Physics of Free-Electron Lasers

One-dimensional FEL Theory

Dinh C. Nguyen & Quinn R. Marksteiner
Los Alamos National Laboratory

US Particle Accelerator School
June 23 – 27, 2014

LA-UR 14-23595

One-dimensional FEL Theory

1. Coupled phase-energy equations
2. Wave equation
3. Charge and current density
4. Slowly-varying amplitude
5. Coupled FEL first-order equations
6. Third-order equation for high-gain FEL
7. Analytic solutions to third-order equation
8. Inclusion of 3D effects

Assumptions in 1D FEL Theory

- Charge density and fields do not depend on the transverse coordinates (x, y), i.e., the electron beam radius is large.
- The electron bunch is long and end effects can be ignored.
- The radiation amplitude grows slowly with z (ignoring 2nd derivative).
- Diffraction of the FEL light is negligible over a gain length.
- The effects of electron beam emittance and energy spread can be ignored.
- 3D effects (diffraction, emittance, and energy spread) can be added later as corrections to the gain length and saturated power.

The Bessell JJ Factor

We mentioned earlier that the oscillatory axial motion of the electrons in a planar undulator causes a reduction in the interaction strength. The reduction is expressed in terms of the difference between J_0 and J_1 Bessel functions of an argument ξ that depends on *K* (see textbook for explanation of $J_0 - J_1$).

$$JJ(\xi) = J_0(\xi) - J_1(\xi)$$

$$\xi = \frac{K^2}{4 + 2K^2}$$

JJ is unity for a helical undulator (no reduction). For a planar undulator, *JJ* becomes less than unity at large *K*. For $K \leq 1$

$$JJ(\xi) \approx 1 - \frac{\xi^2}{4} - \frac{\xi}{2}$$

The modified undulator parameter, *K*-hat, is used in calculations that involve the interaction strength, but not for wavelength calculation.

$$\hat{K} = K \cdot JJ$$

Evolution of Electron Phase

Ponderomotive phase

$$\psi = (k_r + k_u)z - \omega_r t + \varphi_0$$

Taking the derivative with respect to z

$$\frac{d\psi}{dz} = (k_r + k_u) - \frac{\omega_r}{\bar{\upsilon}_z}$$

$$\frac{d\psi}{dz} = (k_r + k_u) - \frac{\omega_r}{c}\left(1 + \frac{1}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)\right)$$

Expressing k_r in term of k_u using resonance condition

$$k_r = k_u \frac{2\gamma_R^2}{1 + \frac{K^2}{2}}$$

$$\frac{d\psi}{dz} = (k_r + k_u) - k_r\left(1 + \frac{1}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)\right)$$

$$\frac{d\psi}{dz} = k_u - k_u \frac{\gamma_R^2}{\gamma^2} = k_u\left(\frac{\gamma^2 - \gamma_R^2}{\gamma^2}\right) \approx 2k_u\left(\frac{\gamma - \gamma_R}{\gamma_R}\right)$$

For small energy deviations

$$\gamma + \gamma_R \approx 2\gamma_R$$

$$\frac{d\psi}{dz} = 2k_u\eta \qquad \text{where} \quad \eta = \frac{\gamma - \gamma_R}{\gamma_R}$$

Evolution of Energy Deviation

Transverse velocity

$$\upsilon_x = \frac{cK}{\gamma}\cos(k_u z)$$

Transverse current

$$j_x = -e\upsilon_x$$

Transverse electric field of radiation

$$E_r(z,t) = E_0\cos(k_r z - \omega_r t)$$

Rate of energy transfer

$$\dot{W} = -e\upsilon_x E_r = -\frac{ecK}{\gamma}\cos(k_u z)\cos(k_r z - \omega_r t + \phi)$$

$$\dot{\gamma}(m_e c^2) = -\frac{ecKE_0}{2\gamma}(\cos\psi + \cos\chi)$$

Rewrite in term of relative energy deviation, η

$$\frac{d\eta}{dt} = -\frac{e\hat{K}E_0}{2\gamma_R^2 m_e c}\cos\psi$$

Rate of change of η with respect to *z*

$$\frac{d\eta}{dz} = -\frac{e\hat{K}E_0}{2\gamma_R^2 m_e c^2}\cos\psi$$

Coupled 1st-order DE Equations

Rate of change of the *j*th electron's phase with respect to *t* (left) and *z* (right)

$$\frac{d\psi_j}{dt} = 2ck_u\eta_j \qquad \frac{d\psi_j}{dz} = 2k_u\eta_j$$

Rate of change of the *j*th electron's energy deviation w.r.t. to *t* (left) and *z* (right)

$$\frac{d\eta_j}{dt} = \frac{-e\hat{K}E_0}{2\gamma_R^2 m_e c}\cos\psi_j \qquad \frac{d\eta_j}{dz} = \frac{-e\hat{K}E_0}{2\gamma_R^2 m_e c^2}\cos\psi_j$$

These two coupled equations can be used to model the phase-space motion of *N* electrons at a constant radiation field amplitude. This applies to the case where a seed laser is used to induce the energy and density modulations, such as the seeded amplifier or the final few passes of an oscillator. In a high-gain FEL, the radiation intensity (and E_0) changes with *t* and *z*.

