CONTROL ROOM ACCELERATOR PHYSICS

Day 2

Introduction to Control Theory

Control Theory

Introduction

- **Control theory** is an interdisciplinary branch of *engineering* and *mathematics* that deals with the behavior of *dynamical systems* with inputs. When one or more output variables of a system needs to follow a certain reference over time, a controller manipulates the inputs to a system to obtain the desired effect on the output of the system.

- A typical objective of a control theory is to calculate proper corrective action for inputs that results in system stability, that is, the system will hold the set point and not oscillate around it.

- Branches of Control Theory
  - Adaptive control: Adapt process and gain parameters to observed behavior
  - Intelligent control: Application of AI techniques in controller design
  - Optimal control: Controller is designed to extremize a given objective
  - Robust control: Design controller to be insensitive to model errors (eg., H^∞ control)
  - Stochastic control: Design control to be insensitive to model uncertainties
  - Energy shaping control: Plant and controller are viewed as energy transformers
  - Self-organized criticality control: Controller is aware of self-organizing systems
  - …

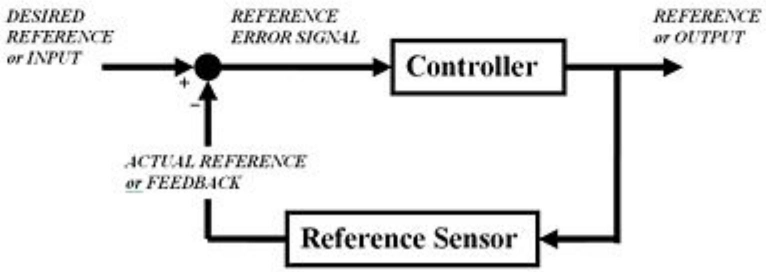
Control Theory

Introduction

- Control theory is relatively young and a very active research area.
- Accelerator control utilizes very little formal control! Wide open.
- Control theory is technical.
  - Dynamical Systems
  - Abstract algebra
  - Functional analysis
  - Signal processing
  - Differential manifolds
  - Abstract algebra (groups, rings, modules, …)
  - Lie groups and algebras
  - Representation theory (representing algebraic structure with matrices)
- Applications in
  - Aerospace
  - Automotive
  - Robotics
  - Biology
  - Economics

Outline

1. Introduction
2. Linear dynamical systems
3. Modeling linear systems
4. Control theory: Stability, observability, controllability
5. Control examples: Perturbation and disturbance rejection

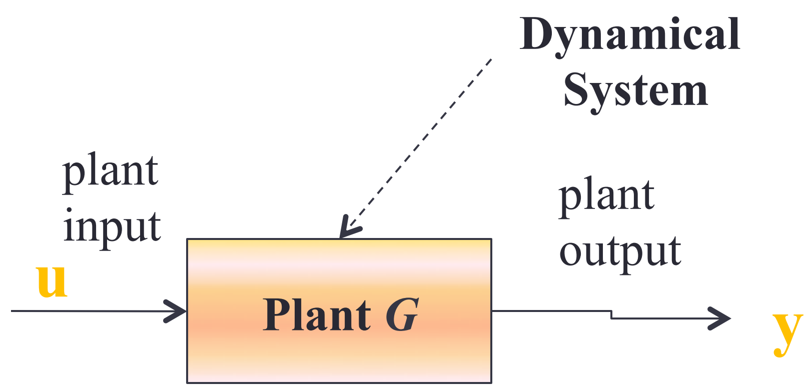


We restrict our attention to the basic tenets of control theory and linear dynamical systems, with some examples found in accelerator control.

Linear Dynamical Systems

Introduction

A Dynamical System is a system **G** that varies in time. Typically it has inputs **u** and outputs **y**. In control theory **G** is usually referred to as a *plant*.



Linear Dynamical Systems are *dynamical systems* **G** where the input **u** and output **y** are linearly related

- If $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ is the input then output **y** is

$$\mathbf{y} = \mathbf{G}(\mathbf{u}_1 + \mathbf{u}_2) = \mathbf{G}(\mathbf{u}_1) + \mathbf{G}(\mathbf{u}_2)$$

$$b_2 \ddot{y}(t) + b_1 \dot{y}(t) + b_0 y(t) = u(t)$$

continuous time

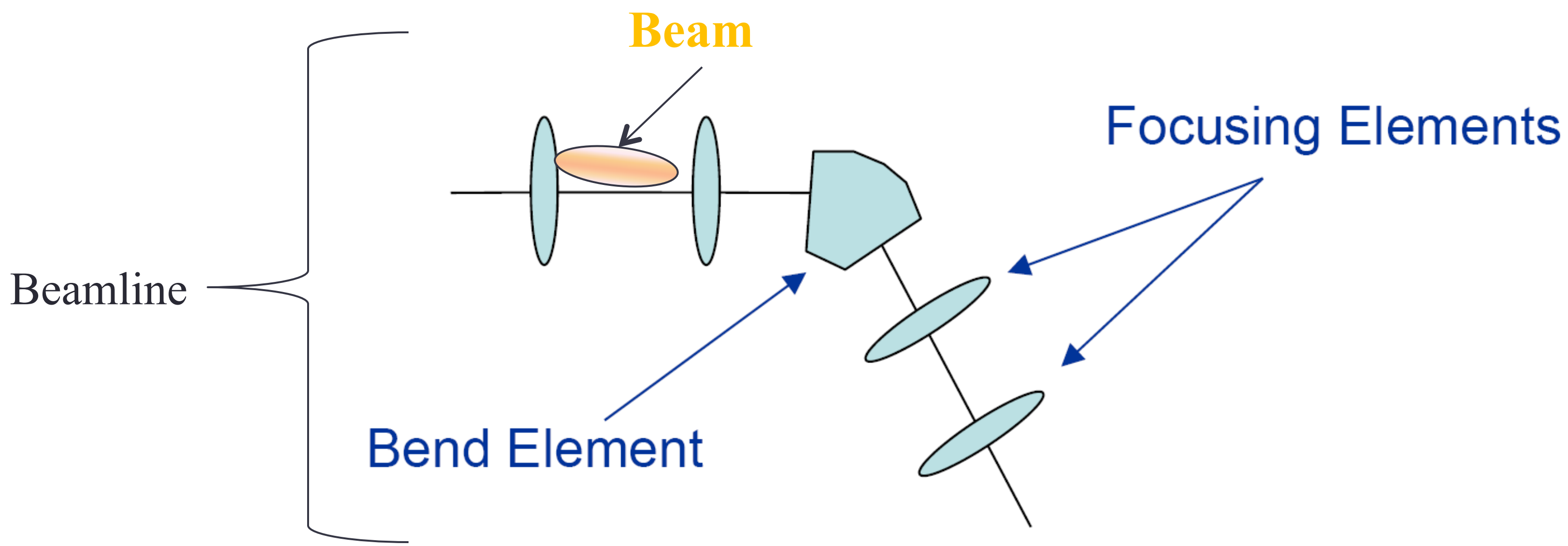
$$b_2 y_{k-2} + b_1 y_{k-1} + b_0 y_k = u_k$$

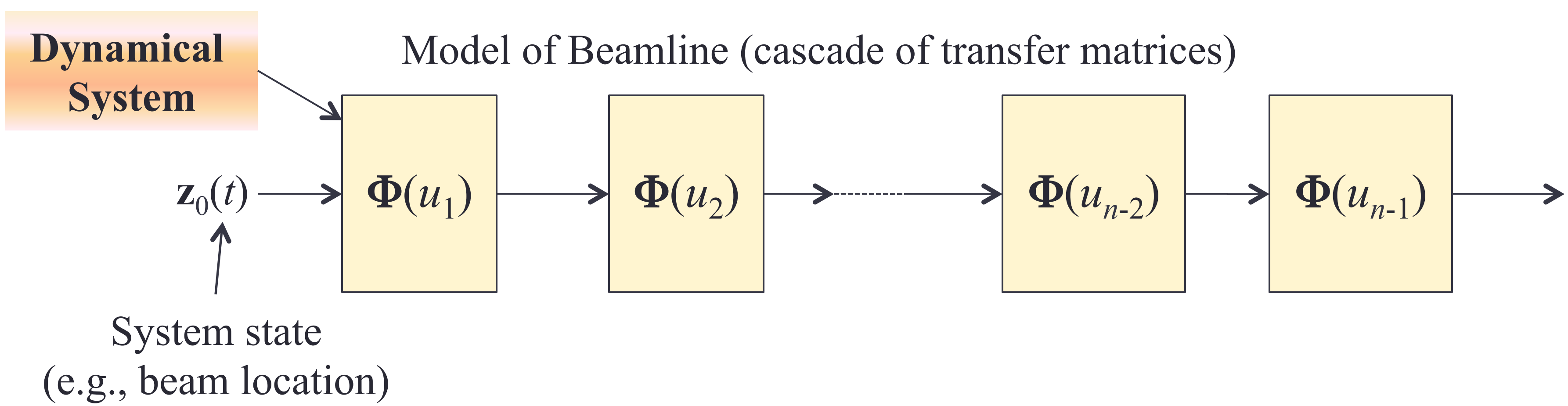
discrete time

- Linear systems may be
  - Continuous time (ODEs)
  - Discrete time (delay equations)
  - Typically these are related!

