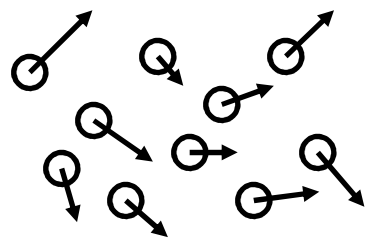
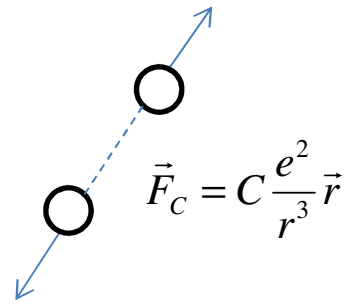
Transverse dynamics, single particle

S. Di Mitri (1.5 hr.)

Magnetic focusing

- Any beam of same-charge particles tend to disperse because of **repulsive Coulomb forces** and initial particles' angular divergence.





- **External transverse focusing** maintains the beam compact and the charge density high. For ultra-relativistic particles, magnetic focusing is more practical and efficient than electric.

  $\vec{F}_L = e\left(\vec{E} + \vec{v} \times \vec{B}\right)$ *is the Lorentz force. To produce the same work of 1 MeV over 1 m, we need E = 1 MV/m or just B = 3 T.*

- An FEL beam delivery system is a sequence of RF and magnetic elements.
  - **Dipole** magnets [$B_y=B_0$] are used in spectrometer lines for beam dump and diagnostics, in magnetic compressors and transfer lines. They determine the beam direction.
  - **Quadrupole** magnets [$B_y=(dB_y/dx)\Delta x$] are in between RF structures, diagnostic stations, transfer lines and undulator. They determine the beam transverse size.
  - **Sextupole** magnets [$B_y=(d^2B_y/d^2x)\Delta x^2$] are rarely used in dispersive regions for linearization of the longitudinal phase space.

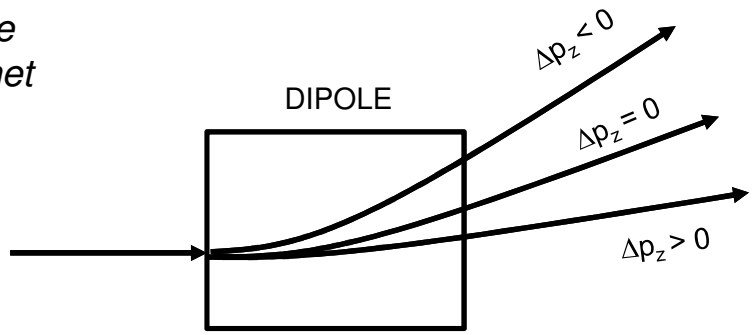
Dipole magnet

- Particles with different longitudinal momentum follow different trajectories (*i.e.*, bending radius) according to:

$$p_z[GeV/c] = 0.2998 \cdot B_y[T] \cdot R[m]$$

- The lateral separation from the reference (*i.e.*, on-energy) trajectory per unit relative energy deviation is the **longitudinal momentum dispersion** function:

Dipole magnet



$$x_\eta(E;s) = \eta_x(s)\frac{\Delta E}{E_0}$$

- When applied to the beam energy spread, η_x detrmines the ***chromatic beam size***. This can be regulated (or made null) along the beam line by controlling η_x.

$$\sqrt{\left\langle x_\eta^2(s)\right\rangle_N} = \left(\eta_x^2(s)\left\langle\frac{\Delta E}{E_0}\right\rangle_N^2\right)^{1/2} = \eta_x(s)\sigma_\delta \equiv \sigma_{x,\eta}(s)$$

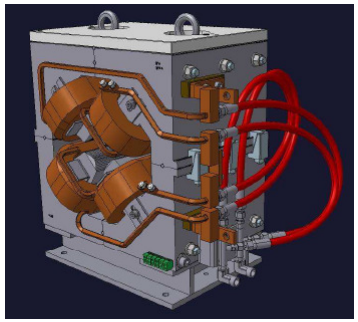
EXERCISE: demonstrate the aforementioned relationship between p_z and B_y. *Hint*: use equation motion for the radial coordinate.

Quadrupole magnet

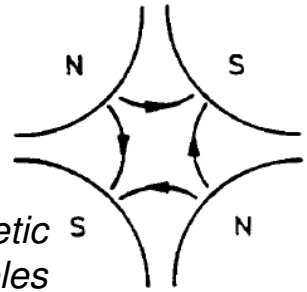
- A quadrupole magnet implies a **transverse force** that is ***linear*** with the particle's **transverse displacement** from the quadrupole magnetic axis.

$$x''(s) = \frac{e}{p_z}\frac{dB_y}{dx}(s)x(s) \equiv kx$$

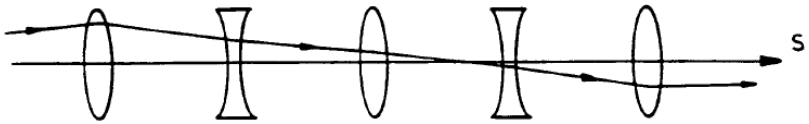
$$k\left[m^{-2}\right] = 0.2998\frac{g[T/m]}{p_z[GeV/c]}$$



Normally-oriented 1uadrupole magnet



Magnetic poles



*An alternating series of focusing and defocusing quads (lenses), not too far each other, leads to **overall** focusing. Same in the other plane.*

- The principle of Alternating Strong Focusing applies to rings as well as to linacs. In practice, use many split magnets of alternating focusing sign to confine the beam in the accelerator. The total **net effect** is ***focusing*** in ***both transverse planes***.

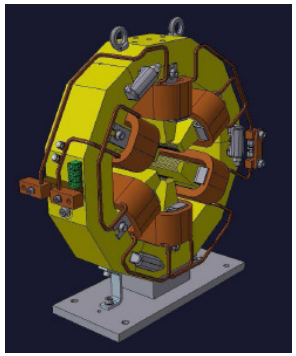
- If we consider the motion of the beam centroid into a **displaced quadrupole magnet**, we find that the beam is (cohrently) **kicked by**:

$$x' = klx$$

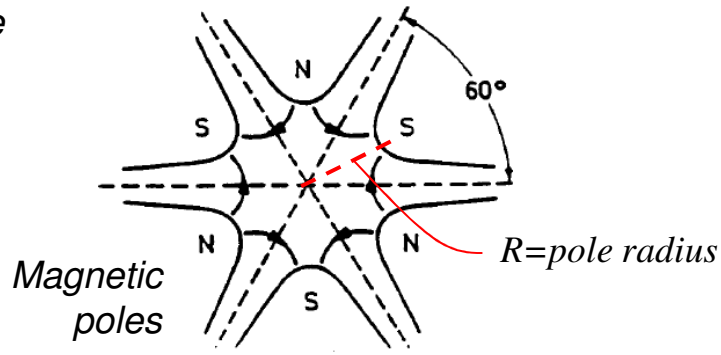
EXERCISE: demonstrate the aforementioned relationship for the linear focusing. *Hint*: start from Lorentz force. Verify that a quadrupole focusing in one plane is *defocusing* in the other.

Multi-pole field expansion

- Higher order magnets (*e.g.*, sextupoles) introduce **nonlinear focusing**, *i.e.* the restoring force goes like x^q, with $q \geq 2$. When used in dispersive regions, they **couple x_β and x_η**.



Sextupole magnet



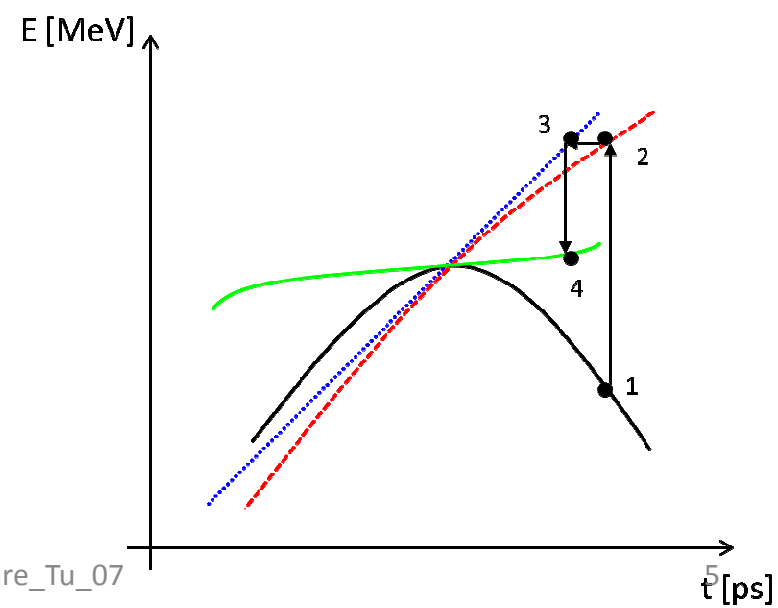
Multipolar field expansion:

$$B_y(x) = \sum_0^n b_n \left(\frac{x}{R}\right)^n_{y=0}$$

$$b_n = \frac{1}{n!}\left(\frac{\partial^n B_y}{\partial x^n}\right)_{y=0} R^n$$

- *Sextupoles* used in *dispersive regions* and in the presence of *correlated energy spread*, can be used to manipulate (*e.g.*, linearize) the longiotudinal phase space.

