Lecture 5

Transverse Beam Optics, Part I

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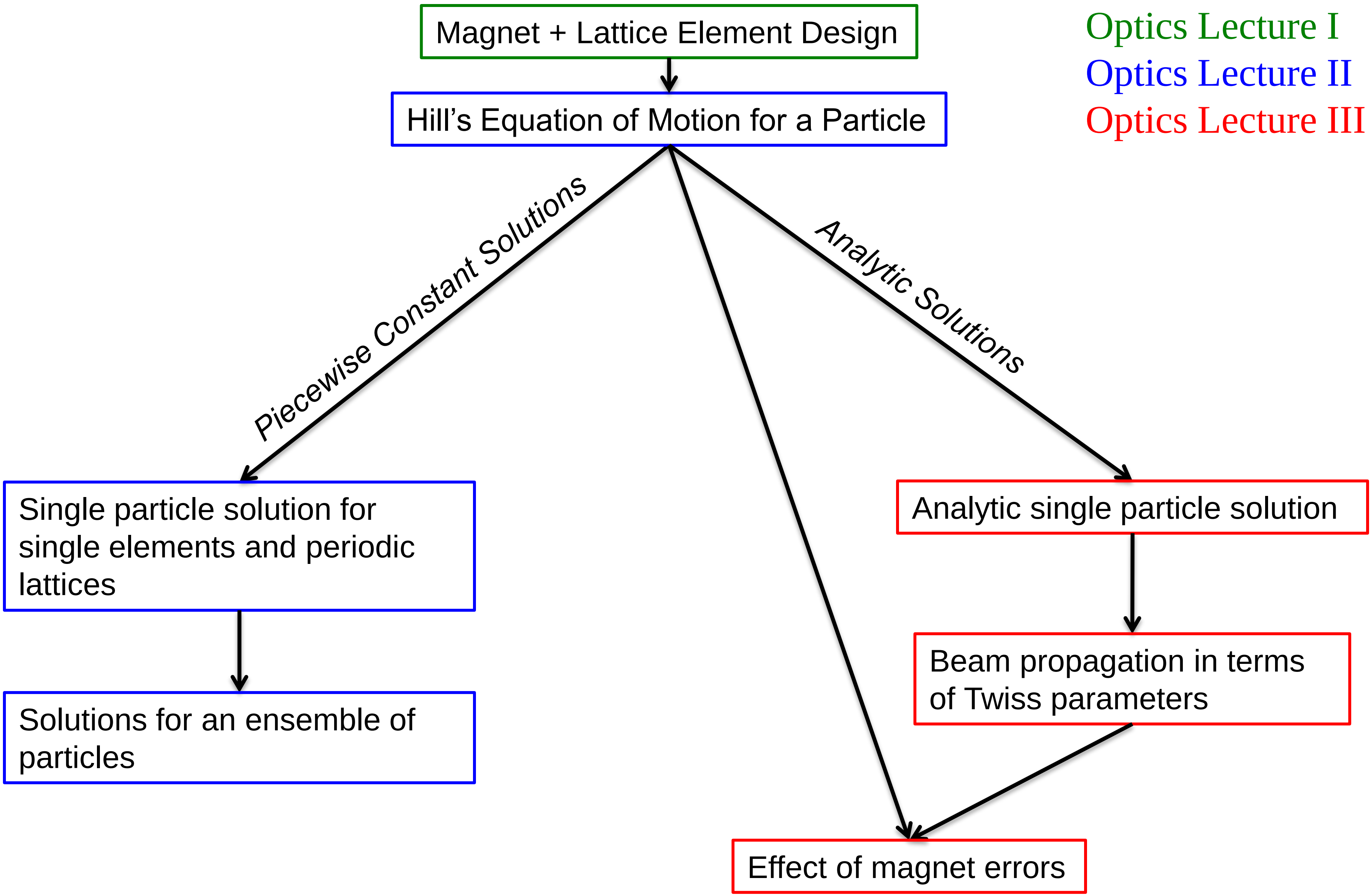
Outline of Optics Lectures I - III

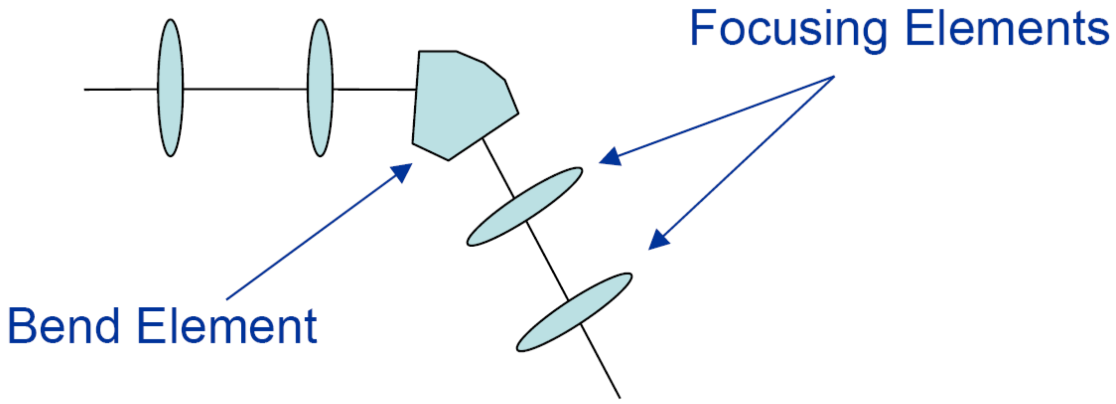


Part I: Bending and Focusing

Definition of Beam Optics

Beam optics: The process of guiding a charged particle beam from A to B using magnets.

An array of magnets which accomplishes this is a *transport system*, or magnetic lattice.



Recall the Lorentz Force on a particle:

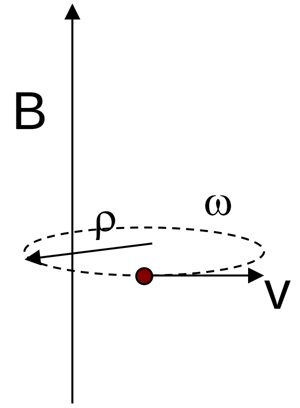
$F = ma = e(E + v \times B) = \gamma m_0 v^2/\rho$, where γm_0 is the relativistic mass.

In magnetic transport systems, typically we have E=0. So,

$F = ma = e(v \times B) = \gamma m_0 v^2/\rho$

Force on a Particle in a Magnetic Field

The simplest type of magnetic field is a constant field. A charged particle in a constant field executes a circular orbit, with radius ρ and frequency ω.



To find the direction of the force on the particle, use the right-hand-rule.

What would happen if the initial velocity had a component in the direction of the field?

Dipole Magnets

A ***dipole magnet*** gives us a constant field, B.