Linear Dynamical Systems

Example: Linear Beam Optics, a Discrete “Time” Case





Linear Dynamical Systems

Note on Modeling Physics and Engineering Problems

- Most linear dynamical systems in physics and engineering are not naturally expressed in the form

  $\mathbf{G}(\mathbf{u}) = \mathbf{y}$ (output as function of input)

  - The output $\mathbf{y}$ is generally a combination of differentials *of itself!*
  - For example, consider the 2nd order linear operator $\mathbf{L}$

$$\mathbf{L}(y) \equiv b_2 \ddot{y}(t) + b_1 \dot{y}(t) + b_0 y(t)$$

  - Then our model appears as

$$\mathbf{L}(y) = u$$

  - This is a linear equation, but the wrong direction! By comparing equations we find (abstractly)

$$\mathbf{y} = \mathbf{G}(\mathbf{u}) = \mathbf{L}^{-1}(\mathbf{u})$$ What does $\mathbf{L}^{-1}$ mean?!

Linear Dynamical Systems

The State Space Representation (Common in Control Theory)

What is **y** = **G**(**u**) = **L**$^{-1}$(**u**) for a linear differential system ?

- Start with our 2nd order linear differential equation

$$b_2\ddot{y}(t)+b_1\dot{y}(t)+b_0 y(t)=u(t), \quad \text{or} \quad \ddot{y}=-\frac{b_1}{b_2}\dot{y}-\frac{b_0}{b_2}y+u$$

- Define our *state variables* x_1 and x_2

$$x_1 \equiv y$$
$$x_2 \equiv \dot{y}$$

- Differentiate $\dot{x}_1=\dot{y}=x_2$; then differentiate x_2 yielding

$$\dot{x}_2=\ddot{y}=-\frac{b_1}{b_2}x_2-\frac{b_0}{b_2}x_1+u$$

- Arranging into **matrix-vector** format

input

$$\frac{d}{dt}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0 & 1\\ -\frac{b_1}{b_2} & -\frac{b_0}{b_2}\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}u$$

This has form

$$\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t)$$

$$y=\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

output $\Longrightarrow$ $\mathbf{y}(t)=\mathbf{C}\mathbf{x}(t)$

Linear Time-Invariant Dynamical System
State Space Representation

- The general representation of an n^{th}-order Linear Time-Invariant (LTI) dynamical system is (for continuous case)

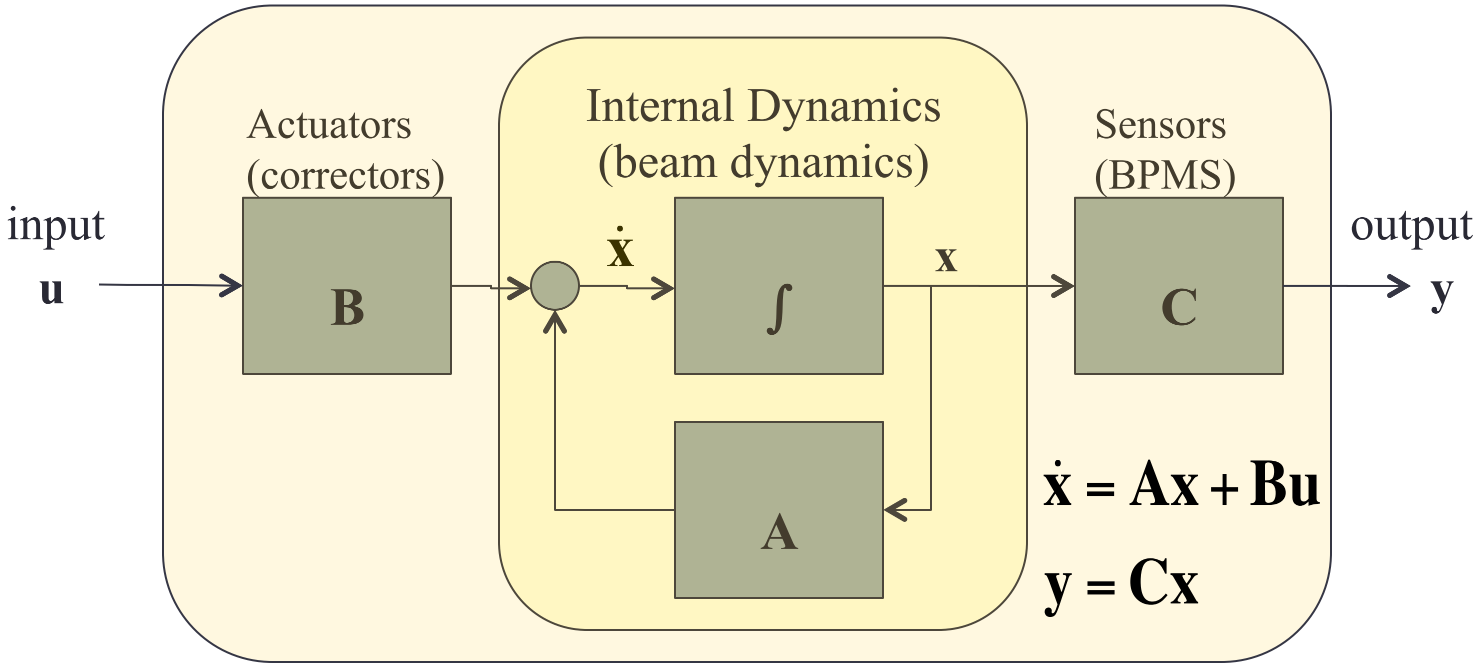
$$\mathbf{y} = \mathbf{G}(\mathbf{u}) \quad \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), & \mathbf{x}(t) \in \mathbf{R}^{n}, \mathbf{y}(t) \in \mathbf{R}^{m}, \mathbf{u}(t) \in \mathbf{R}^{p} \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t), & \mathbf{A} \in \mathbf{R}^{n \times n}, \mathbf{B} \in \mathbf{R}^{n \times p}, \mathbf{C} \in \mathbf{R}^{m \times n}, \mathbf{D} \in \mathbf{R}^{m \times p} \end{cases}$$

  - **A, B, C, D** are matrices of the appropriate dimensions
  - **x** is the *state vector* (plant internal dynamics – e.g., the beam)
  - **y** is the *output vector* (sensor output, what we can observe – e.g., BPM response)
  - **u** is the *input vector* (actuator input – e.g., magnet strengths)
- Often we drop the matrix **D** since we can renormalize output **y**
- The above is called the *state space representation*

LTI Dynamical System

Block Diagram of State Space Representation

- The matrices **A, B, C** determine plant properties
  - Matrix **A** determines the internal plant dynamics
  - Matrix **B** determines how input affects internal dynamics
  - Matrix **C** determines coupling between internal dynamics and output



LTI Dynamical System

Solution to the State Equations

Solution to state vector equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \qquad \mathbf{A} \in \mathbf{R}^{m \times m},\ \mathbf{B} \in \mathbf{R}^{m \times n}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \qquad \mathbf{C} \in \mathbf{R}^{k \times m}$$

is

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}_0 + \int_0^t e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

- This is completely analogous to the scalar case $\dot{x}(t) = ax(t) + bu(t), \quad y(t) = cx(t)$
- Internal dynamics $\mathbf{x}(t)$ are the sum of natural system response $e^{t\mathbf{A}}$ and the natural response *convolved* with the driving term $\mathbf{B}\mathbf{u}(t)$
  - Note $\mathbf{A}$ is square and the matrix exponential function $e^{t\mathbf{A}}$ is well-defined

- Thus, the system dynamics are dictated by the matrix $e^{t\mathbf{A}}$
- The control properties are dictated by matrix $\mathbf{B}$
- The observation properties are dictated by matrix $\mathbf{C}$

LTI Dynamical System

Discrete Case

- For discrete case the state representation looks like

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k$$
$$\mathbf{y}_k = \mathbf{C}\mathbf{x}_k$$

- The solution to the state variable equation is

$$\mathbf{x}_k = \mathbf{A}^k \mathbf{x}_0 + \sum_{i=0}^{k-1} \mathbf{A}^{k-1-i} \mathbf{B}\mathbf{u}_k$$

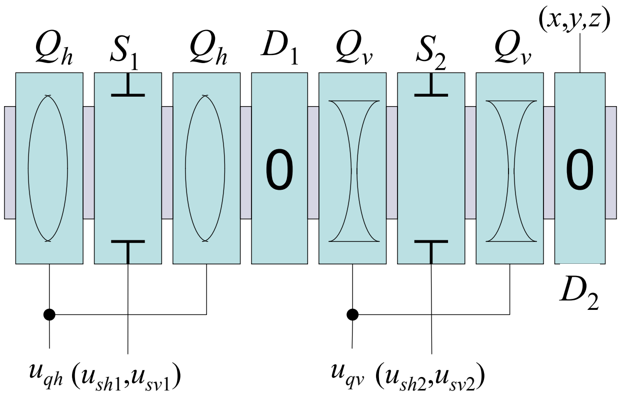
- This solutions is analogous to the continuous case, only the natural response is dictated directly by matrix $\mathbf{A}^k$ rather than $e^{t\mathbf{A}}$, as we shall see.