1. RF curvature
2. Off-crest acceleration (adds linear E-chirp)
3. Sextupole in dispersive region
4. Off-crest acceleration (removes linear E-chirp)



Hill's equation

$$\gamma m_e\left(\ddot{r}-\dot{\theta}^2 r+\frac{\dot{\gamma}}{\gamma}\dot{r}\right)=-e\left(\vec{v}\times\vec{B}\right);$$

- r → x
- expand B up to ***first order*** in x
- d/dt → d/ds
- consider an off-momentum $p_z = \gamma m_e v_z = p_{z,0}(1+\delta)$

$$\underbrace{x''(s)+\frac{\gamma'(s)}{\gamma(s)}x'(s)+\left[k(s)(1-\delta)-\frac{1}{R(s)^2}\right]x(s)}=\frac{\delta}{R(s)}$$

x_β, solution of the homogeneous e.o.m. describes the **betatron oscillations** (below, on-energy and with no acceleration)

x_η, solution of the complete e.o.m. describes the **long. momentum dispersion**, η_x.

SINGLE PARTICLE, LINEAR β-MOTION

$$x_\beta(s)=\sqrt{2J_x\beta_x(s)}\cos\Delta\mu_x \qquad x_\beta'(s)=\frac{dx_\beta}{ds}=-\sqrt{\frac{2J_x}{\beta_x(s)}}\left[\alpha_x(s)\cos\Delta\mu_x+\sin\Delta\mu_x\right]$$

where: $\alpha_x=-\frac{1}{2}\frac{d\beta_x}{ds},\ \gamma_x=\frac{1+\alpha_x^2}{\beta_x},\ \Delta\mu_x(s)=\int_0^s\frac{1}{\beta_x(s')}ds'$

Parameters of **Courant-Snyder** (also **Twiss** functions)

and $$2J_x=\gamma_x x_\beta^2+2\alpha_x x_\beta x_\beta'+\beta_x x_\beta'^2$$

Single particle **Courant-Snyder invariant**

Beam emittance

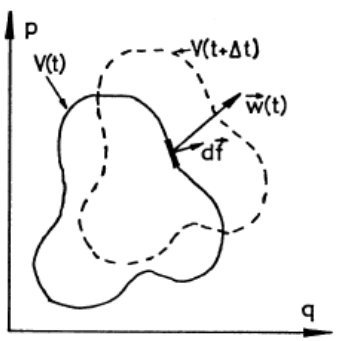
❖ As the particle moves ***along the line*** in the ***linear approximation***, it maps an ellipse in the phase space (x,x') whose axes and slope change point to point, but whose area = $2J_{x,y}(s)$ remains ***constant***. This is the particle ***C-S invariant***.

❖ We now consider the ***ensemble*** of ***particles*** at an ***arbitrary point*** of the line. For a linear motion, particles lye on omothetic ellipses. ***Liouville's theorem*** states that in the absence of "frictional" forces (dissipative or diffusion terms ∝ x' in Hill's eq.), the area of the ***beam ellipse*** is a ***constant*** of the motion. This is the ***beam geometric emittance***.

- Important: Liouville's theorem (area preservation) is still valid for a ***nonlinear motion***!

$d\vec{f}$ ... vector of surface element

$\vec{w}(t)$ ... phase space velocity of surface elemen



$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix}$$

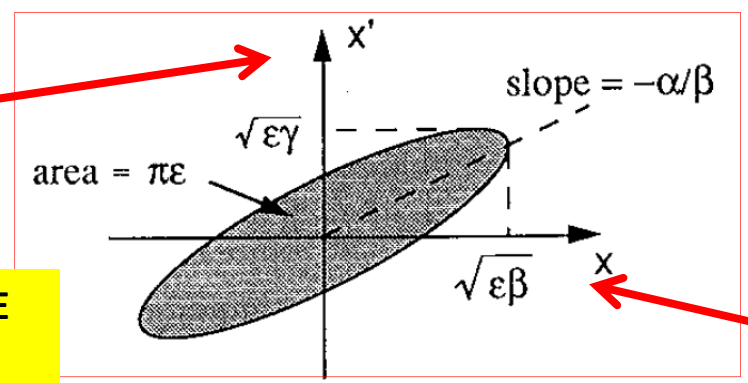
$$\frac{dV(t)}{dt} = \int \vec{w}\cdot d\vec{f} = \int (\vec{\nabla}\cdot\vec{w})\,dv = \int \left(\frac{\partial}{\partial q}\dot{q} + \frac{\partial}{\partial p}\dot{p}\right) dv = 0$$

surface integral → volume integral (Gauss Theorem)

$$\left(\frac{\partial^2 H}{\partial q \partial p} - \frac{\partial^2 H}{\partial p \partial q}\right) = 0$$

(Hamilton)

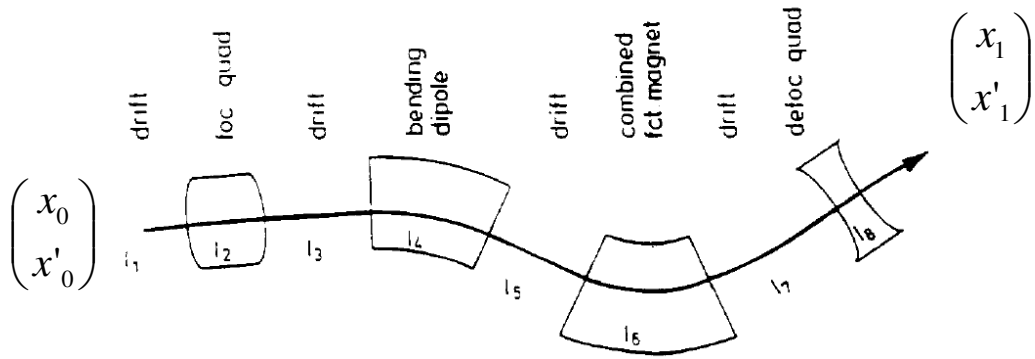
Beam angular divergence



BEAM PHASE SPACE AREA

Beam size

Principal Trajectories



Transfer line made of quadrupoles and dipoles

QUAD

$$M_{Q,x} = \begin{pmatrix} \cos\left(l_q\sqrt{k}\right) & \frac{1}{\sqrt{k}}\sin\left(l_q\sqrt{k}\right) & 0 \\ -\sqrt{k}\sin\left(l_q\sqrt{k}\right) & \sqrt{k}\cos\left(l_q\sqrt{k}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

SBEND

$$M_{D,x} = \begin{pmatrix} \cos\theta & R\sin\theta & R(1-\cos\theta) \\ -\frac{1}{R}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix}_{\text{sector}}$$

From Hill's eq. (no acceleration):

$$\begin{pmatrix} x_0 \\ x'_0 \\ \delta_0 \end{pmatrix} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x'_1 \\ \delta_0 \end{pmatrix}$$

- Transport matrix (C,S = "principal trajectories"; D = η).
- C, S, C', S', D, D' depend on the geometric and magnetic properties of the element.
- Their general form is:

$$C(s) = \cos\left(s\sqrt{k+\frac{1}{R^2}}\right)$$

$$S(s) = \frac{1}{\sqrt{k+\frac{1}{R^2}}}\sin\left(s\sqrt{k+\frac{1}{R^2}}\right)$$

EXERCISE: determine the transport matrix for a quadrupole magnet in *thin lens approximation*, that is $l_q \rightarrow 0$ but $kl_q = const.$

Simplectic matrices

I. **Principal trajectories** (PTs hereafter) are defined with some conventional initial conditions: C(0)=1, S(0)=0, C'(0)=0, S'(0)=1, which make the det(M(0)) ≡ W(0) = 1.

II. Each PT satisies Hill's eq. (by def.). Now assume to have an additional **frictional term** ∝ C', S' and sum the two equations with proper multiplication factors:

$$\begin{array}{r} -S\cdot \\ C\cdot \end{array}\begin{cases} C''+\varsigma C'+KC=0 \\ S''+\varsigma S'+KS=0 \end{cases}$$

$$\overline{(CS''-SC'')+\varsigma(CS'-SC')+K\underbrace{(SC-CS)}_{0}=0;} \quad \Rightarrow \quad W'+\varsigma W=0$$

III. Last eq.states **W(s) = 1 ∀s ⇔ ζ = 0**. Group of **simplectic** matrices satisfies **W = 1** for any algebric manipulation of its members.

IV. Consider the vector product *A = dx × dx'* (area in the phase space).

- It evolves according to the linear transformation:

$$d\vec{x}\cong\left(\frac{dx}{dx_0}dx_0,\frac{dx}{dx'_0}dx'_0\right)\equiv\left(Cdx_0,Sdx'_0\right),\ \ d\vec{x}'\cong\left(\frac{dx'}{dx_0}dx_0,\frac{dx'}{dx'_0}dx'_0\right)\equiv\left(C'dx_0,S'dx'_0\right)$$

- We find **A = W· A_0**, that is a transport matrix with ***unitary determinant* (*e.g.*, simplectic) preserves the *phase space area* in the absence of frictional forces**.

