Equilibrium Electron Beam Sizes

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Radiation Damping – Transverse Directions

- **Emission of a photon does not change the direction of the transverse momentum**

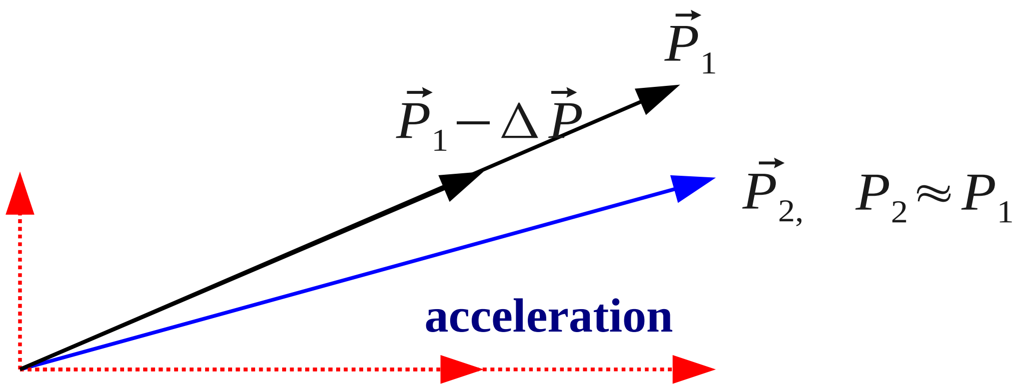
Radiation in a small 1/γ cone: $\Delta \vec{P} \| -\vec{P}$

After radiation, x', y' **do not change:** $x' = \frac{P_x}{P}$ $y' = \frac{P_y}{P}$

$$P_1 \rightarrow P_1 - \Delta P, \quad P_{1x} \rightarrow P_{1x} - \Delta P_x, \quad P_{1y} \rightarrow P_{1y} - \Delta P_y$$

After RF cavity, $(\vec{P}_1 - \Delta \vec{P}) \rightarrow \vec{P}_{2,}$ $P_2 \approx P_1$ $E_2 \approx E_1$

x', y' **are reduced:** $\Delta x' \approx -x' \left(\frac{\Delta P}{P_1}\right), \quad \Delta y' \approx -y' \left(\frac{\Delta P}{P_1}\right)$



Radiation Damping

- **Radiation damping in Longitudinal Directioin**

Radiation loss depends on E^2 (E – energy of electron, U_γ – energy loss per turn)

$$U_\gamma \propto E^2 B^2$$

$$\frac{dU_\gamma}{dE} = \frac{U_\gamma}{E_0}(2+D)$$

$$D = \frac{\oint \frac{\eta}{\rho}\left(2K_1 + \rho^{-2}\right) ds}{\oint \frac{ds}{\rho^2}}$$

- **Radiation damping rates and damping times**

$$\alpha_E = -\frac{1}{2T_0}\frac{U_\gamma}{E_0}(2+D) = -\frac{1}{2T_0}\frac{U_\gamma}{E_0} J_E \qquad \tau_E = \frac{2}{(2+D)}\frac{E_0}{U_\gamma} T_0$$

$$\alpha_x = -\frac{1}{2T_0}\frac{U_\gamma}{E_0}(1-D) = -\frac{1}{2T_0}\frac{U_\gamma}{E_0} J_x \qquad \tau_x = \frac{2}{(1-D)}\frac{E_0}{U_\gamma} T_0$$

$$\alpha_y = -\frac{1}{2T_0}\frac{U_\gamma}{E_0} = -\frac{1}{2T_0}\frac{U_\gamma}{E_0} J_y \qquad \tau_y = 2\frac{E_0}{U_\gamma} T_0$$

- **Damping partitions (independent of the arrangement of magnetic optics)**

$$J_E + J_x + J_y = 4$$

Quantum Excitation

Photon emission process is rapid and discrete:

