Magnetic Optics for Charged Particles

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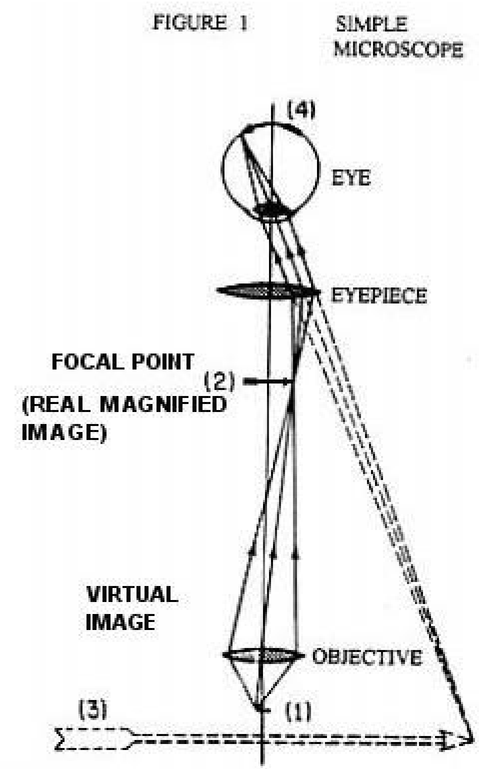
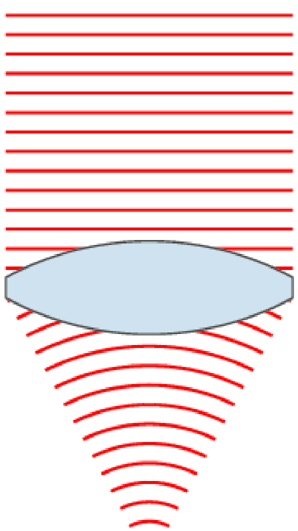
Charged Particle Beam Optics: Light Optics

Light Optics

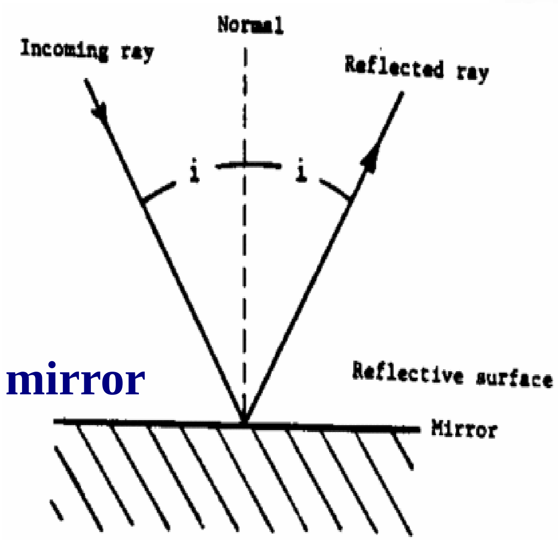


Basic Elements

lens



Flat mirror



www.tedpella.com/mscope_html/BA400full.jpg
www.rifeenergymedicine.com/idafig1.jpg
en.wikipedia.org/wiki/Lens_%28optics%29

Magnets Modeling: Magnetic Multipole Expansion

Impulse Boundary Condition

- **Magnetic field is longitudinally uniform inside the magnet and zero outside**
- **Magnetic field is modeled as a 2D field**

$$\vec{B}=\left(B_x(x,y),B_y(x,y),0\right)=\nabla\times A_z(x,y),\qquad \vec{A}=\left(0,0,A_z(x,y)\right)$$

Solve Laplace Equation: $\nabla^2\Phi=0$

General solution: $\Phi(r,\theta)=B_0\rho_0\sum_{m=0}^{\infty}\frac{r^{n+1}}{n+1}\left(b_n\sin(n+1)\theta+a_n\cos(n+1)\theta\right)$

$$\vec{B}=\nabla\,\Phi(r,\theta),$$

$$B_x=\frac{\partial\Phi}{\partial x}=B_0\rho_0\sum_{m=0}^{\infty}r^n\left(b_n\sin(n\theta)+a_n\cos(n\theta)\right),$$

$$B_y=\frac{\partial\Phi}{\partial y}=B_0\rho_0\sum_{m=0}^{\infty}r^n\left(b_n\cos(n\theta)-a_n\sin(n\theta)\right),$$

$$\boxed{B_y+iB_x=B_0\rho_0\sum_{n=0}^{\infty}(b_n+ia_n)(x+iy)^n,}$$

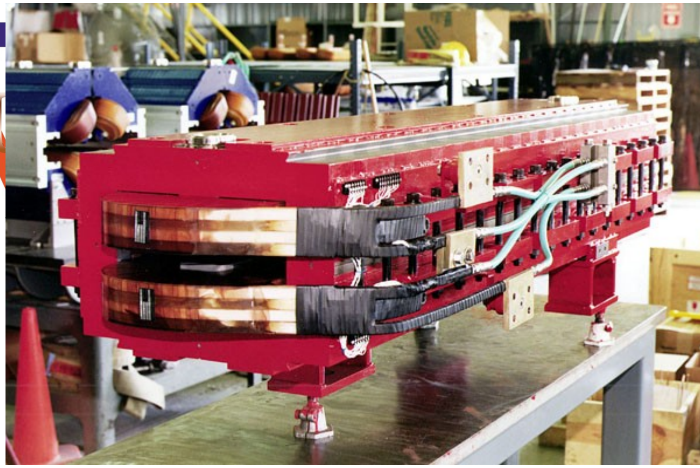
Each term → A single multipole

Normal components: $b_n=\frac{1}{B_0\rho_0}\frac{1}{n!}\frac{\partial^n B_y(x,y)}{\partial x^n}\Big|_{x=0,\,y=0},$

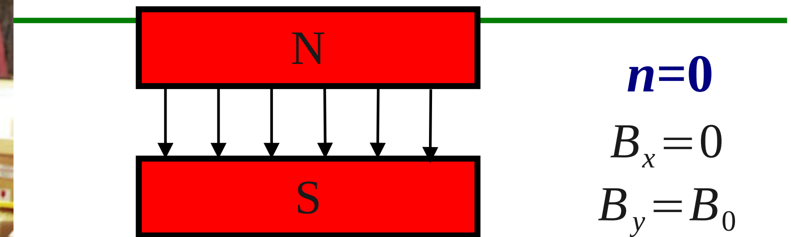
$$B_y(x,-y)=B_y(x,y)$$
$$B_x(x,-y)=-B_x(x,y)$$