Stable transport

Pictures courtesy of S. Milton

1. Impose equality of the the C-S- invariant for x_2, x_1.
2. Use x_2=M(x_1) in terms of Principal Trajectories and substitute into point 1.
3. We can determine M in terms of the Twiss functions:

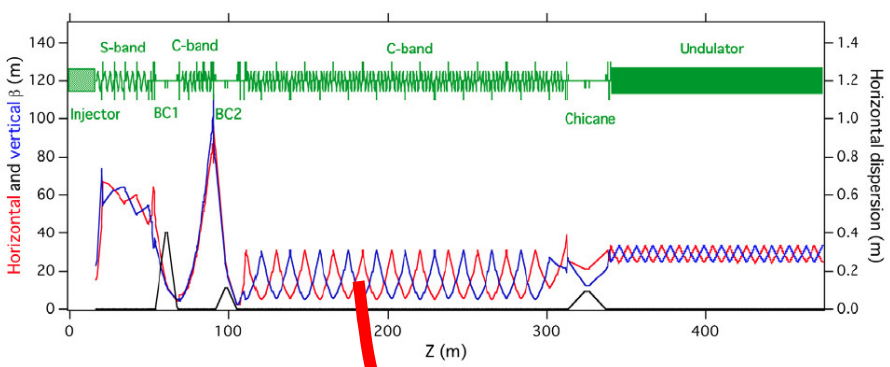
$$M(s_1 \to s_2) = \begin{bmatrix} \cos\Delta\mu_{12} + \alpha_1 \sin\Delta\mu_{12} & \beta_1 \sin\Delta\mu_{12} \\ -\gamma_1 \sin\Delta\mu_{12} & \cos\Delta\mu - \alpha_1 \sin\Delta\mu_{12} \end{bmatrix}$$

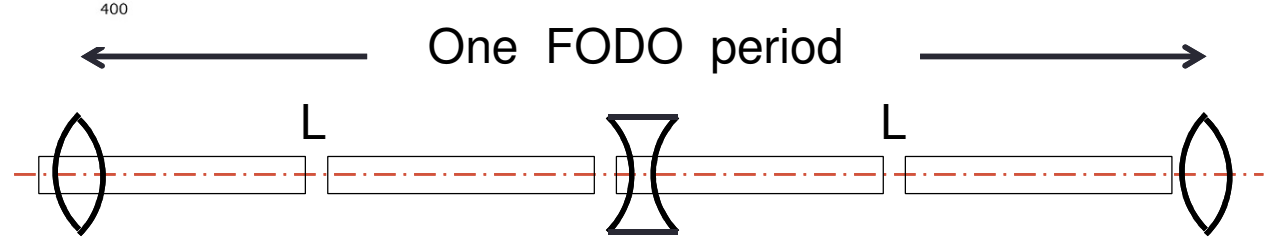
Assume ***standard FODO lattice*** along the main linac and Quads in thin lens approximation.

Stability implies (general property):

$$\left|\frac{1}{2}TrM\right| = \left|\cos\Delta\mu_{12}\right| < 1 \Rightarrow \frac{L^2}{2f_q^2} = \left(kl_q L\right)^2 < 2$$

$$\frac{\beta_{\max}}{\beta_{\min}} = \frac{1+\sin\frac{\Delta\mu_{12}}{2}}{1-\sin\frac{\Delta\mu_{12}}{2}}$$





Which emittance?

IDEAL ACCELERATOR

1. Acceleration gives a term $\propto x'$ in Hill's eq. ($x' \equiv \Delta p_x/p_z \sim 1/\beta\gamma$). Consider *canonical* coordinates (x, p_x) so that $J_x(x,x') \rightarrow J_x(x, p_x)$: The *transverse momentum* is *not* affected by *longitudinal acceleration*, so that the **normalized emittance $\varepsilon^L_{n,x} = \beta\gamma\varepsilon^L_x$** is preserved.
2. Normalized emittance is affected by frictional forces at very low energies (e.g., short-range space charge forces). Once these can be neglected (>50 MeV for ~1 nC high brightness beams), Liouville's theorem predicts that it is preserved all along the accelerator. Accordingly, the **geometric emittance** shrinks like **$\varepsilon^L_x \sim 1/\beta\gamma$**.

REAL ACCELERATOR

3. Unfortunately, frictional forces due to emission of synchrotron radiation (*e.g.*, in dipoles) enlarge the Liouville emittance. Might be the same for other short-range e.m. interactions with boundary. These are called "**collective effects**".

MORE AND MORE REAL...

4. Can single particle effects, like **nonlinear magnetic focusing**, degrade the emittance?
   - Liouville theorem still holds in the presence of nonlinear focusing.
   - However, we do not really measure the beam canonical phase space area, just particles density distribution in the phase space. The distribution statistical parameters (*e.g.*, mean, std, etc..) are used to evaluate a **statistical emittance**.
   - Statistical parameters of the distribution *are* affected by nonlinear forces, so the statistical emittance can be degraded, in contrast to Liouville's theorem.

Statistical emittance

Pictures courtesy of M. Ferrario

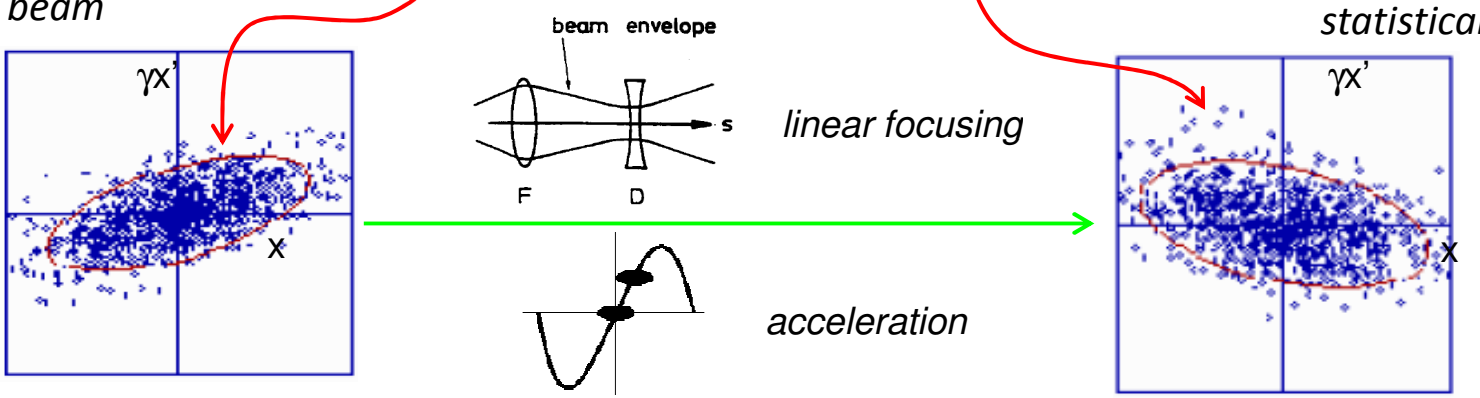
- Statistical geometric emittance, $\varepsilon_x(P)$, is a measure of the *spread* (in transverse position and angular divergence) of a given *fraction* P of beam particles in the phase space (x,x'). Thus, it always relates to the given fraction of charge that is sampled in the phase space.

- Connection between beam C-S parameters and statistical emittance is given by the ***beam matrix***:

$$\varepsilon = \sqrt{\det \varepsilon \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}} \equiv \sqrt{\det \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle xx' \rangle & \langle x'^2 \rangle \end{pmatrix}}$$

C-S parameters map the equivalent beam ellipse.

Beam second order moments map the statistical emittance



Statistical normalized emittance

$$\varepsilon_{n,x} = \gamma \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$

Beam size and divergence

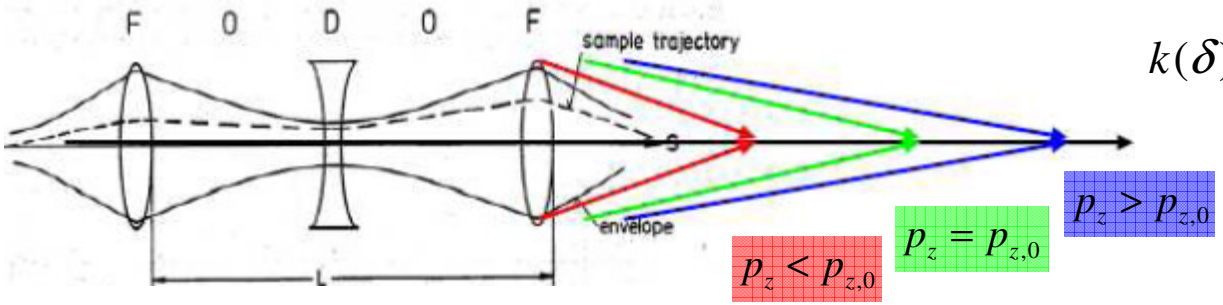
- In the presence of dispersion: $\begin{cases} x = x_\beta + x_\eta = \sqrt{2J_x \beta_x} + \eta_x \delta \\ x' = x'_\beta + x'_\eta = \sqrt{2J_x \gamma_x} + \eta'_x \delta \end{cases}$ STD $\begin{cases} \sigma_x = \sqrt{\varepsilon_x \beta_x + (\eta_x \sigma_\delta)^2} \\ \sigma'_x = \sqrt{\varepsilon_x \gamma_x + (\eta'_x \sigma_\delta)^2} \end{cases}$

Chromatic aberration

Pictures courtesy of
H.Briant, CAS 94-01

The most important source of ε_x degradation from single particle dynamics is *chromatic aberration,* that is the phase advance depends on the particle energy, $\Delta\mu=\Delta\mu(\delta)$.