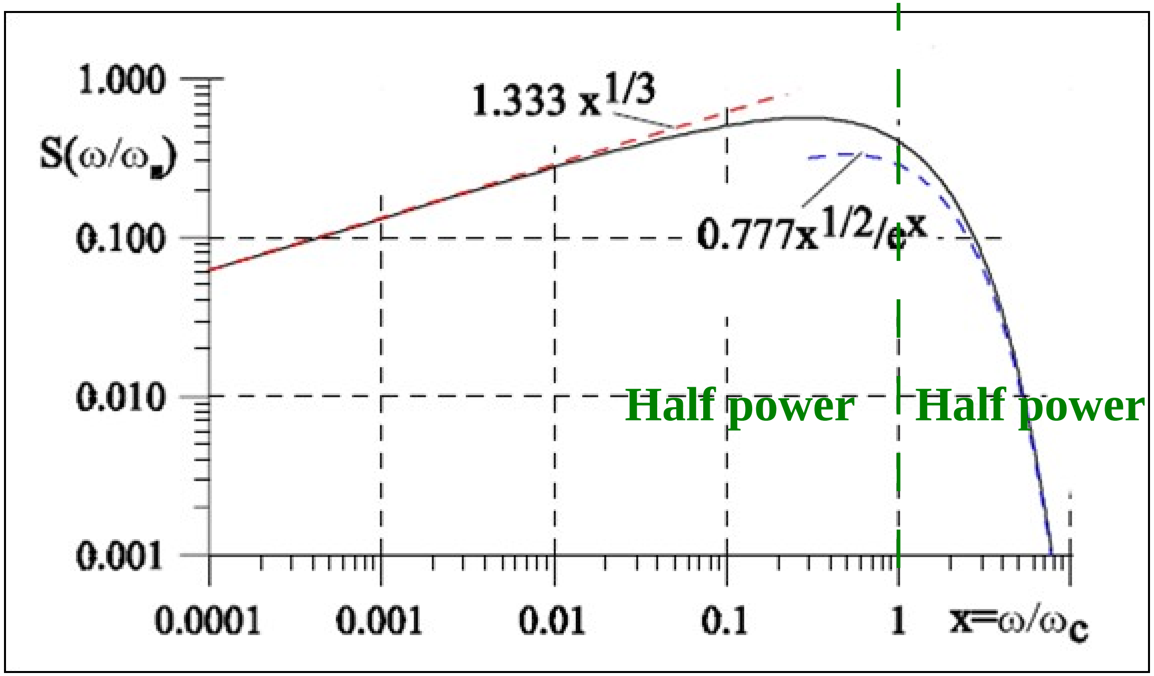
Change energy/momentum, not position

$$f(\varepsilon/\varepsilon_c) = \frac{8}{15\sqrt{3}} \frac{1}{\varepsilon_c} \frac{S(\varepsilon/\varepsilon_c)}{\varepsilon/\varepsilon_c}$$

with $S(x) = \frac{9\sqrt{3}}{8\pi} x \int_x^{\infty} K_{5/3}(y)\,dy$

Critical photon energy:

$$\varepsilon_c[keV] = 0.665\, E^2[GeV^2]\, B[T]$$



Quantum Excitation and Energy Spread and Bunch Length

- **Mean Photon Energy**

$$\langle \epsilon_\gamma \rangle = \frac{1}{N_\gamma} \int_0^\infty \epsilon_\gamma n(\epsilon_\gamma)\ d\epsilon_\gamma = \frac{P_\gamma}{N_\gamma} = \frac{8}{15\sqrt{3}} \epsilon_c$$

- **Mean Photon Energy Squared**

$$\langle \epsilon_\gamma^2 \rangle = \frac{1}{N_\gamma} \int_0^\infty \epsilon_\gamma^2 n(\epsilon_\gamma^2)\ d\epsilon_\gamma = \frac{11}{27} \epsilon_c^2$$

