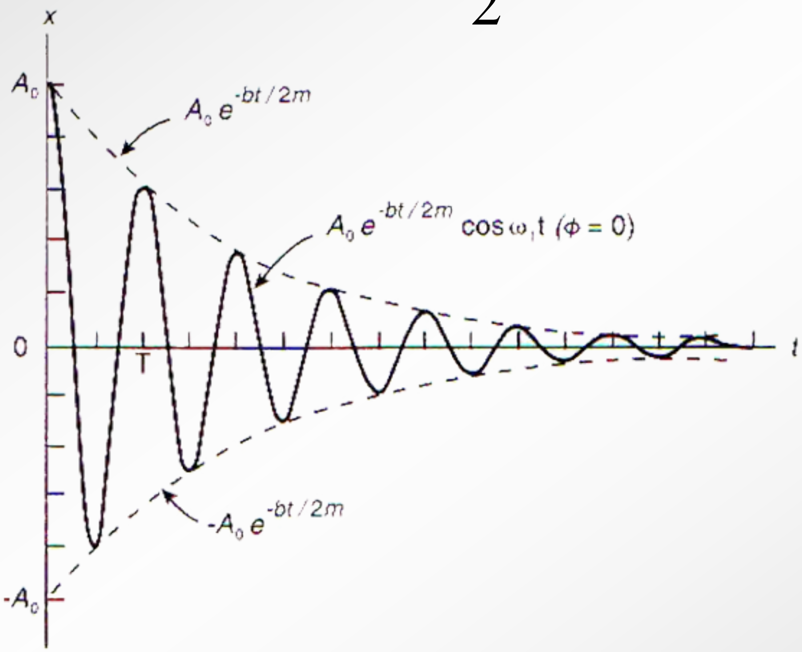
The width is $\dfrac{\Delta\omega}{\omega_0} = \dfrac{R}{\sqrt{L/.C}}$

More basics from circuits - Q

$$Q = \frac{\omega_o \circ \ Energy\ stored}{Time\ average\ power\ loss} = \frac{2\pi \circ Energy\ stored}{Energy\ per\ cycle}$$

and

$$\mathcal{E} = \frac{1}{2} L\, I_o I_o^* \qquad \langle \mathcal{P} \rangle = \langle \mathrm{i}^2(t) \rangle R = \frac{1}{2} I_o I_o^* R_{surface}$$



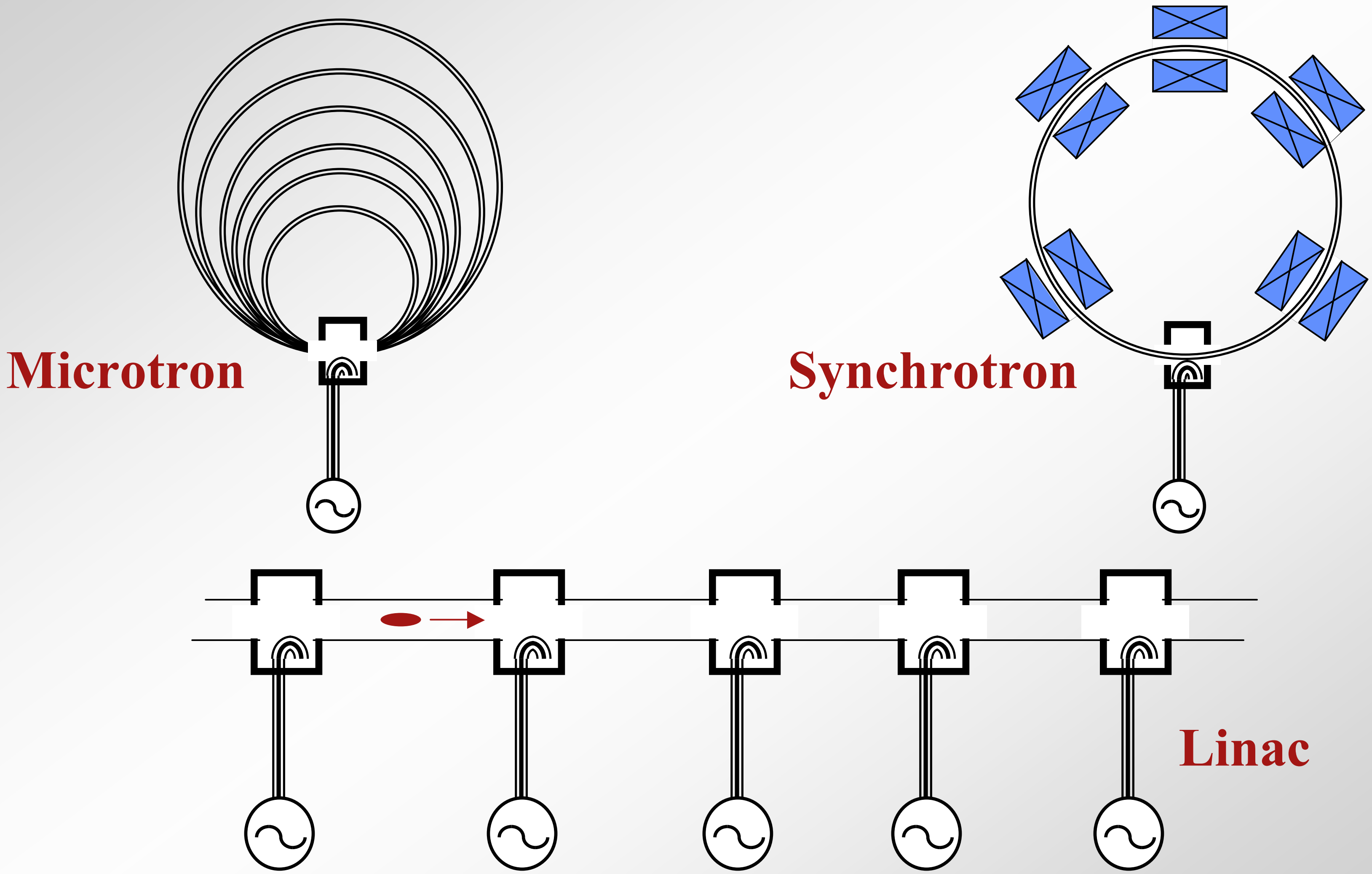
$$\therefore \ Q = \frac{\sqrt{L/C}}{R} = \left(\frac{\Delta\omega}{\omega_o}\right)^{-1}$$