LTI Dynamical System

Example: Discrete State Space Modeling Beam Steering

Say we have Beam Position Monitors (BPMs) as our sensors, then our observables are the coordinates (x,y,z); that is, we do not have access to the full state vector – no momentum components.

Note state vector **z** and observation vector **y**



- State vector $\mathbf{z} = (x, x', y, y', z, z')$
- Observation vector $\mathbf{y} \equiv \begin{pmatrix} x & y & z \end{pmatrix}^T$
- Then $\mathbf{y}_k = \mathbf{C}\mathbf{z}_k(t)$
- where $\mathbf{C} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$
- With $\{\boldsymbol{\Phi}_k\}$ as the transfer matrices, our modeling equations are

$$\mathbf{z}_{k+1} = \Phi_k(u_k)\mathbf{z}_k$$
$$\mathbf{y}_k = \mathbf{C}\mathbf{z}_k$$

These are in the form of the **discrete** state space representation

Control Theory

Application of Control Theory to Dynamical Systems

OUTLINE

- Dynamical System
  - **Stability**
  - **Controllability**
  - **Observability**
- Control
  - Closed-loop/Open-loop
  - Regulator
  - State Observer
  - PID Control

Control Theory

Stability

- Loosely, the *stability* of a dynamical system at equilibrium point $\mathbf{y}_0$ indicates that the output $\mathbf{y}(t)$ always "stays near" $\mathbf{y}_0$ for sufficiently small inputs $\mathbf{u}(t)$.
  - A linear system is *stable* if all bounded inputs **u** produce bounded outputs **y** ($\|\mathbf{u}(t)\| < \infty \Rightarrow \|\mathbf{y}(t)\| < \infty$ all t)
  - If the system eventually returns to $\mathbf{y}_0$, it is said to be *asymptotically stable*: the variables of an asymptotically stable control system do not permanently oscillate.
  - If the response to an input $\mathbf{u}(t)$ neither decays nor grows over time, it is *marginally stable*. The output may oscillate indefinitely about $\mathbf{y}_0$ but will not blow up.
- Otherwise the system is *unstable*.

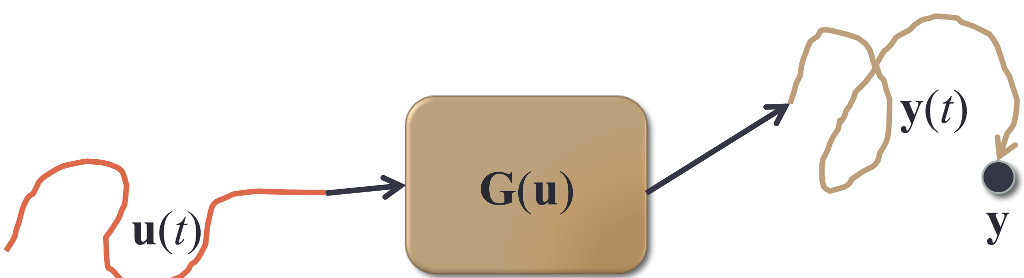
- Examples
  - An ideal pendulum is marginally stable at $\mathbf{y}_0 = 0^\circ$
  - A damped pendulum is asymptotically stable at $\mathbf{y}_0 = 0^\circ$
  - An (inverted) pendulum is unstable at $\mathbf{y}_0 = 180^\circ$
  - However, we can stabilize an inverted pendulum with *active stabilization* $\mathbf{u}(t) = \mathbf{u}[\mathbf{y}(\cdot),t]$
    - "Fly by wire"
    - (Stability is the opposite of maneuverability)

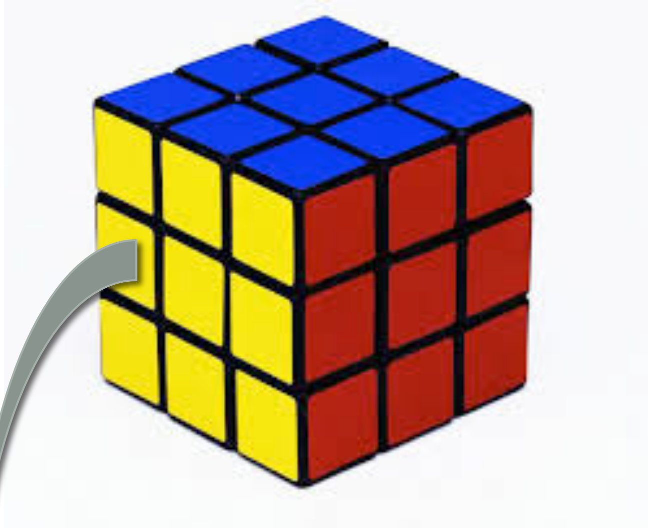
See http://vimeo.com/2952236

Control Theory
Controllability

Roughly speaking, the concept of *controllability* denotes the ability to move a system around in its entire configuration space using only certain admissible inputs. The exact definition varies slightly within the framework or the type of models applied.

For us, *controllability* means there exists at least one valid control strategy $\mathbf{u}(\cdot) : \mathbf{R}_+ \to \mathbf{R}^p$ that brings the system to any output $\mathbf{y} \in \mathbf{R}^m$.





Is the Rubik's cube controllable?

Control Theory

Observability

- Informally, *observability* is a measure of how well the internal states **x** of a system can be inferred by knowledge of its external outputs **y**.

- More formally, a system is said to be *observable* if, for any possible sequence of control vectors $\mathbf{u}(\cdot)$ we can determine the current state $\mathbf{x}(t)$ in finite time $t < \infty$ only by watching the outputs $\mathbf{y}(\cdot)$. Note we know both $\mathbf{u}(t)$ and $\mathbf{y}(t)$ for all t.

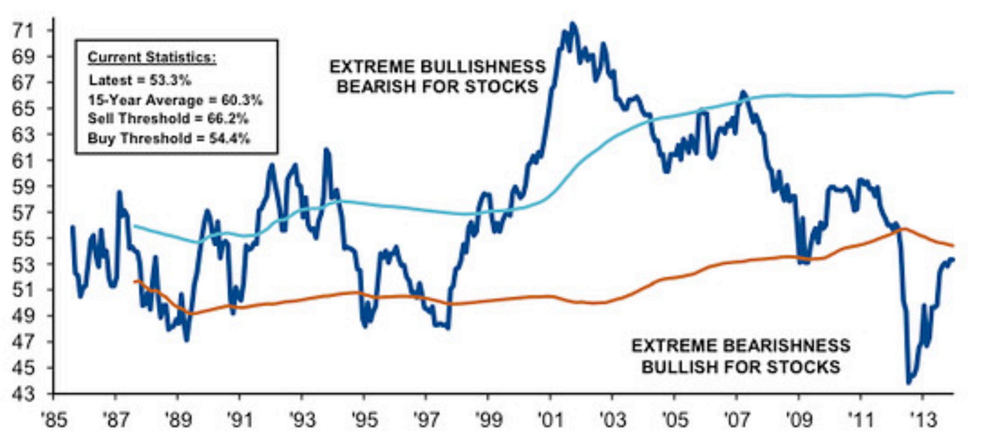
- If a system is not observable, this means the current values of some of its states cannot be determined through output sensors. Their value is unknown to the controller (although they might be estimated through various means).

- The *observability* and *controllability* of a system are mathematical *duals*.

- The concept of observability was introduced by American-Hungarian scientist Rudolf E. Kalman for linear dynamic systems.

Can we determine the internal state of the stock market by observing the S&P ?