- **Energy Spread : balance of radiation damping and quantum excitation**

(A – amplitude of longitudinal motion)

$$\frac{dA^2}{dt} = -\frac{2}{\tau_E} A^2 + \frac{1}{E_0^2} \langle N_\gamma \langle \epsilon_\gamma^2 \rangle \rangle_{storage\ ring}$$

$$I_2 = \oint \frac{ds}{\rho^2}$$

$$I_3 = \oint \frac{ds}{\rho^3}$$

$$I_4 = \oint ds \frac{\eta}{\rho^3} (1 + 2K_1 \rho^2)$$

- **Relative Energy Spread (at the limit of zero current)**

$$\sigma_E^2 = \frac{1}{2} \langle A^2 \rangle = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} \frac{\gamma^2}{J_E} \frac{\langle \frac{1}{\rho^3} \rangle}{\langle \frac{1}{\rho^2} \rangle} = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} \gamma^2 \frac{I_3}{2I_2 + I_4}$$

$$\sigma_E \propto E$$

- **Bunch Length (simple phasespace rotation due to synchrotron oscillation)**

$$\sigma_l = \frac{c\eta_c}{\omega_s} \sigma_E$$

$$\sigma_l \propto E^{3/2} V_{RF}^{-1/2}$$

Quantum Excitation and Transverse Emittance

- **Horizontal Emittance (at the limit of zero current)**

$$I_5 = \oint ds \frac{H(s)}{\rho^3}$$

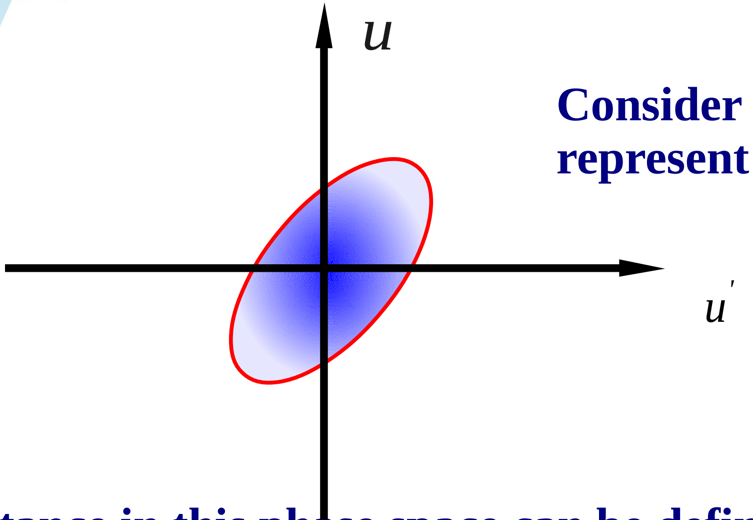
$$\epsilon_x = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} \frac{\gamma^2}{J_x} \frac{\langle \frac{H(s)}{\rho^3} \rangle}{\langle \frac{1}{\rho^2} \rangle} = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} \gamma^2 \frac{I_5}{I_2 - I_4}$$

$$\epsilon_x \propto E^2$$

where, $H(s) = \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta'_x + \beta_x \eta'^2_x$

- **Vertical Emittance**
  - **For a storage ring with perfect mid-plane symmetry, the H-function in the y-direction is zero due to zero vertical dispersion:** $H_y(s) = 0$
  - **Vertical emittance due to a finite opening of radiation angle (1/γ) is very small**
  - **Vertical emittance in the storage ring is typically determined by emittance coupling**
  - **A reasonably large vertical beam size can be desirable in some cases to achieve a longer beam lifetime and to reduce beam instabilities**

Phase Ellipse



Consider the uncoupled transverse motion. (u, u') is used to represent either horizontal or vertical phase space

Emittance in this phase space can be defined as the area occupied by the particles divided by p

$$\epsilon_u = \frac{A_{uu'}}{\pi}, \qquad u = x, y$$

Without acceleration, (u, u') can be used to present the phase space.

From Liouville's theorem, the emittance which is the phase space area is an invariant of the motion.

Transverse emittance therefore is constant for charged particle beams in an magnetic optics.

RMS Emittance

In the case of a real beam with a finite number of particles (N), a RMS emittance can be defined