Skew components: $a_n=\frac{1}{B_0\rho_0}\frac{1}{n!}\frac{\partial^n B_x(x,y)}{\partial x^n}\Big|_{x=0,\,y=0}$

$$B_y(x,-y)=-B_y(x,y)$$
$$B_x(x,-y)=B_x(x,y)$$



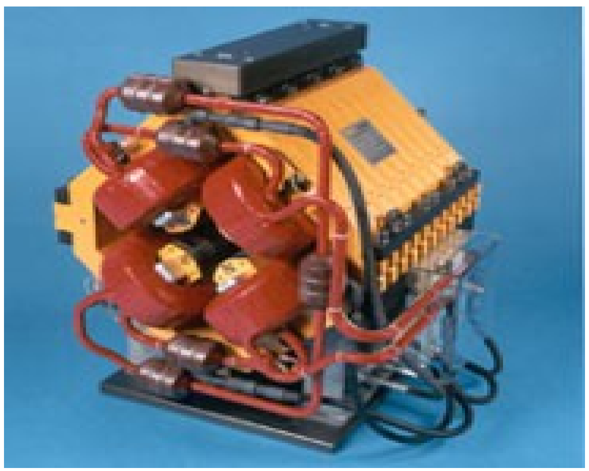
Dipoles



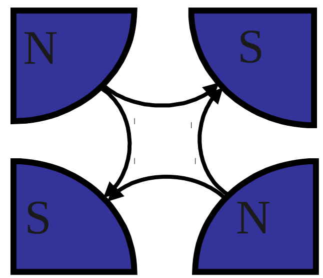
***n*=0**

$B_x = 0$

$B_y = B_0$



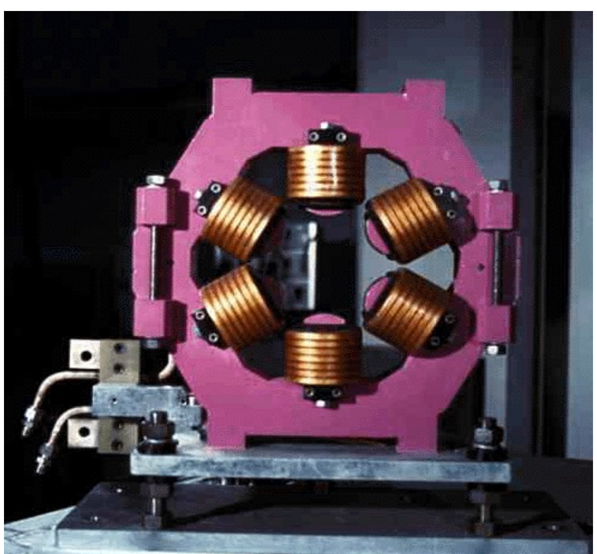
Quadrupoles



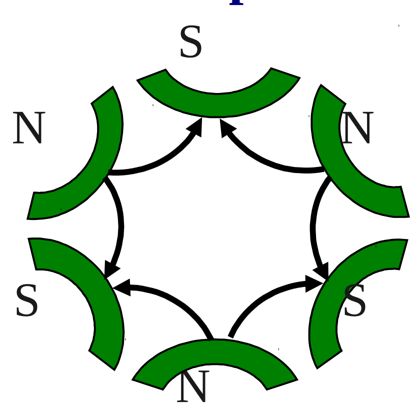
***n*=1**

$B_x = G\,y$

$B_y = G\,x$



Sextupoles



***n*=2**

$B_x = 2\,S\,x\,y$

$B_y = S\,(x^2 - y^2)$

Equation of Motion in Magnetic Field

Charged particles can be guided and confined by electric field and magnetic field

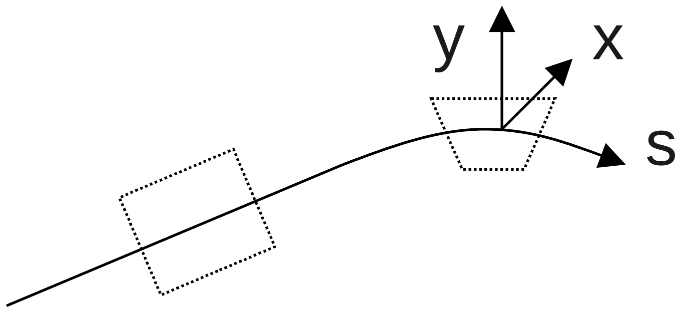
$$\frac{d\vec{p}}{dt} = \frac{d\left(\gamma \vec{\beta} m_0 c\right)}{dt} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

To provide the same amount of force, the magnetic field is easy to realize

- Avalanche electric breakdown in air occurs at a few MV/m
- In conventional magnets, magnetic field can reach about 2 Tesla before becoming fully saturated; even higher magnetic field can be realized in the superconducting magnets
- Magnets can produce a much larger force than the electric field for a relativistic beam
- Most of accelerators use magnetic optics

Coordinate System and Hamiltonian

Coordinate System: Curvilinear Coordinate System:



- Cartesian or Cylindrical Coordinate Systems, depending on the geometry of magnets
- Hamiltonian for the charged particle in a static magnetic field (m = rest mass)

$$H(x, P_x, y, P_y, z, P_z; t) = \sqrt{(P_x - qA_x)^2 c^2 + (P_y - qA_y)^2 c^2 + (P_z - qA_z)^2 c^2 + m^2 c^4}$$

- In Cartesian coordinate, longitudinal coordinate z is used as independent variable for the equivalent new Hamiltonian

$$K(x, p_x, y, p_y, \delta, l; z) = -\sqrt{(1+\delta)^2 - (p_x - a_x)^2 - (p_y - a_y)^2} - a_z$$

where,

$$p_{x,y} = \frac{P_{x,y}}{P_0} = scaled\ canonical\ momentum \qquad a_{x,y,z}(x,y,z) = \frac{qA_{x,y,z}(x,y,z)}{P_0} = scaled\ vector\ potential$$

$$\delta = \frac{P - P_0}{P_0} = scaled\ momemtun\ deviation \qquad l = pathlength\ of\ the\ orbit$$

- Phase-space vector $\vec{X} = \{x, p_x, y, p_y, \delta, l\}$

Figure by David Robin, LBNL

Hamiltonians of Multipoles

- **Impulse Boundary Approximation**

$$K(x, p_x, y, p_y, \delta, l; z) = -\sqrt{(1+\delta)^2 - p_x^2 - p_y^2} - a_z(x, y)$$

- **Paraxial Approximation**

$$K(x, p_x, y, p_y, \delta, l; z) = -\delta + \frac{p_x^2 + p_y^2}{2(1+\delta)} - a_z(x, y)$$

- **Normal dipole magnets**

$$K(x, p_x, y, p_y, \delta, l; z) = -\delta + \frac{p_x^2 + p_y^2}{2(1+\delta)} + K_0 x$$