Boundary conditions for a perfect conductor $\sigma = \infty$

1. If electric field lines terminate on a surface, they do so normal to the surface
   a) any tangential component would quickly be neutralized by lateral motion of charge within the surface.
   b) The E-field must be normal to a conducting surface
2. Magnetic field lines avoid surfaces
   a) otherwise they would terminate, since the magnetic field is zero within the conductor
      i. The normal component of B must be continuous across the boundary for $\sigma \neq \infty$

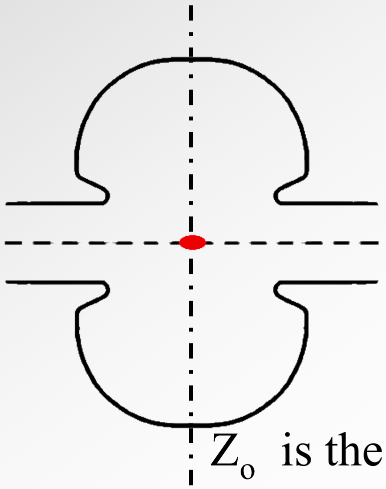
RF-cavities

RF-cavities for acceleration:
The heart of modern accelerators



RF cativties: Basic concepts

- Fields and voltages are complex quantities.
  - For standing wave structures use phasor representation



$$\tilde{V} = Ve^{i\omega t} \quad \text{where} \quad V = |\tilde{V}|$$

At t = 0 particle receives maximum voltage gain

Z_o is the reference plane

- For cavity driven externally, phase of the voltage is

$$\theta = \omega t + \theta_o$$

- For electrons $v \approx c$; therefore $z = z_o + ct$

Basic principles and concepts

- Superposition
- Energy conservation
- Orthogonality (of cavity modes)
- Causality

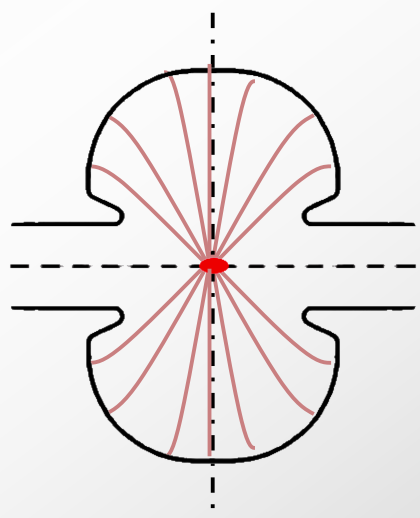
Basic principles:
Reciprocity & superposition