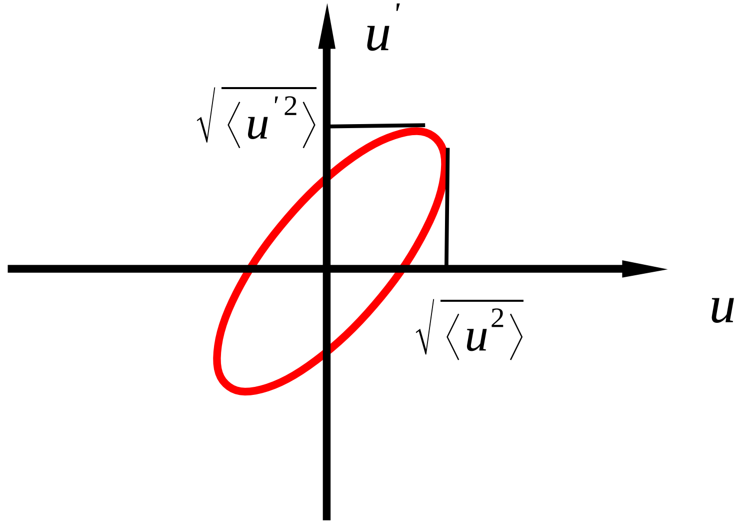
$$\langle u^2\rangle = \frac{1}{N}\sum_{n=1}^{N} u_n^2 \qquad \langle u'^2\rangle = \frac{1}{N}\sum_{n=1}^{N} u_n'^2 \qquad \langle u u'\rangle = \frac{1}{N}\sum_{n=1}^{N} u_n u_n'$$

$$\epsilon_{rms} = \sqrt{\langle u^2\rangle\langle u'^2\rangle - \langle u u'\rangle^2}$$

This allows us to associate **equivalent ellipse** **in the phase space with area** $\pi\,\epsilon_{rms}$

with the real beam distribution

$$\frac{\langle u'^2\rangle}{\epsilon_{rms}} u^2 + \frac{\langle u^2\rangle}{\epsilon_{rms}} u'^2 - 2\frac{\langle u u'\rangle}{\epsilon_{rms}} u u' = \epsilon_{rms}$$



Beam Twiss Parameters

A set of twiss parameters can be defined using the equivalent beam ellipse by comparing the following ellipse expressions

$$\frac{\langle u'^2\rangle}{\epsilon_{rms}}u^2+\frac{\langle u^2\rangle}{\epsilon_{rms}}u'^2-2\frac{\langle u u'\rangle}{\epsilon_{rms}}u u'=\epsilon_{rms}$$