FEL Pendulum Equation

Shift the phase variable by $\pi/2$ and call it ϕ

$$\frac{d\phi}{dt} = 2k_u c\,\eta$$

$$\frac{d\eta}{dt} = -\frac{eE_0\hat{K}}{2m_e c\gamma_R^2}\sin\phi$$

FEL pendulum equation

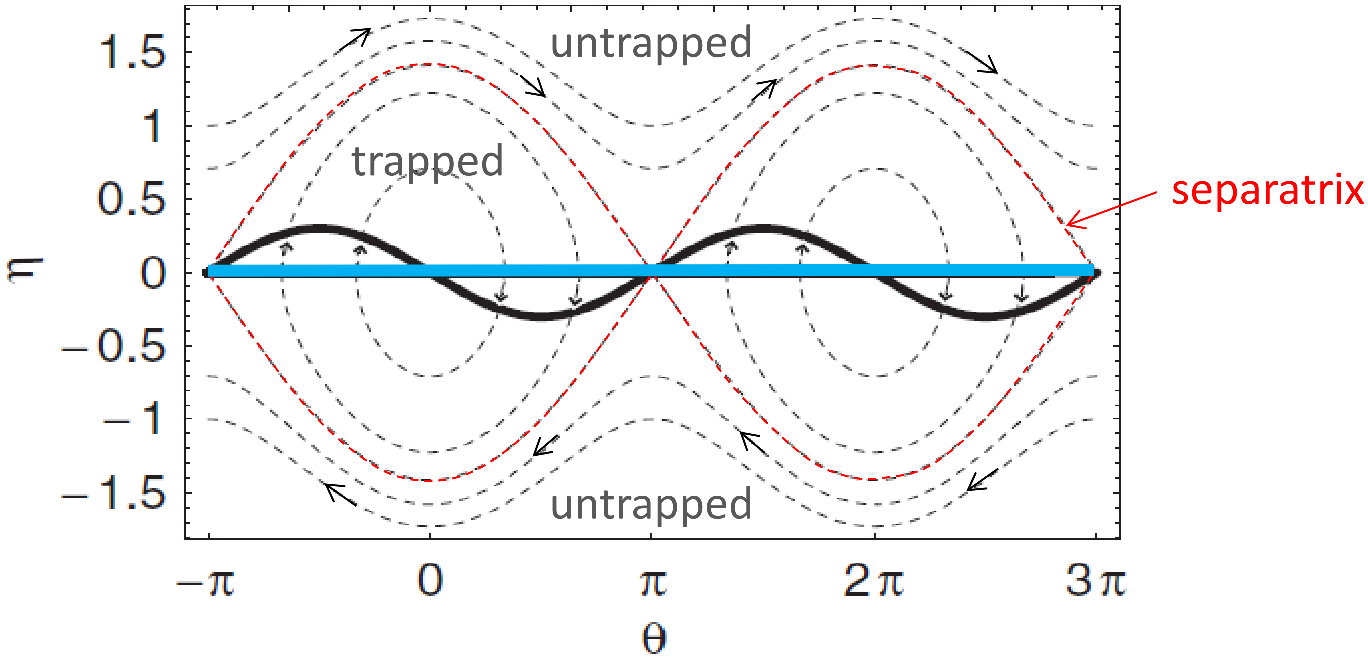
$$\frac{d^2\phi}{dt^2} + \Omega^2 \sin\phi = 0$$

where Ω is the synchrotron oscillation frequency.

$$\Omega^2 = \frac{eE_0\hat{K}k_u}{m_e\gamma_R^2}$$

At one-quarter of the synchrotron period, the electrons have sinusoidal energy modulations. At one-half of the period, FEL bunching is maximum. Electrons become over-bunched after one-half of the synchrotron period.