- **Normal quadrupole magnets**

$$K(x, p_x, y, p_y, \delta, l; z) = -\delta + \frac{p_x^2 + p_y^2}{2(1+\delta)} + \frac{1}{2} K_1 (x^2 - y^2)$$

- **Normal sextupole magnets**

$$K(x, p_x, y, p_y, \delta, l; z) = -\delta + \frac{p_x^2 + p_y^2}{2(1+\delta)} + \frac{1}{3} K_2 (x^3 - 3 x y^2)$$

Cartesian Coordinate System

Equations of Motion: Dipole and Quadrupole

- Linear Equation of motion for a magnet with both dipole and quadrupole field

$$x'' + \left(\frac{1}{\rho_0^2} + K_1\right) x = \frac{\delta}{\rho_0}$$

$$y'' - K_1 y = 0$$

where, $x'' = \frac{d^2 x}{ds^2}, \qquad y'' = \frac{d^2 y}{ds^2}$

$\frac{1}{\rho_0^2}$ **is the weak focusing term produced by the dipole field**

- **Consider a simple harmonic oscillator**

$$u'' + k u = 0, \qquad k = constant$$

- **General solution**

$$u(s) = C(s) u(0) + S(s) u'(0)$$
$$u'(s) = C'(s) u(0) + S'(s) u'(0)$$

where,

$$C(s) = \cos(\sqrt{k}\, s), \qquad S(s) = \frac{1}{\sqrt{k}} \sin(\sqrt{k}\, s), \qquad k > 0$$

$$C(s) = \cosh(\sqrt{|k|}\, s), \qquad S(s) = \frac{1}{\sqrt{|k|}} \sinh(\sqrt{|k|}\, s), \qquad k < 0$$

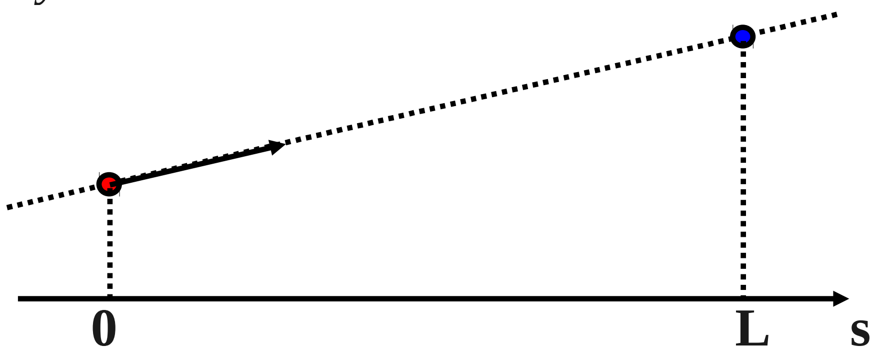
- **Matrix representation**

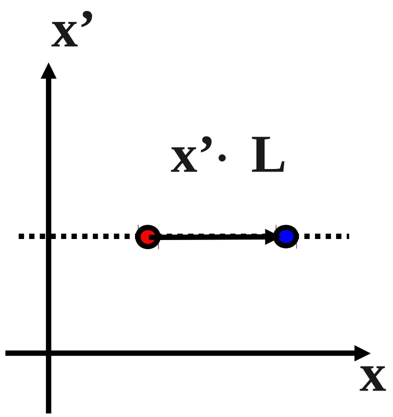
$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} \begin{pmatrix} u(0) \\ u'(0) \end{pmatrix}$$

Matrix Representation: A Drift Space

- For a draft space without magnetic field, $L = s - s_0$

$$x'' = 0$$
$$y'' = 0$$





$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u'_0 \end{pmatrix}$$

$$M_{drift}(s|s_0) = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}, \quad L = s - s_0$$

$$u(s) = u_0 + (s - s_0) u'_0 = u_0 + L u'_0$$
$$u'(s) = u'_0$$

In a drift, the slope of the trajectory remains constant while the position changes linearly with distance

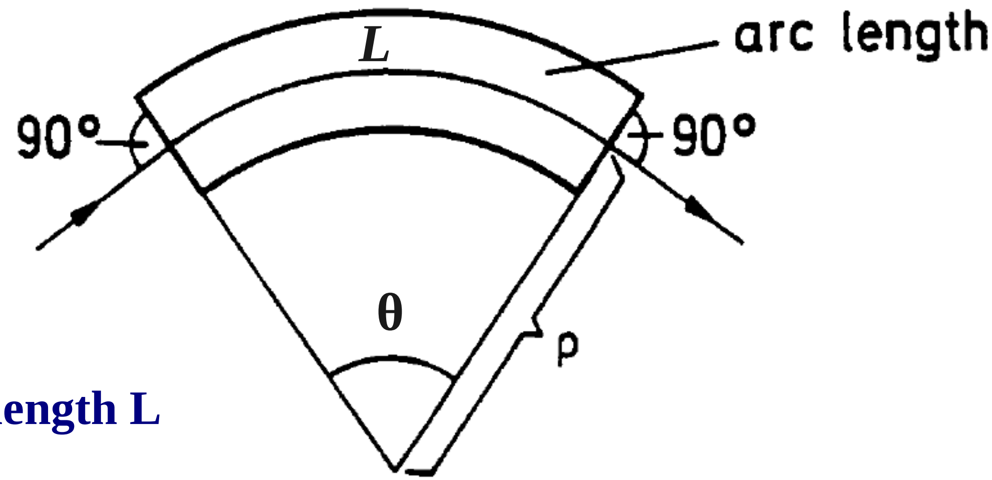
Figure by David Robin, LBNL

Matrix Representation: Sector Dipole

- For a sector dipole magnet

$$x'' + \frac{1}{\rho_0^2} x = \frac{\delta}{\rho_0}$$

$$y'' = 0$$



- The transfer matrix for a sector dipole of arc length L

$$M_{x,sector} = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix} \qquad \theta = \frac{L}{\rho}$$