$$\gamma u^2+2\alpha u u'+\beta u'^2=\epsilon$$

Beam twiss parameters

$$\beta_{beam}=\frac{\langle u^2\rangle}{\epsilon_{rms}}$$

$$\alpha_{beam}=\frac{-\langle u u'\rangle}{\epsilon_{rms}}$$

$$\gamma_{beam}=\frac{\langle u'^2\rangle}{\epsilon_{rms}}$$

$$\beta_{beam}\gamma_{beam}-\alpha_{beam}^2=1$$

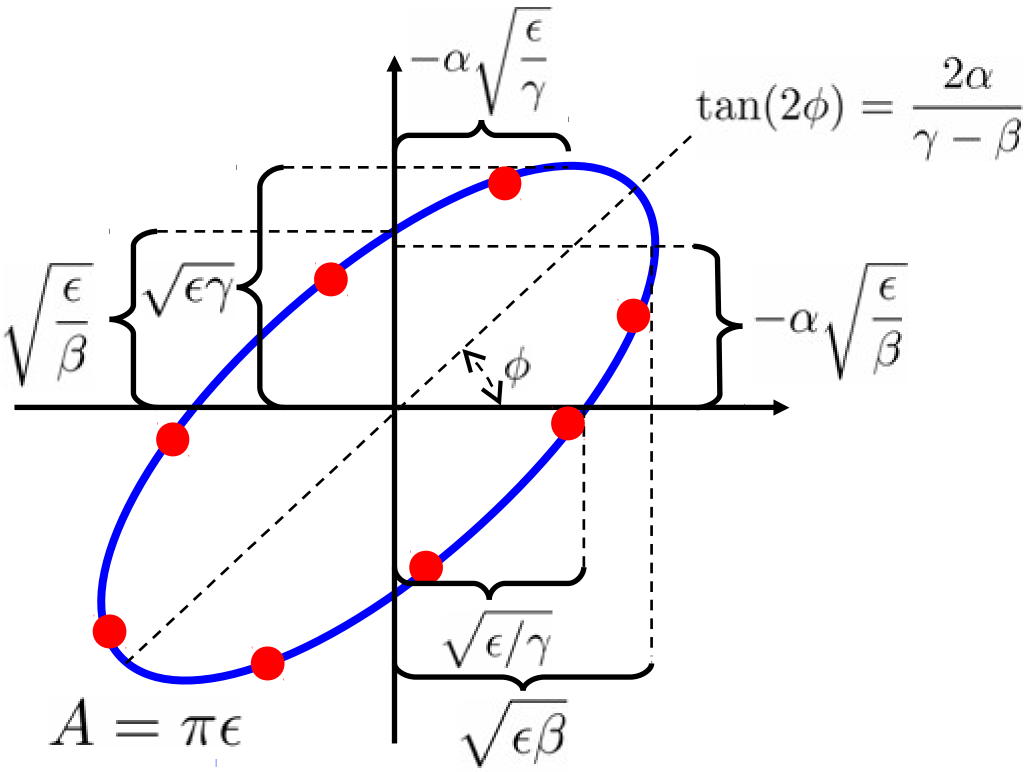


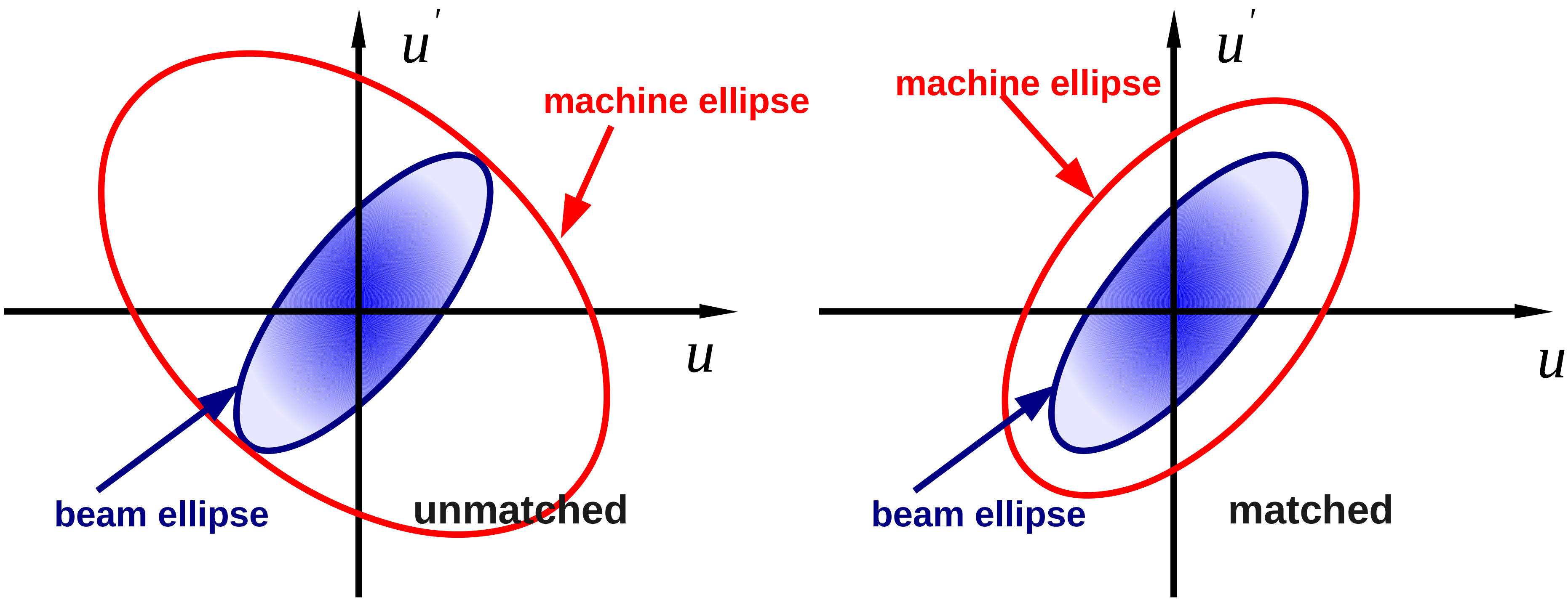
Figure by David Robin,LBNL

Machine Ellipse vs Beam Ellipse

The beam ellipse should be matched to the machine ellipse in order to be effectively transported in the periodic system

