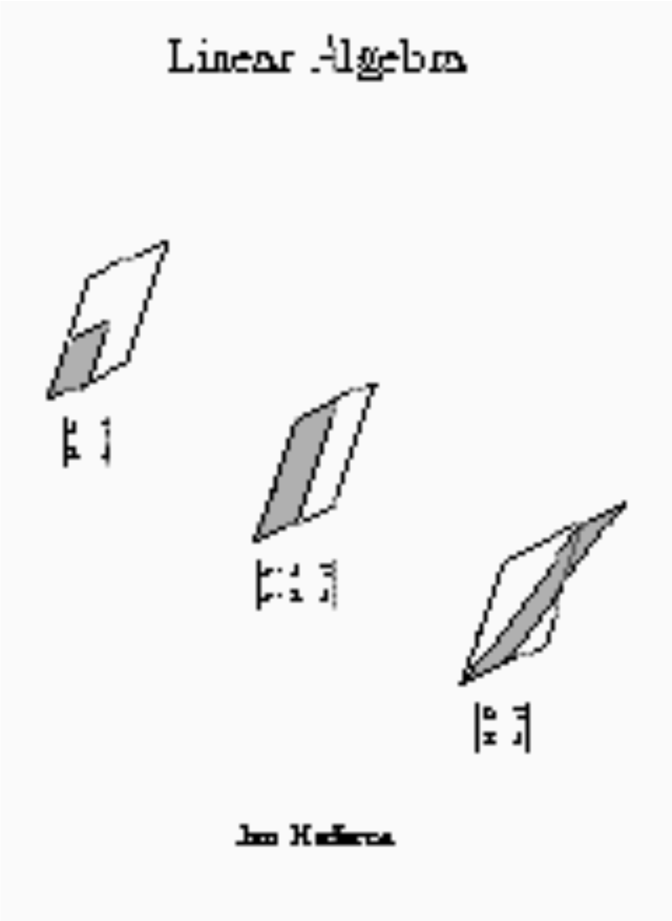
CONTROL ROOM ACCELERATOR PHYSICS

Day 2

Review of Linear Algebra

Outline

1. Introduction
2. Vector spaces
3. Matrices and linear operators
4. Eigenvalues and eigenvectors
5. Diagonalization
6. Singular-value decomposition



Introduction

Motivation

- Linear algebra provides many practical techniques for analysis and solution of linear (and nonlinear) systems. We utilize these established methods in our applications.

- This material should be a review, but we formalize it then show it in the context of accelerator control applications

Introduction

Linear Algebra: Definition and Description

- **Linear algebra** is the branch of mathematics concerned with the study of *vectors*, *vector spaces* (also called *linear spaces*), *linear maps* between vector spaces (also called *linear transformations, linear operators*), and systems of linear equations.

- Here we will think of "linear algebra" loosely as matrices
  - Matrices are "tangible",
  - They are computer friendly
  - Represent linear relations (maps) between finite dimensional vector spaces (e.g., the space of correctors and the space of BPMs)

- Our objective here is to review some basic facts about matrices and demonstrate applications to accelerator control

Introduction

Basic Notation

$\mathbf{Z}$

- The set of integers {…, -1, 0, +1, …}

$\mathbf{R}$

- The set of real numbers

$\mathbf{R}^n$

- The n times Cartesian product of **R**, or the set of "n-tuples"
  - Vectors of the form $\mathbf{x} = (x_1, \ldots, x_n)$

$\mathbf{R}^{m\times n}$

- The set of $m\times n$ matrices having real number elements

$L : \mathbf{R}^m \to \mathbf{R}^n$

- Linear operators mapping elements from $\mathbf{R}^m$ to elements in $\mathbf{R}^n$

$GL(n,\mathbf{R})$

- The set of elements in $\mathbf{R}^{n\times n}$ having nonzero determinant
  - These are the *invertible* matrices (has an inverse for matrix multiplication)

Vector Spaces

Definition and Description

- **Definition**: A *vector space* over the field $\mathbf{R}$ is a set V along with two operations, *vector addition* $+$ and *scalar multiplication* $\bullet$, such that for any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and scalars $r, s \in \mathbf{R}$ we have the axioms:

ugly details

- Associativity of addition: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
- Commutativity of addition: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- Identity element of addition: $\exists\ \mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.
- Inverse elements of addition:
  - For every $\mathbf{v} \in$ V, there exists an element $-\mathbf{v} \in V$, called the *additive inverse* of $\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
- Distributivity of multiplication w.r.t. vector addition: $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
- Distributivity of multiplication w.r.t. field addition: $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$
- Compatibility of scalar and field multiplication: $a(b\mathbf{v}) = (ab)\mathbf{v}$
- Identity element of multiplication: $1\mathbf{v} = \mathbf{v}$, where $1 \in \mathbf{R}$ is the identity.

basic idea

- Informally, "any linear combination of elements in V is also an element in V"
  - That is, If $\mathbf{u}, \mathbf{v} \in V$ and $r, s \in \mathbf{R}$ then $r\mathbf{u} + s\mathbf{v} \in V$

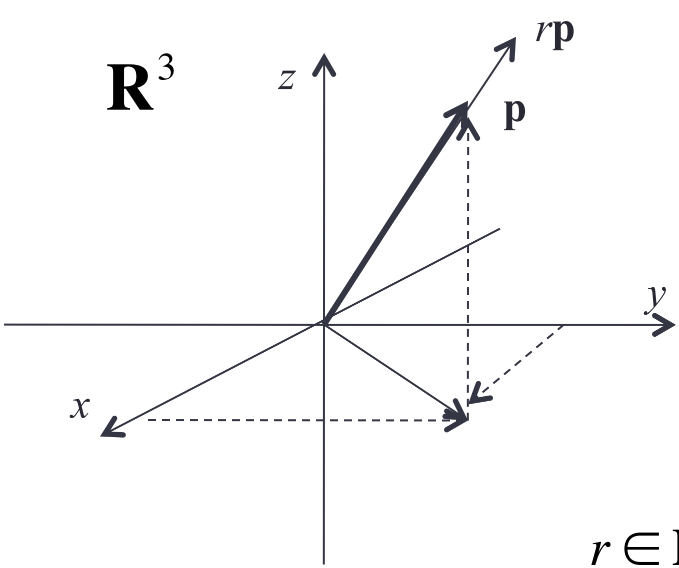
Anything that fits the above requirements is a vector space, and can be treated by linear algebraic methods.

Vector Spaces

The Space $\mathbf{R}^n$

We can consider the space $\mathbf{R}^n$ as the natural extension of the more familiar vectors in Euclidean 3 space.

This is the most common vector space for computer solution.