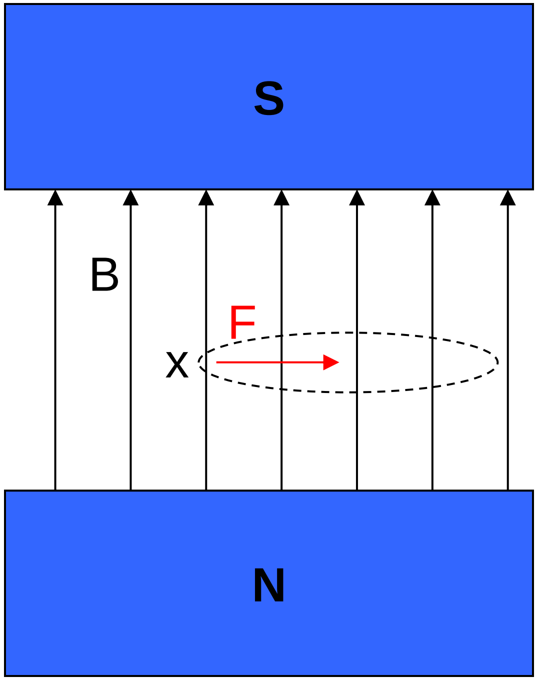
The field lines in a magnet run from North to South. The field shown at right is positive in the vertical direction.

Symbol convention:

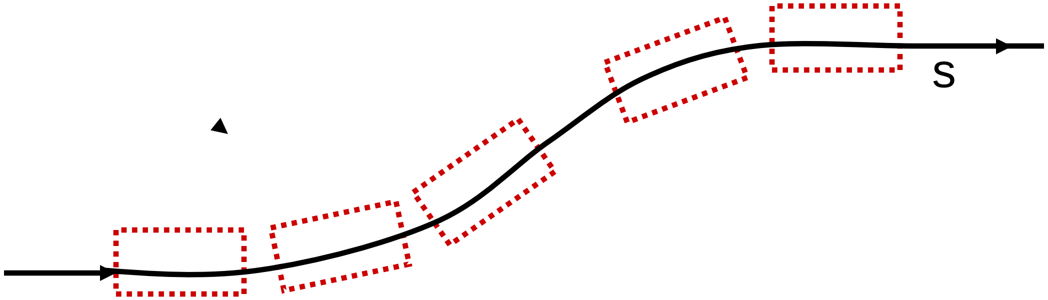
x - traveling into the page,

• - traveling out of the page.

In the field shown, for a positively charged particle traveling into the page, the force is to the right.



In an accelerator lattice, dipoles are used to *bend* the beam trajectory. The set of dipoles in a lattice defines the ***reference trajectory***:



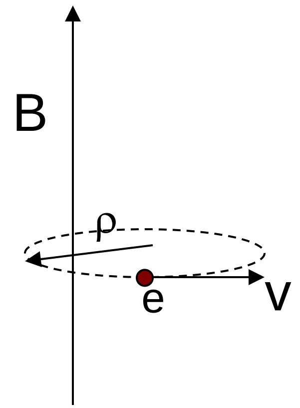
Field Equations for a Dipole

Let’s consider the dipole field force in more detail.

(**Derivations**)

$$F = e(\vec{v} \times \vec{B}) = ma$$

Assuming a uniform B field:



$$evB = \frac{\gamma m_o v^2}{\rho}$$

$$\frac{1}{\rho} = \frac{eB}{\gamma m_o v} = \frac{eB}{p} \quad \text{where } p = \gamma m_o v$$

Recognizing the relationship between energy and momentum:

$$\left.\begin{array}{l} p = \gamma m_0 v \\ E = \gamma m_0 c^2 \end{array}\right\} \rightarrow pc = \beta E$$

Gives the “magnetic rigidity”:

$$\boxed{B\rho = \frac{p}{e}}$$

(Weidemann 2.6)

Field Equations for a Dipole

For a particle of mass m, energy E, and momentum p:

1) The bending radius of the motion of the particle in the dipole field

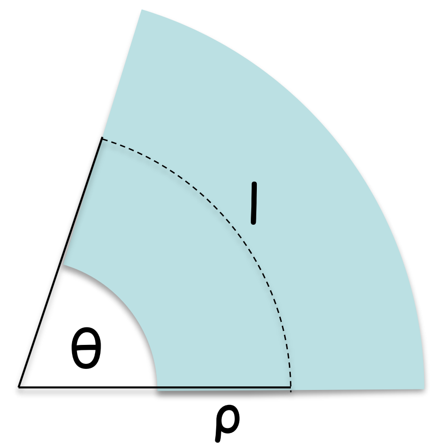
$$\frac{1}{\rho}(m^{-1}) = 0.2998\frac{B(T)}{\beta E(GeV)}$$

(Weidemann 2.8)

2) The total angle of deflection along a path length s:

$$\theta = \int \frac{ds}{\rho}$$

(Weidemann 2.9)

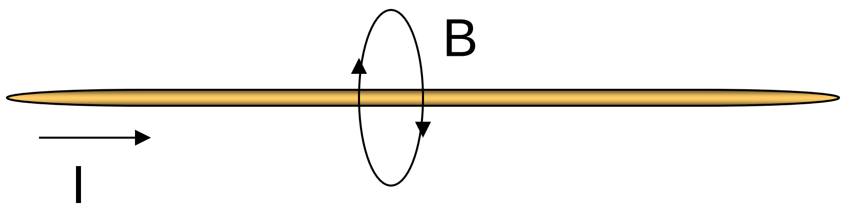


If the field is uniform over the length l,

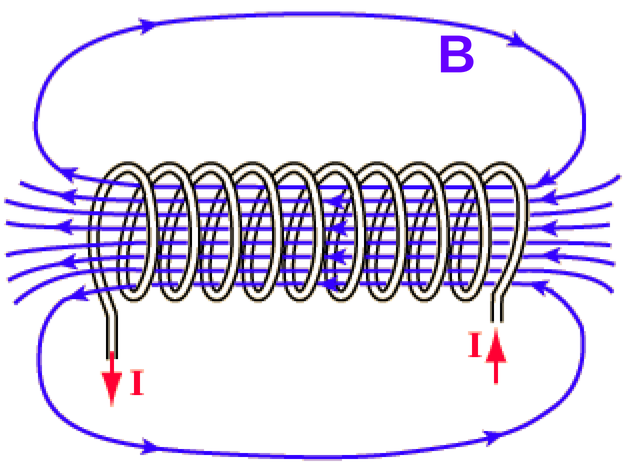
$$\theta = \frac{l}{\rho}$$

Generating a B Field from a Current

Recall that a current in a wire generates a magnetic field B which curls around the wire:



Or, by winding many turns on a coil we can create a strong uniform magnetic field.



The field strength is given by one of Maxwell's equations:

$$\nabla \times \frac{B}{\mu_r} = J$$

$$\mu_r = \frac{\mu_{\text{material}}}{\mu_0}$$

The Dipole Current-to-Field Relationship

In an accelerator dipole magnet, we use current-carrying wires and metal cores of high μ to set up a strong dipole field:

N turns of current *I* generate a field perpendicular to the pole tip surface.