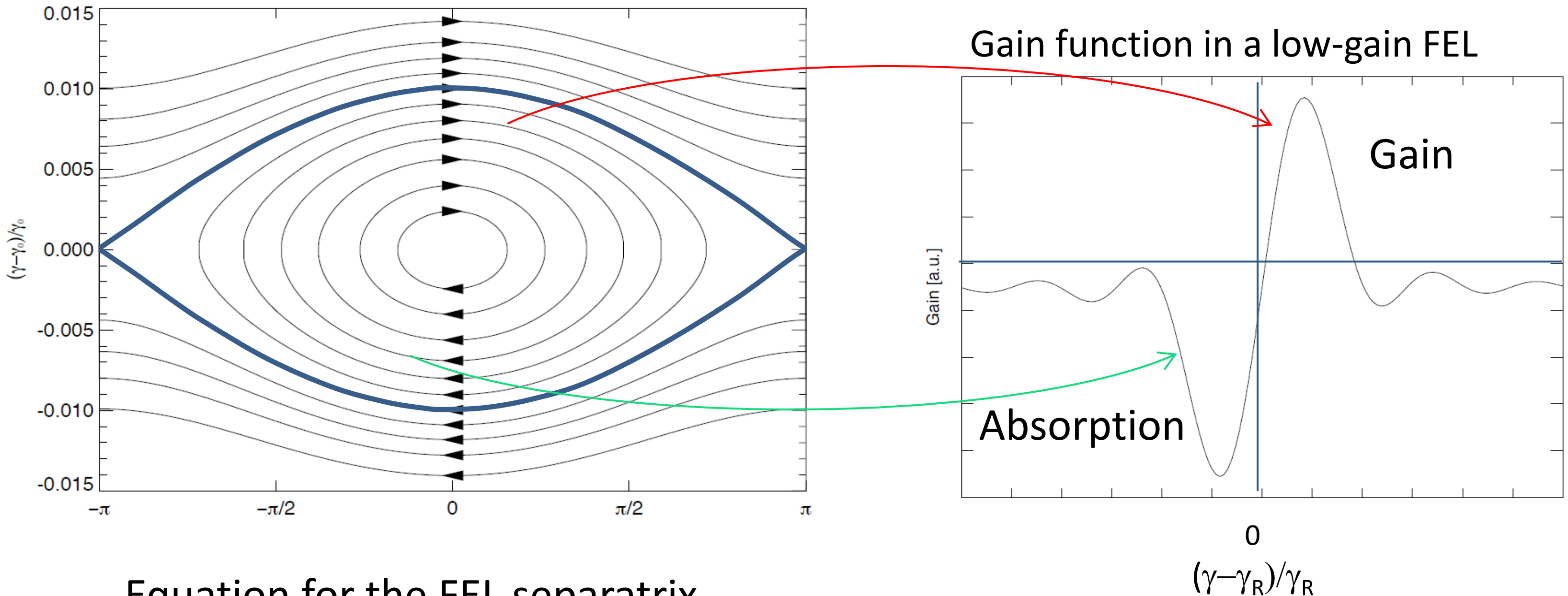
Motions in Phase Space



The initial distribution (blue line) depicts mono-energetic electrons at time = 0. At ¼ of the synchrotron period, the electron distribution develops energy modulations (black line). Electrons trapped inside the separatrix (red) execute close-orbit motions. Electrons outside the separatrix flow over and under the separatrix. These electrons also provide gain (linear regime) until the separatrix grows in height and capture the untrapped electrons, causing them to execute trapped electron phase-space motions (nonlinear regime).

Separatrix

The separatrix or bucket (blue) separates the trapped (closed) orbits from the untrapped (open) orbits. Electrons with energy above the resonant energy γ_R move lower and provide FEL gain. Electrons below γ_R absorb FEL radiation power.



Equation for the FEL separatrix

$$\eta_S(\psi) = \pm \frac{\Omega}{k_u c}\cos\left(\frac{\phi}{2}\right)$$

FEL Small-signal Gain

The small-signal gain is calculated from the energy gained by the radiation (also equal to the energy lost by the electrons) in each pass divided by the radiation energy. The small-signal gain as function of the initial energy detuning Δ is derived using second-order perturbation theory.

$$\phi(t) = \phi_0(t) + \varepsilon\phi_1(t) + \varepsilon^2\phi_2(t) + \ldots$$

$$\eta(t) = \eta_0(t) + \varepsilon\eta_1(t) + \varepsilon^2\eta_2(t) + \ldots$$

Start with zeroth-order solutions

$$\eta_0(t) = \Delta = const.$$

$$\phi_0(t) = 2ck_u\Delta + \vartheta_0$$

To the first-order ε^1 the equations are

$$\dot{\phi}_1(t) = 2ck_u\eta_1$$

$$\dot{\eta}_1(t) = -\Omega^2 \sin\phi_0 = -\Omega^2 \sin(2ck_u\Delta + \vartheta_0)$$

2^{nd}-order Perturbation Theory

Solutions to the 1^{st} order equations

$$\eta_1(t) = \frac{\Omega^2}{2ck_u\Delta}\left[\cos(2ck_u\Delta\cdot t) - \cos\vartheta_0\right]$$

$$\phi_1(t) = \frac{\Omega^2}{\Delta}\left[\frac{\sin(2ck_u\Delta\cdot t) - \sin\vartheta_0}{2ck_u\Delta} - t\cos\vartheta_0\right]$$

The phase-averaged energy deviation is zero for the 1^{st} order.

$$\eta(t) = \Delta + 0 + \varepsilon^2\eta_2(t) + \ldots$$

To the first-order ε^2 the equations are

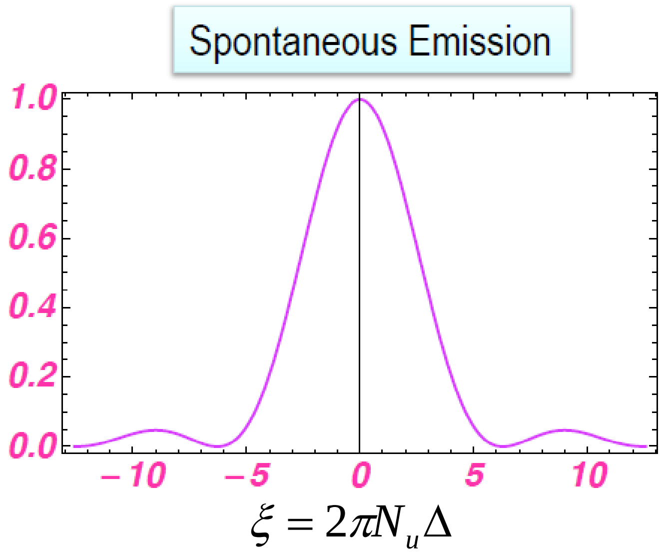
$$\dot{\phi}_2(t) = 2ck_u\eta_2$$

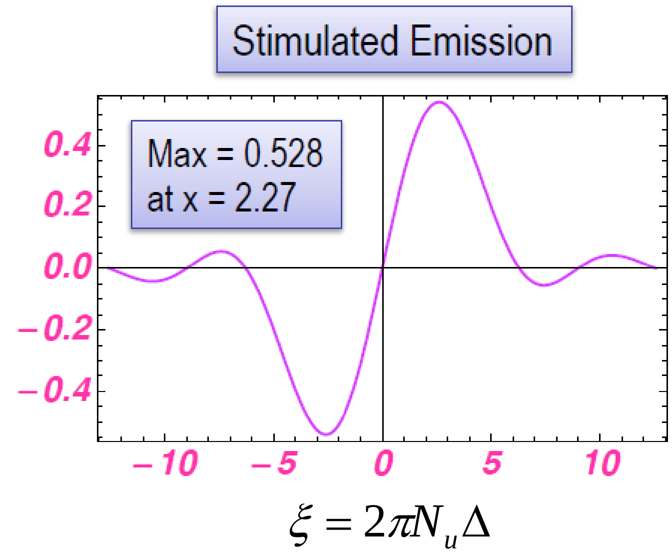
$$\dot{\eta}_1(t) = -\Omega^2\phi_1(t)\cos(2ck_u\Delta + \vartheta_0)$$