Chart 1: Sell Side Consensus Indicator (as of December 2013)



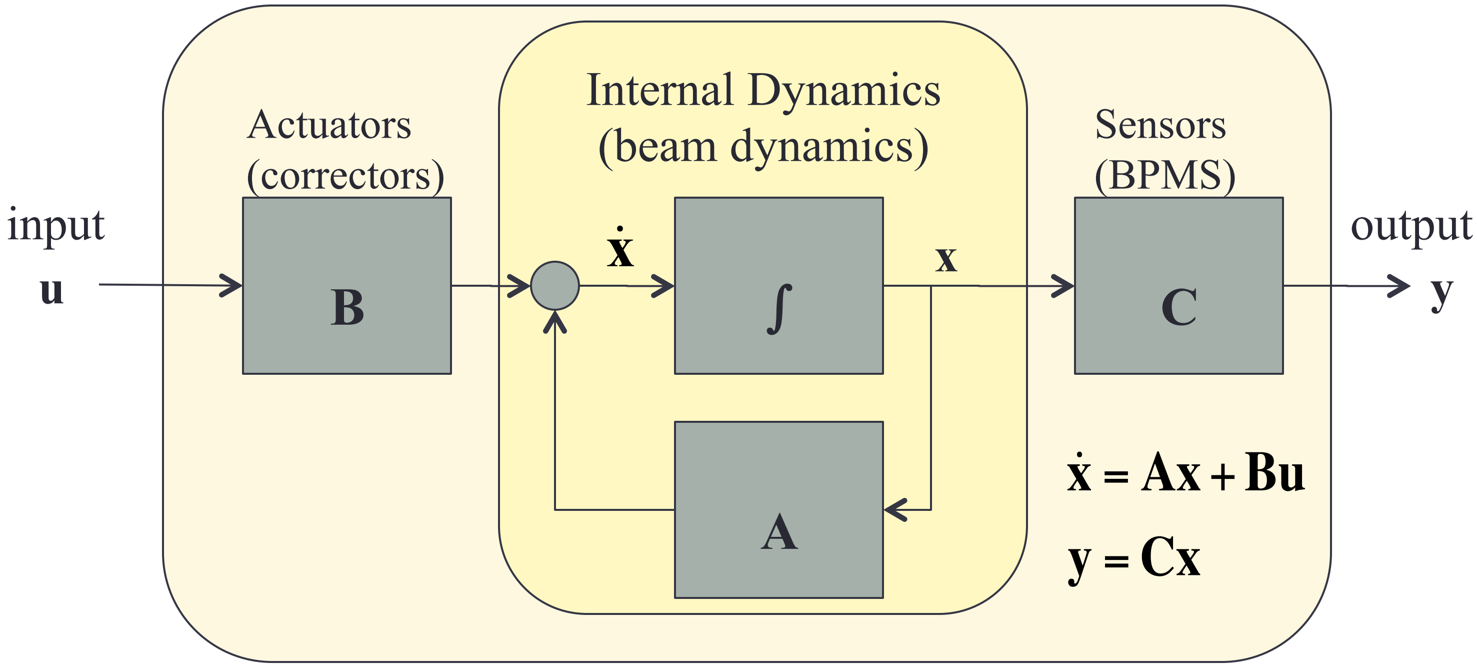
Source: BofA Merrill Lynch US Equity & Quant Strategy

Note: Buy and Sell signals are based on rolling 15-year +/- 1 standard deviations from the rolling 15-year mean. A reading above the blue line indicates a Sell signal and a reading below the red line indicates a Buy signal.

Control Theory

Stability, Controllability, and Observability of an LTI System

- The matrices **A, B, C** determine plant properties
  - Matrix **A** determines *stability* (we cover this)
  - Matrices **A** and **B** determine *controllability* (outputs we can reach)
  - Matrices **A** and **C** determine *observability* (watching the output says?)



Control Theory

An Example LTI System: Continuous Scalar System

- Consider the scalar case $\dot{x}(t) = ax(t) + bu(t)$
  - $x(t), y(t), a, b, c \in \mathbf{R}$ $y(t) = cx(t)$

- "Solution" to $\dot{x}(t) = ax(t) + bu(t)$ is

$$x(t) = e^{at}x_0 + \int_0^t e^{a(t-\tau)} bu(\tau)\,d\tau$$

  - Differentiate to prove it (Exercise)
  - Internal dynamics are the sum of natural response $e^{at}x_0$ at initial condition x_0, plus the natural response *convolved* ("folded") with the driving term $bu(t)$.

- Finally, system solution $y(t)$ is proportional to $x(t)$,

$$y(t) = cx(t) = ce^{at}x_0 + c\int_0^t e^{a(t-\tau)} bu(\tau)\,d\tau$$

$$= ce^{at}\left[x_0 + b\int_0^t e^{-a\tau} u(\tau)\,d\tau\right]$$

Control Theory

Stability, Controllability, and Observability of Example System

System

$$\dot{x} = ax + bu$$

$$y = cx$$

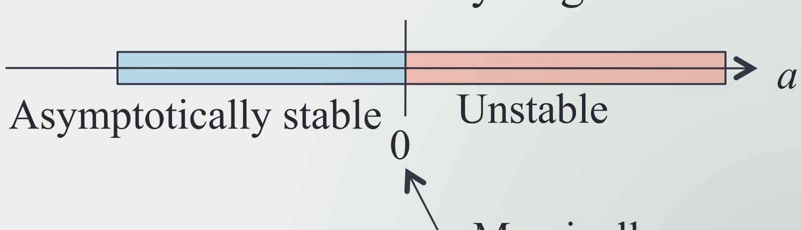
With solution

$$y(t) = ce^{at}x_0 + c\int_0^t e^{a(t-\tau)}bu(\tau)\,d\tau$$

Stability

- For natural response [e^{at} term]
  - $a > 0$ e^{at} unbounded
  - $a = 0$ e^{at} is stable
  - $a < 0$ e^{at} decays

Scalar Stability Region



Stable system iff $a \leq 0$

Stability with Inputs

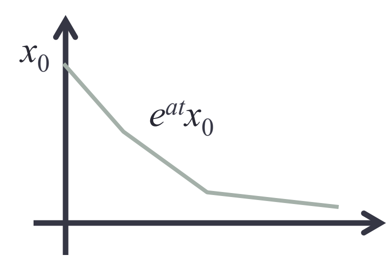
- Consider driven response ($u \neq 0$)
  - Note $(t - \tau) > 0$ so $e^{a(t-\tau)}$ amplifies or attenuates $u(t)$ according to whether $a > 0$ or $a < 0$, respectively.
  - Thus, system is stable iff $a \leq 0$.
  - Stability dominated by natural response e^{at}

Observability

- So long as $c \neq 0$

Controllability

- So long as $b \neq 0$



Infer about Matrix-Vector Systems

- Look at eigenvalues/singular values of **A**, **B**, **C**
  - Positive eigenvalues of **A** indicate instability
  - Zero singular values of **B** indicate uncontrollable modes
  - Zero singular values of **C** indicate unobservable modes

Control Theory

Stability, Controllability, and Observability of the General LTI System

Solution to vector equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \qquad \mathbf{A} \in \mathbf{R}^{m\times m},\ \mathbf{B} \in \mathbf{R}^{m\times n}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \qquad \mathbf{C} \in \mathbf{R}^{k\times m}$$

is

$$\mathbf{y}(t) = \mathbf{C}e^{t\mathbf{A}}\mathbf{x}_0 + \mathbf{C}\int_0^t e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

- This is completely analogous to the scalar case
- Again, internal dynamics are the sum of natural response $e^{t\mathbf{A}}$ plus the natural response *convolved* with the driving term $\mathbf{Bu}(t)$
  - Note $\mathbf{A}$ is square and the matrix exponential function $e^{t\mathbf{A}}$ is well-defined
- The state dynamics are dictated by the matrix $\mathbf{H}(t) = e^{t\mathbf{A}}$
- The control dynamics are dictated by matrix **B**
- The observation dynamics are dictated by matrix **C**
  - Positive eigenvalues of **A** indicate instability
  - Zero singular values of **B** indicate uncontrollable modes
  - Zero singular values of **C** indicate unobservable modes

Control Theory

Stability of Continuous LTI Systems

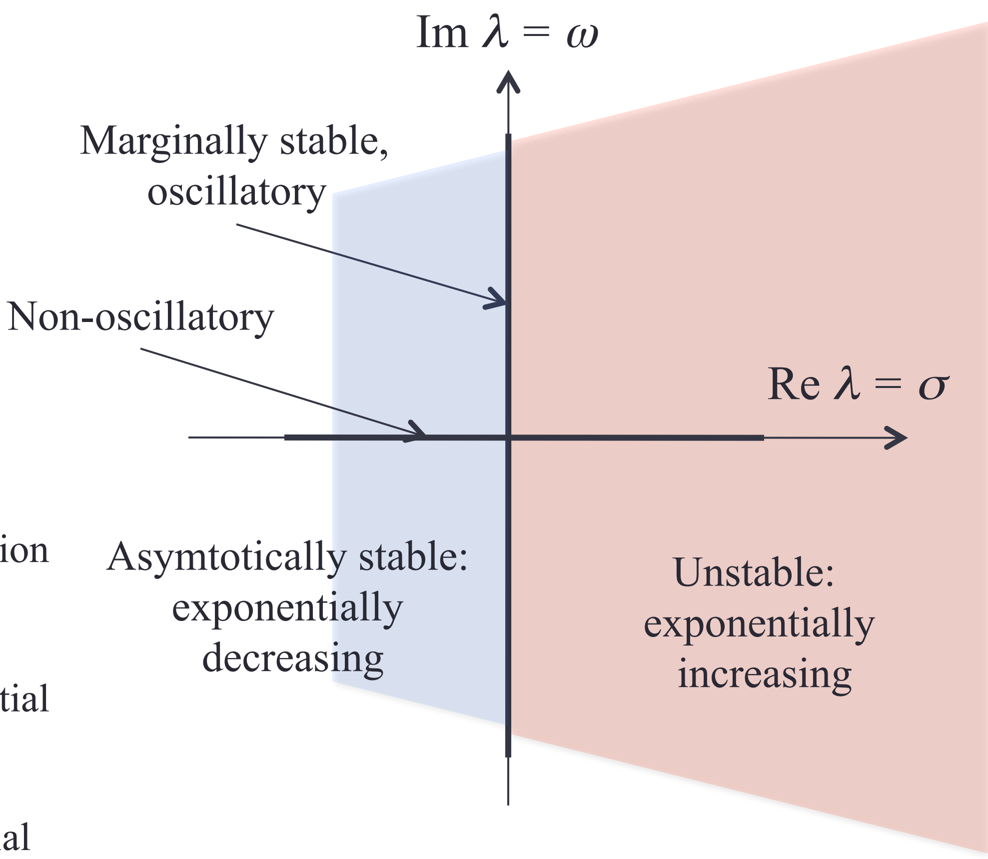
- Let λ be an eigenvalue of $\mathbf{A}$
  - Dynamics of eigenmode $\mathbf{e}_\lambda$ are determined by $e^{t\lambda}$
- λ on the complex plane

$$\text{Let}\quad \lambda = \sigma + j\omega \quad \sigma, \omega \in \mathbf{R}$$

then

$$e^{t\lambda} = e^{\sigma t}\cos\omega t + ie^{\sigma t}\sin\omega t$$

  - A finite imaginary part of λ ($\omega \neq 0$) implies oscillation
  - A zero imaginary part of λ ($\omega = 0$) implies none
  - A negative real part of λ ($\sigma < 0$) indicates exponential decay – asymptotically stable
  - A positive real part of λ ($\sigma > 0$) indicates exponential growth – unstable