Using Maxwell's equation for B, we can derive the relationship between B in the gap, and I in the wires:

$$I_{coil} = NI_{wire} = \frac{G(m)B_{\perp}(T)}{\mu_0}$$

(Wiedemann 2.13)

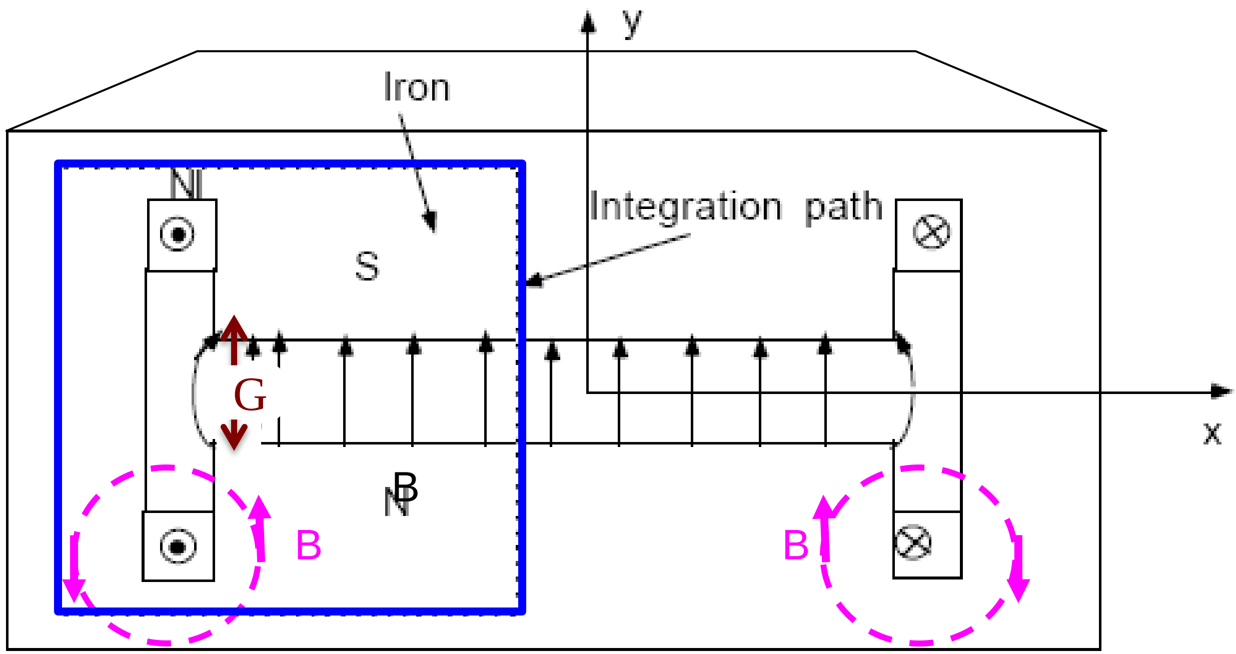
Dipole Current-to-Field Derivation

Begin with Ampere's Law:

$$\int \nabla \times \frac{B}{\mu_r} = \oint \frac{B}{\mu_r}\cdot dl = I_{enc}$$

where $I_{enc} = 2I_{coil}$

(**Derivations**)



Pure dipole: NI turns/pole

≈ 0 since μ_{iron} is large

$$\oint \frac{B}{\mu_r}\cdot dl = \frac{1}{\mu_0}\oint B_y\,dy + \frac{1}{\mu_{iron}}\oint B_y\,dy = \frac{2GB_y}{\mu_0}$$

$$\boxed{I_{coil} = NI_{wire} = \frac{GB_y}{\mu_0}}$$

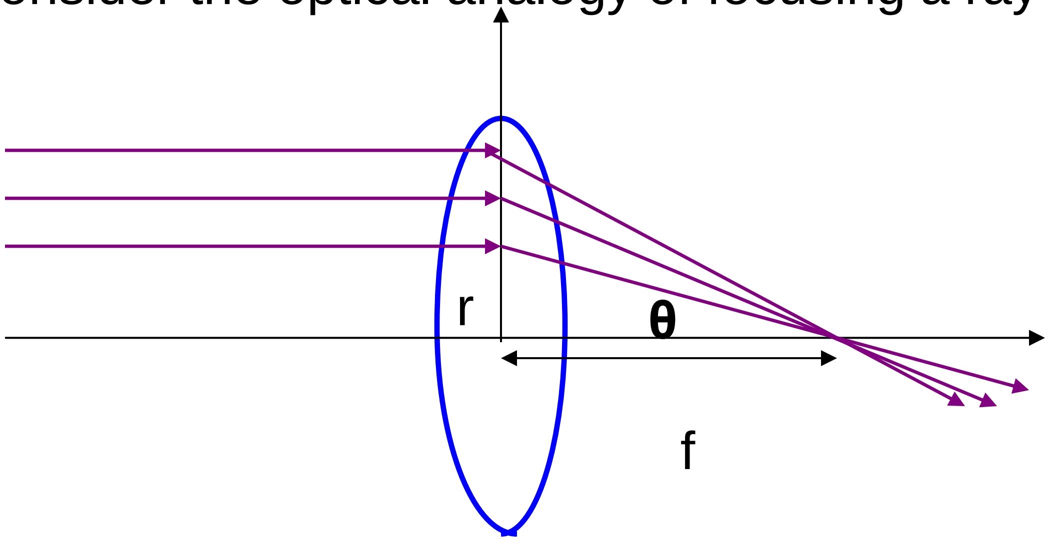
(Wiedemann 2.13)

Optical Analogy for Focusing

We have seen that a dipole produces a constant field that can be used to bend a beam.

Now we need something that can focus a beam. *Without focusing, a beam will naturally diverge.*

Consider the optical analogy of focusing a ray of light through a lens:



The rays come to a focus at the focal point, *f*. The focusing angle depends on the distance from center, r.

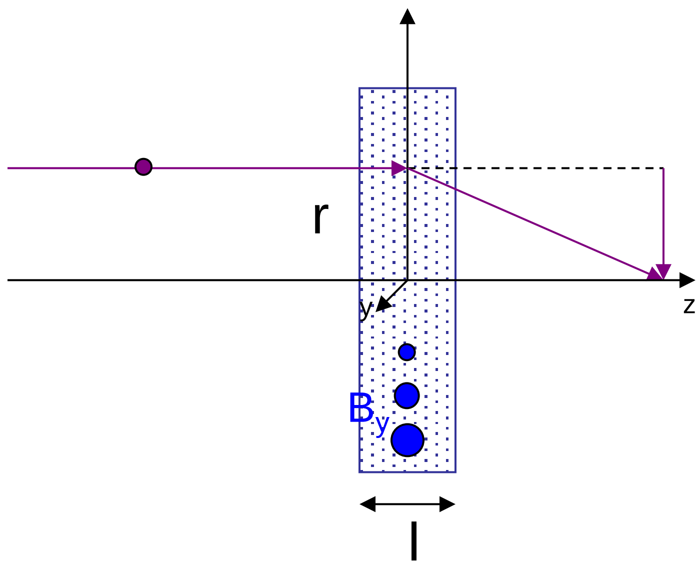
The farther off axis, the stronger the focusing effect! The dependence is ***linear*** *for small x.*

$$\tan\theta = \frac{x}{f}$$

$$\theta \approx \frac{x}{f}, \text{ for small } x$$

Focusing Particles with Magnets

Now consider a ***magnetic lens***. This magnet has a field which increases in strength with distance from the axis.