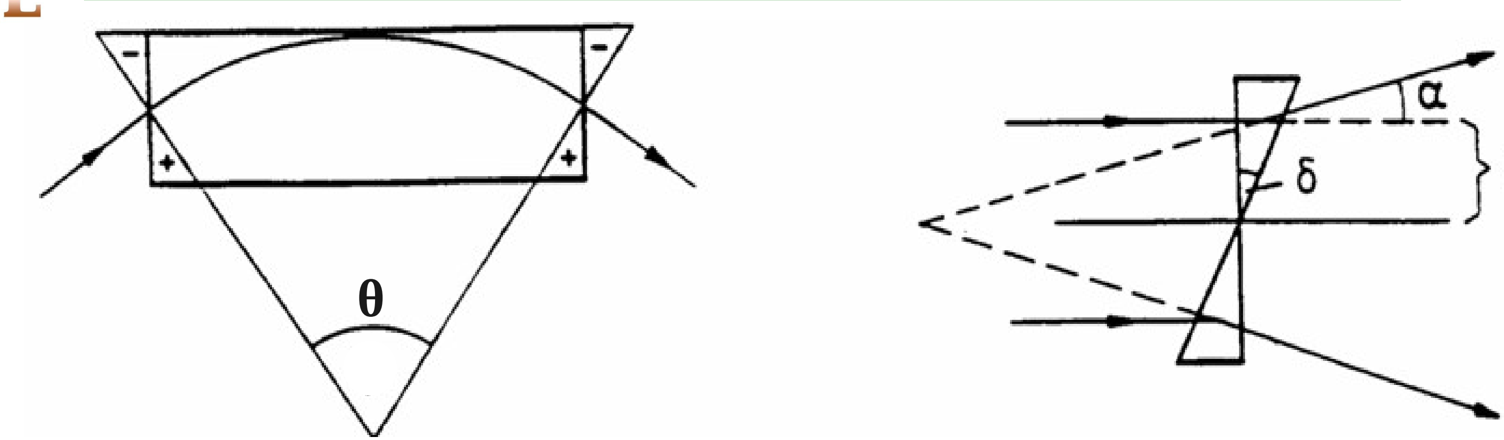
Focusing

$$M_{y,sector} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

Drift

This is a hard-edge model, neglecting the edge focusing.

Matrix Representation: Rectangular Dipole



- **The rectangular dipole of length L can be considered as a sector dipole sandwiched by entrance and exit wedges**

$$M_{x,edge} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\rho}\tan\alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \alpha = entrance/exit\ angle$$

$$M_{rect} = M_{exit} \cdot M_{sector} \cdot M_{entrance}$$

For symmetric trajectory, and a small bending angle q << 1

$$M_{x,rect} = \begin{pmatrix} 1 & \rho\sin\theta & \rho(1-\cos\theta) \\ 0 & 1 & 2\tan\frac{\theta}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

Drift

$$M_{y,rect} = \begin{pmatrix} 1-\theta\tan\frac{\theta}{2} & \rho\theta \\ -\frac{1}{\rho}(2-\theta\tan\frac{\theta}{2})\tan\frac{\theta}{2} & 1-\theta\tan\frac{\theta}{2} \end{pmatrix}$$

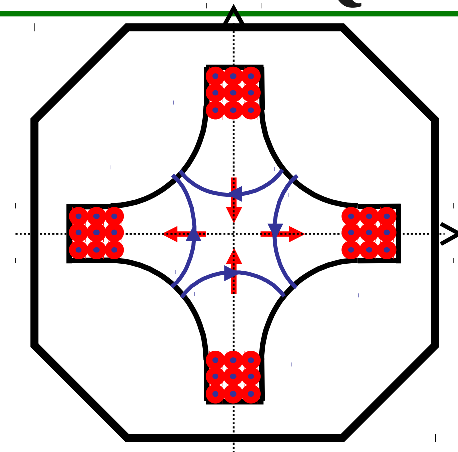
Focusing

Figure by David Robin, LBNL

Matrix Representation: A Quadrupole

- **Equations of motion:**

$$x'' + K_1 x = 0$$
$$y'' - K_1 y = 0$$



$$B_x = G\,y = B_0 \rho_0 K_1 y$$
$$B_y = G\,x = B_0 \rho_0 K_1 x$$
$$K_1 = \frac{G}{B_0 \rho_0}; \qquad units: m^{-2}$$

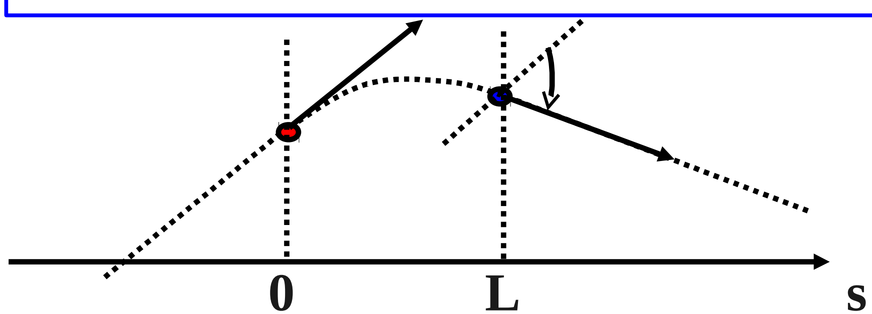
- **The transfer matrix for a focusing quadrupole of length L**

$$M_x(s|s_0) = \begin{pmatrix} \cos(\sqrt{K_1}L) & \frac{1}{\sqrt{K_1}}\sin(\sqrt{K_1}L) \\ -\sqrt{K_1}\sin(\sqrt{K_1}L) & \cos(\sqrt{K_1}L) \end{pmatrix}$$

$K_1 > 0,$ *horizontal focusing; vertical defocusing*

$$M_y(s|s_0) = \begin{pmatrix} \cosh(\sqrt{K_1}L) & \frac{1}{\sqrt{K_1}}\sinh(\sqrt{K_1}L) \\ \sqrt{K_1}\sinh(\sqrt{K_1}L) & \cosh(\sqrt{K_1}L) \end{pmatrix}$$