Control Theory
Controllability of LTI Systems

- The LTI state solution

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}_0 + \int_0^t e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

- Recall system is *controllable* iff we can steer to any output $\eta \in \mathbf{R}^n$
  - There exist a $\mathbf{u}_1(\bullet)$ and $t_1 > 0$ such that $\mathbf{y}[t_1;\mathbf{u}_1(t_1)] = \boldsymbol{\eta}$.

- LTI controllability criteria:
  1. Matrix $e^{t\mathbf{A}}\mathbf{B}$ has full row rank (true?)
  2. Augmented matrix $[\mathbf{B} \mid \mathbf{AB} \mid \ldots \mid \mathbf{A}^{n-1}\mathbf{B}]$ has full row rank
     - That is, there are n linearly independent columns
  3. Show 2. implies 1. (hint, expand $e^{t\mathbf{A}}$)

Control Theory
Observability of an LTI System

- The LTI output solution

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}_0 + \int_0^t e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

- Recall system is *observable* iff we can deduce the system state $\mathbf{x} \in \mathbf{R}^m$ by observing input $\mathbf{u}(\bullet)$ and output $\mathbf{y}(\bullet)$ for finite $t>0$.

- LTI observability criteria:
  1. Matrix $\mathbf{C}e^{t\mathbf{A}}$ has full row rank
  2. Augmented matrix $[\mathbf{C}^T \mid (\mathbf{C}\mathbf{A})^T \mid \ldots \mid (\mathbf{C}\mathbf{A}^{k\text{-}1})^T]^T$ has full row rank
     - That is, there are n linearly independent columns
  3. Show 2. implies 3. (hint, expand $e^{t\mathbf{A}}$)

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \vdots \\ \mathbf{C}\mathbf{A}^{k-1} \end{bmatrix}$$

Control Theory

The Discrete LTI Dynamical System

- The state solution to discrete LTI system

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k$$
$$\mathbf{y}_k = \mathbf{C}\mathbf{x}_k$$

is

$$\mathbf{x}_k = \mathbf{A}^k\mathbf{x}_0 + \sum_{i=0}^{k-1}\mathbf{A}^{k-1-i}\mathbf{B}\mathbf{u}_k$$

- The important point is that the natural response is dictated by matrix power $\mathbf{A}^k$ rather than the matrix exponential $e^{t\mathbf{A}}$ as before.

- The state dynamics are dictated by the matrix $\mathbf{H}_k = \mathbf{A}^k$
  - By diagonalizing we have $\mathbf{A}^k = \mathbf{T}\mathbf{\Lambda}^k\mathbf{T}^{-1}$
  - The dynamics are controlled by eigenvalue powers λ^k rather than $e^{t\lambda}$

Stability of LTI System

Discrete Systems

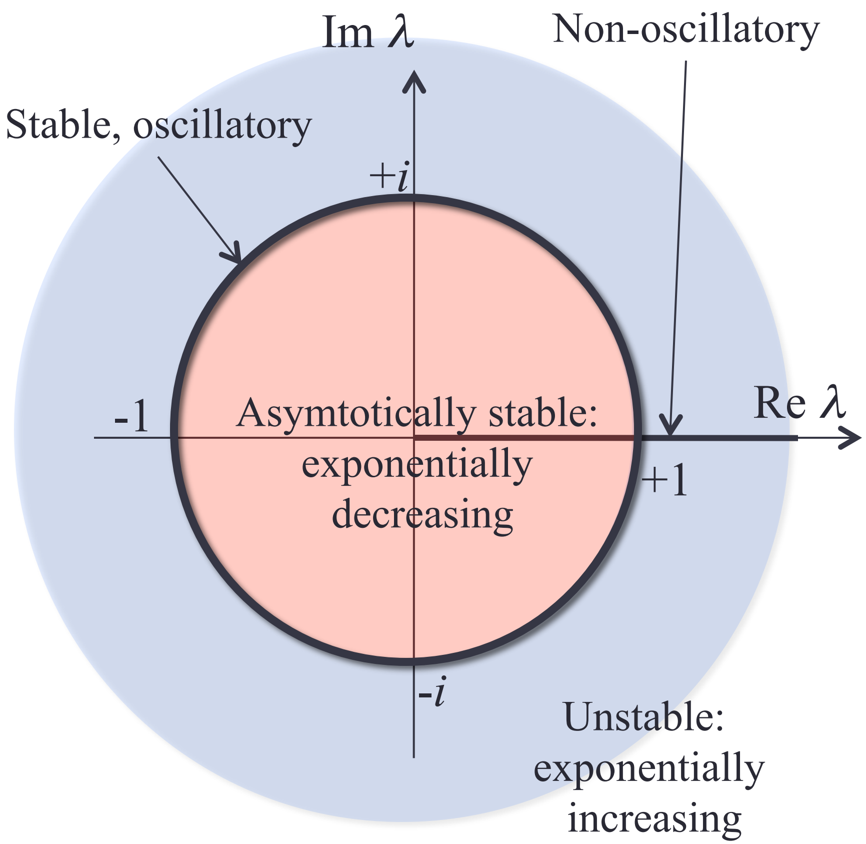
- Let λ be an eigenvalue of **A**
  - Dynamics of eigenmode for λ are determined by value of λ^k
- Magnitude-angle representation of λ on the complex plane

Let $\lambda = \rho e^{j\theta} \qquad \rho, \theta \in \mathrm{R}$

  - That is, $\rho = |\lambda|$, and θ =ang λ
- This yields

$$\lambda^k = \rho^k e^{jk\theta} = \rho^k \cos k\theta + i\rho^k \sin k\theta$$

  - $\theta = 0$ indicates no oscillation
  - $|\lambda| = 1$ indicates (marginally) stable oscillation
  - $|\lambda| < 1$ indicates exponential decay
  - $|\lambda| > 1$ indicates exponential growth



Looks like a Z transform

Control Theory

Controllability and Observability of a Discrete LTI System

- State solution to the discrete LTI system

$$\mathbf{x}_k = \mathbf{A}^k \mathbf{x}_0 + \sum_{i=0}^{k-1} \mathbf{A}^{k-1-i} \mathbf{B} \mathbf{u}_i$$

- The system is *controllable* iff we can steer to any state $\eta \in \mathbf{R}^n$
  - There exists a $k > 0$ and sequence $\{\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{k-1}\}$ such that $\mathbf{y}_k = \boldsymbol{\eta}$.

- LTI controllability criteria:
  1. Matrix $(\mathbf{I}+\mathbf{A}+\ldots+\mathbf{A}^{m-1})\mathbf{B}$ has full row rank
  2. Augmented matrix $[\mathbf{B} \mid \mathbf{AB} \mid \ldots \mid \mathbf{A}^{m-1}\mathbf{B}]$ has full row rank
     - That is, there are n linearly independent columns

- Observability is analogous to the continuous case
  - Augmented matrix $[\mathbf{C}^T \mid (\mathbf{CA})^T \mid \ldots \mid (\mathbf{CA}^{k-1})^T]^T$ has full row rank