Addition of vectors

$$\mathbf{p} = (p_x, 0, 0) + (0, p_y, 0) + (0, 0, p_z) = (p_x, p_y, p_z)$$

Multiplication by scalars

$$r\mathbf{p} = r(p_x, p_y, p_z) = (rp_x, rp_y, rp_z)$$

$$r \in \mathbf{R}$$
$$\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n$$
$$\mathbf{y} = (y_1, \ldots, y_n) \in \mathbf{R}^n$$

Vector addition $\Rightarrow$ $\mathbf{x} + \mathbf{y} = \left(x_1 + y_1, \ldots, x_n + y_n\right)$

Scalar multiplication $\Rightarrow$ $r\mathbf{x} = \left(rx_1, \ldots, rx_n\right)$

Extension to $\mathbf{R}^n$

Vector Spaces

Examples

- Although $\mathbf{R}^n$ is the vector space of our primary concern, there are many others

  - The set $F(\mathbf{R})$ of real valued functions on the real line $\mathbf{R}$, $\{f : \mathbf{R} \rightarrow \mathbf{R}\}$
  - The set $F([0,1])$ of real functions on interval $[0,1]$, $\{f : [0,1] \rightarrow \mathbf{R}\}$
    - Note $F([0,1])$ is a vector subspace of $F(\mathbf{R})$

  - The set $\mathbf{R}^\infty$ of infinite sequences $\{r_1, r_2, r_3, \ldots\}$, $r_i \in \mathbf{R}$
  - The set $F(\mathbf{Z}_+)$ of real functions on the positive integers $\mathbf{Z}_+$ $\{f : \mathbf{Z}_+ \rightarrow \mathbf{R}\}$
    - Vector spaces $\mathbf{R}^\infty$ and $F(\mathbf{Z}_+)$ are isomorphic (the same thing)

  - The set $C^n(\mathbf{R})$ of continuous functions on $\mathbf{R}$ $\{f : \mathbf{R} \rightarrow \mathbf{R} \mid d^n f/dx^n < \infty\}$
  - The set $C^\infty(\mathbf{R})$ of smooth functions on $\mathbf{R}$ $\{f : \mathbf{R} \rightarrow \mathbf{R} \mid d^n f/dx^n < \infty \text{ all } n\}$

  - Note the subspace structure $C^\infty(\mathbf{R}) \subseteq C^n(\mathbf{R}) \subseteq F(\mathbf{R})$

Vector Spaces

Basis Sets

- A *basis* for vector space V is a (possibly infinite) subset $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots\}$ of V such that any $\mathbf{v} \in V$ can be expressed as a *linear combination* of basis vectors.
  - If $\mathbf{v} \in V$ then $\exists\, r_1, r_2, r_3, \ldots \in \mathbf{R}$ such that $\mathbf{v} = r_1\mathbf{e}_1 + r_2\mathbf{e}_2 + r_3\mathbf{e}_3 + \ldots$

- All basis sets of V have the same cardinality, called the *dimension* of V
  - $\dim V = |\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots\}|$

- Examples:
  - $(1,\ldots,0), (0, 1, \ldots,0), \ldots, (0, 0, \ldots,1)$ form the *standard* basis for $\mathbf{R}^n$.
  - $\{1, x, x^2, \ldots, x^{n-1}\}$ is a basis for the subspace of f in $C^\infty([0,1])$ such that $d^nf/dx^n = 0$
  - The eigenvectors of any symmetric, pos. def. matrix $\mathbf{A} \in \mathbf{R}^{n\times n}$ form a basis of $\mathbf{R}^n$.
  - The first n eigenfunctions $\{\psi_1, \psi_2, \ldots, \psi_n\}$ of quantum mechanical operator $\mathbf{A}$ form a basis for the subspace containing the first n observables of A.
  - $\{1, \cos 2\pi ft, \cos 4\pi ft, \ldots, \cos (n\text{-}1)\pi ft\}$ is a basis for the set of all even, band-limited functions with period $1/f$, and cutoff frequency nf.

All these vector spaces are isomorphic!

(As a Hilbert space they are not)

Linear Operators Between Vector Spaces

Matrix Representations

A *linear map* $L : U \rightarrow V$ between vector spaces U and V has the defining property

$L(r\mathbf{x}+s\mathbf{y}) = rL(\mathbf{x}) + sL(\mathbf{y})$ for vectors $\mathbf{x},\mathbf{y} \in U$ and scalars $r,s \in \mathbf{R}$.

(Linear maps respect the algebra of vector spaces!)

- Given any basis set $\{\mathbf{e}_i\} \subseteq U$, by linearity L is completely defined by where it maps basis vectors $\{\mathbf{e}_i\} \in U$, that is by $\{L(\mathbf{e}_i)\} \in V$
  - Any $\mathbf{u} \in U$ can be expressed as $\mathbf{u} = r_1\mathbf{e}_1+\ldots+r_m\mathbf{e}_m$ some $\{r_i\} \subseteq \mathbf{R}$
  - Then $L(\mathbf{u}) = L(r_1\mathbf{e}_1+\ldots+r_m\mathbf{e}_m) = r_1L(\mathbf{e}_1) + \ldots + r_mL(\mathbf{e}_m)$
  - Is $\{L(\mathbf{e}_1), \ldots, L(\mathbf{e}_m)\}$ a basis set for V ?

- A matrix $\mathbf{A}$ in $\mathbf{R}^{m\times n}$ is, in a natural way, a linear map between the vector spaces $\mathbf{R}^m$ and $\mathbf{R}^n$ under the usual matrix multiplication
  - Once again $\mathbf{Au} = \mathbf{A}(r_1\mathbf{e}_1+\ldots+r_m\mathbf{e}_m) = r_1\mathbf{Ae}_1 + \ldots + r_m\mathbf{Ae}_m$
  - When is $\{\mathbf{Ae}_1, \ldots, \mathbf{Ae}_m\}$ a basis set for $\mathbf{R}^n$?

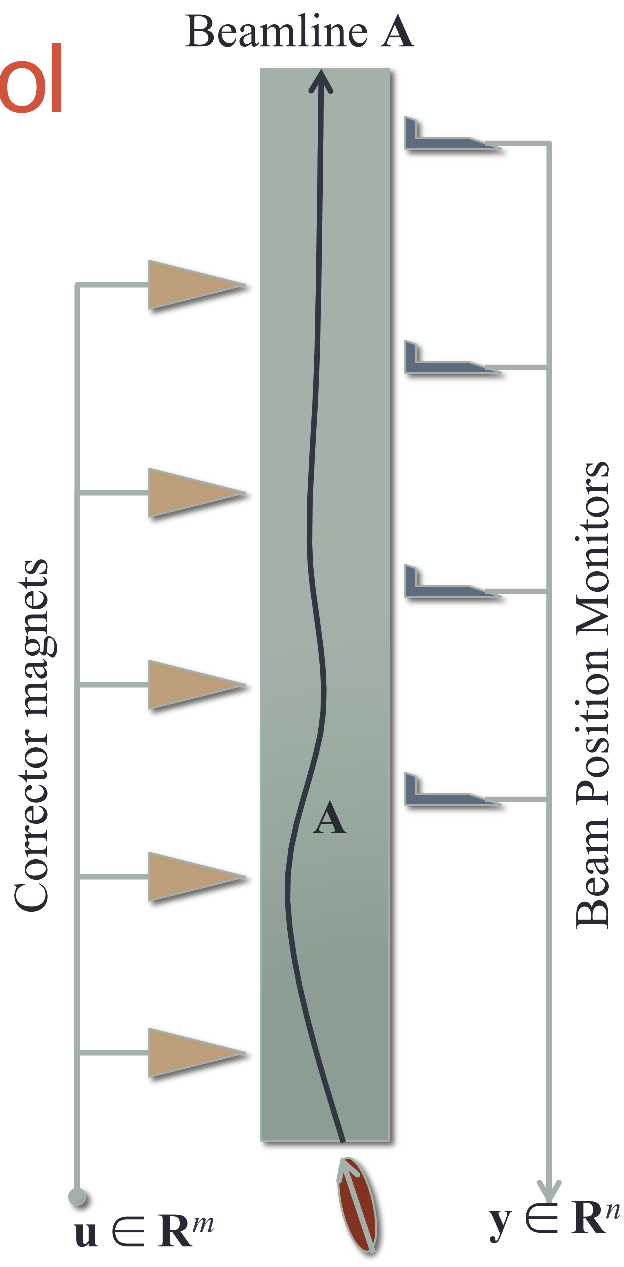
Matrices in Accelerator Control

Maps Between Sensors and Actuators

- A matrix **A** in $\mathbf{R}^{m\times n}$ can represent a linear map between the vector spaces $\mathbf{R}^m$ and $\mathbf{R}^n$