❖ If you can kick the beam, the beam can kick you

==>

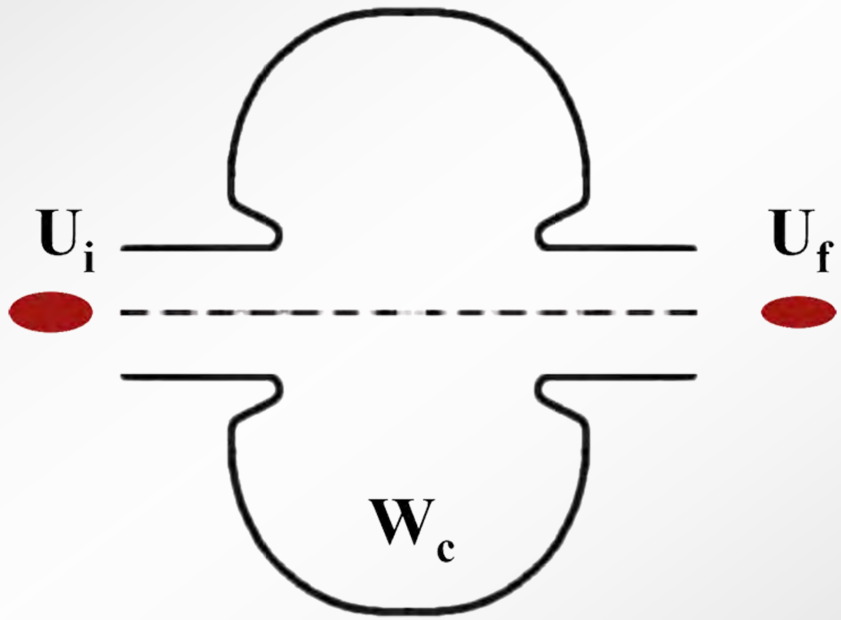
$$\text{Total cavity voltage} = V_{generator} + V_{beam\text{-}induced}$$

$$\text{Fields in cavity} = \mathbf{E}_{generator} + \mathbf{E}_{beam\text{-}induced}$$



Basic principles: Energy conservation

- Total energy in the particles and the cavity is conserved
  - Beam loading



$$\Delta W_c = U_i - U_f$$

Basics: Orthogonality of normal modes

- Maxwell's equations are linear
  - The EM field is NOT a source of EM fields
- Therefore,
  - Each mode in the cavity can be treated independently in computing fields induced by a charge crossing the cavity.
  - The total stored energy is equals the sum of the energies in the separate modes.
  - The total field is the phasor sum of all the individual mode fields at any instant.

Basic principles: Causality

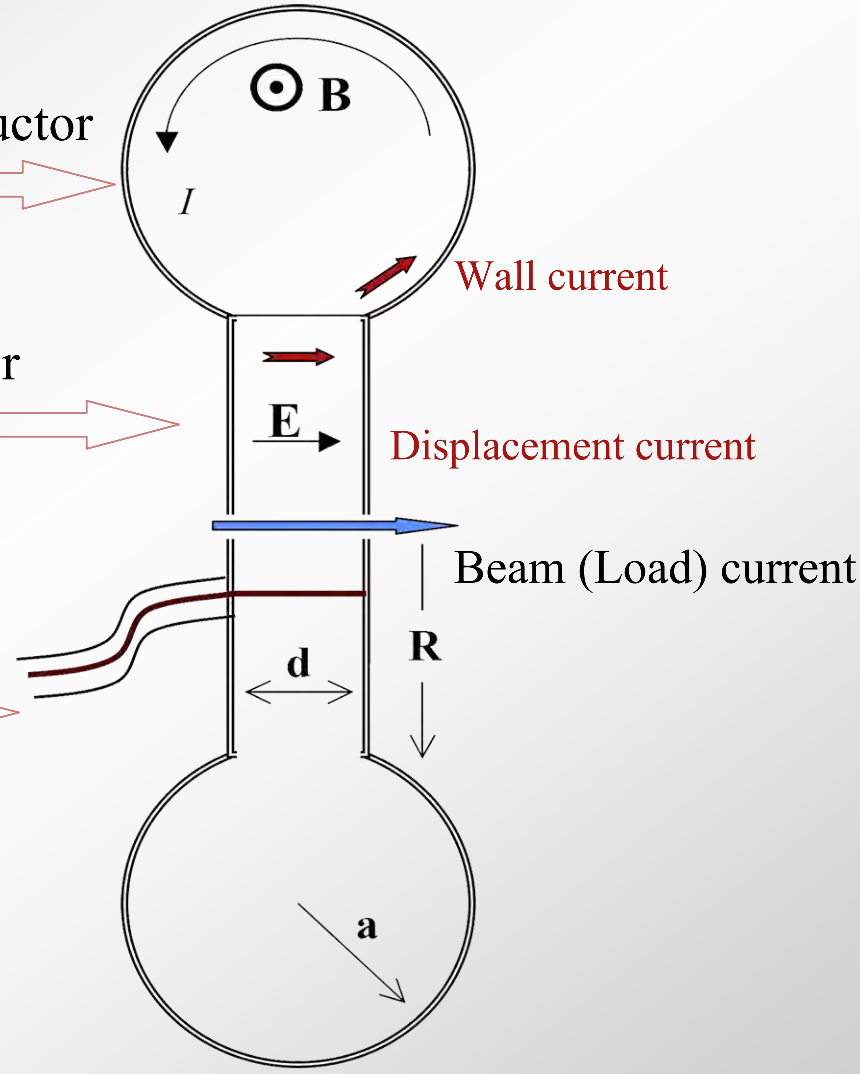
- No disturbance ahead of a charge moving at $v \approx c$
- In a mode analysis of the growth of beam-induced fields, field must vanish ahead of the moving charge *for each mode*