In practice, the effective quadrupole strength acting on the particle depends on its energy:



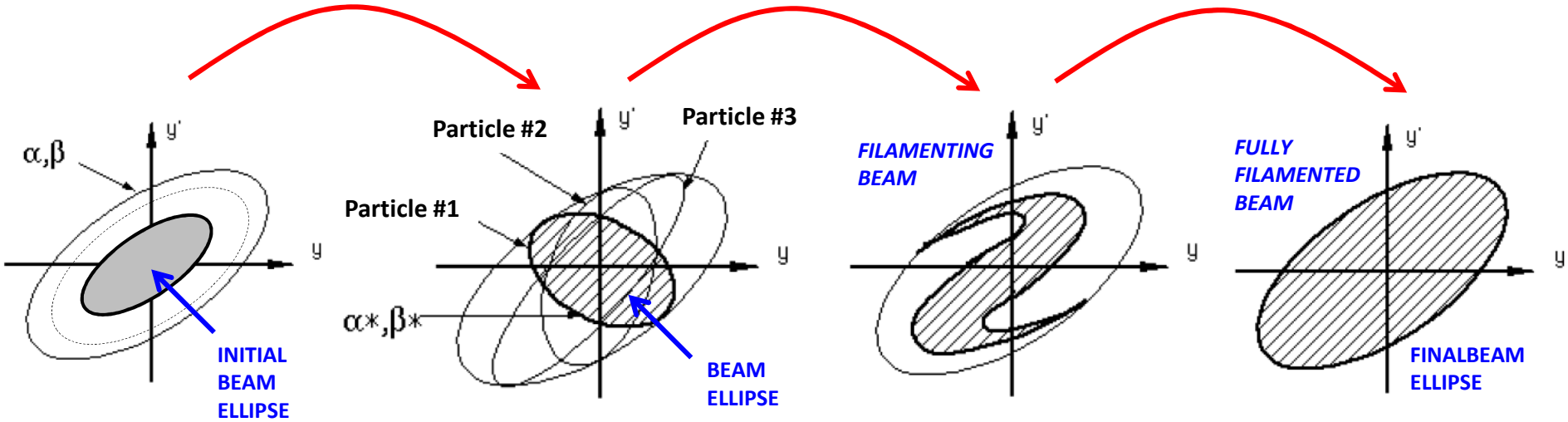
$$k(\delta) = \frac{eg}{p_z} \cong \frac{eg}{p_{z,0}}\left(1+\frac{\Delta p_z}{p_{z,0}}\right) = k_0 + k_0\delta$$

Particles at different energies are mapped onto different ellipses.

Chromaticity,
after one β-period

After many "rotations"

Emittance growth



Aberration-induced emittance growth

We assume to perturb the particle distribution with a local error kick $Q^2 = \langle \Delta x'^2 \rangle$. The *istantaneously perturbed* emittance can be estimated with the beam matrix:

$$\tilde{\varepsilon} = \sqrt{\det\begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle xx' \rangle & \langle x'^2 \rangle + \langle \Delta x'^2 \rangle \end{pmatrix}} = \sqrt{\det \varepsilon \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma + Q^2 \end{pmatrix}} = \sqrt{\varepsilon^2 \left(1 + \frac{\beta Q^2}{\varepsilon}\right)}$$

If the perturbation is small enough, we can write: $\frac{\Delta\varepsilon}{\varepsilon} \cong \frac{1}{2}\frac{\beta}{\varepsilon} Q^2$

The *chromatic kick* error in a quadrupole is $\Delta x' = k\delta l\, x$, hence:

$$\boxed{\frac{\Delta\varepsilon}{\varepsilon} \cong \frac{1}{2}(\beta k l)^2 \sigma_\delta^2}$$

QUAD. CHROMATIC ABERRATION

Now assume a sextupolar field component in a non-dispersive region (real magnets always have magnetic field errors due to finite pole size, manufacturing errors, etc.).

The *geometric kick* error by a sextupole is $\Delta x' = m\, l\, x^2$, hence:

$$\boxed{\frac{\Delta\varepsilon}{\varepsilon} \cong \frac{1}{2}(\beta m l)^2 \sigma_x^2}$$

SEXT. GEOMETRIC ABERRATION

EXERCISE: evaluate the emittance growth induced by a sextupolar chromatic aberration (beam size is assumed to be dominated by the chromatic contribution).

Magnetic field tolerances

- Every real magnet includes ***systematic*** and ***random*** **field errors**, both due to the finite magnet dimension and mechanical tolerances. The formers are constrained by symmetries of the nominal field pattern. The latters may cover all orders of the field expansion.
- The magnets should be manufactured in a way that field components higher than the nominal should be small enough to avoid beam emittance dilution. Same approach as before applies. We assume perfectly aligned magnets.

Quadrupole component (n=1) in a Dipole magnet (n=0):

$$k_{1,0} = \frac{eg_{1,0}}{p_{z,0}} = \frac{\theta}{Rl}\left|\frac{b_1}{b_0}\right| \Rightarrow Q_{1,0} = \frac{\theta}{R}\eta\delta\left|\frac{b_1}{b_0}\right| \Rightarrow \left.\frac{\Delta\varepsilon}{\varepsilon}\right|_{1,0} \cong \frac{\beta}{2\varepsilon}\left(\frac{\theta}{R}\eta\sigma_\delta\frac{b_1}{b_0}\right)^2 \le 1\% \Rightarrow \boxed{\left|\frac{b_1}{b_0}\right| \le \frac{1}{\theta}\frac{R}{\eta\sigma_\delta}\sqrt{\frac{\Delta\varepsilon}{\varepsilon}\frac{2\varepsilon}{\beta}}}$$

Quadrupole-like strength in a dipole

Quadrupole-like *chromatic* kick error

RMS emittance growth (tolerance)

Magnetic field tolerance (chromatic aberration)

Sextupole component (n=2) in a Quadrupole magnet (n=1):

$$k_{2,1} = \frac{em_{2,1}}{p_{z,0}} = \frac{2k_1}{R}\left|\frac{b_2}{b_1}\right| \Rightarrow Q_{2,1} = \frac{2k_1 l}{R}x^2\left|\frac{b_2}{b_1}\right| \Rightarrow \left.\frac{\Delta\varepsilon}{\varepsilon}\right|_{2,1} \cong \frac{\beta}{2\varepsilon}\left(\frac{2k_1 l}{R}x^2\frac{b_2}{b_1}\right)^2 \le 1\% \Rightarrow \boxed{\left|\frac{b_2}{b_1}\right| \le \frac{1}{k_1 l}\frac{R}{\varepsilon\beta}\sqrt{\frac{\Delta\varepsilon}{\varepsilon}\frac{2\varepsilon}{\beta}}}$$

Sextupole-like strength in a dipole

Sextupole-like kick error

RMS emittance growth (tolerance)

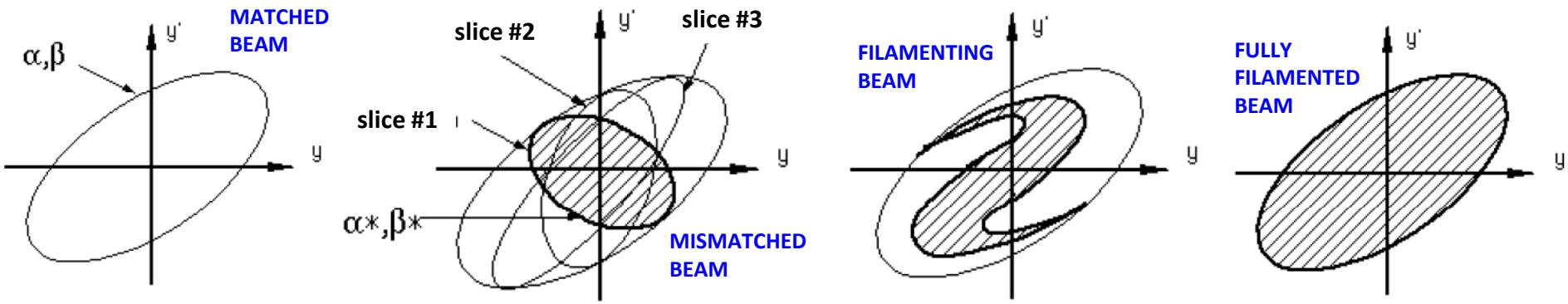
Magnetic field tolerance (geometric aberration)

Optics mismatch

Pictures courtesy of
H.Briant, CAS 94-01

Same emittance growth as due to aberrations is expected to happen if the beam is ***injected*** into the line with an ***already mismatched ellipse***. We can distinguish two practical cases:

1) The entire beam phase space is mapped onto an ellipse which is different from the design one. Thus, particles may explore ***wider phase space amplitudes*** that excite aberrations.
2) Longitudinal portions of the bunch –***slices***– are mismatched each other. Thus, the projected phase space area is wider than that for the entirely matched beam (see below).