$$\frac{\langle u'^2\rangle}{\epsilon_{rms}} u^2 + \frac{\langle u^2\rangle}{\epsilon_{rms}} u'^2 - 2\frac{\langle u u'\rangle}{\epsilon_{rms}} u u' = \gamma_{beam} u^2 + 2\alpha_{beam} u u' + \beta_{beam} u'^2 = \epsilon_{rms}$$

$$\gamma u^2 + 2\alpha u u' + \beta u'^2 = \epsilon$$



Matched case

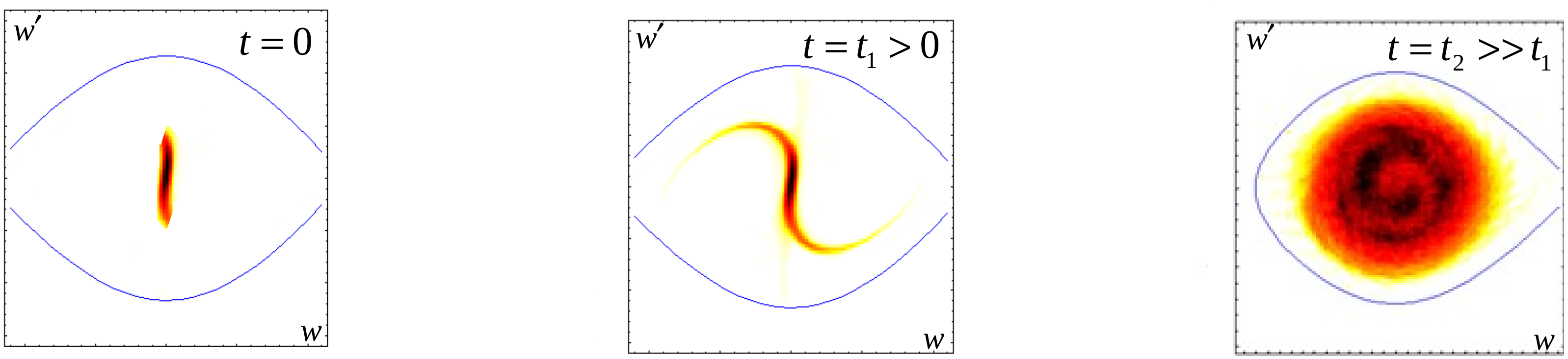
$$\beta_{beam} = \beta\,, \quad \alpha_{beam} = \alpha\,, \quad \gamma_{beam} = \gamma$$

Nonlinear Forces and Filamentation

For a Hamiltonian system, the emittance is conserved due to Liouville's theorem

However, when nonlinear forces act on the system, e.g. nonlinear magnetic fields, space charge force, the rms emittance is not conserved

Filamentation: a nonlinear mixing of the particle distribution in the phase space



Filamentation will typically lead to the growth of the rms emittance while the emittance according Liouville's theorem is supposedly to remain unchanged

Importance of Emittance in Accelerator Applications

What determines the emittance

- Emittance is an invariant in Hamiltonian system: transport lines, linear accelerators, heavy particles rings with no substantial radiation damping. Initial preparation of the beam determines its emittance.
- Emittance is determined by the balance of synchrotron radiation emission (quantum excitation) and radiation damping in the electron and positron storage rings.

Importance of Emittance in Accelerator Applications

- Synchrotron light sources: smaller emittance means higher brightness
- Colliders: smaller emittance results in higher luminosity
- FELs: a large emittance will degrade the FEL gain, requiring to increase the length of FEL undulators to achieve the same gain and saturation

Importance of Emittance in Accelerator Applications

Price of a low emittance storage ring

- Low emittance usually needs a strong focusing, which desires a set of strong sextupoles to compensate its large negative chromaticity. The nonlinear forces introduced by these sextupoles may significantly reduce the dynamic aperture. Many techniques have been developed to optmize the nonlinear beam dynamics of a storage ring.
- A small emittance ring may have a short Touschek lifetime (when the emittance is extremely small, the Touschek lifetime becomes longer again).
- The error tolerance on magnets is tight for a small emittance ring.
- The requirement on the power supply stability is higher as the beam size is small.

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