Basic components of an RF cavity

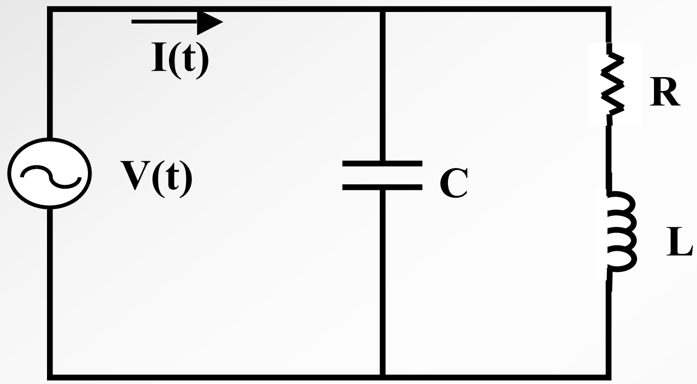
Outer region: Large, single turn Inductor

Central region: Large plate Capacitor

Power feed from rf - generator



We have already solved this circuit
Lumped circuit analogy of resonant cavity



$$Z(\omega) = \left[j\omega C + (j\omega L + R)^{-1} \right]^{-1}$$

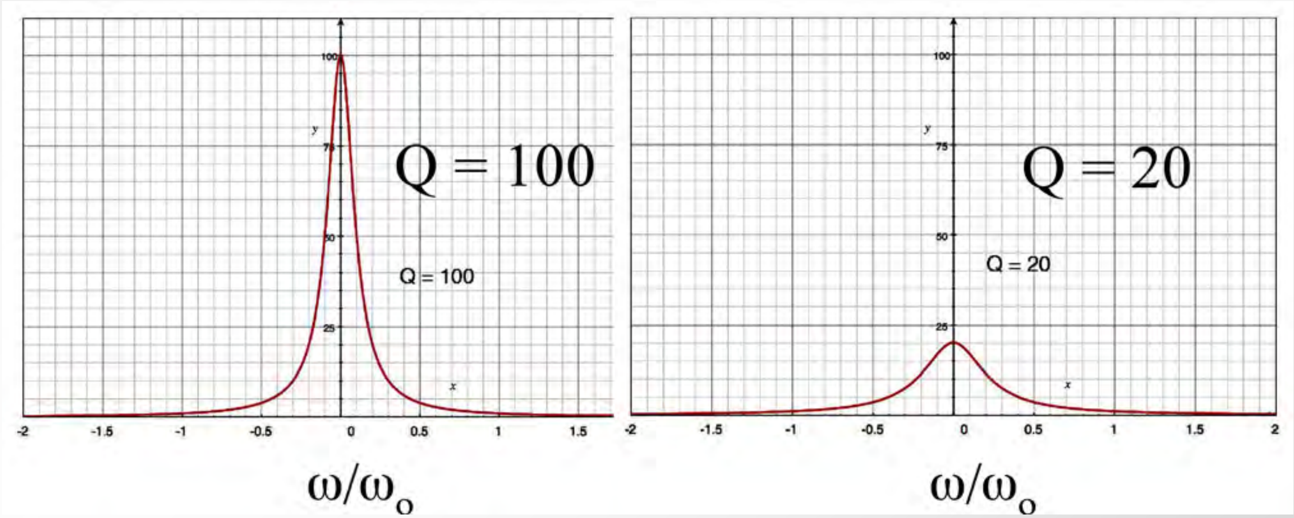
$$Z(\omega) = \frac{1}{j\omega C + (j\omega L + R)^{-1}} = \frac{(j\omega L + R)}{(j\omega L + R)\, j\omega C + 1} = \frac{(j\omega L + R)}{(1 - \omega^2 LC) + j\omega RC}$$

The resonant frequency is $\omega_o = 1/\sqrt{LC}$

Q of the lumped circuit analogy

Converting the denominator of Z to a real number we see that

$$| Z(\omega) | \sim \left[\left(1 - \frac{\omega^2}{\omega_o^2} \right)^2 + (\omega RC)^2 \right]^{-1}$$