Mismatch parameter, *B* (normal linac operation usually requires B < 1.1):

$$B = \frac{1}{2}\left(\beta\tilde{\gamma} - 2\tilde{\alpha}\alpha + \gamma\tilde{\beta}\right) \geq 1$$

B also describes the ***RMS emittance dilution*** due to filamentation of a mismatched beam, independently from the specific charge distribution in (x,x'): $\tilde{\varepsilon} = B\varepsilon$

Note that RMS emittance can still be approximately preserved (B≈1), while the *total* emittance (100% particles in phase space) is not. The total emittance dilution factor is:

$$\zeta = B + \sqrt{B^2 - 1}, \quad \tilde{\varepsilon}_{100\%} = \zeta\varepsilon_{100\%}$$

Optics sensitivity to focusing errors

- We start from a general form of the ***final* optics mismatch** due to generic *focusing errors* $k_i\tau_i$ distributed along the accelerator:

$$B(\tau) \cong 1 + \frac{1}{2}\left[\left(\sum_{i=1}^{N} k_i\beta_i l_i\tau_i \cos(2\Delta\mu_i)\right)^2 + \left(\sum_{i=1}^{N} k_i\beta_i l_i\tau_i \sin(2\Delta\mu_i)\right)^2\right]$$

- Consider one error (quadrupole) at the time and average over $\Delta\mu$:

$$B_q(\tau) \cong 1 + \frac{1}{2}(k\beta l\tau)_q^2 = 1 + \xi_q,$$

where ξ can be defined as the optics **sensitivity to quadrupole errors**. If *T=B–1* (e.g., 5%) is the *tolerance* on the *final* optics mismatch induced by *N* error kicks (e.g., 100), then *on average* ξ_q should be smaller than $T/\sqrt{N}$ (0.5%) at each quadrupole location.

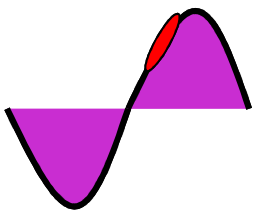
- Notice that if $k\tau$ is a focusing error that can lead to emittamce growth (e.g., $\tau=\delta$), then ξ_q and T become, resepctively, the **sensitivity** and **tolerance** on the final **emittance dilution**:

$$\frac{\Delta\varepsilon}{\varepsilon} \cong \frac{1}{2}\sqrt{\sum_{i=1}^{N}(\beta_i k_i l_i \sigma_{\delta,i})^4} \equiv \sqrt{\sum_{i=1}^{N}\xi_i^2} \le T$$

- **Optimization loops** in codes can be used to search quadrupole strengths that both satisfy all optics constraints and, at the same time, **minimize the optics sensitivty to focusing errors**.

Large energy spread

❖ A rather large energy spread (~1% RMS) is typically imposed to the e-beam with RF off-crest phasing to execute magnetic bunch length compression. Such an energy spread is usually produced with a few accelerating structures, starting from ~0.1% level. After compression, it is adiabatically damped by on-crest acceleration along tens of meters.

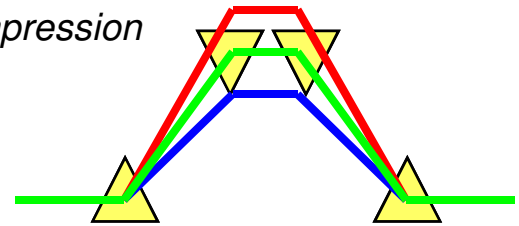


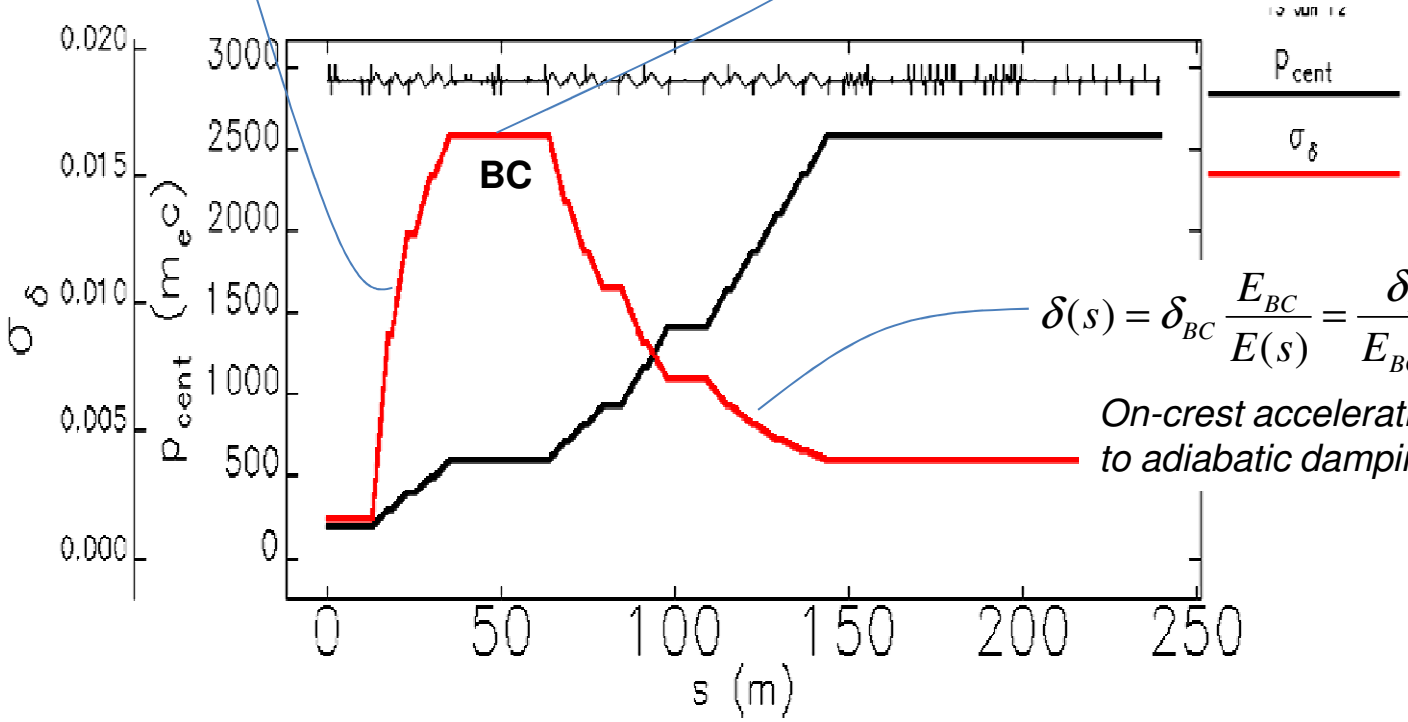
Off-crest acceleration generates energy chirp

$$h = \frac{2\pi}{\lambda_{RF}} \frac{eV_0 \cos\varphi_{RF}}{E_0 + eV_0 \sin\varphi_{RF}}$$