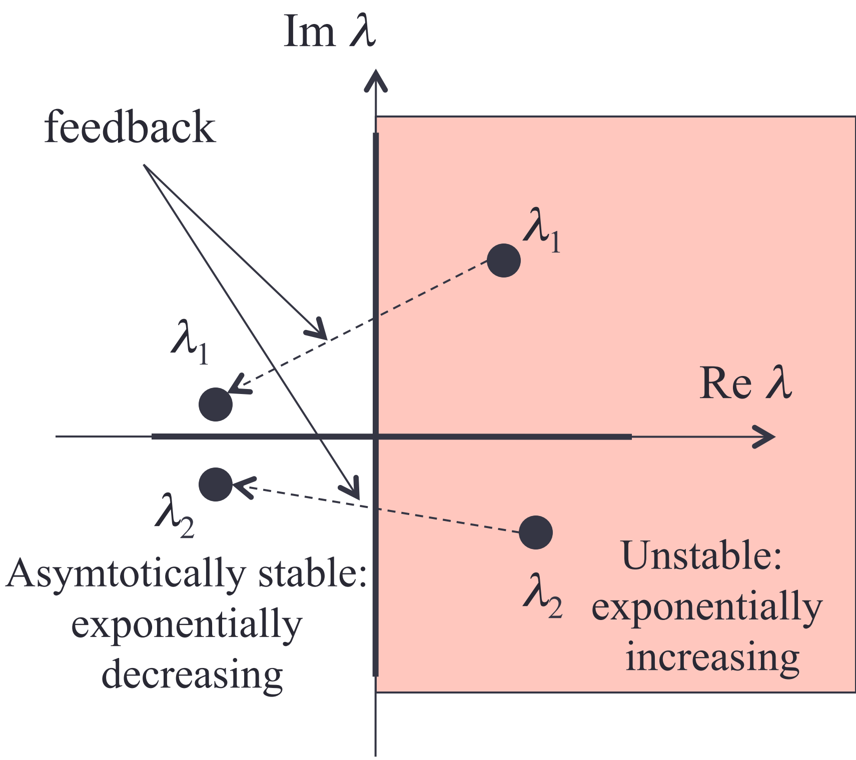
Classical Control Example

Stabilization: The Role of Feedback

- Typically in classical control we pick a feedback control law of the form

$$\mathbf{u}(t) = \mathbf{K}\mathbf{y}(t)$$

- The matrix **K** is chosen to move the system response (eigenvalues λ_1 and λ_2 of **A**) from the right half-plane to the left half-plane
  - Location in **C** determines feedback response
  - Open loop stability depends upon $e^{t\mathbf{A}}$
  - Closed loop stability depends upon $e^{t(\mathbf{A}+\mathbf{KCB})}$
  - Show this!



Note: "Classical Control" refers to the use of transfer functions in frequency domain to design feedback loops. Modern methods are more general, more applicable, and do not necessarily involve feedback

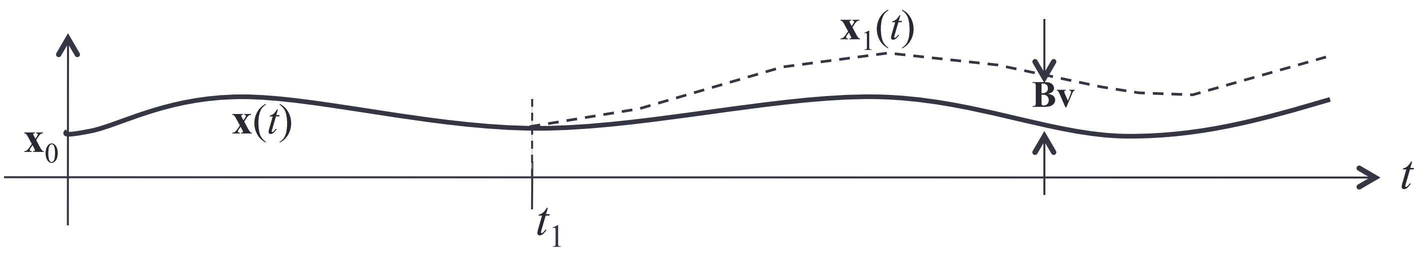
Classic Control Example:
Disturbance Rejection

- What if control signal $\mathbf{u}(t)$ is perturbed by a small amount $\delta\mathbf{u}(t)$?
  - That is $\mathbf{u} \to \mathbf{u} + \delta\mathbf{u}$
  - Let us choose a special perturbation

$$\delta\mathbf{u}(t) = \begin{cases} 0 \in \mathbf{R}^m & \text{for} \quad t < t_1 \\ \mathbf{v} \in \mathbf{R}^m & \text{for} \quad t > t_1 \end{cases}$$

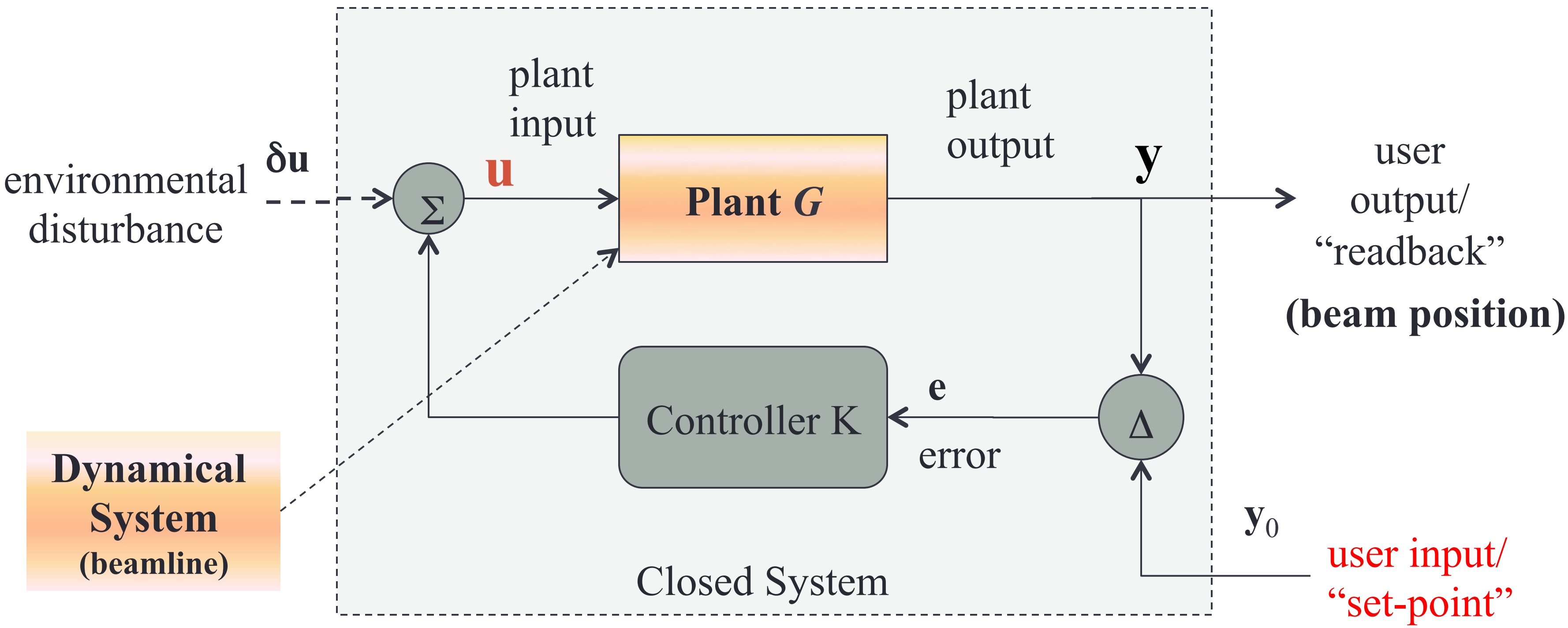
  - Then the (open loop) perturbed response is given by

$$\mathbf{x}_1(t) = \begin{cases} \mathbf{x}(t) & \text{for} \quad t < t_1 \\ \mathbf{x}(t) + \int_{t_1}^{t} e^{(t-\tau)\mathbf{A}} d\tau\, \mathbf{Bv} & \text{for} \quad t > t_1 \end{cases}$$



Classical Control Example

Linear Regulator with Disturbance Rejection



$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \qquad \mathbf{A} \in \mathbf{R}^{m\times m},\ \mathbf{B} \in \mathbf{R}^{m\times n}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \qquad \mathbf{C} \in \mathbf{R}^{k\times m}$$

$$\mathbf{e}(t) = \mathbf{y}(t) - \mathbf{y}_0$$

$$\mathbf{u}(t) = \mathbf{K}\mathbf{e}(t) \qquad \mathbf{K} \in \mathbf{R}^{m\times k}$$

Controller
- How do we get this to work???
  - How fast to react?
  - Ignore certain frequencies?
  - etc.