The width is $\frac{\Delta\omega}{\omega_0} = \frac{R}{\sqrt{L/.C}}$

Translate circuit model to a cavity model:
Directly driven, re-entrant RF cavity

Outer region: Large, single turn Inductor

$$L = \frac{\mu_o \pi a^2}{2\pi(R+a)}$$

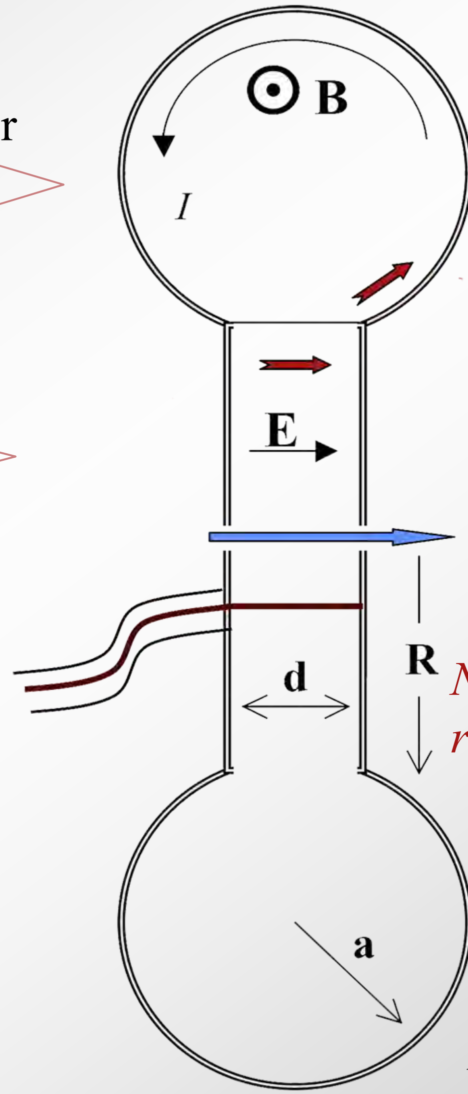
Central region: Large plate Capacitor

$$C = \varepsilon_o \frac{\pi R^2}{d}$$

$$\omega_o = 1\Big/\sqrt{LC} = c\left[\frac{2((R+a)d}{\pi R^2 a^2}\right]^{1/2}$$

Q – set by resistance in outer region

$$Q = \sqrt{L/C}\Big/R$$

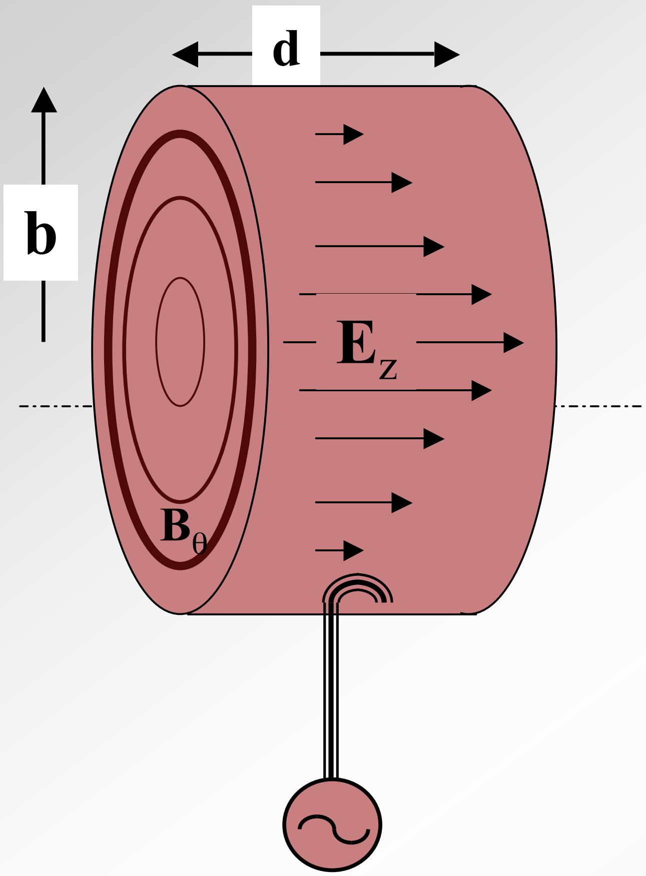


Expanding outer region raises Q

Narrowing gap raises shunt impedance

Source: Humphries, Charged Particle Accelerators

Properties of the RF pillbox cavity



$$\sigma_{walls} = \infty$$

- We want lowest mode: with only $\mathbf{E}_z$ & $\mathbf{B}_\theta$
- Maxwell's equations are:

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(rB_\theta\right) = \frac{1}{c^2}\frac{\partial}{\partial t}E_z \quad \text{and} \quad \frac{\partial}{\partial r}E_z = \frac{\partial}{\partial t}B_\theta}$$

- Take derivatives

$$\frac{\partial}{\partial t}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rB_\theta\right)\right] = \frac{\partial}{\partial t}\left[\frac{\partial B_\theta}{\partial r} + \frac{B_\theta}{r}\right] = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}$$