Question: Please give the matrices for $K_1<0$



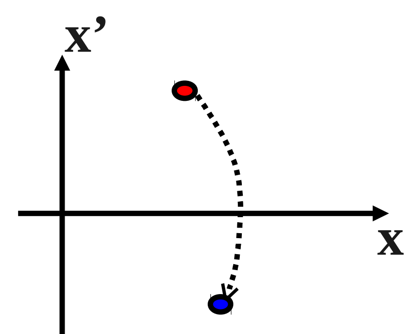


Figure by David Robin, LBNL

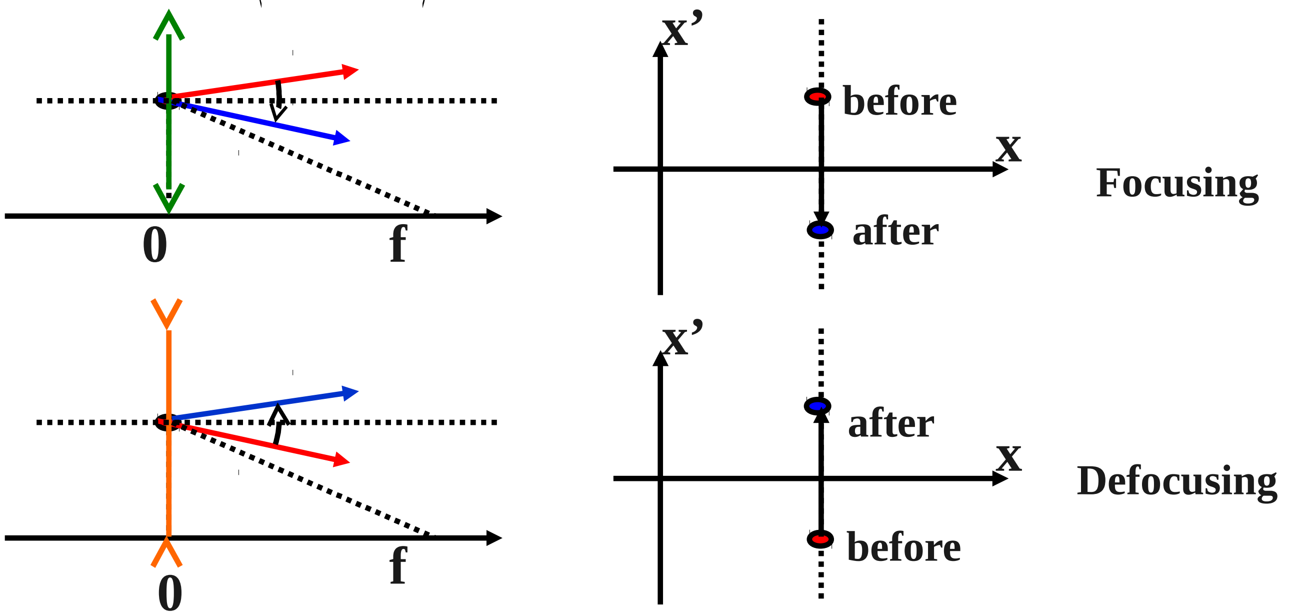
Matrix Representation: A Thin Lens

- When the length of a quadrupole is very short, it can be considered as a thin lens

$$L \to 0$$
$$K_1 \to \infty$$
$$K_1 L \to constant = -\frac{1}{f}$$

- The transfer matrix for a thin lens

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_0' \end{pmatrix} \qquad \begin{matrix} f > 0, & focusing \\ f < 0, & defocusing \end{matrix}$$

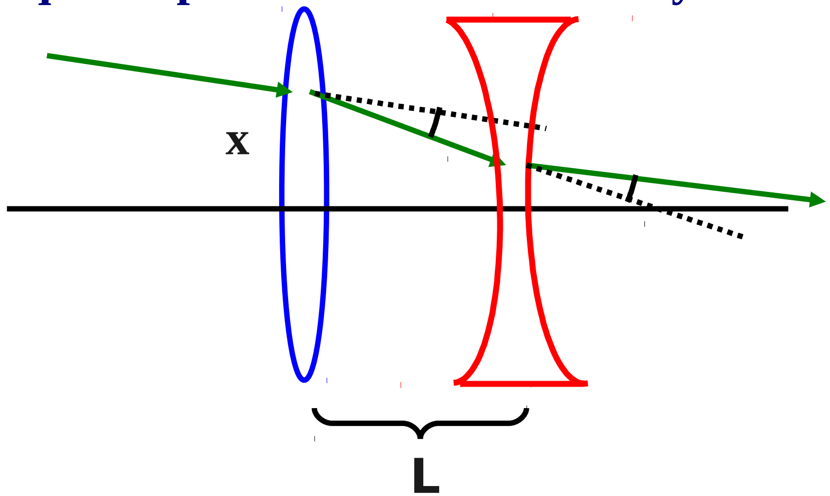


After a thin lens, the position remains unchanged while the slope reduces (focusing) or increases (defocusing)

Figure by David Robin, LBNL

Matrix Representation: A Doublet

Consider a quadrupole doublets modeled by two thin lens, f_1 and f_2



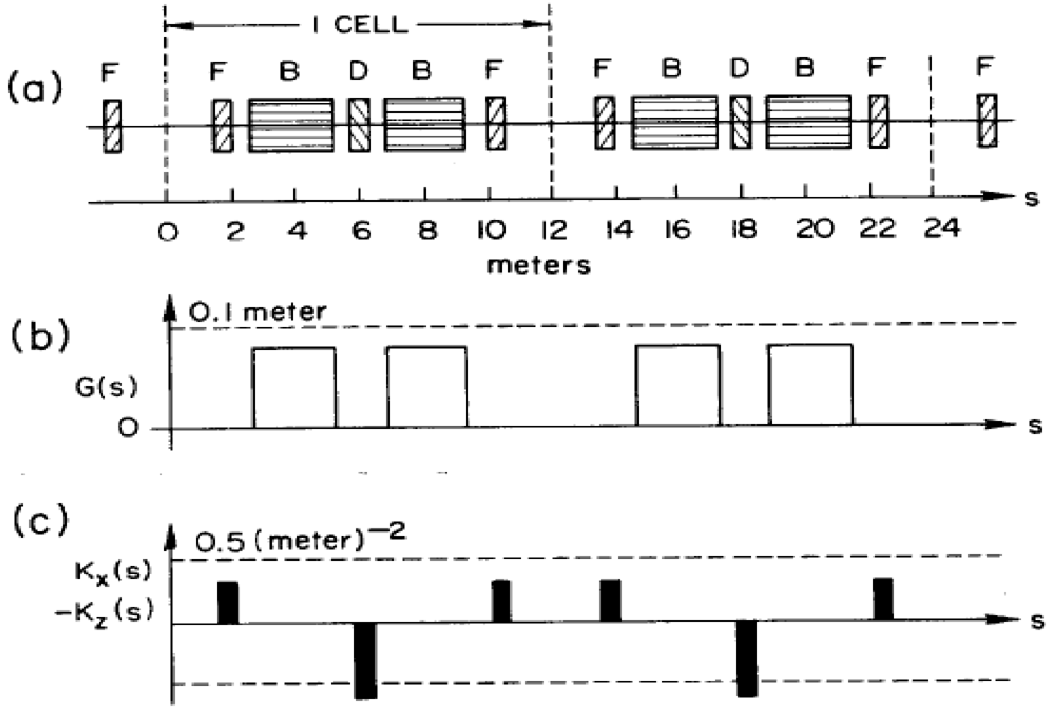
$$M_{doublet} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{L}{f_1} & L \\ -\frac{1}{f_{eff}} & 1-\frac{L}{f_2} \end{pmatrix} \qquad f_{eff} = -\frac{f_1 f_2}{L-(f_1+f_2)}$$

- **Alternating gradient, $f_1 > 0$ and $f_2 < 0$,** $$f_{eff} = \frac{|f_1 f_2|}{L-(f_1+f_2)}, \qquad L < f_1, L > (f_1 - |f_2|)$$