  - Accelerator Control Example:
    - Elements of $\mathbf{R}^m$ are *corrector magnet* strengths (m mag.'s)
    - Elements of $\mathbf{R}^n$ are *BPM readbacks* (n BPMs)
    - Then **A** in $\mathbf{R}^{m\times n}$ is the *response matrix*

  - Note three cases concerning response matrix **A**:
    - $m > n$ Domain $\mathbf{R}^m$ is "bigger" than range $\mathbf{R}^n$ (more correctors than BPMs)
    - $m < n$ Domain $\mathbf{R}^m$ is "smaller" than range $\mathbf{R}^n$ (more BPMs than correctors)
    - $m = n$ The matrix **A** is square (equal numbers of correctors and BPMs)



Matrices in Accelerator Control

Analyzing Relationship Between Sensors and Actuators

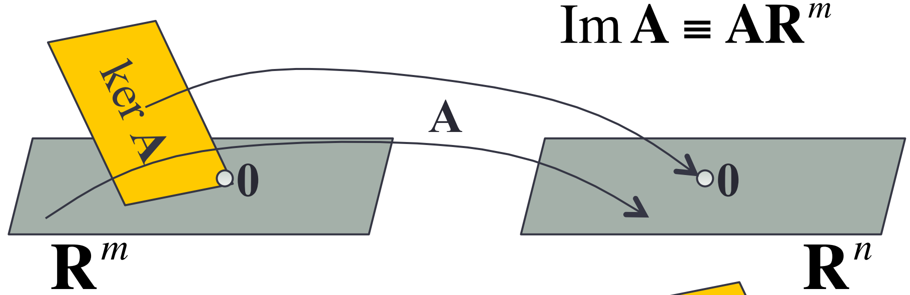
Assume **A** has full rank throughout

$$\ker \mathbf{A} \equiv \left\{ \mathbf{x} \in \mathbf{R}^n \mid \mathbf{A}\mathbf{x} = 0 \right\}$$

$$\operatorname{Im} \mathbf{A} \equiv \mathbf{A}\mathbf{R}^m$$

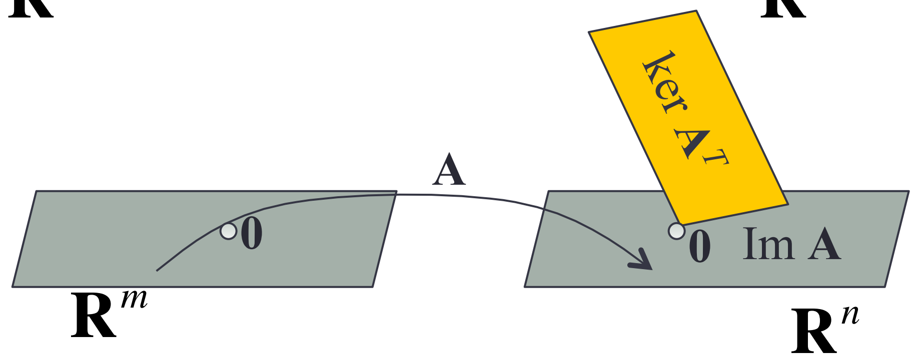
Case 1: $m>n$ $\quad \ker \mathbf{A} \neq 0$

We can get to all beam positions. But, corrector settings are not unique (and they can "fight" each other).



Case 2: $m<n$ $\quad \operatorname{Im} \mathbf{A} \neq \mathbf{R}^n$

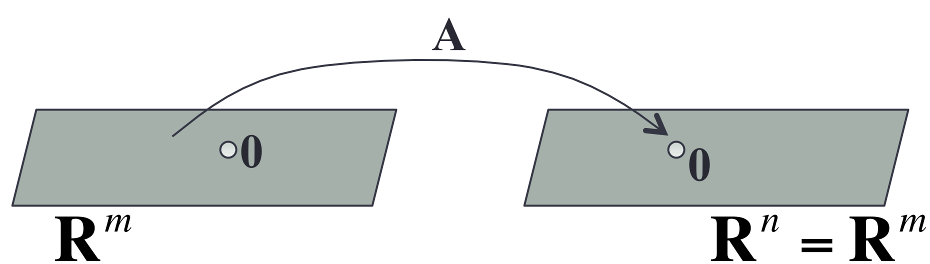
We cannot get to all the beam positions. We don't have enough correctors.



Case 3: $m=n$

$$\ker \mathbf{A} = \mathbf{0}, \quad \operatorname{Im} \mathbf{A} = \mathbf{R}^n$$

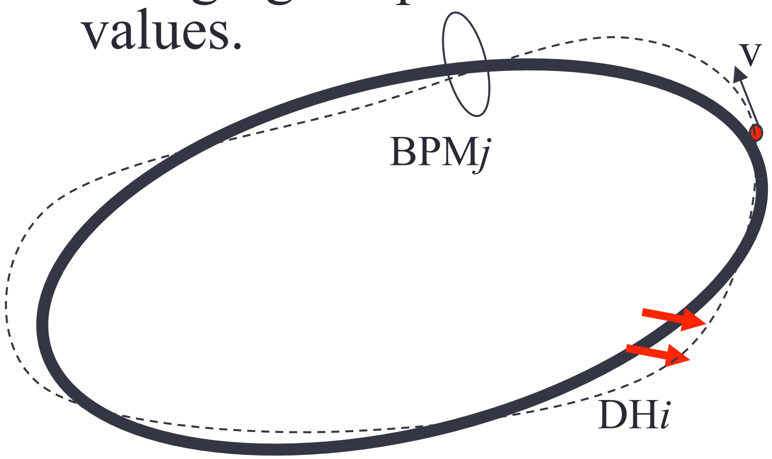
We can steer to all beam positions. Each has a unique corrector setting.



Matrices in Accelerator Control

Matrices are Ubiquitous in Beam Physics

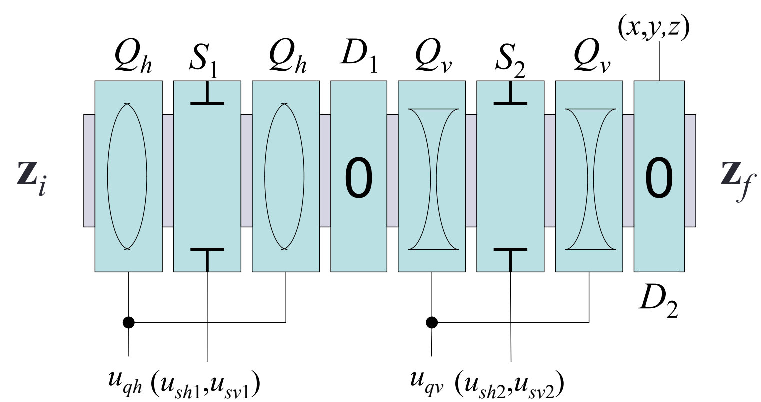
Determining the perturbations in the closed orbit of a ring when changing a dipole corrector values.