For a field which increases linearly with r, the resulting kick will also increase linearly with r.

We can solve for the focal length and focusing strength of this system:

(**Derivations**)

$$k[\mathrm{m}^{-2}] = \frac{1}{fl} = \frac{0.2998 g(\mathrm{T/m})}{bE(\mathrm{GeV})} = \text{focusing strength where } \mathrm{g} = \frac{\mathrm{dB_y}}{\mathrm{dx}}$$

(Wiedemann 2.23)

Derivation: Focal length of a Focusing Magnet

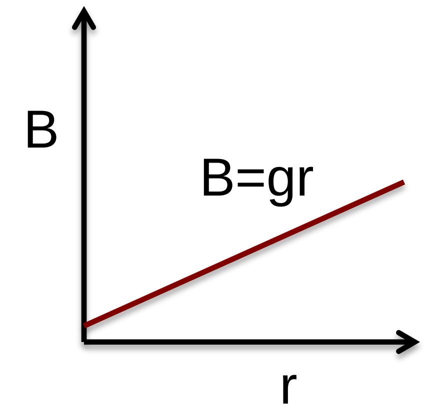
Recall from earlier the relations for bend angle and radius: (**Derivations**)

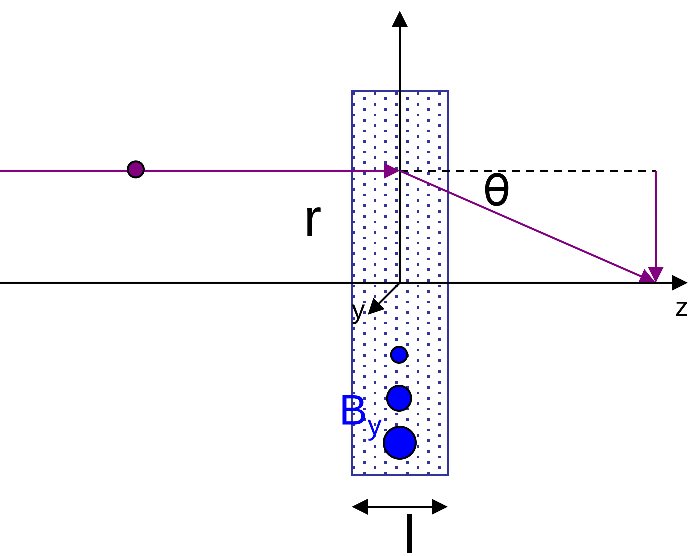
$$q = \frac{l}{r}, \text{ and } \frac{1}{r}(m^{-1}) = 2.998\frac{B(\mathrm{T})}{bE(\mathrm{GeV})}$$

$$Þ \quad q = \frac{l}{r} = 2.998\frac{lB(\mathrm{T})}{bE(\mathrm{GeV})}$$

Assume the field increases linearly with radius then:

$$q = \frac{l}{r} = 2.998\frac{g(\mathrm{T/m})rl}{bE(\mathrm{GeV})}$$





Consider our field like a optical lens of focal length *f*. Then,

$$q » \frac{r}{f}$$

And relating, we get the focal length of the magnet:

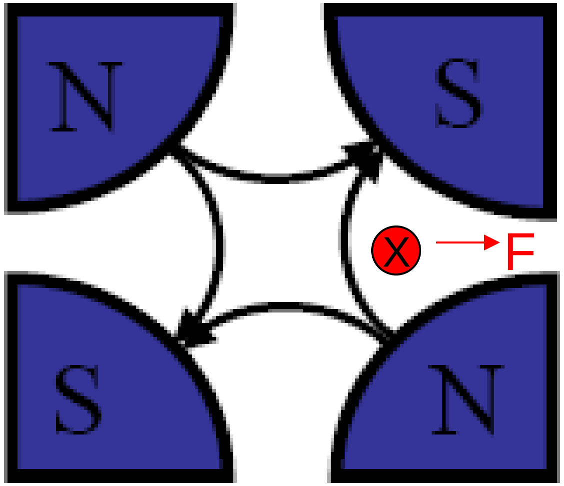
$$\frac{1}{f} = 2.998\frac{g(\mathrm{T/m})l}{bE(\mathrm{GeV})}$$

Quadrupole Magnets

A **quadrupole magnet** imparts a force proportional to distance from the center. This magnet has 4 poles:

Consider a positive particle traveling into the page (into the magnet field).

According to the right hand rule, the force on a particle on the right side of the magnet is to the right, and the force on a similar particle on left side is to the left.



This magnet is horizontally defocusing. A distribution of particles in x would be defocused!

What about the vertical direction?
-> A quadrupole which defocuses in one plane focuses in the other.

Quadrupole Current-to-Field Equations

As with a dipole, in an accelerator we use current-carrying wires wrapped around metal cores to create a quadrupole magnet:

The field lines are denser near the edges of the magnet, meaning the field is stronger there.

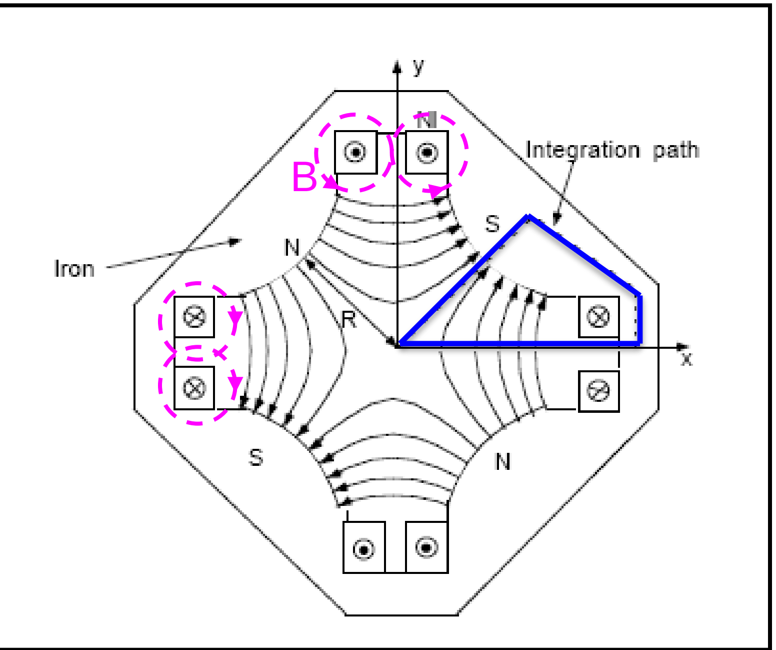
The strength of B_y is a function of x, and visa-versa. The field at the center is zero!