Classical Control Example

Design of Linear Regulator

- Recall governing equations for continuous LTI plant with linear feedback **K**

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \qquad \mathbf{A} \in \mathbf{R}^{m\times m},\ \mathbf{B} \in \mathbf{R}^{m\times n}$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \qquad \mathbf{C} \in \mathbf{R}^{k\times m}$$
$$\mathbf{e}(t) = \mathbf{y}(t) - \mathbf{y}_0$$
$$\mathbf{u}(t) = \mathbf{K}\mathbf{e}(t) \qquad \mathbf{K} \in \mathbf{R}^{n\times k}$$

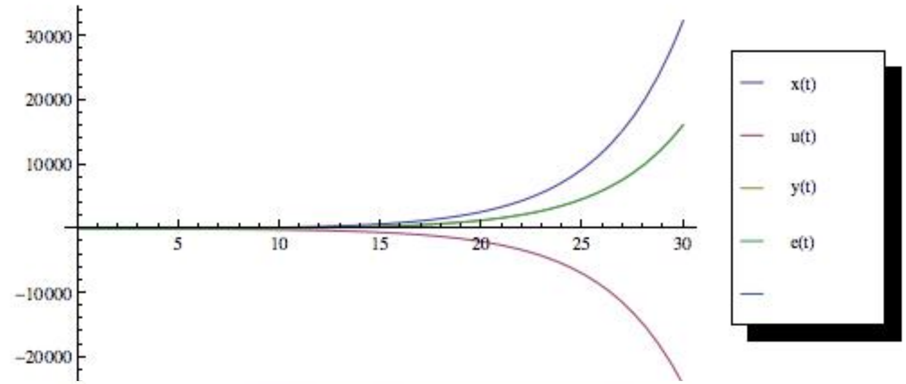
$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{K}\mathbf{e}(t)\\ &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{K}\left(\mathbf{C}\mathbf{x}(t) - \mathbf{y}_0\right)\\ &= (\mathbf{A} + \mathbf{B}\mathbf{K}\mathbf{C})\mathbf{x}(t) - \mathbf{B}\mathbf{K}\mathbf{y}_0\end{aligned}$$

Stability governed by matrix **A** + **BKC**

- Let's try a scalar example
  - $a = 1$, $b = 1$
  - $c = ½$,
  - Choose
    - $y_0 = 2$
    - $\delta u = 3$
  - Feedback
  - $k = ??$

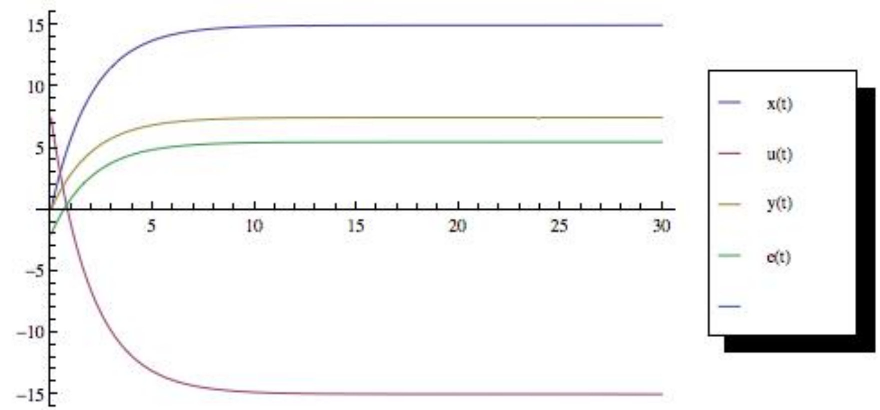
Pick k = -3/2

$a+bkc$ = 1+1(-3/2)½ = ¼



Pick k = -3

$a+bkc$ = 1+1(-3)½ = -½



Classical Control – A Final Word

The Transfer Function

- Transfer function $\mathbf{H}(s)$ of plant $\mathbf{G}$ is input/output relation in frequency domain
  - Take Laplace transform $\int \cdot\, e^{-st}dt$ of everything in sight (s is on the right half plane)

$$\begin{aligned} s\mathbf{x}(s) &= \mathbf{Ax}(s) + \mathbf{Bu}(s) \\ \mathbf{y}(s) &= \mathbf{Cx}(s) \end{aligned} \quad \Rightarrow \quad \mathbf{y}(s) = \mathbf{C}(\mathbf{A} - s\mathbf{I})^{-1}\mathbf{Bu}(s) \quad \Rightarrow \quad \mathbf{H}(s) \equiv \mathbf{C}(\mathbf{A} - s\mathbf{I})^{-1}\mathbf{B}$$

- Stability of $\mathbf{H}(s)$ requires that s cannot be an eigenvalue of $\mathbf{A}$
  - This will not happen if all eigenvalues are on the left half plane as before!

- Lets apply this to the perturbed regulator

Loop equations $\begin{aligned} \mathbf{e} &= \mathbf{Hu} - \mathbf{y}_0 \\ \mathbf{u} &= \delta\mathbf{u} + \mathbf{Ke} \end{aligned}$ have solution $\begin{pmatrix} \mathbf{e} \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} 0 & (\mathbf{I} - \mathbf{HK})^{-1}\mathbf{H} \\ (\mathbf{I} - \mathbf{KH})^{-1}\mathbf{K} & (\mathbf{I} - \mathbf{KH})^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{y}_0 \\ \delta\mathbf{u} \end{pmatrix}$

Thus, the regulator is stable if $(\mathbf{I} - \mathbf{KH})^{-1}$ exists and is stable for all s
(or equivalently $(\mathbf{I} - \mathbf{KH})^{-1}$)

- It can be shown this is equivalent to existence of $\begin{pmatrix} \mathbf{I} & -\mathbf{K}(s) \\ -\mathbf{H}(s) & \mathbf{I} \end{pmatrix}^{-1}$ for $\mathrm{Re}(s) > 0$

Control and LTI Systems

Review

- Most LTI system can be put into state variable form
  - First-order, n-dimension matrix-vector ODE or difference equation
- For continuous case stability is determined by the matrix exponential $e^{t\mathbf{A}}$
- For discrete case stability is determined by the matrix power $\mathbf{A}^k$
- Stability $\Rightarrow$ Where are the eigenvalues of $\mathbf{A}$!?
- Closed loop stability depends upon $\mathbf{A} + \mathbf{BKC}$

Supplemental Material

- More detail on Linear System theory

Accelerator Control

Controlling a "Deadbeat System"

- Controlling accelerator systems is atypical
  - Individual shots cannot be controlled (speed of light)
  - Shots are correlated
  - Noise
  - Discrete systems in location index k
  - Continuous systems in time t
  - Sampled systems in time t_k

- Many applications require "deadbeat controllers"
  - Feedback control that steers system to set-point in fewest number of steps

  - Matching
  - Orbit correction
  - Twiss parameter observer

Accelerator Control

A Beam Position State Observer

Accelerator Control

Perturbations: Control and Orbit Difference

- The former type of perturbation (of the control signal) is of the type used for **orbit difference** applications
  - A magnet value along the beamline is perturbed from its nominal value.
  - The perturbed orbit remain identical to the nominal orbit until it reaches the perturbed magnet, from there it diverges according to the effects of the magnet
  - By subtracting the nominal trajectory from the perturbed trajectory we can observe the first-order response of the magnet
  - We may perform the same procedure using a model of the beamline then compare the two magnet responses. Such a tool is valuable in diagnosing beamline irregularities.

Linear Dynamical Systems

Motivation

- The material on stability and control is important for…
  - Light sources, where beam positions must be maintained to tight tolerances
  - RF systems where, for example, resonant tuning is essential in a highly disruptive environment

- We restrict the analysis to linear dynamical systems, however…

  Most beamlines are designed to be linear systems
  - At least most be treated as such
  - The XAL online model is designed using these principles.

The State Representation

Putting Linear Differential Equations in State Variable Form

Earlier we questioned the meaning of $\mathbf{y} = \mathbf{G}(\mathbf{u}) = L^{-1}(\mathbf{u})$, well here it is…

- Start with our 2nd order linear differential equation

$$b_2\ddot{y}(t) + b_1\dot{y}(t) + b_0 y(t) = u(t), \quad \text{or} \quad \ddot{y} = -\frac{b_1}{b_2}\dot{y} - \frac{b_0}{b_2}y + u$$

- Define our *state variables* x_1 and x_2

$$x_1 \equiv y$$
$$x_2 \equiv \dot{y}$$

- Differentiate x_2 yielding

$$\dot{x}_2 = \ddot{y} = -\frac{b_1}{b_2}x_2 - \frac{b_0}{b_2}x_1 + u$$

- Arranging into matrix-vector format

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{b_1}{b_2} & -\frac{b_0}{b_2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

This has form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

Eq: Constant Coefficient ODE (cont.)

- Thus, 2nd order equation in standard form has the state representation

$$b_2\ddot{y}(t)+b_1\dot{y}(t)+b_0 y(t)=u(t)$$

$$\Rightarrow$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)+\mathbf{D}\mathbf{u}(t)$$

where $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\dfrac{b_1}{b_2} & -\dfrac{b_0}{b_2} \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\mathbf{C} = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $\mathbf{D} = 0$

•The state representation is generally easier to solve on the computer.

•There is a plethora of literature on the state representation properties.

The Matrix Exponential

What is e^{tA}?