$$\mathbf{A} = \begin{pmatrix} \frac{\partial BPM_1}{\partial DH_1} & \cdots & \frac{\partial BPM_1}{\partial DH_m} \\ \vdots & & \vdots \\ \frac{\partial BPM_n}{\partial DH_1} & \cdots & \frac{\partial BPM_n}{\partial DH_m} \end{pmatrix}$$

Response matrices

Propagating the beam coordinates $\mathbf{z}_i$ at one location of the beamline to another (downstream) location $\mathbf{z}_f$.



$$\mathbf{z}_f = \Phi(\mathbf{u})\mathbf{z}_i \qquad \Phi \in \mathbf{R}^{6\times 6}$$

Transfer matrices

Matrix Decomposition

Eigenvalues, Factorization, Diagonalization

- Because matrices model important aspects of beam physics and accelerator control, it is instructive to look at their structure
  - The *eigenvectors* and *eigenvalues* of a matrix decompose the vector spaces into the natural "modes" of the system
  - Matrix *factorization* techniques decompose a matrix into constituent factors with special structure
  - Matrix *diagonalization* is a special type of factorization which identifies the eigen modes of a matrix
  - We cover two explicit methods of factoring a matrix, eigenvalue decomposition and singular-value decomposition, both are diagonalization processes.
- We focus on response matrices throughout the discussion, however, the material also applies to transfer matrices, linearized dynamics, coupled equations, etc.

Eigenvectors and Eigenvalues
The Natural Modes of A Square Matrix

- For a *square* matrix $\mathbf{A}$ in $\mathbf{R}^{n\times n}$ we can often find special vectors $\mathbf{e}$ in $\mathbf{R}^n$ so that

$$\mathbf{Ae} = \lambda\mathbf{e}$$

  where $\lambda \neq 0$ is a scalar (either real or complex)
  - Any such vector $\mathbf{e}$ is called an *eigenvector* of $\mathbf{A}$
  - Any such scalar λ is called an *eigenvalue* of $\mathbf{A}$

- $\mathbf{A}$ does not change the direction of $\mathbf{e}$, only the length!
  - $\mathbf{A}$ acts like an amplifier on $\mathbf{e}$ with gain λ
  - Eigenvectors $\mathbf{e}_1$, $\mathbf{e}_2$, … are the natural, (decoupled), modes of $\mathbf{A}$
  - What if we could decompose *all* of $\mathbf{R}^n$ into eigenvectors of $\mathbf{A}$? (see homework)

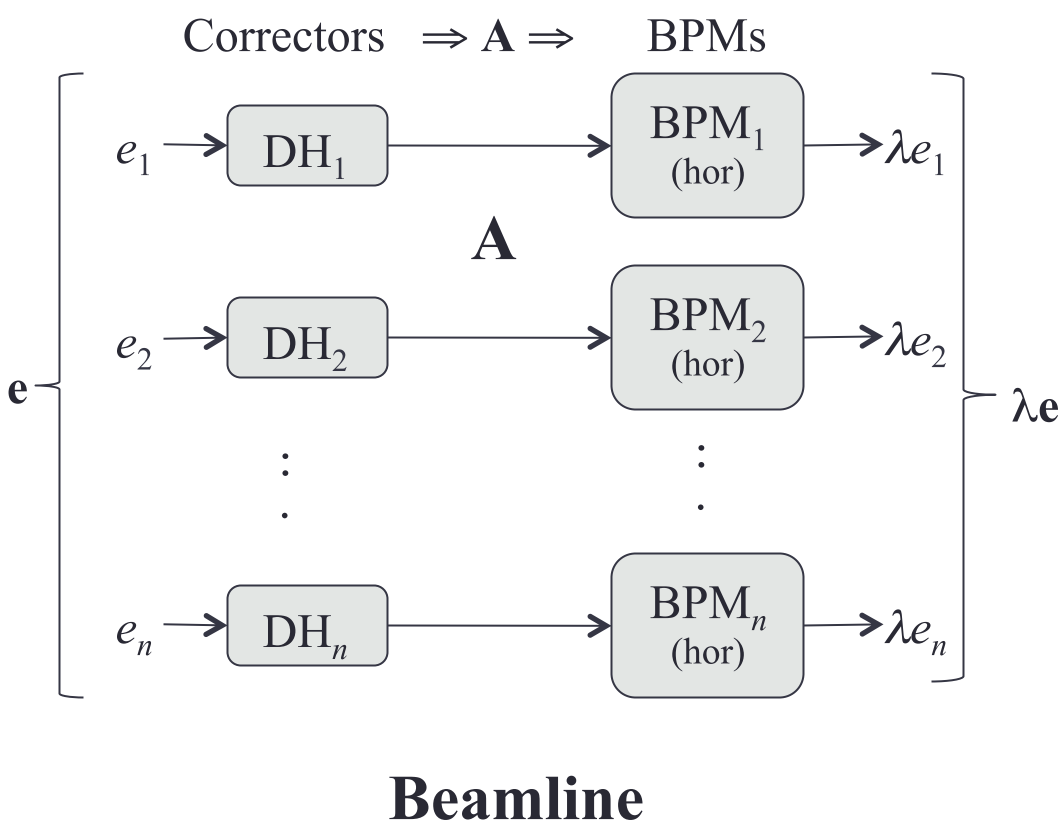
Eigenvectors and Eigenvalues

Example: Examining Orbit Position System Response

Say we find an eigenvector $\mathbf{e} = (e_1, e_2, \ldots, e_n)$ for the response matrix $\mathbf{A}$ so that

$$\mathbf{Ae} = \lambda\mathbf{e}$$

- Drive each dipole corrector DH_i with the strength e_i at the eigenvector coordinate i
- Each BPM_i behaves as if it is directly connected to DH_i
- The response (beam positions) are simply amplified by eigenvalue λ



Called a *natural* (*uncoupled*) *mode* of the system

Matrix Factorizations

Examples:

- Algebraic analogy:
  - The integers $\mathbf{Z}$ have prime factorizations, e.g., 42 = 2•3•7
  - Real numbers have only trivial factorizations (every real number is a factor of any other real number).

- Matrix $\mathbf{A}$ in $\mathbf{R}^{m\times n}$ admits many factorizations (or *decompositions*)
  - $\mathbf{A} = \mathbf{LU}$ $(m{=}n)$, $\mathbf{L}$ upper triangular, $\mathbf{U}$ lower triangular ("LU" decomposition)
  - $\mathbf{A} = \mathbf{QR}$, $\mathbf{Q} \in SO(m)$, $\mathbf{R}$ symmetric, positive definite (polar decomp.)
  - $\mathbf{A} = \mathbf{T\Lambda T}^{-1}$ $(m{=}n)$, $\mathbf{T} \in GL(n, \mathbf{R})$, $\mathbf{\Lambda} \in \mathrm{diag}(\mathbf{R}^{n\times n})$ (eigenvalue decomp.)
  - $\mathbf{A} = \mathbf{UDV}^T$, $\mathrm{U} \in SO(m)$, $\mathbf{D} \in \mathrm{diag}(\mathbf{R}^{n\times n})$, $\mathrm{V} \in SO(n)$ (sing. value decomp.)

- We cover eigenvalue decomposition and the more general singular value decomposition in detail.

Matrix Factorization

Factoring into Natural Modes

Sometimes a square matrix **A** in $\mathbf{R}^{n\times n}$ can be factored as

$$\mathbf{A} = \mathbf{T}\boldsymbol{\Lambda}\mathbf{T}^{-1}$$

Where **T** is in $GL(n,\mathbf{R})$ (invertible) and $\boldsymbol{\Lambda} = \mathrm{diag}\{\lambda_1,\ldots,\lambda_n\}$ in $\mathbf{R}^{n\times n}$.