For the 2^{nd} order, the phase-averaged energy deviation is

$$\langle\eta_2(t)\rangle_{\vartheta_0} = \frac{\Omega^4 ck_u T^3}{(2ck_u\Delta\cdot T)^3}\left[\cos(2ck_u\Delta\cdot T) + \frac{(2ck_u\Delta\cdot T)}{2}\sin(2ck_u\Delta\cdot T) - 1\right]$$

Madey Theorem





Small-signal gain (stimulated emission) function of a low-gain FEL

$$G(x) = -\frac{\Omega^4 c k_u T^3}{4} \frac{d}{d(\xi)} \left(\frac{\sin^2(\xi)}{\xi^2} \right)$$

<u>Madey theorem</u>: the line shape of the small-signal gain of a low-gain FEL is the negative derivative of the spontaneous emission curve, a sinc-square function.

Wave Equation in 1D Theory

One-dimension approximation of the wave equation driven by a transverse complex current of electrons oscillating along the x direction in the undulator

$$\left(\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\tilde{E}(z,t)=\mu_0\frac{\partial\tilde{j}_x}{\partial t}$$

Assume a trial solution to the wave equation in the complex form

$$\tilde{E}_x(z,t)=\tilde{E}_x(z)\exp[i(k_r z-\omega_r t)]$$

We also assume the seed radiation is polarized in the x direction, the same as the electron oscillatory motion in a planar undulator. The x component of the radiation field interacts with the complex transverse current due to electrons' velocity in x. This interaction causes the radiation amplitude to vary with z, especially in a high-gain FEL where the amplitude grows exponentially with z.

We must now find the rate of change of radiation field amplitude with respect to z, unlike the situation in slide 7 where the radiation field amplitude is constant.

Slowly Varying Amplitude (SVA)

Plugging the trial solution into the wave equation

$$\left[-k_r^2 + 2ik_r\tilde{E}'(z) + \tilde{E}''(z) + \frac{\omega_r^2}{c^2}\right]\exp\left[i\left(k_r z - \omega_r t\right)\right] = \mu_0 \frac{\partial \tilde{j}_x}{\partial t}$$

The complex amplitude varies slowly with z so we can drop the 2nd derivative

$$\frac{d\tilde{E}_x}{dz} = -\frac{i\mu_0}{2k_r}\cdot\frac{\partial \tilde{j}_x}{\partial t}\exp\left[-i\left(k_r z - \omega_r t\right)\right]$$

Transverse current density is related to the longitudinal current density by

$$\tilde{j}_x = \tilde{j}_z \frac{K}{\gamma}\cos\left(k_u z\right)$$

Rate of change of field amplitude with respect to z

$$\frac{d\tilde{E}_x}{dz} = -\frac{i\mu_0\hat{K}}{2k_r\gamma}\cdot\frac{\partial \tilde{j}_z}{\partial t}\exp\left[-i\left(k_r z - \omega_r t\right)\right]\cos\left(k_u z\right)$$

Field Amplitude & j_1 Current

The longitudinal current density has two components, the static (DC) and time-varying (AC) components.

$$\tilde{j}_z = j_0 + \tilde{j}_1(z)\exp(i\theta)$$

$$\tilde{j}_z = j_0 + \tilde{j}_1(z)\exp\left[i(k_r + k_u)z - \omega_r t\right]$$

Taking the partial derivative with respect to time

$$\frac{\partial \tilde{j}_z}{\partial t} = -i\omega_r \tilde{j}_1(z)\exp\left[i(k_r + k_u)z - \omega_r t\right]$$

Rate of change of the field amplitude with z is proportional to the j_1 current, i.e., the first harmonic Fourier component of the AC current.

$$\frac{d\tilde{E}_x}{dz} = -\frac{\mu_0 c\hat{K}}{4\gamma_R} \cdot \tilde{j}_1$$

Longitudinal Distribution

Electron longitudinal distribution $$S(\psi)=\sum_{n=1}^{N}\delta(\psi-\psi_n)$$

Complex Fourier coefficients

$$c_k=\frac{1}{\pi}\int_0^{2\pi}S(\psi)\exp(-ik\psi)d\psi$$

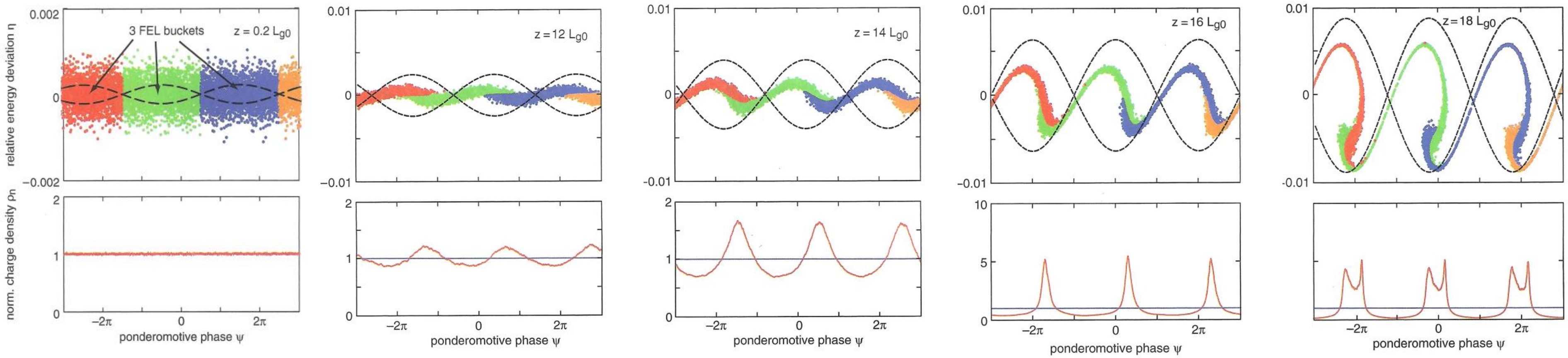
Expanding the distribution in complex Fourier coefficients