Bunch length compression

$$\Delta z = R_{56} \Delta E/E$$



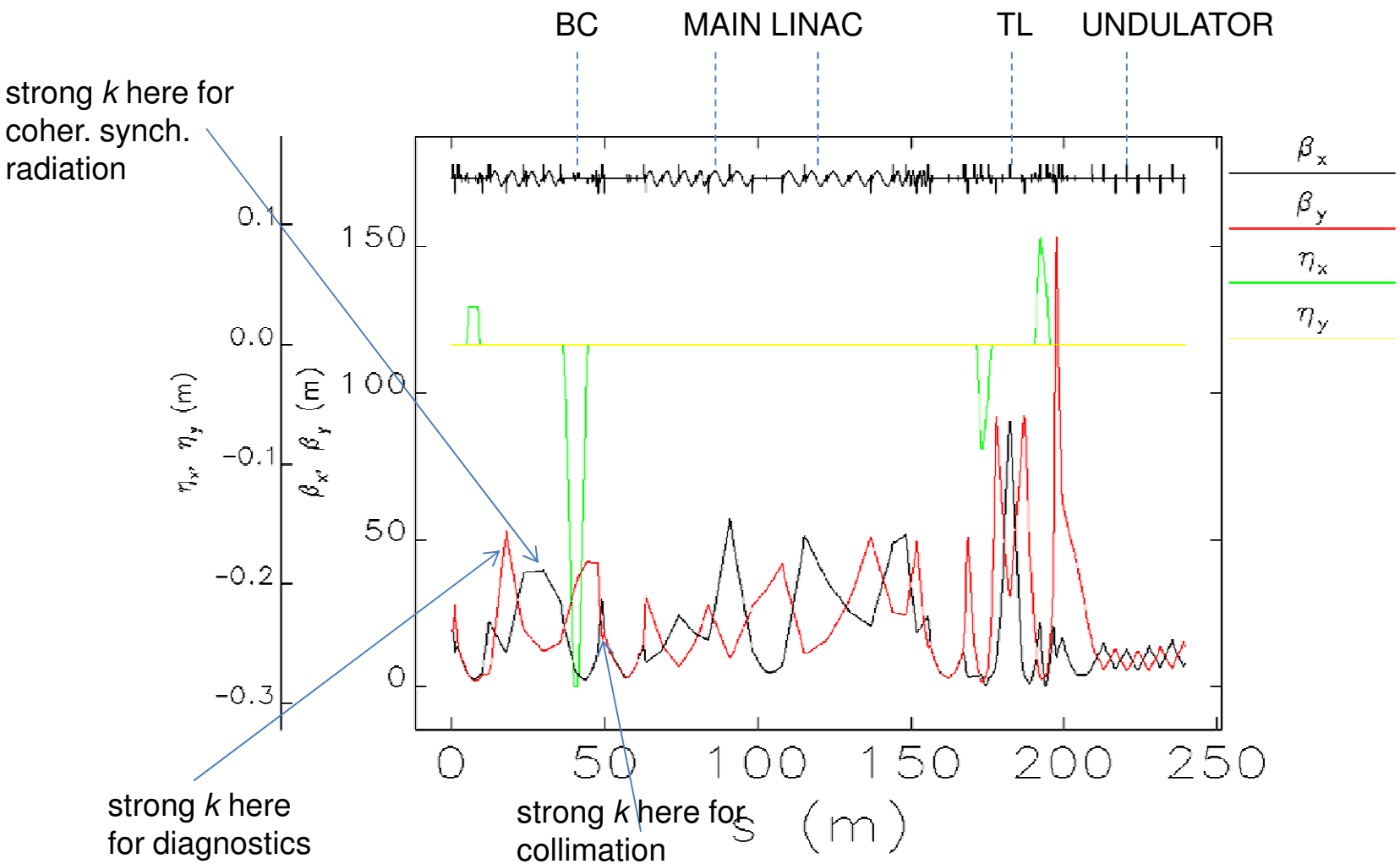


$$\delta(s) = \delta_{BC} \frac{E_{BC}}{E(s)} = \frac{\delta_{BC} E_{BC}}{E_{BC} + eGs}$$

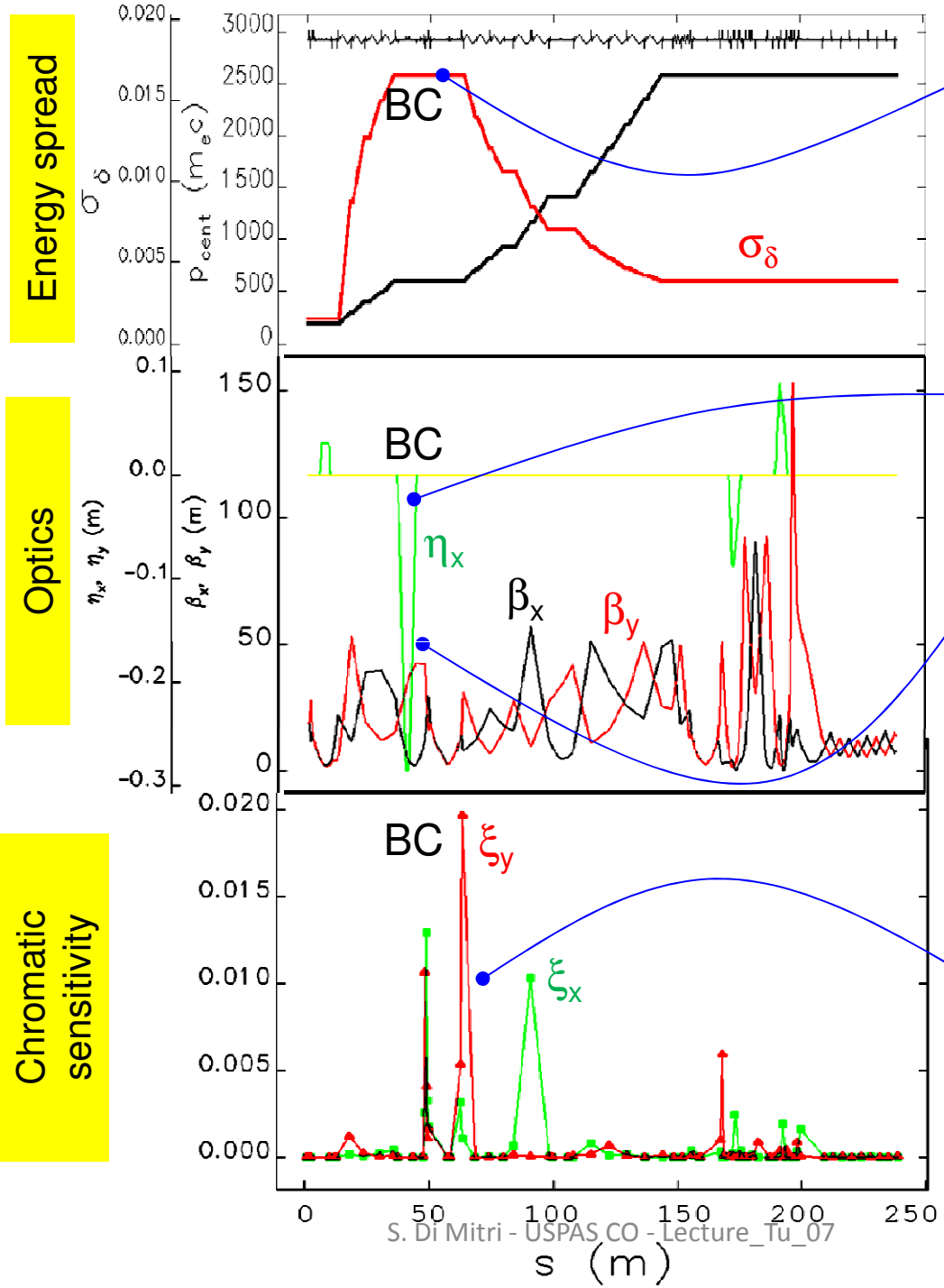
On-crest acceleration leads to adiabatic damping

Strong focusing

❖ At the same time, strong focusing ($kl \geq 0.5\text{m}^{-1}$) is adopted for particles confinement in small iris structures, collimators, to optimize the resolution of diagnostic stations and to counteract collective effects.



Put all together……



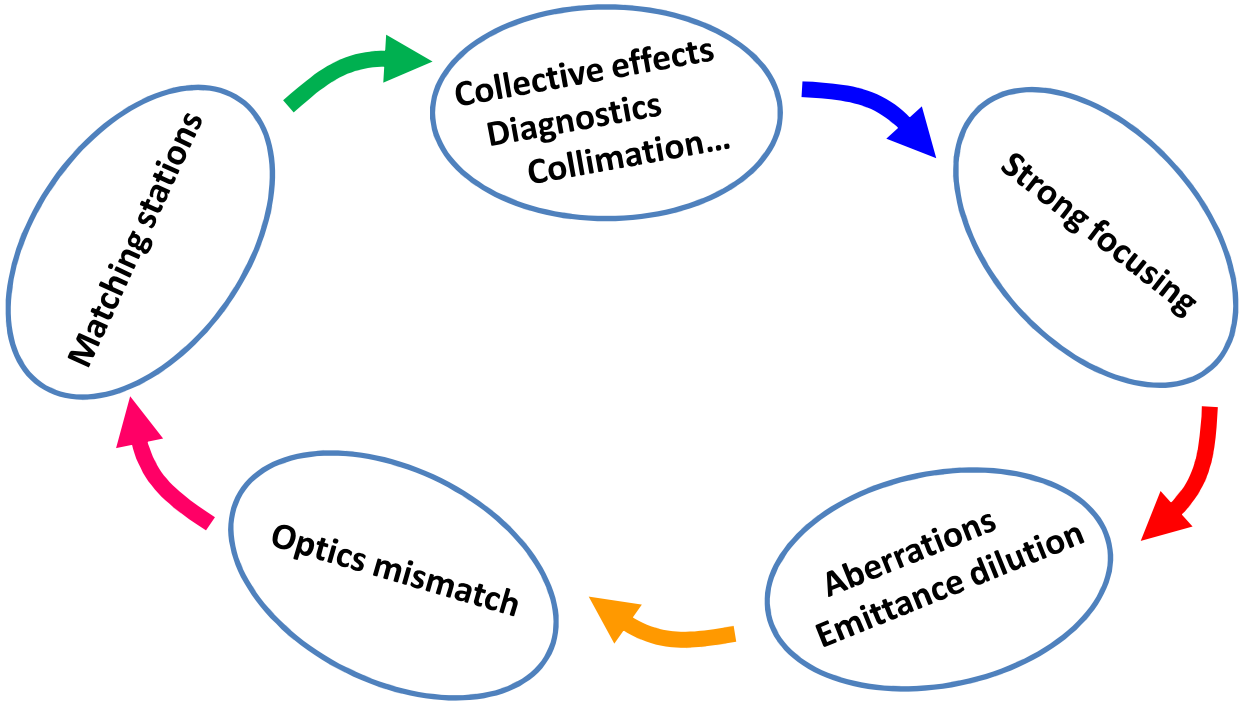
σ_δ=1.7% over σ_t=2.8ps gives an energy chirp h=20m^{-1}.