- **A special case, $f_1 = -f_2 = f$,** $$\frac{1}{f_{eff}} = \frac{L}{f^2}$$

Figure by David Robin, LBNL

Hill's Equation and Piecewise Focusing



$$x'' + K_x(s)\, x = 0$$

$$y'' + K_y(s)\, y = 0$$

$$K_{x,y}(s+C) = K_{x,y}(s)$$

Hill's equation for x or y motion $u'' + K(s)\, u = 0 \qquad K(s+C) = K(s)$

General solution $$u(s) = \sqrt{\epsilon}\sqrt{\beta(s)}\cos(\phi(s)+\phi_0)$$

where

$\beta(s)$ **is the beta-function satisfying** $\frac{1}{2}\beta\beta'' - \frac{1}{4}(\beta')^2 + \beta^2 K(s) = 1$

$$\beta(s+C) = \beta(s)$$

$\phi(s)$ **is the betatron phase** $\frac{d\phi(s)}{ds} = \frac{1}{\beta(s)}$

$$\phi(s) = \int_{s_0}^{s} \frac{ds}{\beta(s)} + \phi_0 \qquad \phi(s+C) = \phi(s) + \mu$$

M. Sands, SLAC-121 (UC-28) (1970)

Hill's Equation and Piecewise Focusing

$$u(s)=\sqrt{\epsilon}\sqrt{\beta(s)}\cos(\phi(s)+\phi_0)$$

$$u'(s)=\frac{\sqrt{\epsilon}\beta'}{2\sqrt{\beta(s)}}\cos(\phi(s)+\phi_0)-\frac{\epsilon}{\sqrt{\beta}}\sin(\phi(s)+\phi_0)$$

ϵ **is a constant determined by the initial condition**

Defining a set of Twiss parameters or lattice functions:

Machine Twiss parameters

$$\beta(s)$$

$$\alpha(s)\equiv-\frac{1}{2}\frac{d\beta(s)}{ds}$$

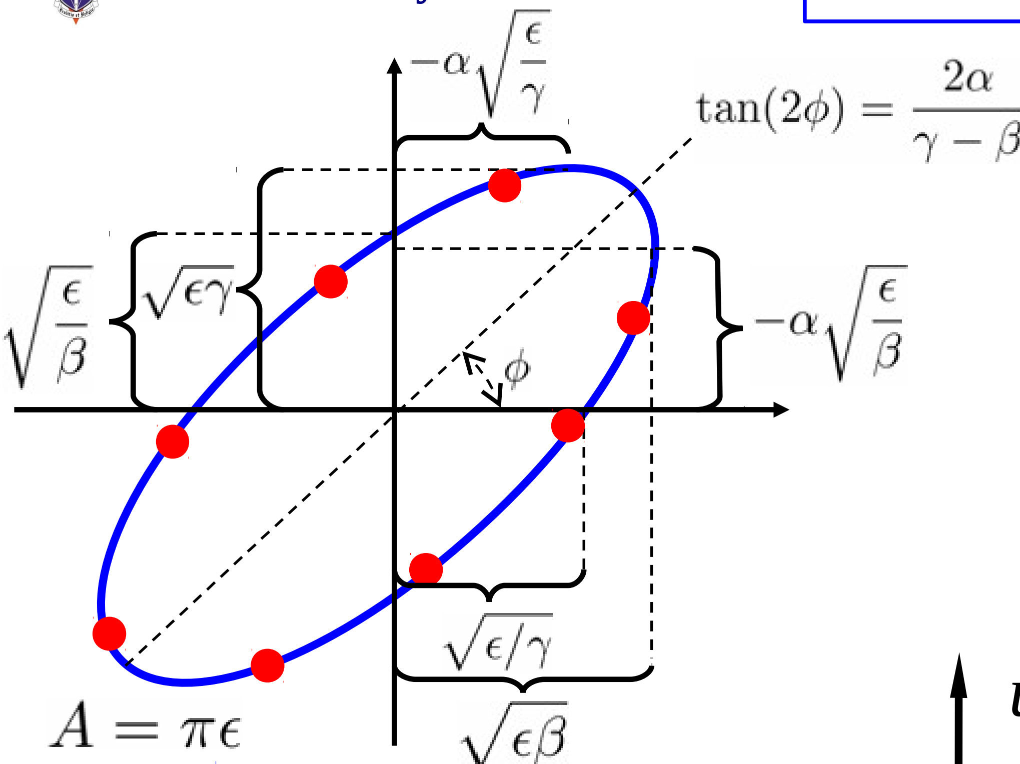
$$\gamma(s)\equiv\frac{1+(\alpha(s))^2}{\beta(s)}$$

$$1+\alpha^2 = \beta\gamma$$

$$u'(s)=\frac{-\sqrt{\epsilon}}{\sqrt{\beta(s)}}\left(\alpha(s)\cos(\phi(s)+\phi_0)+\sin(\phi(s)+\phi_0)\right)$$

Courant-Snyder Invariatn and Phase Ellipse

Courant-Synder Invariant $\gamma(s)u^2(s)+2\alpha(s)u(s)u'(s)+\beta u'^2(s)=\epsilon$

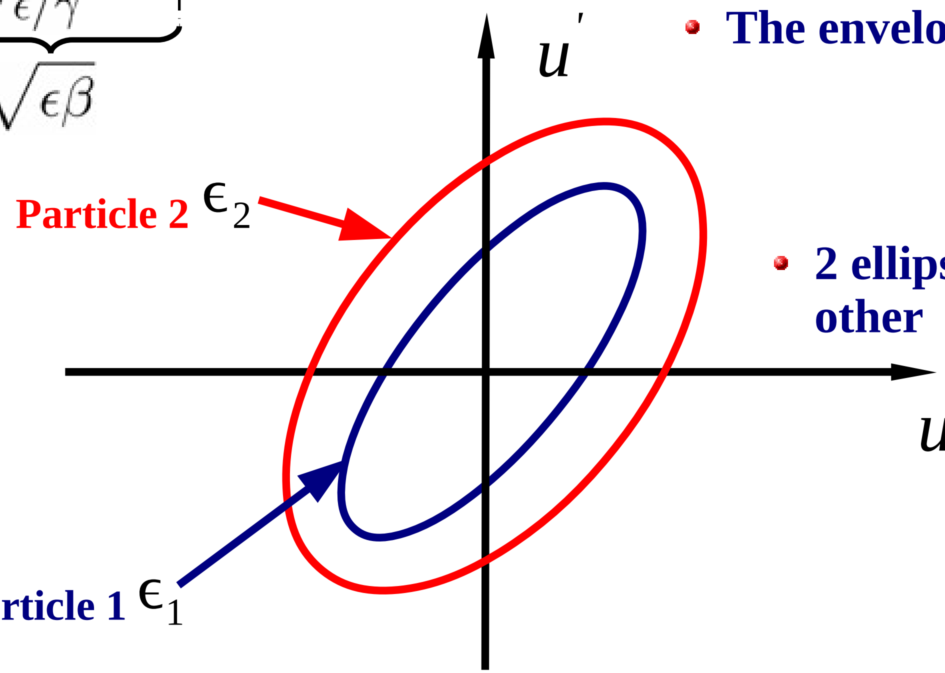


- **Machine ellipse: Twiss parameters shared by betatron motion of individual particles**
- **Area of the ellipse is p e; e determines amplitude of betatron motion for a particle**
- **The envelope of the motion is**