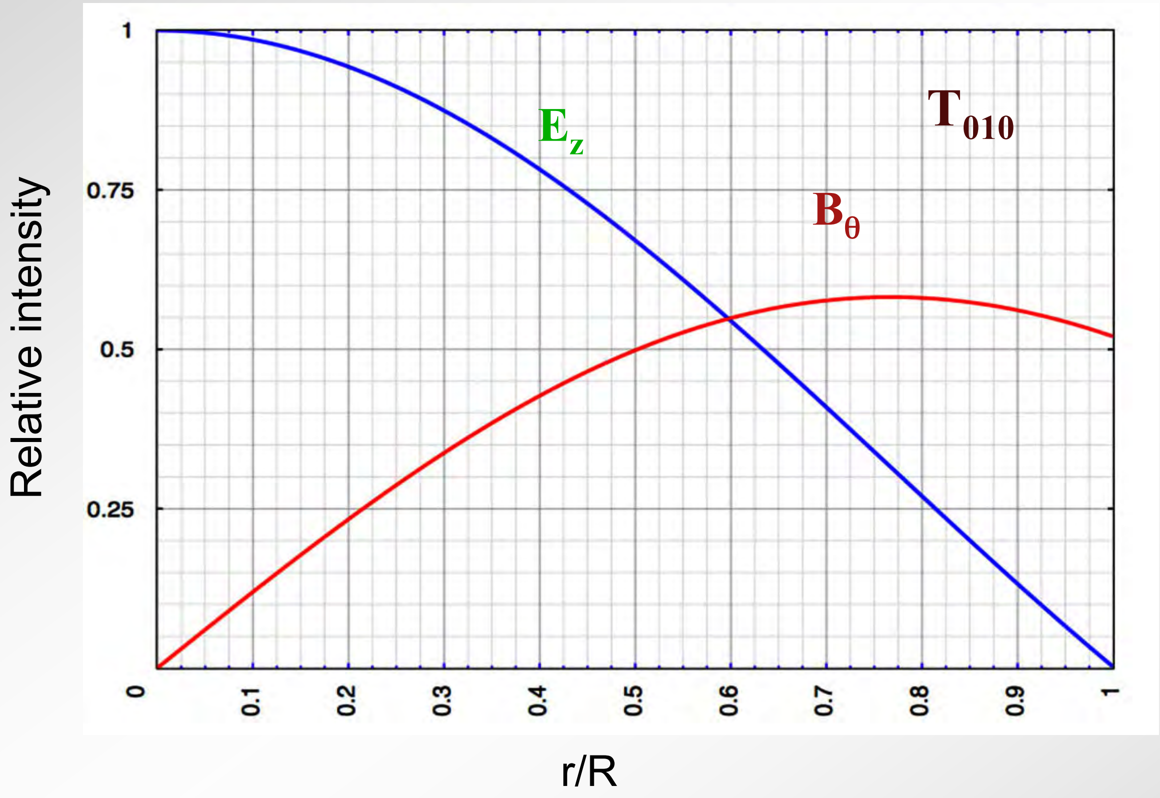
$$\frac{\partial}{\partial r}\frac{\partial E_z}{\partial r} = \frac{\partial}{\partial r}\frac{\partial B_\theta}{\partial t}$$

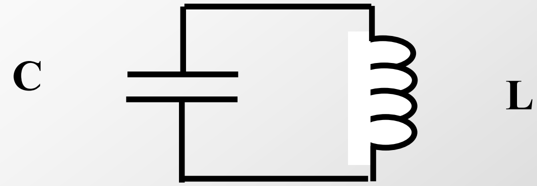
==> $$\boxed{\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r}\frac{\partial E_z}{\partial r} = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}}$$

For a mode with frequency ω

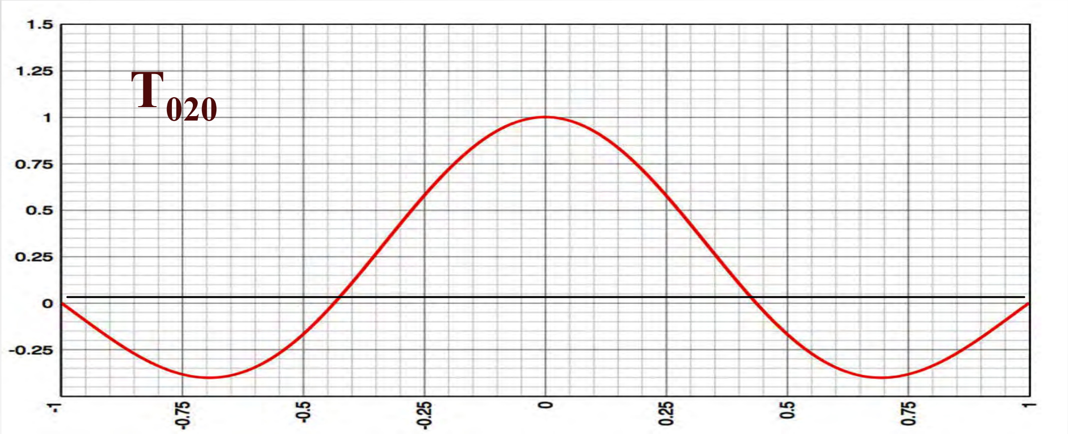
- $$E_z(r,t) = E_z(r)\ e^{i\omega t}$$
- Therefore, $$E_z'' + \frac{E_z'}{r} + \left(\frac{\omega}{c}\right)^2 E_z = 0$$
  - (Bessel's equation, 0 order)
- Hence, $$\boxed{E_z(r) = E_o\ J_o\left(\frac{\omega}{c} r\right)}$$
- Apply boundary condition for conducting walls, $E_z(R) = 0$, therefore $$\boxed{\frac{2\pi f}{c} b = 2.405}$$