$$S(\psi)=\frac{c_0}{2}+\mathrm{Re}\left(\sum_{k=1}^{\infty}c_k\exp(ik\psi)\right)$$

Fourier coefficient of the 1st harmonic

$$c_1=\frac{1}{\pi}\sum_{n=1}^{N}\exp(-i\psi_n)$$

Evolution of Current Density



Initial DC current density

$$j_0 = -\frac{ec}{Ak_r}\frac{c_0}{2}$$

First harmonic current density

$$\tilde{j}_1 = -\frac{ec}{Ak_r}c_1$$

First harmonic current is proportional to the correlation of the phases of *N* electrons.

$$\tilde{j}_1 = j_0 \frac{2}{N}\sum_{n=1}^{N}\exp(-i\psi_n)$$

Universal Coupled First-order Differential Equations

Rate of change of the ponderomotive phase of the n^{th} electron

$$\frac{d\psi_n}{dz} = 2k_u\eta_n$$

Rate of change of the relative energy deviation of the n^{th} electron

$$\frac{d\eta_n}{dz} = -\frac{e}{m_0c^2\gamma_R}\mathrm{Re}\left\{\left[\frac{\hat{K}\tilde{E}_x}{2\gamma_R} - \frac{i\mu_0c^2}{\omega_r}\cdot\tilde{j}_1\right]\exp(i\psi_n)\right\}$$

Rate of change of radiation field amplitude
$$\frac{d\tilde{E}_x}{dz} = -\frac{\mu_0c\hat{K}}{4\gamma_R}\tilde{j}_1$$

Rate of change of the j_1 current amplitude
$$\tilde{j}_1 = j_0\frac{2}{N}\sum_{n=1}^{N}\exp(-i\psi_n)$$

Third-order Equation

Follow the discussion on pp. 52 – 56 of the 2nd edition of the textbook.

Third-order equation

$$\frac{\tilde{E}_x'''}{\Gamma^3} + 2i\frac{\eta}{\rho_{FEL}}\frac{\tilde{E}_x''}{\Gamma^2} + \left[\frac{k_p^2}{\Gamma^2} - \left(\frac{\eta}{\rho_{FEL}}\right)^2\right]\frac{\tilde{E}_x'}{\Gamma} - i\tilde{E}_x = 0$$

Growth rate

$$\Gamma = \frac{4\pi}{\lambda_u}\rho_{FEL}$$

FEL rho parameter

$$\rho_{FEL} = \frac{1}{2\gamma}\left(\frac{\hat{K}}{k_u\sigma_b}\right)^{\frac{2}{3}}\left(\frac{I_{pk}}{I_A}\right)^{\frac{1}{3}}$$

Plasma wave-number

$$k_p = \sqrt{\frac{2\lambda_r}{\lambda_u}\frac{\omega^*}{c}}$$

Plasma frequency in beam frame

$$\omega^* = \sqrt{\frac{n_e e^2}{\gamma_R \varepsilon_0 m_e}}$$

Third-order Equation without Plasma Oscillation

Plasma oscillation is important in long-wavelength FELs (Raman regime). In many of today's high-gain FEL, the plasma wavenumber is much smaller than the growth rate, and thus can be ignored.

In FEL where $\frac{k_p}{\Gamma} << 1$ we can rewrite the 3rd order equation as

$$\frac{\tilde{E}_x'''}{\Gamma^3} + 2i\frac{\eta}{\rho_{FEL}}\frac{\tilde{E}_x''}{\Gamma^2} - \left(\frac{\eta}{\rho_{FEL}}\right)^2\frac{\tilde{E}_x'}{\Gamma} - i\tilde{E}_x = 0$$

Special case: mono-energetic electron beam at resonant energy, i.e., $\eta = 0$

$$\tilde{E}_x''' - i\Gamma^3\tilde{E}_x = 0$$

We have arrived at a simple third-order differential equation.