Using Maxwell's equation for B, we can derive the relationship between B in the gap, and I in the wires:

$$I_{coil} = \frac{1}{2\mu_0} g(\mathrm{T/m}) R^2 (\mathrm{m})$$

(Wiedemann 3.23)

Quadrupole Current-to-Field Derivation



(**Derivations**)

$$\int \nabla \times \frac{B}{\mu_r} = \oint \frac{B}{\mu_r} \cdot dl = I_{enc}$$

Recall that for quadrupole field increases linearly with radius:

$$B = gr$$

≈ 0 since μ_{iron} is large

=0 for perp vectors

$$\oint \frac{B}{\mu_r} \cdot dl = \frac{1}{\mu_0} \int B_r \, dr + \frac{1}{\mu_{iron}} \int B \cdot dl_{iron} + \frac{1}{\mu_0} \int B_y \cdot dx$$

$$= \frac{1}{\mu_0} \int_0^R gr \, dr = \frac{gR^2}{2\mu_0}$$

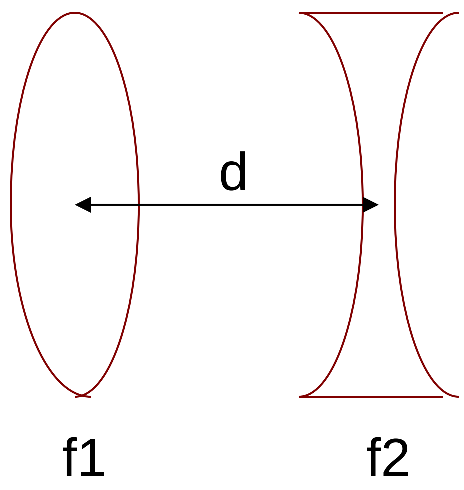
$$I_{enc} = \frac{gR^2}{2\mu_0}$$

(Wiedemann 3.23)

Focusing Using Arrays of Quadrupoles

Quadrupoles focus in one plane while defocusing in the other. So, how can this be used to provide net focusing in an accelerator?

Consider again the optical analogy of two lenses, with focal lengths f1 and f2, separated by a distance d:



The combined f is:

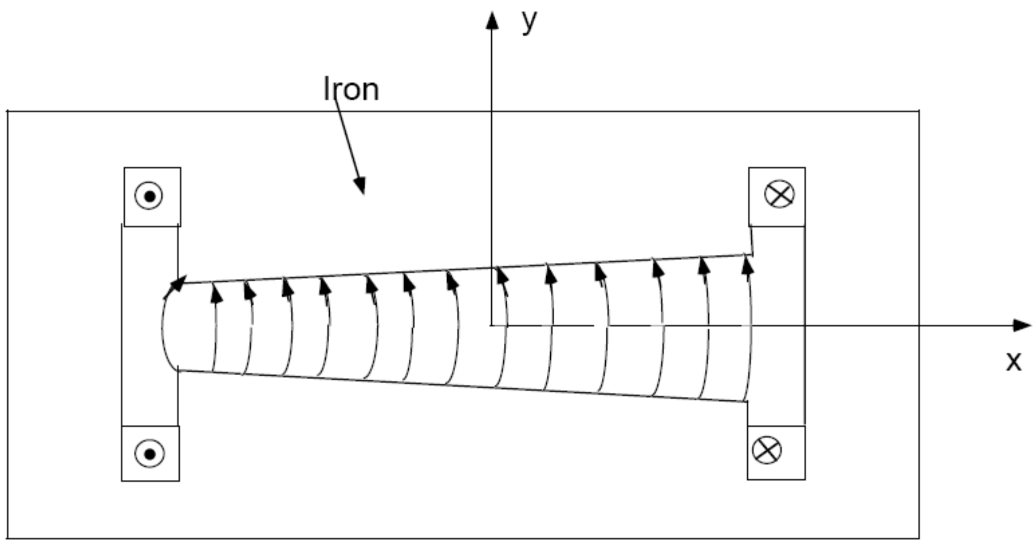
$$\frac{1}{f_{combined}} = \frac{1}{f1} + \frac{1}{f2} - \frac{d}{f1 f2}$$

What if f1 = -f2?

The net effect is *focusing* (positive), 1/f = d/(f1f2)

Other Types of Magnets

Many other types of magnets are used in an accelerator. For instance, gradient magnets are a type of "combined function" magnet which bend and focus simultaneously:



The B field in this magnet has both quadrupole and dipole components.

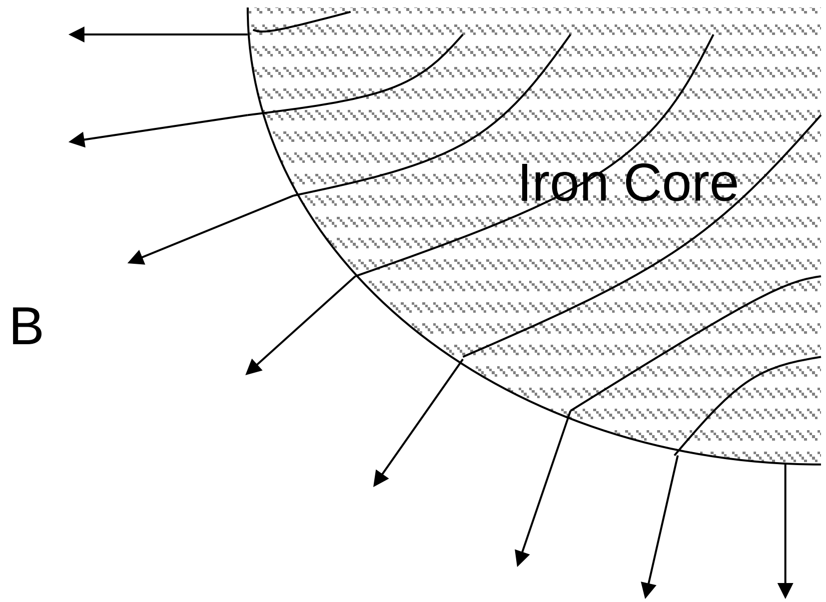
Another type of magnet is the solenoid, shown previously, which focuses in the radial direction.

Part II: Arbitrary Order Magnets and Material Properties

B-Fields at the Pole Tips

Electromagnets are composed of ferromagnetic poles surrounded by current carrying wires that set up the B field in the material.

Below saturation of iron or similar material, the field lines on the vacuum side are always perpendicular to the pole tip surface:



Magnetic lines may have both || and ⊥ path inside the material, but outside, only the field ⊥ to the surface survives. To get as strong of a field in the gap as possible, we should try to make the ⊥ piece inside as large as possible.

Below saturation, we can add the B field any way we want inside the material. By setting the pole tip geometry perpendicular to the B field components at any location, we can get any of the desired multipoles.

Magnet design concepts

1. What should the pole shape of a dipole look like?
2. What should the pole shape of a quadrupole look like?

We know that the B field will be perpendicular to the pole tip surface.

We can draw these out before we do any math…

Laplace's Equation

So far we have derived the B fields for two types of magnets (dipole and quadrupole). It would be very useful for us to have a general expression to represent the B field *and* for the pole tip of an arbitrary order magnet.