$$\sqrt{\epsilon\beta(s)}$$

- **The envelope of u'(s)**

$$\sqrt{\epsilon\gamma(s)}$$

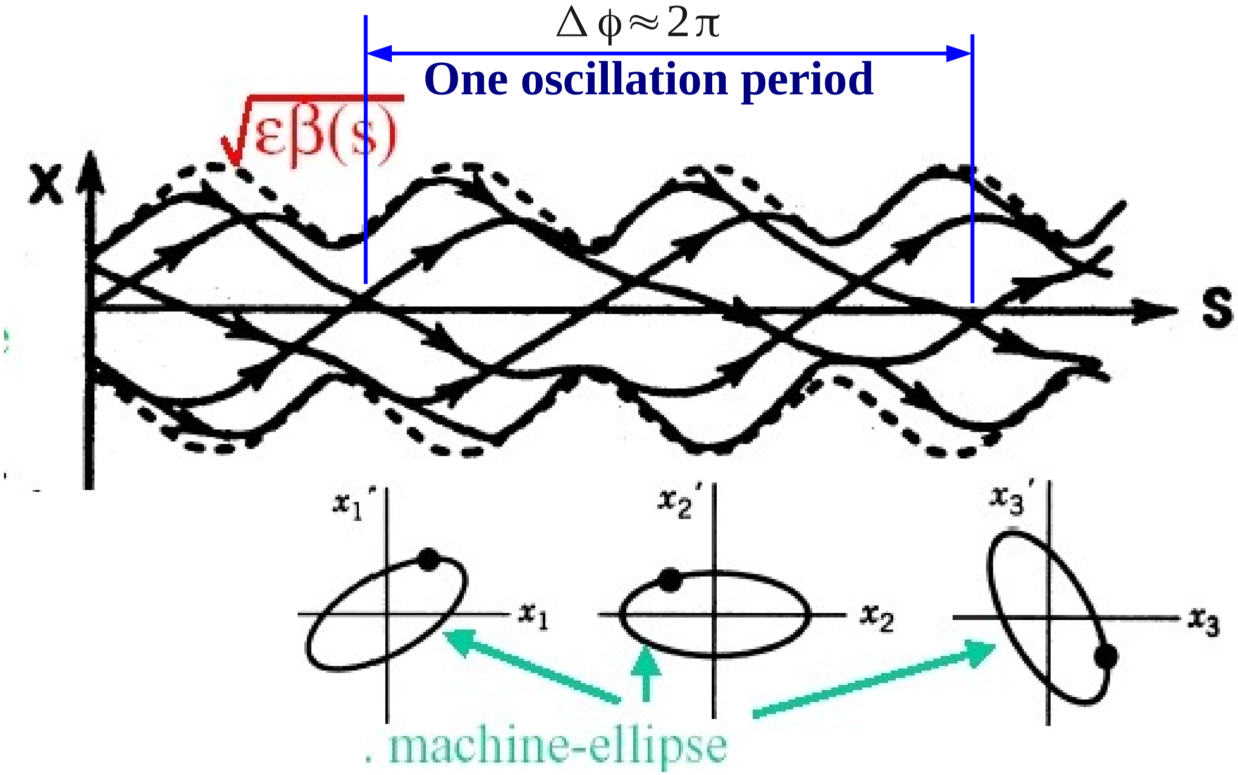


- **2 ellipses are similar to each other**

Figure by David Robin, LBNL

Beam Envelope

Beam envelope, beta function, and amplitude of motion



Question: How many beam position monitors are needed in a storage ring?

M. Sands, SLAC-121 (UC-28) (1970)

Transport of Machine Ellipse

Transport of Twiss parameters using the transfer matrix

$$\begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_2=\begin{pmatrix}R_{11}^2 & -2R_{11}R_{12} & R_{12}^2\\ -R_{11}R_{21} & 1+2R_{12}R_{21} & -R_{12}R_{22}\\ R_{21}^2 & -2R_{21}R_{22} & R_{22}^2\end{pmatrix}\begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_1$$

$$M_{1\rightarrow 2}=\begin{pmatrix}R_{11} & R_{12}\\ R_{21} & R_{22}\end{pmatrix}$$

The transfer matrix can be expressed in terms of twiss parameters

$$M_{1\rightarrow 2}=\begin{pmatrix}\sqrt{\frac{\beta_2}{\beta_1}}\left(\cos\phi_{12}+\alpha_1\sin\phi_{12}\right) & \sqrt{\beta_1\beta_2}\sin\phi_{12}\\ -\frac{(\alpha_2-\alpha_1)\cos\phi_{12}+(1+\alpha_1\alpha_2)\sin\phi_{12}}{\sqrt{\beta_1\beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}}\left(\cos\phi_{12}-\alpha_2\sin\phi_{12}\right)\end{pmatrix}$$

where $\phi_{12}=\phi_2-\phi_1$

Changes of Twiss Parameters

- **A Drift Space** $M_{1\to 2} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$
$$\alpha(s) = \alpha_0 - \gamma_0 s$$
$$\gamma(s) = \gamma_0$$

Phase advance in a drift space starting from the waist a = 0

$$\beta(s) = \beta_0\left(1 + \left(\frac{s}{\beta_0}\right)^2\right)$$

$$\Delta\phi_{0\to s} = \int_0^s \frac{ds}{\beta(s)} = \arctan\left(\frac{s}{\beta_0}\right)$$