FEL Cubic Equation

Using the trial solution of the form $\tilde{E}_x(z) = A \cdot \exp[\alpha z]$

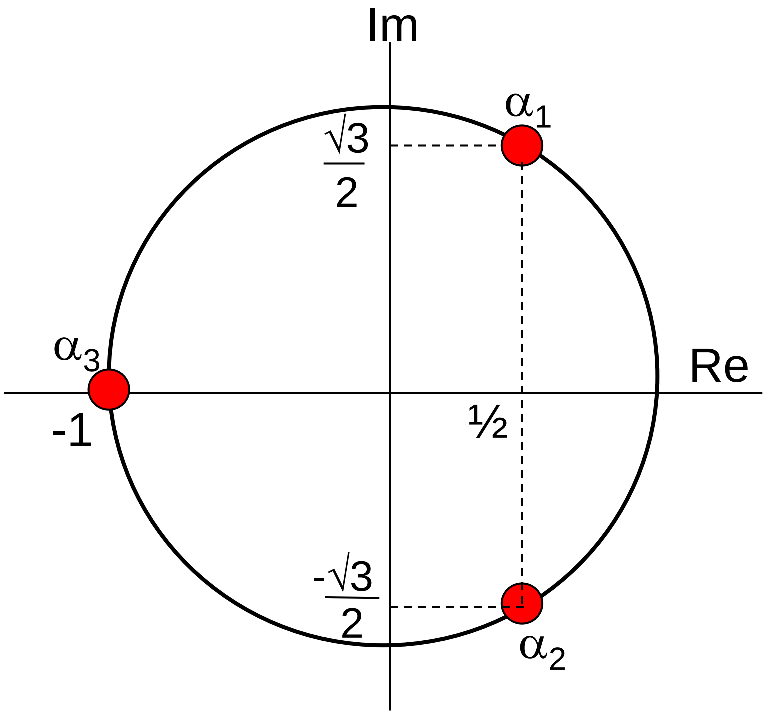
one obtains the cubic equation $\alpha^3 = i\Gamma^3$

Three roots of the cubic equation:

$$\alpha_1 = \frac{\left(i + \sqrt{3}\right)}{2}\Gamma$$ Growing mode

$$\alpha_2 = \frac{\left(i - \sqrt{3}\right)}{2}\Gamma$$ Decaying mode

$$\alpha_3 = -i\Gamma$$ Oscillatory mode



Lethargy Region

In the first 2-3 gain lengths, the growing, decaying and oscillating modes compete with one another. The sum of three electric fields grows slowly with z.

$$\tilde{E}_x(z) = \frac{E_{in}}{3} \cdot \left[\exp\left(\frac{\left(i + \sqrt{3}\right)\Gamma}{2} z \right) + \exp\left(\frac{\left(i - \sqrt{3}\right)\Gamma}{2} z \right) + \exp(-i\Gamma z) \right]$$

$$\left|\tilde{E}_x(z)\right|^2 \approx \frac{\left|E_{in}\right|^2}{9} \cdot \left[4\cosh^2\left(\frac{\sqrt{3}}{2} \Gamma z \right) + 4\cosh\left(\frac{\sqrt{3}}{2} \Gamma z \right) \cos\left(\frac{3}{2} \Gamma z \right) + 1 \right]$$

At z > 3 gain lengths, the exponentially growing mode dominates and the radiation field and intensity as functions of z can be written as

$$\left|\tilde{E}_x(z)\right| \approx \frac{E_{in}}{3} \cdot \exp\left[\frac{\sqrt{3}\Gamma}{2} z \right]$$

$$\left|\tilde{E}_x(z)\right|^2 \approx \frac{\left|E_{in}\right|^2}{9} \cdot \exp\left[\sqrt{3}\Gamma z \right]$$

Long Undulator, High-gain FEL

FEL power grows exponentially with z

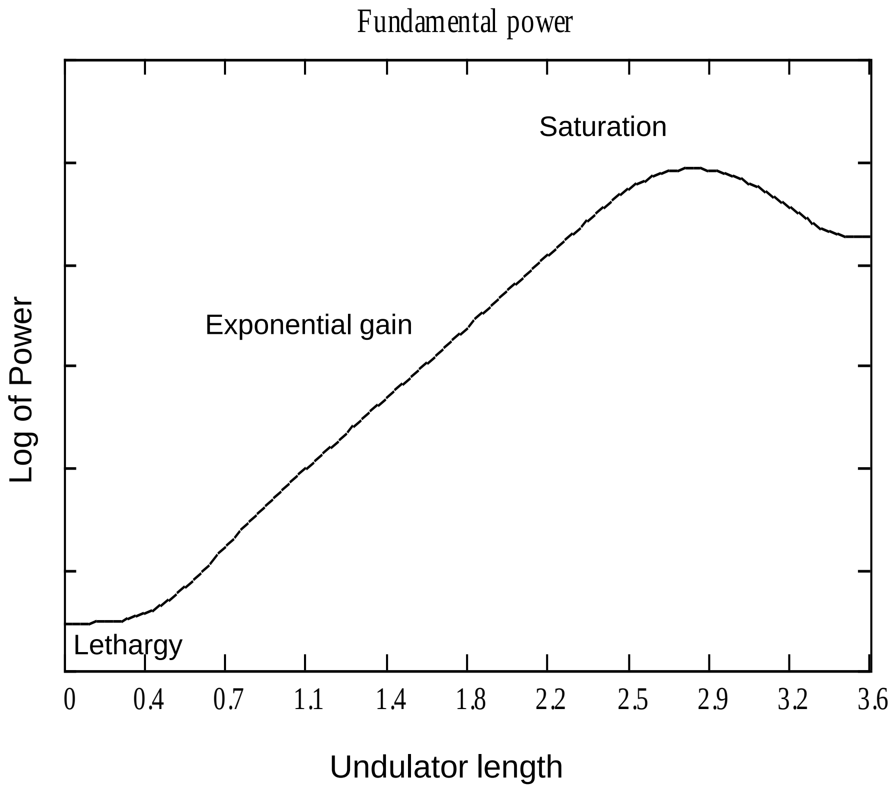
$$P(z) \approx \frac{P_0}{9} \exp\left(\frac{z}{L_{G0}}\right)$$

Power gain length if no 3D effects

$$L_{G0} = \frac{\lambda_u}{4\pi\sqrt{3}\rho}$$

FEL power at saturation (no 3D effects)

$$P_{sat} = \frac{\rho I_b E_b}{e}$$



At saturation, the electrons have undergone one-half of the synchrotron period. The electrons begin to absorb the FEL radiation, but the electrons' phase space becomes chaotic so the FEL power is not significantly reduced but oscillates about an average non-zero value. The synchrotron oscillation period is a measure of the FEL bucket height.