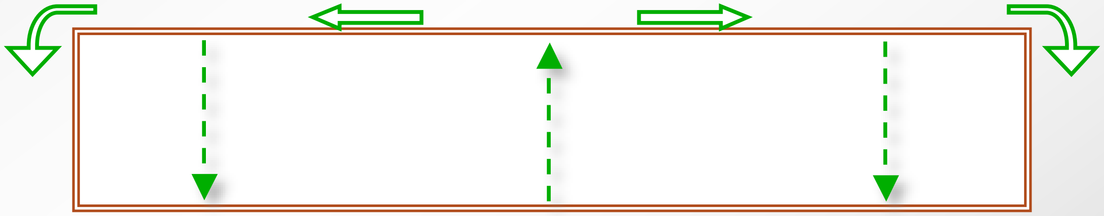
E-fields & equivalent circuit: T_{on1o} mode

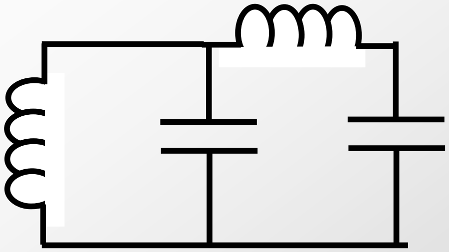




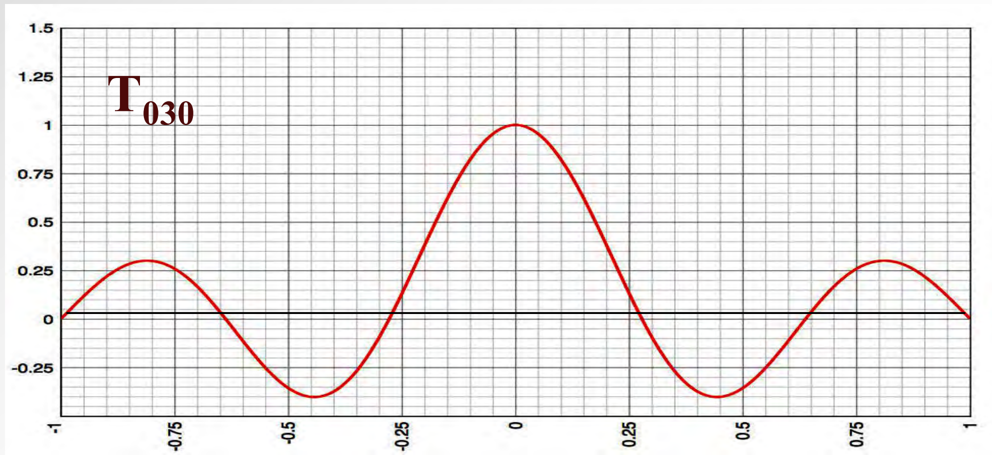
E-fields & equivalent circuits for T_{o2o} modes