R_{56}=-40mm leads to C=5.

Strong focusing is needed after BC to recover large β-oscillations and to re-match the beam to the next diagnostic and colimation station.

The sensitivity tends to follow the β-amplitude.
Particularly high after the compressor.

Optics matching: why, where.



- **Optics matching stations** are usually placed:
  - at the ***injector exit***, because SC-forces make the beam optics less predictable;
  - in front of ***laser heater*** for optimum e-beam/laser transverse pverlap;
  - in front of ***diagnostic stations*** to improve the measurement resolution;
  - in front of ***magnetic compressors*** to counteract CSR-emittance growth;
  - in front of ***undulator***, for optimum e-beam/photons coupling, thus to maximize the FEL amplification.

RF focusing

Assume a TW-CG structure, transit time factor = 1. E_z has now explicit radial dependence. Maxwell eqs. for t-dependent e.m. field:

$$\nabla\cdot\vec{E} = \frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{\partial E_z}{\partial z} = 0$$

$$(\nabla\cdot\vec{B})_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\varphi) = \frac{1}{c^2}\frac{\partial E_z}{\partial t}$$

$$\Rightarrow \quad E_r \cong -\frac{r}{2}\frac{\partial E_z}{\partial z} \qquad B_\varphi \cong \frac{r}{2c^2}\frac{\partial E_z}{\partial t}$$

and also make us of:

$$dE(z,t) = \frac{\partial E(z,t)}{\partial z}dz + \frac{\partial E(z,t)}{\partial t}dt$$

$$\Downarrow$$

$$\frac{\partial E_z}{\partial z} = \frac{dE_z}{dz} - \frac{\partial E_z}{\partial t}\frac{dt}{dz}$$

$$E_z = E_{z,0}\cos(\phi);\ \ \phi = k_z ct$$

In conclusion:

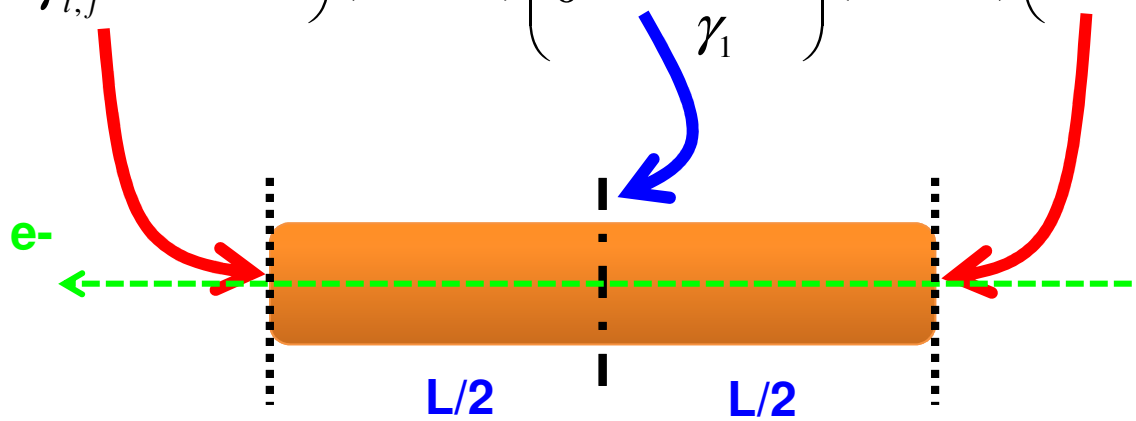
$$F_r = q(E_r - \dot{z}B_\theta) = -\frac{q}{2}r\left[\frac{\partial E_z(z,t)}{\partial z} - \frac{\beta_z}{c}\frac{\partial E_z(z,t)}{\partial t}\right] = -\frac{q}{2}r\left[\frac{d}{dz} - \frac{k}{\beta\gamma^2}\frac{\partial}{\partial\phi}\right]E_z(z,\phi)$$

- Neglect term ~γ^2 and consider a ***static*** $E_{z,0}$ through a gap long l_g: $F_r = -\frac{qE_{z,0}r}{2l_g} \approx -\frac{(qE_{z,0})^2}{2\beta\gamma m_e c^2}r$
- For $E_z = E_{z,0}\cos\phi$ at the structure's ***edges***: $\Delta r' = \frac{\Delta p_r}{p_z} \cong \frac{F_r(\phi)dt}{p_z} \cong \mp\frac{qE_{z,0}\cos(\phi)}{2\beta_{i,f}^2\gamma_{i,f}m_e c^2}r$
- Previous case for a ***cell-to-cell*** focusing model gives: $F_{r,eff} = \frac{\eta(\phi)}{4}\frac{(qE_{z,0})^2}{2\beta_i\gamma_i m_e c^2}r$ $\quad \eta(\phi)\approx 0, TW$ $\quad \eta(\phi)\approx 1, SW$
- Term ~γ^2 provides ***RF phase*** focusing : $F_r(\phi) = -\frac{qk_z r}{2\beta\gamma^2}E_{z,0}\sin(\phi)$

Transport matrix

- Cell-to-cell (also "ponderomotive" or "body-focus) and edge focusing describe the fringe field effect inside and at the edge of the structure, respectively.
- In the following, we will consider TW structures, at energies > 100 MeV.
- Transport matrix for acceleration with pseudo-canonical coordinates (x, x') is *not* simplectic $\Rightarrow$ automatically includes *adiabatic damping*.

$$\begin{pmatrix} x_1 \\ x'_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{qE_{z,0}\cos(\phi)}{2\gamma_{i,f}mc^2} & 1 \end{pmatrix} \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{\gamma_0}{\gamma'}\ln\frac{\gamma_1}{\gamma_0} \\ 0 & \frac{\gamma_0}{\gamma_1} \end{pmatrix} \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{qE_{z,0}\cos(\phi)}{2\gamma_{i,f}mc^2} & 1 \end{pmatrix} \bullet \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$



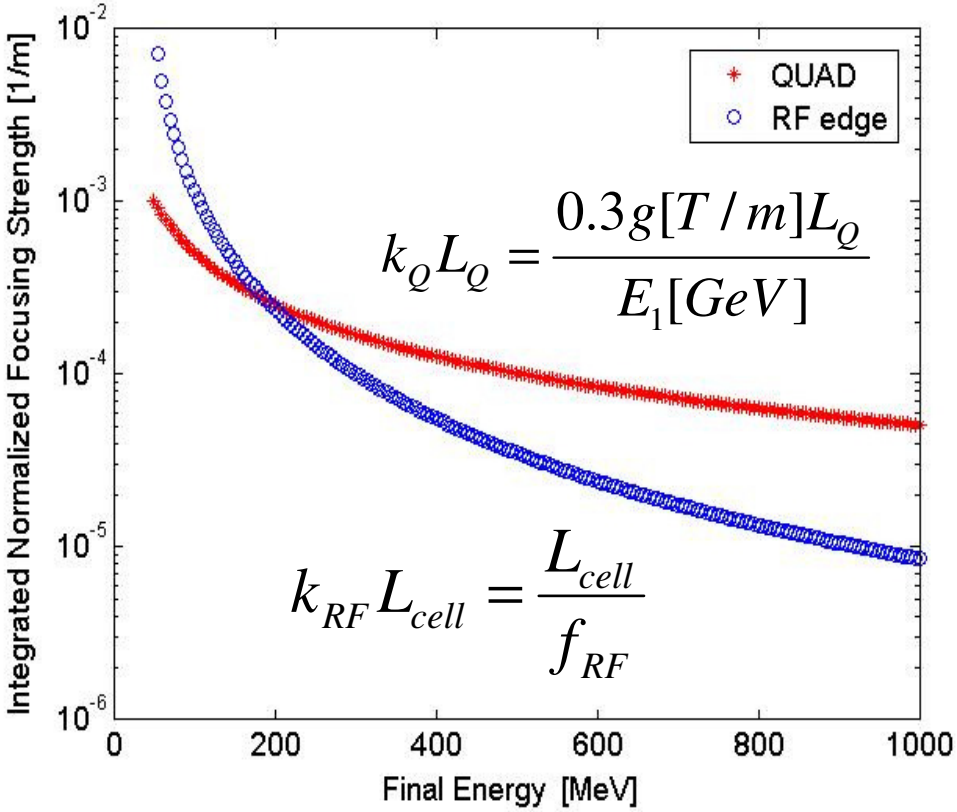
$$\gamma' = \frac{\gamma_1 - \gamma_0}{L} \cong \frac{qE_{z,0}\cos(\phi)}{m_e c^2}$$

RF vs. magnetic focusing

- L → 0, the ***RF focal length*** is: $f_{RF} \approx \frac{1}{M_{21}} = \frac{-1}{\frac{\gamma'}{\gamma_0\gamma_1}\left[\gamma_1 - \gamma_0 - \gamma_0 \ln\frac{\gamma_1}{\gamma_0}\right]} \rightarrow \frac{-1}{\frac{\gamma'}{\gamma_1}\ln\frac{\gamma_1}{\gamma_0}}$

Focusing at the entrance dominates over defocusing at the exit

The overall focusing is damped by acceleration.