Optical Diffraction

The rms radius of a focused electron beam is determined by its un-normalized (geometric) emittance and the average β function of the focusing optics (β is the electron beam's property analogous to the photon beam's Rayleigh length. One needs to match the average focusing β to the electron beam's β).

$$\beta_{ave} = \frac{\sigma_b^2}{\varepsilon_u}$$

Optical diffraction is measured by the radiation Rayleigh length, which is given by the square of the electron beam's rms radius in the undulators divided by the photon beam's emittance, $\lambda/4\pi$.

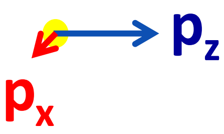
$$z_R = \frac{4\pi\sigma_b^2}{\lambda}$$

For diffraction 3D effect to be small, the gain length must be shorter than the Rayleigh length

$$L_G \leq z_R$$

Electron Beam Emittance

x and z momenta at low energy

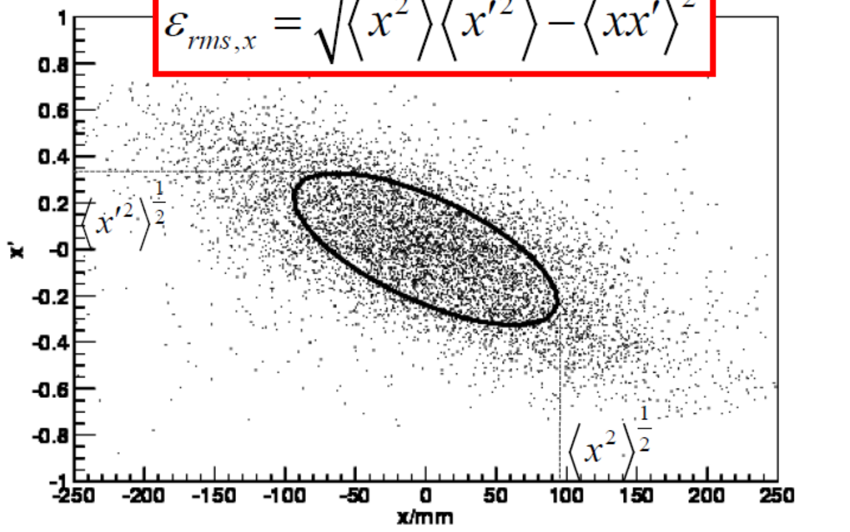


$$x' = \frac{p_x}{p_z}$$

x and z momenta at high energy

Acceleration reduces x’ by boosting p_z

$$\varepsilon_{rms,x} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2}$$

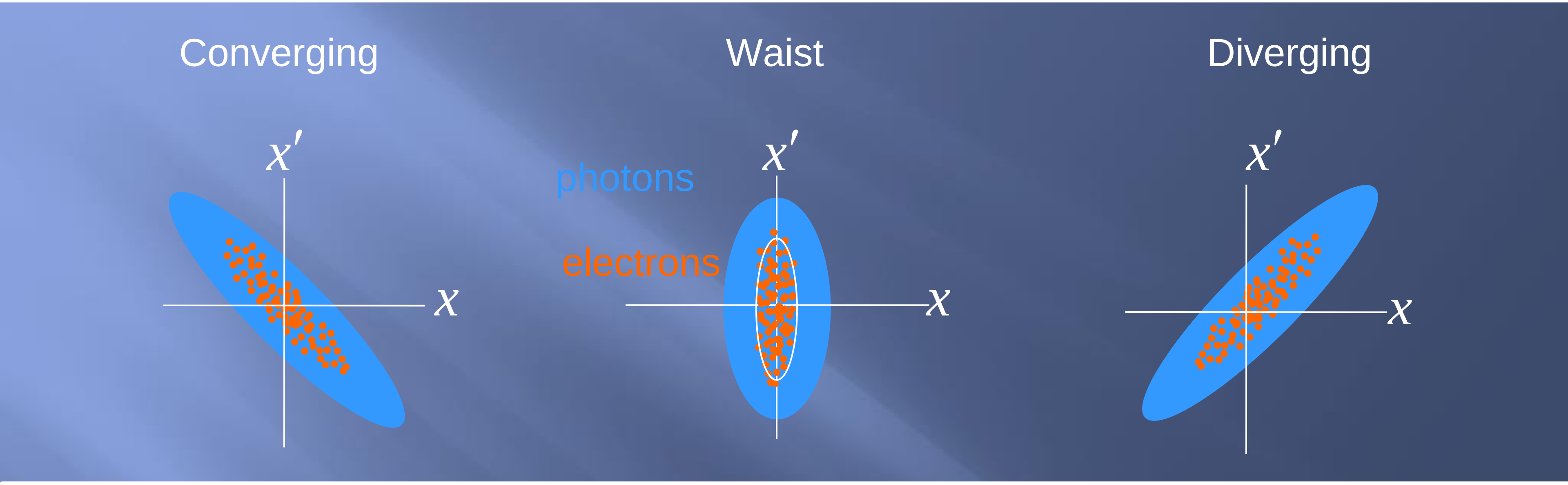


Normalized emittance is the phase space area of the beam in its rest frame.
Un-normalized (geometric) emittance is the phase space area in Lab frame,

$$\varepsilon_u = \frac{\varepsilon_n}{\beta\gamma}$$

where β ~ 1 for highly relativistic beams.