Maximum phase advance

$$\Delta\phi_{-\infty\to\infty} \to \pi$$

- **A Thin Lens** $M_{1\to 2} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$

$$\beta_2 = \beta_1$$
$$\alpha_2 = \alpha_1 + \frac{\beta_1}{f}$$
$$\gamma_2 = \gamma_1 + \frac{2\alpha_1}{f} + \frac{\beta_1}{f^2}$$

$$\Delta\phi = 0$$

Evolution of Phase Ellipses

Drift space vs thin lens quadrupole

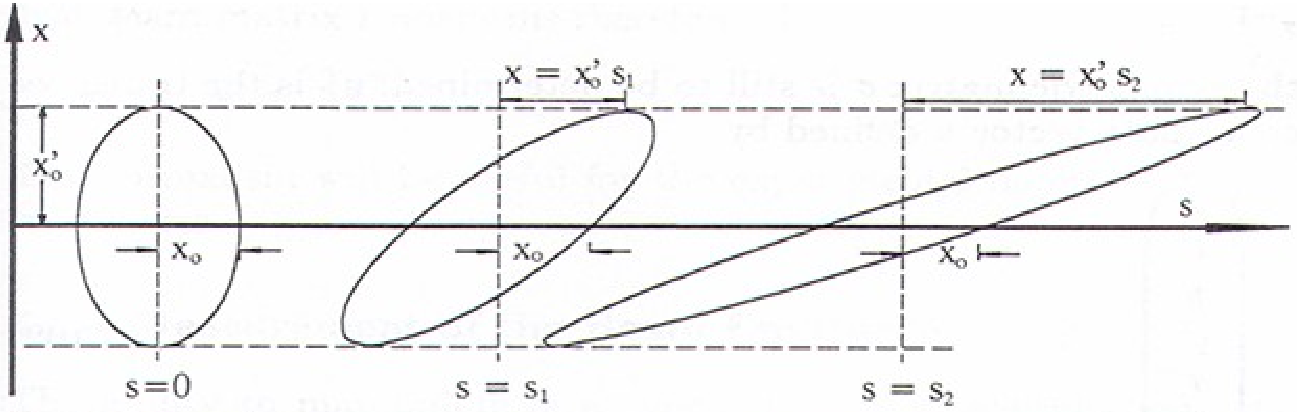


Fig. 5.23. Transformation of a phase space ellipse at different locations along a drift section

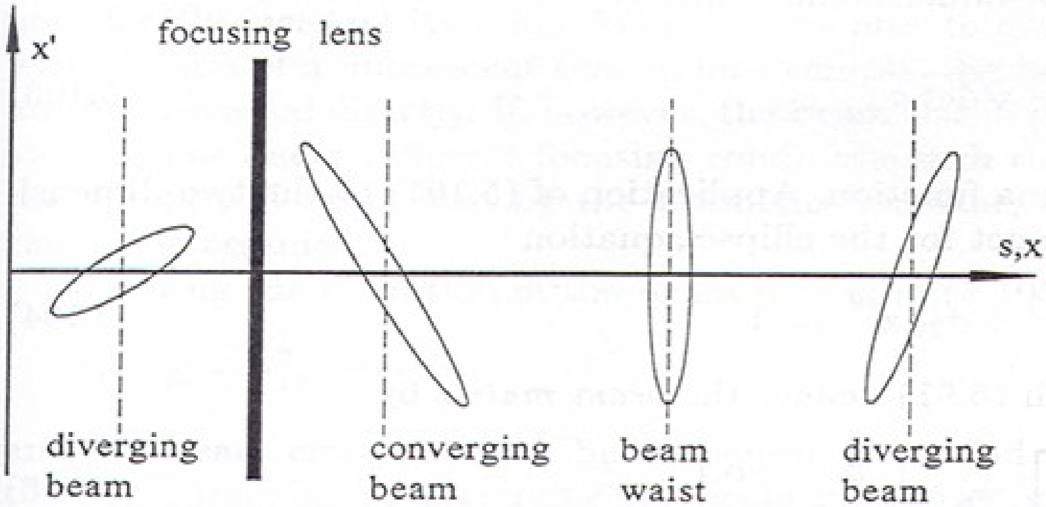
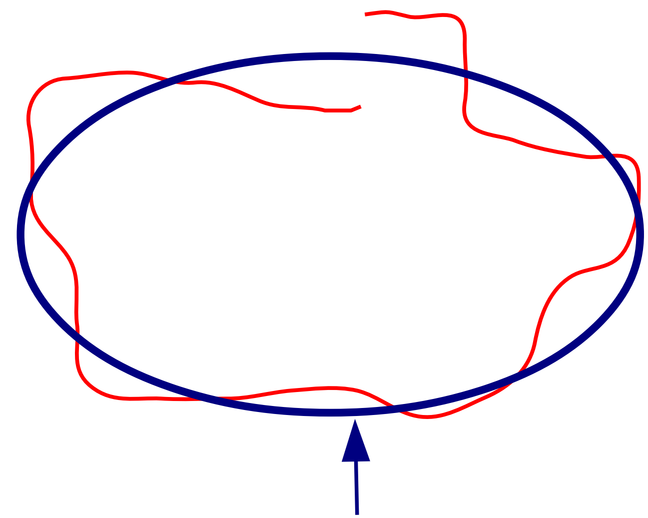


Fig. 5.24. Transformation of a phase ellipse due to a focusing quadrupole. The phase ellipse is shown at different locations along a drift space downstream from the quadrupole.

Maps and One-Turn Maps

A map is a functional relationship which associate the final phase space vector to the initial phase space vector of the charged particle

$$\vec{X}_0 = \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ \delta \\ l \end{pmatrix}_{initial} \xrightarrow{\text{Map}} \vec{X} = \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ \delta \\ l \end{pmatrix}_{final}$$



Closed orbit

- **A transfer matrix is a linear map**
- **A one-turn matrix is a linear one-turn map**
- **Higher order maps can be constructed using Lie transformations or Lie maps**
- **A one-turn map can be generated by tracking a particle with a small deviation with respect to the closed orbit for one turn**

$$X_k = \sum_{j=1}^{6} R_{kj} X_{0j} + \sum_{j,l=1}^{6} T_{kjl} X_{0j} X_{0l} + \cdots$$

One-turn Matrix

- **Applying the periodic condition, the one-turn matrix can be written as**

$$M_{one-turn} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu + \alpha\sin\mu \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$$

where $\mu = \phi_{s+C} - \phi_s$ **is the one-turn betatron phase advance**

- **Computing tune and twiss parameters**

$$\cos\mu = \frac{R_{11} + R_{22}}{2} \qquad tune, \nu = \frac{\mu}{2\pi}$$

$$\beta = \frac{R_{12}}{\sin\mu} \qquad \alpha = \frac{R_{11} - \cos\mu}{\sin\mu} \qquad \gamma = \frac{-R_{21}}{\sin\mu}$$

- **Stability Condition for linear betatron motion**

$$|Tr\, M_{one-turn}| = |2\cos\mu| < 2$$ **or** $$\left|\frac{R_{11} + R_{22}}{2}\right| < 1$$

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