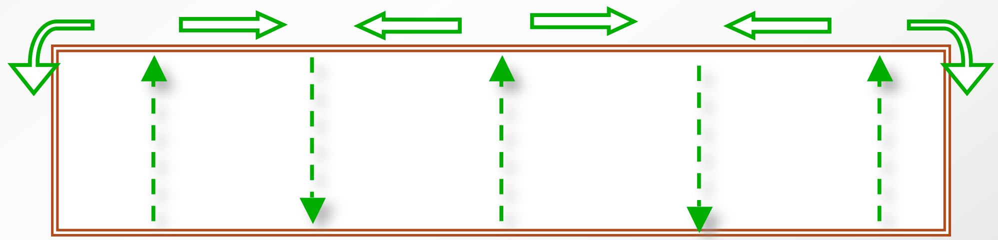


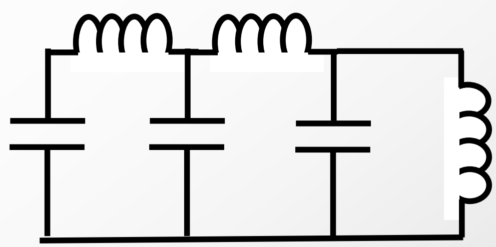




E-fields & equivalent circuits for T_{ono} modes

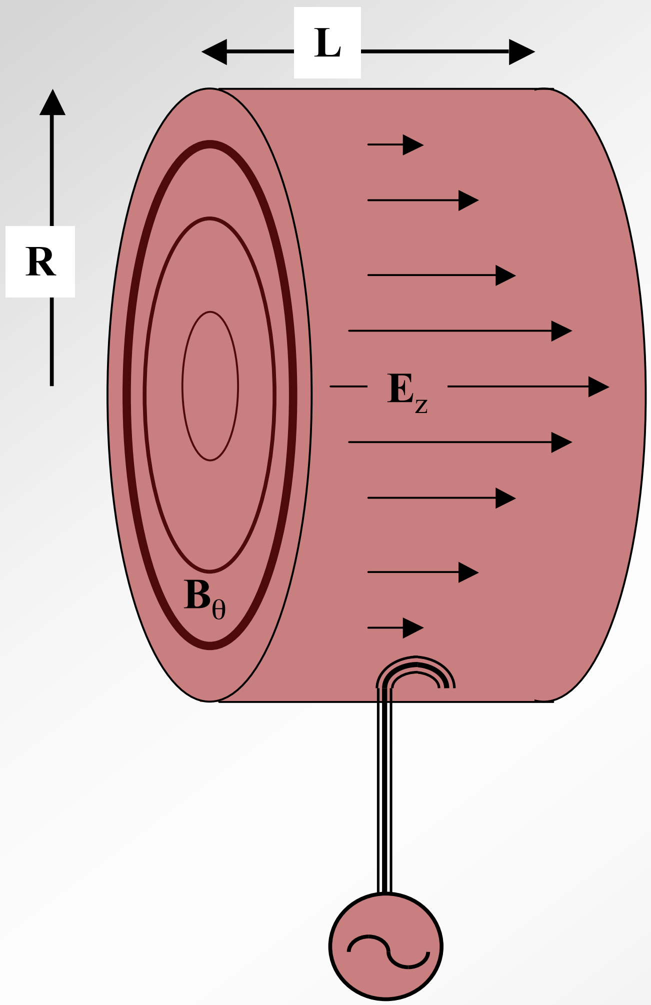






T_{0n0} has n coupled, resonant circuits; each L & C reduced by 1/n

Simple consequences of pillbox model



- Increasing R lowers frequency

  ==> Stored Energy, $\mathcal{E} \sim \omega^{-2}$

- $\mathcal{E} \sim E_z^2$

- Beam loading lowers E_z for the next bunch

- Lowering ω lowers the fractional beam loading

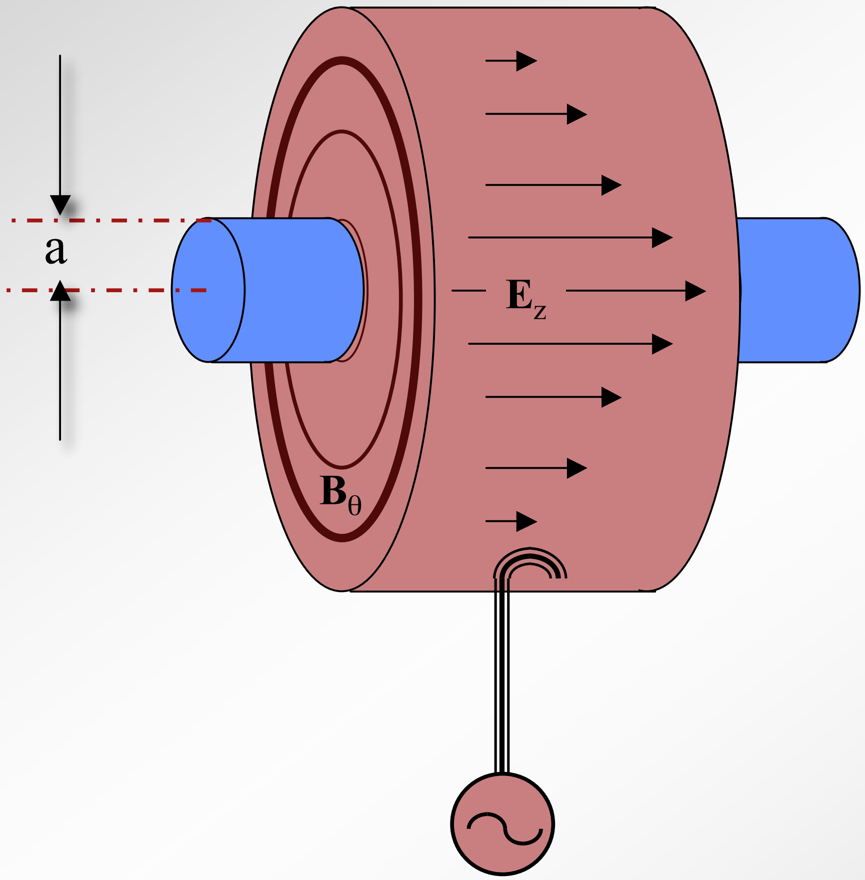
- Raising ω lowers $Q \sim \omega^{-1/2}$

- If time between beam pulses,

  $T_s \sim Q/\omega$

  almost all $\mathcal{E}$ is lost in the walls

The beam tube complicates the field modes
(& cell design)



- Peak E no longer on axis
  - $E_{pk} \sim 2 - 3 \times E_{acc}$
  - FOM = E_{pk}/E_{acc}
- ω_o more sensitive to cavity dimensions
  - Mechanical tuning & detuning
- Beam tubes add length & €'s w/o acceleration
- Beam induced voltages ~ a^{-3}
  - Instabilities