$E_1 - E_0$ = 50 MeV / structure
L_{srt} = 3 m, L_{cell} = 30 mm
g = 1.7 T/m
L_Q = 0.1 m

Coupler cell RF kick

Pictures courtesy of M. Dal Forno

- Geometric asymmetries of the input/output **coupler cells** may contribute with transverse electric field kicks that affect the beam trajectory and size, with dipole, quadrupole and higher order E_z dependence on the particle offset.

1. **Coupler acc. field with ampl. & phase y-gradient, dipole approximation**

$$E_z(y,t) = \left(E_{z,0} + \Delta E_{z,0}\frac{y}{2a}\right)\cos\left(\phi_s + \Delta\phi_c\frac{y}{2a} + \omega_{rf}\Delta t\right)$$

2. **Panofsky – Wenzel theorem**

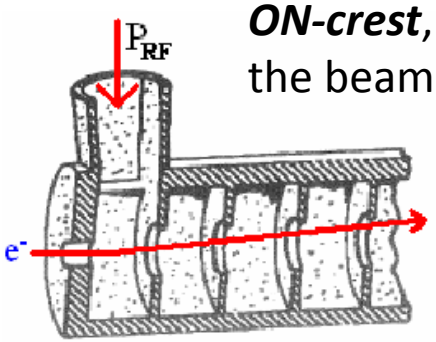
$$\Delta y' = \frac{\Delta p_y}{p_z} = -i\frac{e}{k_{rf}\,p_z c}\int_0^{l_{cell}} \nabla_\perp E_z\,dz = \ldots$$

expand for $\omega\Delta t \ll 1$

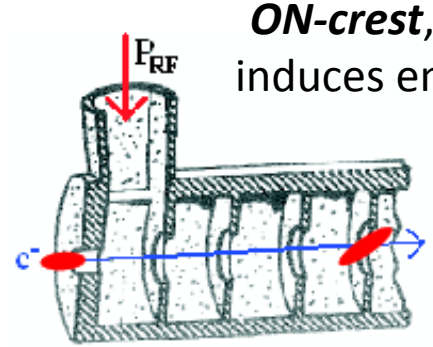
$$\ldots \cong \frac{e l_{cell}}{2 a k_{rf}\,p_z c}\left[E_{z,0}\Delta\phi_c\cos\phi_s - \Delta E_{z,0}\sin\phi_s\right] + k_{rf}\Delta z\left[E_{z,0}\Delta\phi_c\sin\phi_s + \Delta E_{z,0}\cos\phi_s\right]$$

centroid kick **head-tail kick**

ON-crest, phase error "kicks" the beam centroid.



ON-crest, amplitude error induces emittance growth.



Impact on the beam motion

- The **input coupler** effect typically **dominates** because:
  - beam is at lower energy,
  - accelerating field at the entrance is not attenuated yet.

- **Trajectory (mi)steering** can be compensated with steering magnets in proximity of the accelerating structure.
  - However, a beam passing off-axis in the structure can excite transverse wakefields (see next lectures). Use *feed-forward* steering or put steerers *on* the structure.

- For on-crest acceleration (typical in injector), the **head-tail** induced **emittance growth** (from equation previous slide) is:

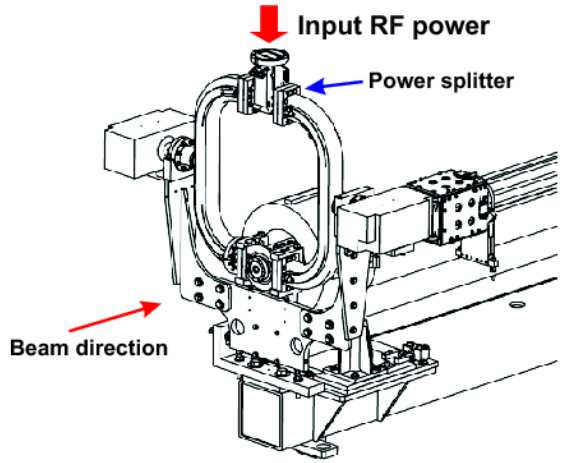
$$\varepsilon_y = \sqrt{\langle y^2 \rangle \langle y'^2 \rangle - \langle yy' \rangle^2} \approx \sqrt{\sigma_{y,0}^2 \left(\sigma_{y',0}^2 + \langle \Delta y'^2 \rangle \right)} \overset{on-crest}{\cong} \sqrt{\varepsilon_{y,0}^2 + \frac{\sigma_{y,0}^2}{4a^2} \left(\frac{e \Delta E_{z,0} l_{cell}}{p_z c} \right)^2 \sigma_z^2}$$

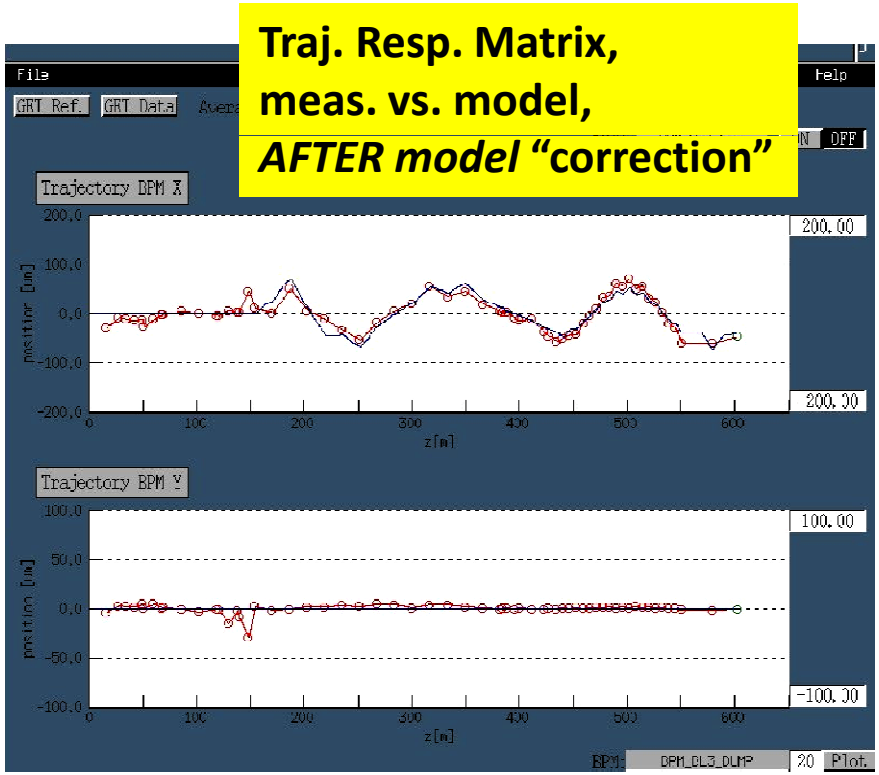
Spurious RF focusing

Pictures courtesy of
T. Hara, M. Dal Forno

- Special coupler designs ("racetrack" cell shaping, symmetric RF waveguide, cell tuning) are usually adopted to get rid of the dipolar and possibly of the quadrupolar field component.

- Residual effects, typically quadrupolar, have to be taken into account as a "correction factor" in the modeling (matrix) of RF focusing.





RF focusing in ELEGANT code

- **TWLA**: $2\pi/3$ CG, edge focusing (opt.), numerical integration.

- **RFCA**: π SW, edge focusing (opt.), body-focus (opt.), matrix (single-kick approx. by default), *N_KICKS*, *PHASE_REFERENCE*.
  - Also good for TW-CG, with body-focus turned off.
  - "N_KICKS = XX" is equivalent to a split structure. Used for numerical integration of wakes (*e.g.*, geometric, LSC, etc.) in a long structure.
  - For the one-structure model, just use: N_KICKS=0, PHASE_REFERENCE = 0.

- **RFCA *split in units*** (*e.g.*, for particle dynamics inside a long structure).
  - Each unit has to be long (multiple of) λ_{RF}.
  - Proper focusing for a TW-like structure is given by setting: N_KICKS = 1, END1_FOCUS = 1 and END2_FOCUS = 1 in each unit (inner focusing is cancelled out and only that at the edges remains).
  - Set PHASE_REFERENCE=*n*, with *n* integer and unique for each unit (otherwise the units will be individually phased, which could cause unphysical result).

!! Warning!! In old Elegant versions, Twiss functions are computed correctly only for N_KICKS = 0 !!