Assumptions for a general accelerator magnet:

1) There is a material-free region for passage of particles.
2) The magnet is long enough that we can ignore components of B in the z direction, and treat only the (x,y) plane.
3) Fields are calculated in a current-free region ($\nabla \times B = 0$), so there is a scalar potential V such that $B = \nabla V$

Putting these together with $\nabla \cdot B = 0$, we arrive at Laplace's equation in free space:

$$\boxed{\Delta V = 0}$$

Properties of Solutions to Laplace's Equation

$$\Delta V = 0$$

$$\Delta V = \frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} = 0 \quad (\text{2D Cartesian})$$

$$\Delta V = \frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} + \frac{1}{r^2}\frac{d^2V}{d\varphi^2} = 0 \quad (\text{2D Cylindrical})$$

What does a solution to Laplace's equation provide?

1) Any electromagnetic potential, V, which satisfies Laplace's equation can be visualized using a set of equipotential lines (in 2D) or equipotential surfaces (in 3D).

1) The B field can easily be derived by taking the gradient of V:

$$B = -\nabla V_B(x, y)$$

1) This is mathematically equivalent to the problem of electrostatics for E fields in charge-free regions.

Solution to Laplace's Equation

If we initially adopt a cylindrical coordinate system (r,ϕ,z) for the solution, V, then we can guess a solution for the potential in the form of a Taylor expansion.

Consider features of the magnets we have seen so far (dipole and quadrupole):

1. The factor of p/e is always present.
2. The dependence of field on position increases with magnet order. So B goes as r^n .
3. The angular repetition of poles increases with magnet order.

(**Derivations**)

A general solution which meets these requirements:

$$V(r,\phi,z) = \frac{-p}{e}\sum_{n>0}\frac{r^n}{n!}A_n(z)e^{in\phi}$$

A_n are coefficients to be found.

(Wiedemann 3.3)

Example: The Dipole Field

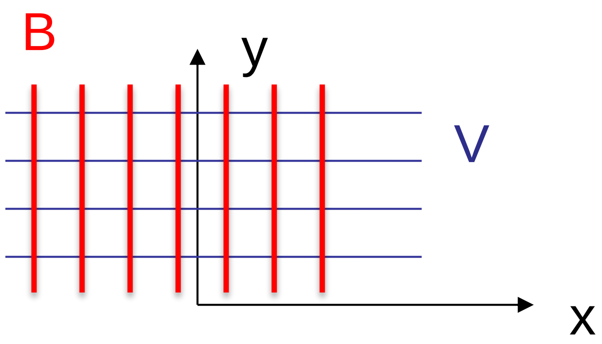
Example: Expand the potential for the n=1 case, imaginary component. Then find the field from the potential.

(**Derivations**)

$$V_1 = \frac{-p}{e} A_{01} y$$

$$B_1 = -\nabla V_1 = \frac{p}{e} A_{01} \hat{y},$$

$$A_{01} = \kappa = \frac{1}{\rho}$$



We find that n=1 gives a dipole field. The coefficient A_1-B_1 is the dipole strength found earlier (κ=1/ρ). Note that for the normal case, only the vertical field (horizontal bending) is present.

Example: The Quadrupole Field

<u>Another Example:</u> Now expand the n=2 case:

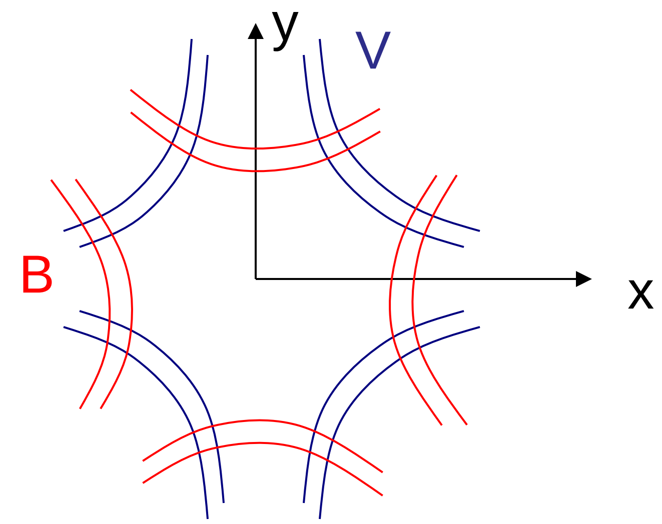
(**Derivations**)

$$V_2 = \frac{-p}{e} A_{11} xy$$

$$B_x = -\frac{dV_2}{dx} = \frac{p}{e} A_{11} y; \quad B_y = -\frac{dV_2}{dy} = \frac{p}{e} A_{11} x$$

$$A_{11} = \frac{e}{p} \frac{dB_y}{dx} = k$$

Equipotential lines
of constant xy.



These are the equations for a normal quadrupole, which we derived earlier. We can associate the coefficient A_2-B_2 with the quadrupole strength, k.

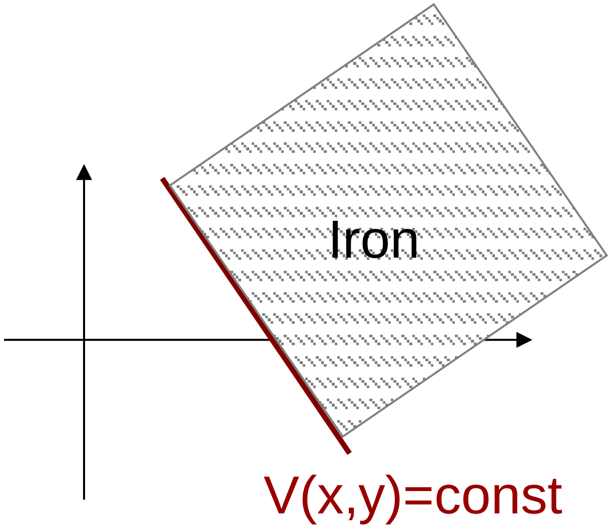
Magnet Design – Pole Tip Geometry

How do we design a real magnet for a specific multipole component?

As seen earlier, our solution to Laplace's equation, V, gives us the equipotential lines for any particular multipole. Since $B = \nabla V$, the field is perpendicular to the equipotential surfaces. Because B is also perpendicular to the surface of a ferromagnetic material, such as iron, the surface is an equipotential surface. Therefore, we design the ferromagnetic "pole tip" to match the equipotential surface of the desired multipole.

The equation for the equipotential surface becomes the equation for the pole tip geometry.