- **Λ** is called the *spectral matrix* (with spectrum $\{\lambda_1,\ldots,\lambda_n\}$)
- **T** is called the *modal matrix* and is composed of eigenvectors
- When this condition is satisfied, i.e. when $\mathbf{A} = \mathbf{T}\boldsymbol{\Lambda}\mathbf{T}^{-1}$, we say **A** is *diagonalizable*

- The matrix **T** describes the coupling modes between the correctors and the BPMs.
- The matrix **Λ** describes the gains between these natural couplings
- For example….

Matrix Factorization

Decoupling Beam Position Response

- Our corrector space DH is isomorphic to $\mathbf{R}^n$, that is, $DH \cong \mathbf{R}^n$
  - The vector $\mathbf{x} \in DH$ represents the set of corrector strengths $x_1, x_2, \ldots, x_n$

- Our response space BPM is isomorphic to $\mathbf{R}^n$, that is, $BPM \cong \mathbf{R}^n$
  - The vector $\mathbf{y} \in BPM$ represents the set of beam positions $y_1, y_2, \ldots, y_n$

- The response matrix $\mathbf{A} \in \mathbf{R}^{n \times n}$ relates $\mathbf{y}$ to $\mathbf{x}$, that is,

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

$$\mathbf{A} = \begin{pmatrix} \frac{\partial BPM_1}{\partial DH_1} & \cdots & \frac{\partial BPM_1}{\partial DH_m} \\ \vdots & & \vdots \\ \frac{\partial BPM_n}{\partial DH_1} & \cdots & \frac{\partial BPM_n}{\partial DH_m} \end{pmatrix}$$

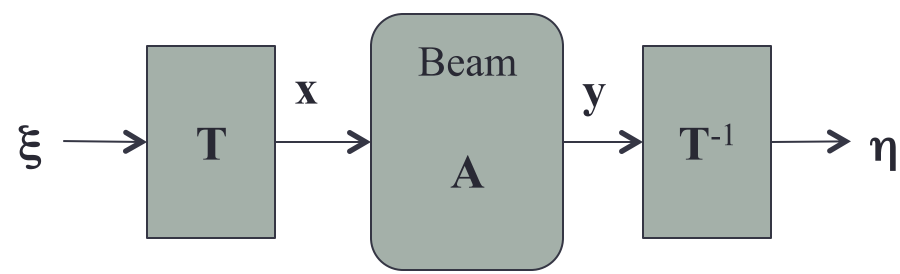
- If $\mathbf{A}$ factors as $\mathbf{A} = \mathbf{T}\boldsymbol{\Lambda}\mathbf{T}^{-1}$, then $\boldsymbol{\Lambda} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$
  - Set $\boldsymbol{\xi} \equiv \mathbf{T}^{-1}\mathbf{x}$ and $\boldsymbol{\eta} \equiv \mathbf{T}^{-1}\mathbf{y}$

    yielding

$$\boldsymbol{\eta} = \boldsymbol{\Lambda}\boldsymbol{\xi}$$

    or the n scalar equations

$$\eta_i = \lambda_i \xi_i \quad \text{for } i = 1, \ldots, n$$



Design of a response decoupler

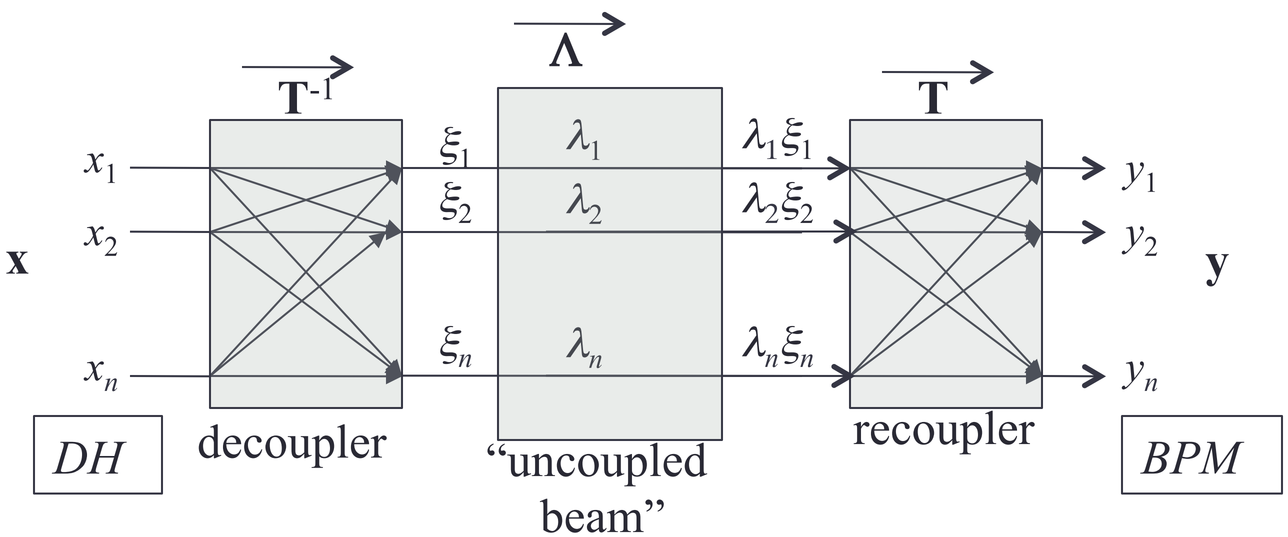
The restrictive part here is requiring the same number of correctors and BPMs

Matrix Diagonalization

Decoupling Beam Position Response (Special Case $m=n$)

Matrix diagonalization decouples our corrector space $DH \cong \mathbf{R}^n$ and our response space $BPM \cong \mathbf{R}^n$ into the natural modes of (square) response matrix $\mathbf{A}$

- Again, transform our inputs $\mathbf{x}$ to the *uncoupled* inputs $\xi \equiv \mathbf{T}^{-1}\mathbf{x}$
- Let $\mathbf{s}_i \equiv (0,\ldots, 1,\ldots, 0) \in \mathbf{R}^n$ be the i^{th} standard basis vector
- Let $\mathbf{x} = \mathbf{e}_i$ and note (accept) $\mathbf{e}_i \equiv \mathbf{T}\mathbf{s}_i = \text{col}_i\, \mathbf{T} \in DH$
  - $\mathbf{T}\mathbf{s}_i$ is the i^{th} eigenmode. Or equivalently, $\mathbf{T}^{-1}\mathbf{e}_i = \mathbf{s}_i$ transforms eigenvectors to standard basis vectors.
  - Recall each $\mathbf{x} \in DH$ decomposes as $\mathbf{x} = \xi_1\mathbf{e}_1 + \ldots + \xi_n\mathbf{e}_n$ (Hwk - prove this!)
- Output vector $\mathbf{y}_i = \lambda_i\mathbf{e}_i = \lambda_i\mathbf{T}\mathbf{s}_i \in BPM$ is the response to eigenvector input $\mathbf{e}_i$.



$$\mathbf{y} = \mathbf{T}\Lambda\mathbf{T}^{-1}\mathbf{x}$$

Singular Value Decomposition (SVD)

Generalizing Eigenvalue Decomposition

Any matrix $\mathbf{A} \in \mathbf{R}^{m\times n}$ may be factored as follows:

$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^T$$

where $\mathbf{U} \in \mathbf{R}^{m\times m}$, $\mathbf{D} = \mathrm{diag}\{\sigma_1,\ldots,\sigma_n\} \in \mathbf{R}^{m\times n}$, $\mathbf{V} \in \mathbf{R}^{n\times n}$

- The numbers $\{\sigma_1,\ldots,\sigma_n\}$ are the *singular values* of **A**
  - They are non-negative real numbers
  - They are generalizations of eigenvalues for square matrices
  - Specifically, $\sigma_i = \left[\lambda_i(\mathbf{A}^T\mathbf{A})\right]^{1/2}$ for $n \le m$ and $\sigma_i = \left[\lambda_i(\mathbf{A}\mathbf{A}^T)\right]^{1/2}$ for $n \ge m$
- Matrix **U** and **V** have special properties
  - $\mathbf{V}^T\mathbf{V} = \mathbf{I} \in \mathbf{R}^{n\times n}$, that is, it is *orthogonal*, in set $SO(n)$
  - $\mathbf{U}^T\mathbf{U} = \mathbf{I} \in \mathbf{R}^{m\times m}$, that is, it is *orthogonal,* in set $SO(m)$
    - (note $\mathbf{U}^T\mathbf{U} = \mathbf{I}$ implies $\mathbf{U}^T = \mathbf{U}^{-1}$)

NOTE: No requirements on the number of correctors and BPMs

Singular Value Decomposition (SVD)

Generalizing Diagonalization

- The columns of $\mathbf{V}$ are called the *right singular vectors* of $\mathbf{A}$
- The columns of $\mathbf{U}$ are called the *left singular vectors* of $\mathbf{A}$

Again consider the matrix-vector equation

$$\mathbf{y} = \mathbf{A}\mathbf{x} = \mathbf{U}\mathbf{D}\mathbf{V}^T\mathbf{x}$$

and perform the substitutions

- $\boldsymbol{\xi} = \mathbf{V}^T\mathbf{x}$
- $\boldsymbol{\eta} = \mathbf{U}^T\mathbf{y}$

We then have the (almost) equivalent decouple system

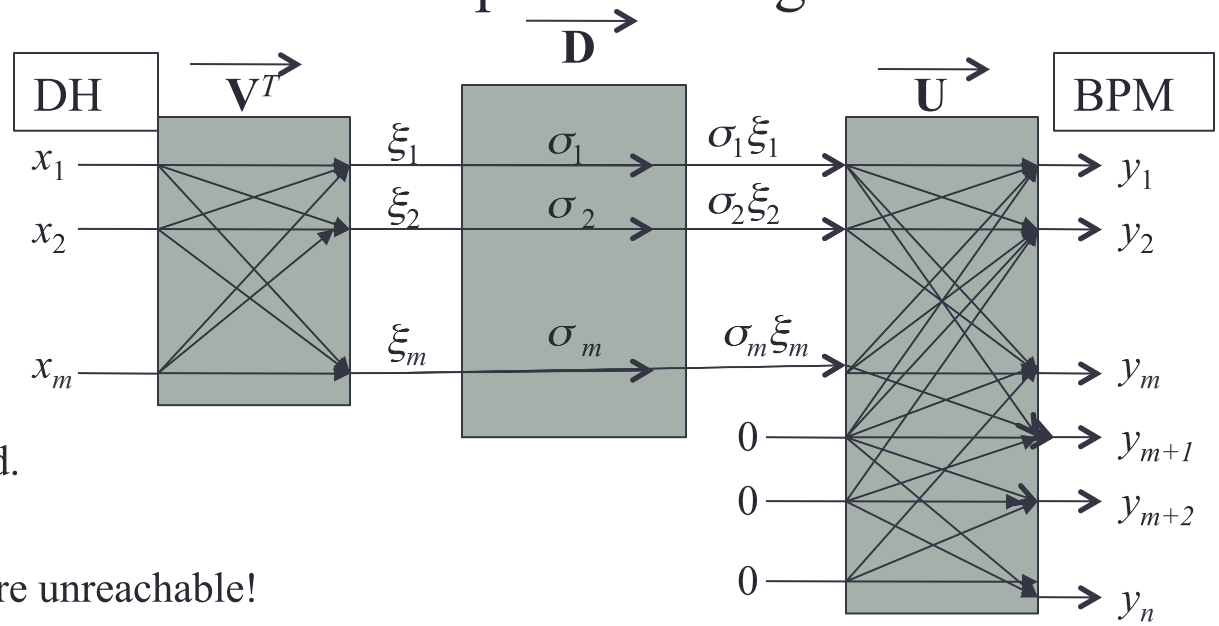
$$\boldsymbol{\eta} = \mathbf{D}\boldsymbol{\xi}$$

Like the case of a diagonalizable $\mathbf{A}$, this equation is completely decoupled $\Rightarrow$ $\eta_i = \sigma_i \xi_i$

Singular Value Decomposition (SVD)

Generalizing Diagonalization

Returning to the corrector/BPM example with singular-value decomposition



Clearly this situation is analogous to diagonalizable case, however…
We must be careful!

- Some inputs may not be connected.

- We have outputs $\{\mathbf{y}\} \subseteq \mathbf{R}^n$ that are unreachable!

- This example was for the case where $m<n$, we have analogous results for $m>n$
"Dead inputs" $\{\mathbf{x}\} \subseteq \mathbf{R}^m$ that do nothing

SVD is an important part of Model-Independent Analysis in high-level beam control

Singular Value Decomposition

Summary

- We can factor *any* matrix $\mathbf{A} \in \mathbf{R}^{m \times n}$ as $\mathbf{A} = \mathbf{UDV}^T$
  - This factorization alone provides an enormous amount of information about a system represented by

$$\mathbf{y} = \mathbf{Ax}$$

- Matrix **V** tells you coupling of inputs (i.e., the x_i)
  - Corrector combinations that have effect

- Matrix **U** tells you coupling of outputs (i.e., the y_i)
  - BPMs combinations that are likely/unlikely

- The diagonal matrix **D** provides …
  - The gains (i.e., the σ_i) for the system
  - Internal system degeneracy – "zero" singular value

Matrix Exponential

A Matrix Function

- Functions of (square) matrices are common in analysis
  - For example, sin($\mathbf{A}$), log($\mathbf{A}$), exp($\mathbf{A}$), for $\mathbf{A} \in \mathbf{R}^{n \times n}$
  - These functions may seem strange, but they are well-defined by the *Taylor series* for their scalar function sibling (matrix powers are well-defined)

- Of particular importance for us is the matrix exponential $e^{t\mathbf{A}}$
  - The scalar t representing time
  - Occurs frequently in linear dynamical systems as the natural response
  - Defined according to Taylor series

$$e^{t\mathbf{A}} \equiv I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \ldots$$

  - Show $e^{t\mathbf{A}}$ is well defined for all $\mathbf{A} \in \mathbf{R}^{n \times n}$ given that e^a is well defined for all $a \in \mathbf{R}$