- Say square matrix **A** admits a diagonalization

$$\mathbf{A} = \mathbf{T}\boldsymbol{\Lambda}\mathbf{T}^{-1}, \qquad \mathbf{T} \in GL(n,\ ) \subset {}^{n\times n}$$
$$\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots)$$

- For example,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$$

- Then $e^{t\mathbf{A}}$ has a very simple form

$$e^{t\mathbf{A}} = e^{t\mathbf{T}\boldsymbol{\Lambda}\mathbf{T}^{-1}} = \mathbf{I} + t\mathbf{T}\boldsymbol{\Lambda}\mathbf{T}^{-1} + \frac{t^2}{2}\left(\mathbf{T}\boldsymbol{\Lambda}\mathbf{T}^{-1}\right)^2 + \frac{t^3}{2\cdot 3}\left(\mathbf{T}\boldsymbol{\Lambda}\mathbf{T}^{-1}\right)^3 + \cdots$$
$$= \mathbf{T}\left(\mathbf{I} + t\boldsymbol{\Lambda} + \frac{t^2}{2}\boldsymbol{\Lambda}^2 + \frac{t^3}{2\cdot 3}\boldsymbol{\Lambda}^3 + \cdots\right)\mathbf{T}^{-1}$$
$$= \mathbf{T}e^{t\boldsymbol{\Lambda}}\mathbf{T}^{-1}$$

- The explicit form of $e^{t\boldsymbol{\Lambda}}$ is very easy to compute…

Formulae:

$$e^a = 1 + a + \frac{1}{2}a^2 + \frac{1}{1\cdot 2\cdot 3}a^3 + \ldots$$

$$\left[\mathbf{T}\boldsymbol{\Lambda}\mathbf{T}^{-1}\right]^n = \mathbf{T}\boldsymbol{\Lambda}^n\mathbf{T}^{-1}$$

Matrix Exponential

Exponential of a Diagonal Matrix

- Let $\mathbf{\Lambda}$ be diagonal with entries $\{\lambda_i\}$, then

$$e^{t\mathbf{\Lambda}} = \sum_{u=0}^{\infty} \frac{t^i}{i!}\mathbf{\Lambda}^i = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{t^i}{i!}\lambda_1^i & & & \\ & \sum_{n=0}^{\infty} \frac{t^i}{i!}\lambda_2^i & & \\ & & \ddots & \\ & & & \sum_{n=0}^{\infty} \frac{t^i}{i!}\lambda_n^i \end{pmatrix} = \begin{pmatrix} e^{t\lambda_1} & & & \\ & e^{t\lambda_2} & & \\ & & \ddots & \\ & & & e^{t\lambda_n} \end{pmatrix}$$

- The stability (behavior) of $e^{t\mathbf{\Lambda}}$ and, hence, $e^{t\mathbf{A}} = \mathbf{T}e^{t\mathbf{\Lambda}}\mathbf{T}^{-1}$ is completely determined by the eigenvalues of $\mathbf{A}$ according to $e^{t\lambda}$

- The matrix $\mathbf{T}$ determines coupling between these natural modes

Matrix Exponential
Notes on the General Case

- Say $\mathbf{A}$ is not diagonalizable
  - A Jordon form always exists so that $\mathbf{A} = \mathbf{T\Lambda T}^{-1}$ where $\mathbf{\Lambda}$ is the Jordan block diagonal matrix ($\mathbf{\Lambda}$ has 1's to the right of the diagonal)
  - Once again $e^{t\mathbf{A}} = \mathbf{T}e^{t\mathbf{\Lambda}}\mathbf{T}^{-1}$
  - The exponential $e^{t\mathbf{\Lambda}}$ is not as easy to compute this time but the qualitative results are the same.
    - ($\mathbf{\Lambda}$ is triangular and we have terms like $\lambda^k e^{t\lambda}$ floating around)
    - The eigenvalues $\{\lambda_i\}$ of $\mathbf{A}$ (the diagonal of $\mathbf{\Lambda}$) determine the stability of the system according to $e^{t\lambda}$
    - The matrix $\mathbf{T}$ determines the coupling between the natural modes of the system

Matrix Exponential

Notes on the General Case (continued)

- Singular value decomposition does not work for decomposing the matrix exponential
  - Factor $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^T$ where $\mathbf{D}$ is the diagonal matrix of singular values and now $\mathbf{U}$ and $\mathbf{V}$ are both in $SO(n)$ since $\mathbf{A}$ is square
  - However, since $\mathbf{V}^T\mathbf{U} \neq \mathbf{I}$ in general, $e^{t\mathbf{A}} \neq \mathbf{U}e^{t\mathbf{D}}\mathbf{V}^T$
    - For example, $(\mathbf{U}\mathbf{D}\mathbf{V}^T)^2 = (\mathbf{U}\mathbf{D}\mathbf{V}^T)\,(\mathbf{U}\mathbf{D}\mathbf{V}^T) \neq \mathbf{U}\mathbf{D}^2\mathbf{V}^T$
- However, Jordan decomposition always exist and allow for generalized eigenvalues, i.e., eigenvalues with value zero.
  - The natural modes for zero eigenvalues are called the *center manifold* of the dynamical system

The Matrix Exponential

Existence

- Define exponential of a *square* matrix **A** by Taylor series

$$e^{t\mathbf{A}} \equiv \sum_{n=0}^{\infty} \frac{t^n}{n!}\mathbf{A}^n = \mathbf{I} + t\mathbf{A} + \frac{t^2}{2}\mathbf{A}^2 + \frac{t^3}{2\cdot 3}\mathbf{A}^3 + \cdots \qquad \text{(well-defined operations)}$$

- Note for any **A** and $t < \infty$ that $\left\| \frac{t^n}{n!}\mathbf{A}^n \right\| \xrightarrow[n\to\infty]{} 0$ (well-defined values)

- In fact $\|\mathbf{A}\| \le \lambda_{\max} = \max \Lambda(\mathbf{A})$ for induced norm $\|\cdot\|$

and

$$\left\|e^{t\mathbf{A}}\right\| = \left\|\sum_{n=0}^{\infty} \frac{t^n}{n!}\mathbf{A}^n\right\| \le \|\mathbf{I}\| + t\|\mathbf{A}\| + \frac{t^2}{2}\left\|\mathbf{A}^2\right\| + \frac{t^3}{2\cdot 3}\left\|\mathbf{A}^3\right\| + \cdots$$

$$\le 1 + t\lambda_{\max} + \frac{t^2}{2}\lambda_{\max}^2 + \frac{t^3}{2\cdot 3}\lambda_{\max}^3 + \cdots$$

$$= e^{t\lambda_{\max}}$$

Linear Systems:

A Note on the Time-Varying Case

- The homogeneous solution $\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}_0$ is the generalization of the one-dimensional ODE $\dot{x}(t) = ax(t); \quad x(0) = x_0$

- If the coefficient a is a function of time, $a = a(t)$, then the solution to the scalar ODE is

- This does not generalize! $x(t) = e^{\int_0^t a(\tau)d\tau} x_0$
  - The solution to $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t); \ \mathbf{x}(0) = \mathbf{x}_0$ is given by

    $\mathbf{x}(t) = \mathbf{\Phi}(t,0)\mathbf{x}_0$ where $\mathbf{\Phi}(t,0)$ is given by the Peano-Baker series

$$\mathbf{\Phi}(t,0) = \mathbf{I} + \int_0^t \mathbf{A}(t_1)dt_1 + \int_0^t \mathbf{A}(t_1)\int_0^{t_1} \mathbf{A}(t_2)dt_2 dt_1 + \int_0^t \mathbf{A}(t_1)\int_0^{t_1} \mathbf{A}(t_2)\int_0^{t_2} \mathbf{A}(t_3)dt_3 dt_2 dt_1 + \cdots$$

  - (It is possible to define $L_t(a) \equiv \int_0^t a(\tau)d\tau$ and say $\mathbf{\Phi}(t,0) = e^{L_t}\left(\mathbf{A}(t)\right)$)

Transfer Function Approach

Relationship of **H**(t) and $\hat{\mathbf{H}}(s)$

- Recall that in frequency domain **x** and **u** are related by

$$\hat{\mathbf{x}}(j\omega) = \hat{\mathbf{H}}(j\omega)\mathbf{B}\hat{\mathbf{u}}(j\omega)$$

where $\hat{\mathbf{H}}(j\omega)$ is the Laplace transform of $e^{t\mathbf{A}}$ evaluated at $s = j\omega$.

$$\hat{\mathbf{H}}(s) = \int_0^\infty e^{-st} e^{t\mathbf{A}} dt = \int_0^\infty e^{-t(s\mathbf{I}-\mathbf{A})} dt = -(s\mathbf{I}-\mathbf{A})^{-1} e^{-t(s\mathbf{I}-\mathbf{A})}\Big|_{t=0}^{t=\infty} = (s\mathbf{I}-\mathbf{A})^{-1} \quad \text{for} \quad \mathrm{Re}\, s > \max \Lambda(\mathbf{A})$$

- If **A** is diagonalizable, i.e., $\mathbf{A} = \mathbf{T}\Lambda\mathbf{T}^{-1}$, then

$$\hat{\mathbf{H}}(s) = \mathbf{T}^{-1}(s\mathbf{I}-\Lambda)^{-1}\mathbf{T}$$

- Thus, $\hat{\mathbf{H}}(s)$ has "poles" at $s = \{\lambda_i\} = \Lambda(\mathbf{A})$
- By the Residue Theorem and Laplace Transform
  - If the poles are in the right half plane $\hat{\mathbf{H}}(j\omega)$ does not exist (is unstable)
  - If the poles are in the left half plane $\hat{\mathbf{H}}(j\omega)$ exists for all ω (is stable)