(**Examples** - Dipole and Quadrupole)



The Solution in Cartesian Coordinates

In practice, it will be more convenient to rewrite the solution in Cartesian coordinates, and to separate the real and imaginary pieces.

$$\mathrm{Re}[V_n(x,y)] = \frac{-p}{e}\sum_{m=0}^{n/2} A_{n-2m,2m}\frac{x^{n-2m}}{(n-2m)!}\frac{y^{2m}}{(2m)!}$$

Skew n-pole

$$\mathrm{Im}[V_n(x,y)] = \frac{-p}{e}\sum_{m=1}^{(n+1)/2} A_{n-2m+1,2m-1}\frac{x^{n-2m+1}}{(n-2m+1)!}\frac{y^{2m-1}}{(2m-1)!}$$

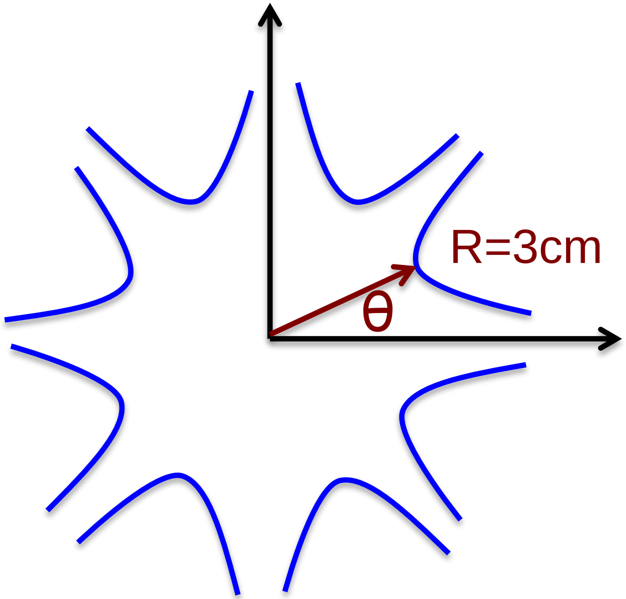
Normal n-pole

(Wiedemann 3.28)

The real and imaginary pieces correspond to different physical orientations of the magnets – “skew” (real) and “normal” (imaginary). We are usually more interested in the “normal” magnets, because they decouple the linear motion in x and y.

Example – Octupole Pole Tip Profile

(Weidemann problem 3.10) Derive the equation for the pole profile of an iron dominated upright octupole with a bore radius R=3 cm.



Normal (Upright) Magnetic Fields

Lowest the orders for normal B field from expansion.
(For more, see Tables 3.3 and 3.4, Wiedemann.)

$$\text{Dipole: } \frac{e}{p}B_x = 0 \;;\; \frac{e}{p}B_y = \kappa_x$$

$$\text{Quadrupole: } \frac{e}{p}B_x = ky \;;\; \frac{e}{p}B_y = k\text{x}$$

$$\text{Sextupole: } \frac{e}{p}B_x = mxy \;;\; \frac{e}{p}B_y = \frac{1}{2}m(x^2 - y^2)$$

A general expression for the normal “strength parameters”, (κ, k, m, etc…)

$$s_n = \frac{e}{p}\left.\frac{\partial^{n-1}B_y}{\partial x^{n-1}}\right|_{\substack{x=0\\y=0}}$$

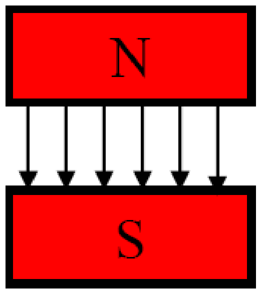
$$s_n = \frac{0.2999}{\beta E}\left.\frac{\partial^{n-1}B_y}{\partial x^{n-1}}\right|_{\substack{x=0\\y=0}}$$

(Wiedemann 3.32, 3.33)

Other n-Pole Magnets

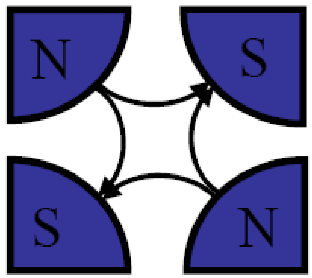
The general equation for B allows us to write the field for any n-pole magnet. Examples of upright magnets:

n=1: Dipole



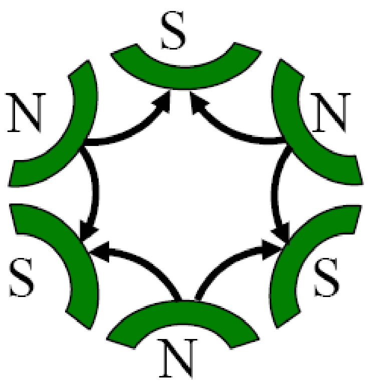
180° between poles

n=2: Quadrupole



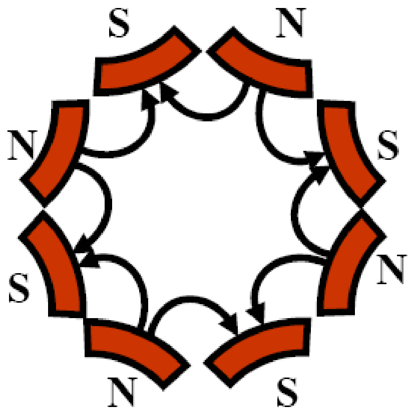
90° between poles

n=3: Sextupole



60° between poles

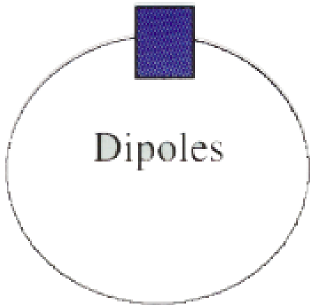
n=4: Octupole

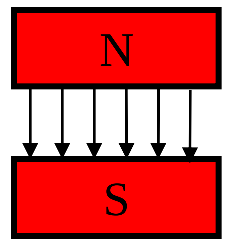


45° between poles

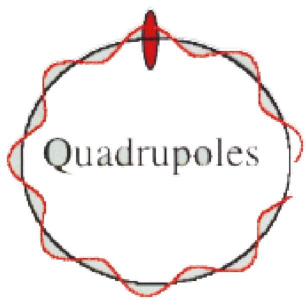
- In general, poles are 360°/2n apart.
- The skew version of the magnet is obtained by rotating the upright magnet by 180°/2n.

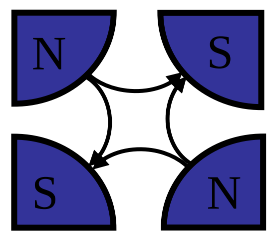
n-Pole Uses



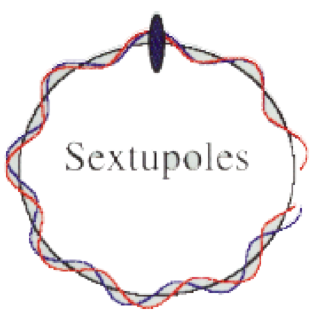


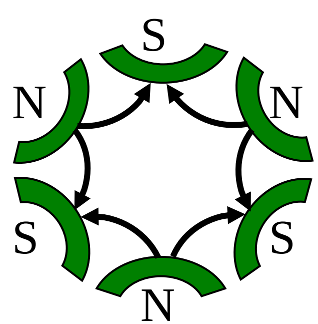
Bending (following reference trajectory)





Focusing the beam





“Chromatic compensation”