- Say $\mathbf{A}$ is diagonalizable where $\mathbf{A} = \mathbf{T\Lambda T}^{-1}$
  - Show that $e^{t\mathbf{A}} = \mathbf{T}e^{t\mathbf{\Lambda}}\mathbf{T}^{-1}$
  - The exponential $e^{t\mathbf{\Lambda}}$ is easy to compute, $e^{t\mathbf{\Lambda}} = \text{diag}\{e^{t\lambda 1}, \ldots, e^{t\lambda n}\}$
  - Jacobi identity

$$\det e^{\mathbf{A}} = e^{\operatorname{tr}\mathbf{A}}$$

Control Theory

A Preview

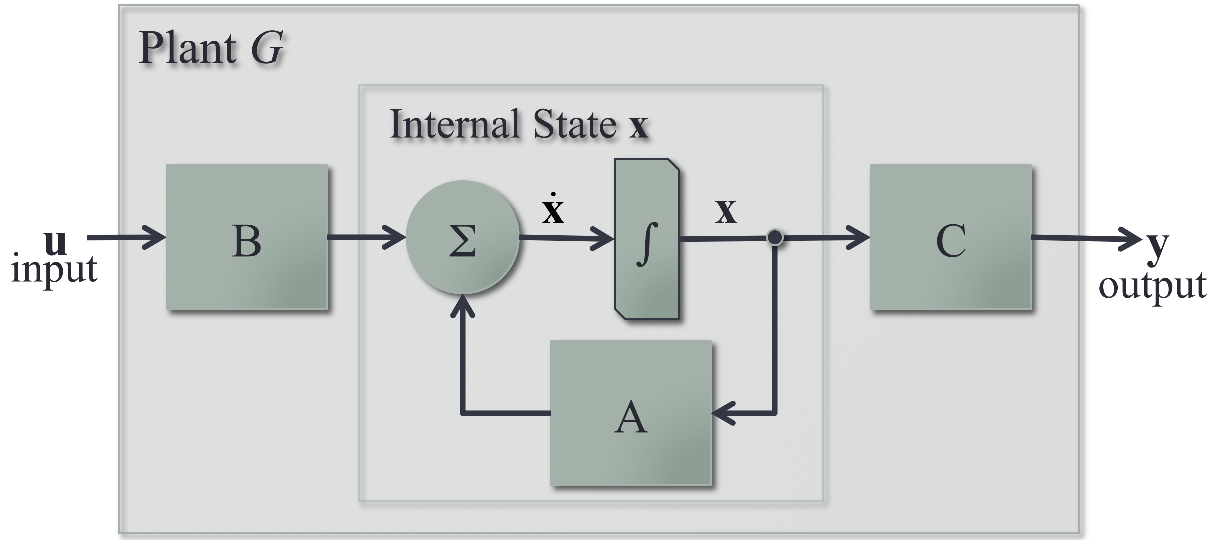
Given a system (a "plant") *G,* control theory starts with the basic questions:
Is the system…

- Stable: is the output **y** bounded for bounded inputs **u**?
  - Is $||\mathbf{y}||<\infty$ for $||\mathbf{u}||<\infty$?
- Observable: can we deduce internal state **x** by observing the inputs **u** and outputs **y**?
- Controllable: can we steer the system to any arbitrary output **y**?
  - (There exists a $\mathbf{u}(t)$ to do so?)

Then we can ask how to *stabilize*, how to *observe*, how to *control*

Linear, continuous-time plant *G*

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t)$$



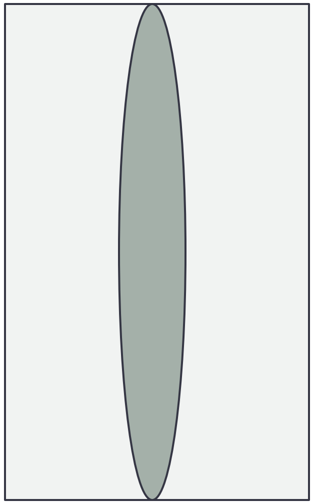
Linear Beam Optics

A Preview

Linear systems and the matrix exponential play a crucial part in linear beam optics and, consequently, the XAL online model

- In linear beam optics beamline elements are modeled by matrices $\mathbf{\Phi} \in Sp(6)$.
  - $Sp(6)$ is The group of 6x6 *symplectic* matrices.
- These matrices are formed from the exponential of another matrix **G**
- The matrix **G** represents the equations of motion
- The XAL online model is based upon these ideas

Focusing Quadrupole n



$$\mathbf{z}(s) = \mathbf{\Phi}_n(s)\mathbf{z}_0$$

$$\mathbf{\Phi}_n(s) \equiv e^{s\mathbf{G}_n}$$

$$\mathbf{z}'(s) = \mathbf{G}_n(s)\mathbf{z}(s)$$

Linear Algebra

Summary

- Matrices can be treated as linear operators between finite dimensional vector spaces, in particular, the spaces $\mathbf{R}^n$
- A square matrix $\mathbf{A}$ often has eigenvalues and eigenvectors that characterize the action of $\mathbf{A}$ upon vector space $\mathbf{R}^n$
- If a matrix $\mathbf{A}$ can be diagonalized as $\mathbf{A} = \mathbf{T\Lambda T}^{-1}$ then its action can be completely decoupled
- *Any* matrix $\mathbf{A} \in \mathbf{R}^{m\times n}$ may be factored according to $\mathbf{A} = \mathbf{UDV}^T$ where $\mathbf{D}$ is the matrix of singular values
  - This factorization, although not as straightforward, also characterizes the action of $\mathbf{A}$ upon $\mathbf{R}^n$ (and domain $\mathbf{R}^m$)

Supplementary Material

- More details on Linear Algebra

Matrix Diagonalization

Factoring into Natural Modes

Sometimes a square matrix $\mathbf{A}$ in $\mathbf{R}^{n\times n}$ can be factored as

$$\mathbf{A} = \mathbf{T}\boldsymbol{\Lambda}\mathbf{T}^{-1}$$

Where $\mathbf{T}$ is in $GL(n,\mathbf{R})$ and $\boldsymbol{\Lambda} = \mathrm{diag}\{\lambda_1,\ldots,\lambda_n\}$ in $\mathbf{R}^{n\times n}$.

- $\boldsymbol{\Lambda}$ is called the *spectral matrix* (with spectrum $\{\lambda_1,\ldots,\lambda_n\}$)
- $\mathbf{T}$ is called the *modal matrix*
- When this condition is satisfied, i.e., when $\mathbf{A} = \mathbf{T}\boldsymbol{\Lambda}\mathbf{T}^{-1}$, we say $\mathbf{A}$ is *diagonalizable*

Fact: If a square matrix $\mathbf{A}$ in $\mathbf{R}^{n\times n}$ has n unique eigenvalues $\{\lambda_1, \ldots,\lambda_n\}$ then it can be factored as above

- In fact, a suitable $\mathbf{T}$ can be formed by augmenting all the eigenvectors as columns. Specifically,

$$\mathbf{T} = \left(\mathbf{e}_1 \middle| \quad \mathbf{e}_2 \middle| \quad \cdots \quad \middle| \mathbf{e}_n\right)$$

Matrix Diagonalization

Interpretation

Matrix diagonalization decouples $\mathbf{R}^n$ into the natural modes of $\mathbf{A}$

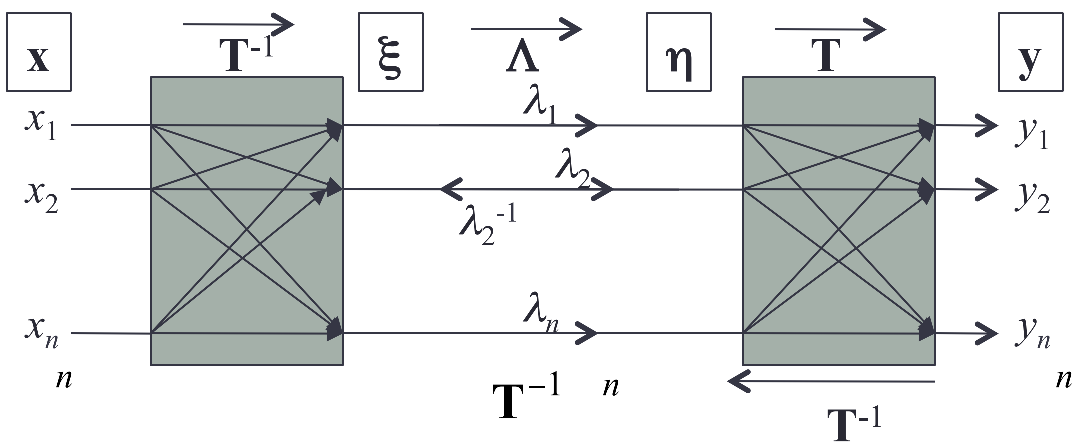
- Instead of working in $\mathbf{R}^n$, we work in the space $\mathbf{T}^{-1}\mathbf{R}^n$!

- To make this less abstract consider equation $\mathbf{y} = \mathbf{A}\mathbf{x}$ and, for example,
  - Think of $\mathbf{A}$ as a multiple-input, multiple-output, coupled amplifier.
  - Instead of using parameters $\mathbf{x}$ and $\mathbf{y}$, use $\boldsymbol{\xi} \equiv \mathbf{T}^{-1}\mathbf{x}$ and $\boldsymbol{\eta} \equiv \mathbf{T}^{-1}\mathbf{y}$
  - Everything decouples as $\xi_i = \lambda_i \eta_i$ (transform back when you're done)

$$\mathbf{y} = \mathbf{T}\boldsymbol{\Lambda}\mathbf{T}^{-1}\mathbf{x}$$

$$\mathbf{T}^{-1}\mathbf{y} = \boldsymbol{\Lambda}\mathbf{T}^{-1}\mathbf{x}$$

$$\boldsymbol{\eta} = \boldsymbol{\Lambda}\boldsymbol{\xi}$$



Matrix Diagonalization

Special Case: Symmetric, Positive Definite Matrix

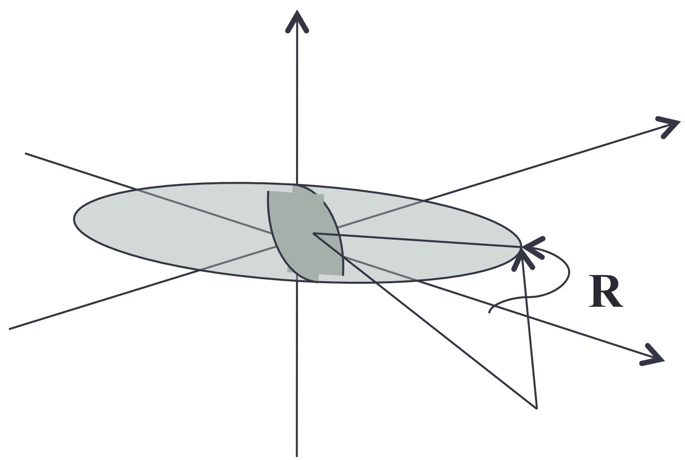
Fact: A positive-definite ($\lambda_i > 0$, for each i), symmetric ($\mathbf{A} = \mathbf{A}^T$), square matrix $\mathbf{A}$ in $\mathbf{R}^{n\times n}$ can always be diagonalized as

$$\mathbf{A} = \mathbf{R}\Lambda\mathbf{R}^T$$

Where $\mathbf{R}$ is in the special orthogonal group $SO(n)$ in $\mathbf{R}^{n\times n}$.

For any element $\mathbf{R}$ of $SO(n)$ in $\mathbf{R}^{n\times n}$

- $\mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{I}$ where $\mathbf{I}$ is the identity matrix
- From the above, $\mathbf{R}^{-1} = \mathbf{R}^{\mathrm{T}}$, e.g., just like a rotation in 3-space
- det $\mathbf{R} = 1$



A appears as a hyper-ellipsoid with semi-axes $\{\lambda_1,\ldots,\lambda_n\}$ rotated by a (generalized) angle **R** in hyper-Euclidean n space

Singular Value Decomposition

Generalizing Diagonalization

Any matrix $\mathbf{A} \in \mathbf{R}^{m\times n}$ may be factored as follows:

$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^T$$

Note the similarity with the special case when $\mathbf{A}$ is symmetric and positive-definite.

where $\mathbf{U} \in \mathbf{R}^{m\times n}$, $\mathrm{D} = \mathrm{diag}\{\sigma_1,\ldots,\sigma_n\}$, $\mathbf{V} \in SO(n) \in \mathbf{R}^{n\times n}$

- The numbers $\{\sigma_1,\ldots,\sigma_n\}$ are the *singular values* of $\mathbf{A}$
  - They may be any (complex) number, *including zero*!
  - They are generalizations of eigenvalues for square matrices
- Matrix $\mathbf{U}$ has the special property that it is “partially orthogonal”
  - $\mathbf{U}^T\mathbf{U} = \mathbf{I} \in \mathbf{R}^{n\times n}$
  - Note that $\mathbf{V}^T\mathbf{V} = \mathbf{I} \in \mathbf{R}^{n\times n}$ because $\mathbf{V} \in SO(n)$

Singular Value Decomposition

Generalizing Diagonalization

- The columns of **V** are called the *right singular vectors* of **A**
- The columns of **U** are called the *left singular vectors* of **A**

Again consider the matrix-vector equation

$$\mathbf{y} = \mathbf{A}\mathbf{x} = \mathbf{U}\mathbf{D}\mathbf{V}^T\mathbf{x}$$

and perform the substitutions

- $\boldsymbol{\xi} = \mathbf{V}^T\mathbf{x}$
- $\boldsymbol{\eta} = \mathbf{U}^T\mathbf{y}$

We then have the (almost) equivalent equation

$$\boldsymbol{\eta} = \mathbf{D}\boldsymbol{\xi}$$

Like the case of a diagonalizable **A**, this equation is completely decoupled $\Rightarrow$ $\eta_i = \sigma_i \xi_i$

Vector Space: Basis Sets

- A *basis* for vector space V is a (possibly infinite) subset $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots\}$ of V such that any $\mathbf{v} \in V$ can be expressed as a *linear combination* of basis vectors.
  - If $\mathbf{v} \in V$ then $\exists\, r_1, r_2, r_3, \ldots \in \mathbf{R}$ such that $\mathbf{v} = r_1\mathbf{e}_1 + r_2\mathbf{e}_2 + r_3\mathbf{e}_3 + \ldots$
- All basis sets of V have the same cardinality, called the *dimension* of V
  - $\dim V = |\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots\}|$
- Examples:
  - (1,…,0), (0, 1, …,0), …, (0, 0, …,1) form the *standard* basis for $\mathbf{R}^n$.
  - $\{1, x, x^2, \ldots, x^{n-1}\}$ is a basis for the vector subspace $\{f \in C^\infty([0,1]) \mid d^nf/dx^n = 0\}$
  - $\{1, \cos 2\pi ft, \cos 4\pi ft, \ldots, \cos (n\text{-}1)\pi ft\}$ is a basis for the set of all even, band-limited functions with period $1/f$, and cutoff frequency nf.
  - The eigenvectors of any symmetric, pos. def. matrix $\mathbf{A} \in \mathbf{R}^{n\times n}$ form a basis of $\mathbf{R}^n$.

All these vector spaces are isomorphic!