Magnet examples

Dipole

Quadrupole

Sextupole

Realistic Magnetic Fields

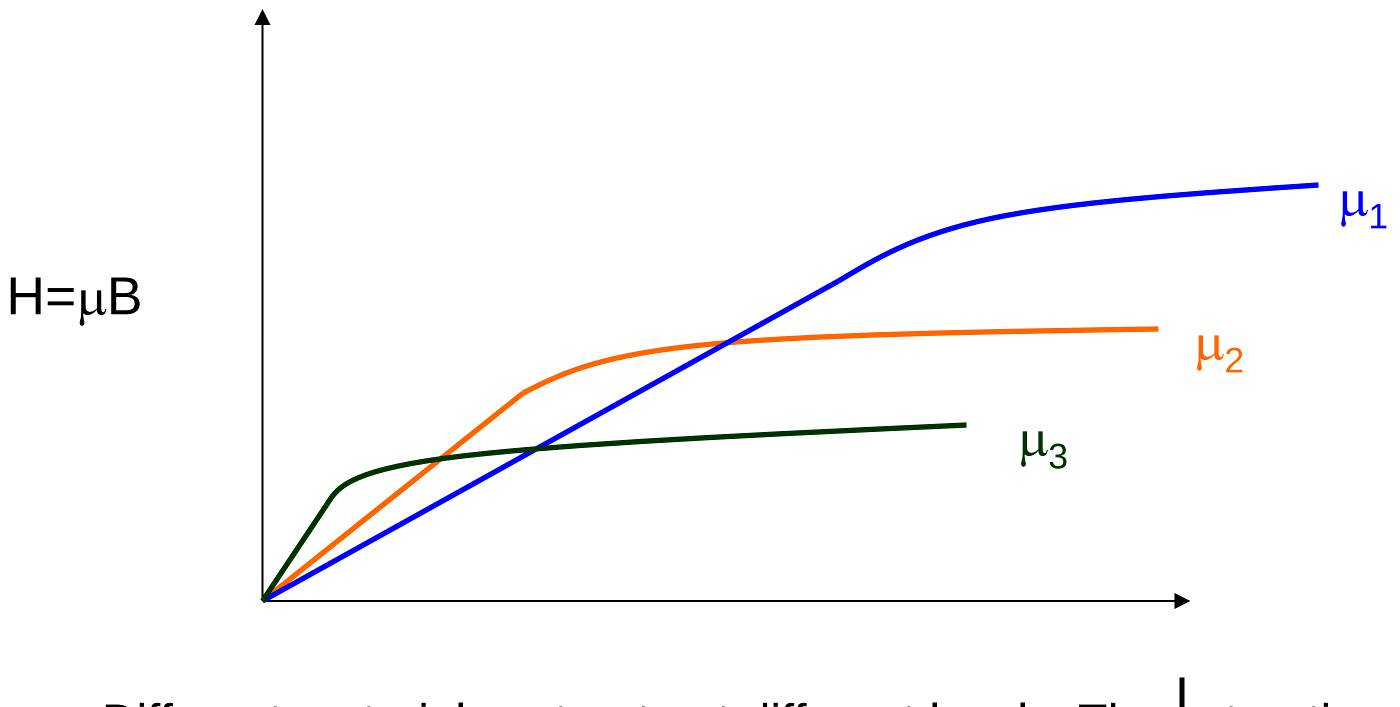
In a “separated function” accelerator lattice, the magnets are designed to fulfill specific duties: Dipoles bend the beam, quadrupoles focus the beam, etc.

However, there is no such thing as a perfect n-pole magnet! All magnets have at least small contributions from other multipoles besides the main multipole.

For separate function magnets, we desire the field strength parameters, s_n, of the unwanted components to be on the level of 10^{-4} or less.

Saturation of Magnetic Materials

In a non-saturated field, the relationship between field strength, B, and driving current, I, is linear. Above saturation, an increase in current does not generate a corresponding increase in field:

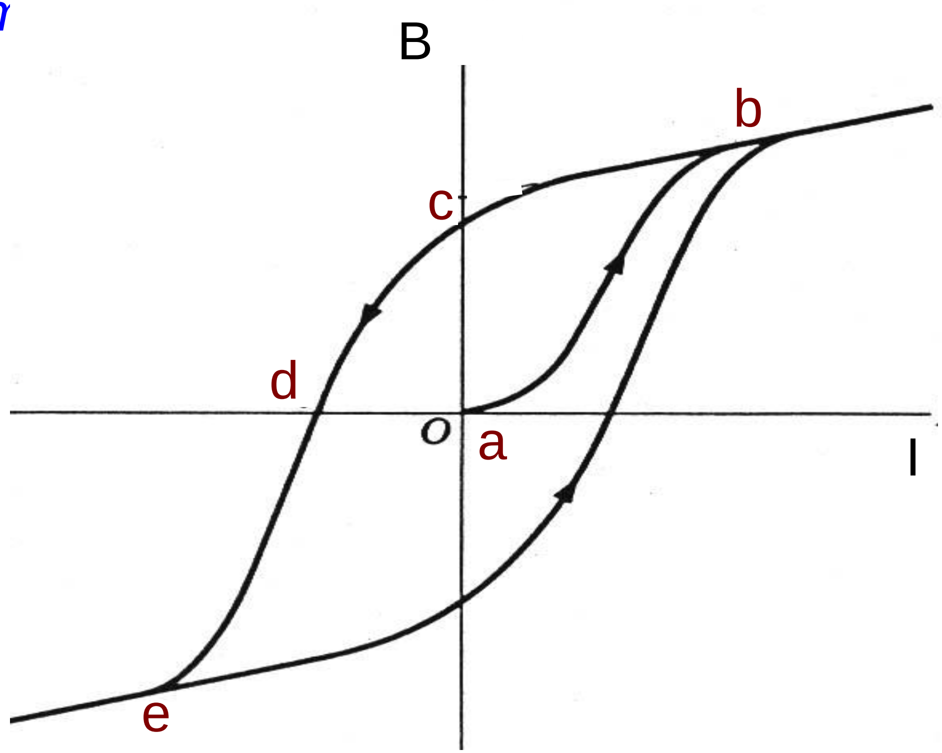


- Different materials saturate at different levels. The saturation phenomenon limits iron-dominated magnets to ~2T.
- Superconducting magnets can give fields up to 6 – 10T.

Hysteresis and Magnet Cycling

An external B-field, created by a current I, creates a B-field in iron by aligning tiny internal dipoles (electron spins) in the material. Saturation occurs when all dipoles are already aligned.

However, if the current and external field are dropped to zero, the material remains partially magnetized. *This gives rise to “hysteresis” and the need for n*



a - start point
b - saturation
c - residual magnetization
d - B=0
e – saturation with –B

Summary

Summary:

1) First, we found the equations for dipole and quadrupole magnets, and analyzed the resulting force on the particle: We found that dipoles are used to bend particles along the “reference trajectory”, and quadrupoles are used to focus particles.
2) Second, we found the current to field equations for dipoles and quadrupoles.
3) Third, we found that we could derive the equations for the B fields for any accelerator magnet from a general form.
4) Finally, we discussed the basic principles of magnet design.

Now we have the complete equations for B. We also have the equation for the force on a particle due to these fields: $F=q(v \times B)$

We can now write the equation of motion for a particle in an accelerator!