Required Transverse Emittance



Photon beam emittance $\frac{\lambda}{4\pi}$

Un-normalized emittance $\varepsilon_u \le \frac{\lambda}{4\pi}$

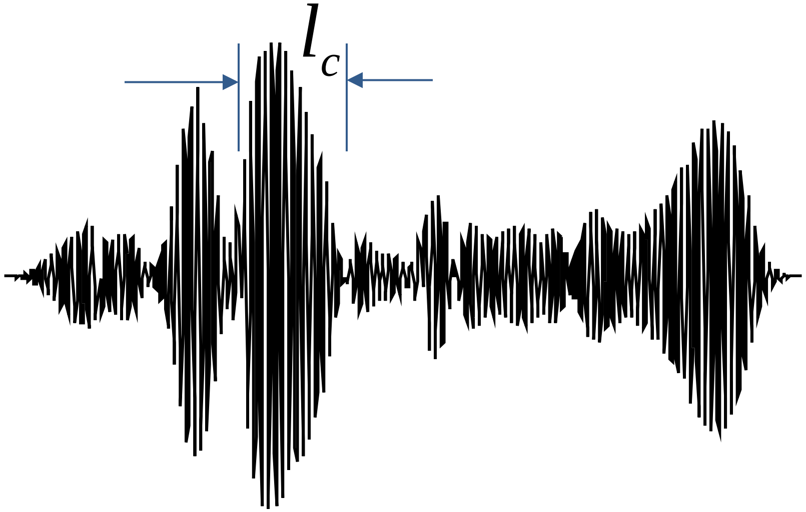
For emittance 3D effect to be small, the electron beam's geometric emittance must be smaller than the photon beam emittance.

Required Energy Spread

Electrons must maintain the same axial velocity during the coherence length

$$l_c \approx N_c \lambda$$

$$N_c \approx \frac{1}{4\pi\rho}$$



where N_c = # of wavelengths in a coherence length

= √3 times the # of periods in one gain length

For 3D effect due to energy spread to be small, the relative rms energy spread must be less than ρ

$$\frac{\sigma_\gamma}{\gamma} \leq \frac{1}{4\pi N_c} \longrightarrow \frac{\sigma_\gamma}{\gamma} \leq \rho$$

Ming-Xie Parameterization

Diffraction

$$L_{1D} \leq z_R \longrightarrow X_d = \frac{L_{1D}}{z_R}$$

Emittance

$$\varepsilon_u \leq \frac{\lambda}{4\pi} \longrightarrow X_\varepsilon = \frac{L_{1D}}{\beta_{ave}} \frac{4\pi\varepsilon_u}{\lambda_r}$$

Energy spread

$$\frac{\sigma_\gamma}{\gamma} \leq \frac{1}{4\pi N_c} \longrightarrow X_\gamma = \frac{4\pi L_{1D}}{\lambda_u} \frac{\sigma_\gamma}{\gamma}$$

Ming-Xie parameters

Ming-Xie parameters should be less than 1 to minimize 3D effects.

3D Gain Length

3D gain length

$$L_G = L_{1D}\, F\left(\frac{L_{1D}}{Z_R}, \frac{L_{1D}}{\beta_{av}} \frac{4\pi\varepsilon}{\lambda_r}, \frac{4\pi L_{1D}}{\lambda_u} \frac{\sigma_\gamma}{\gamma}\right)$$

Increase in gain length due to diffraction, emittance and energy spread

$$F\left(X_d, X_\varepsilon, X_\gamma\right) = 1 + a_1 X_d^{a_2} + a_3 X_\varepsilon^{a_4} + a_5 X_\gamma^{a_6}$$

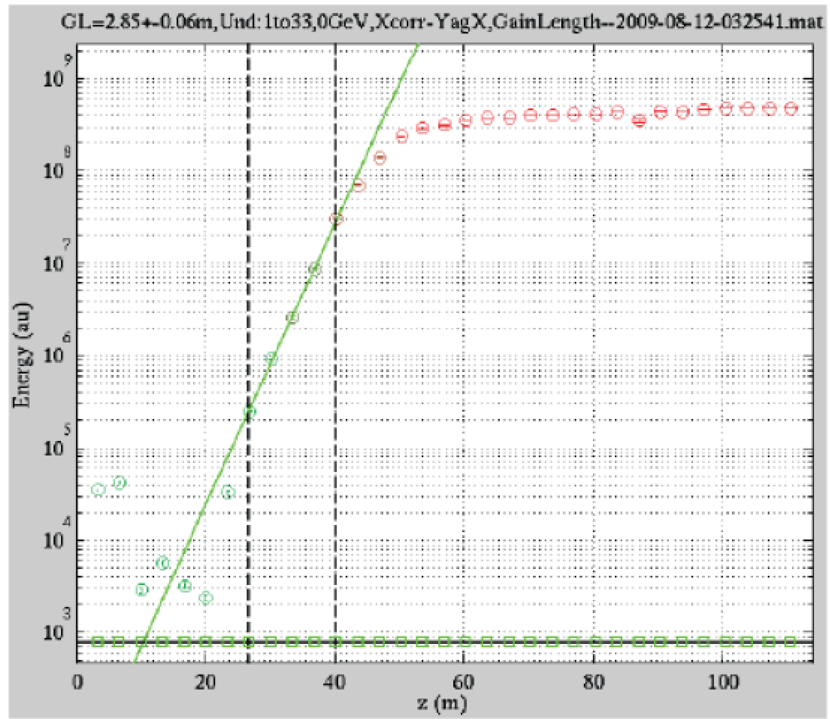
$$+ a_7 X_\varepsilon^{a_8} X_\gamma^{a_9} + a_{10} X_d^{a_{11}} X_\gamma^{a_{12}} + a_{13} X_d^{a_{14}} X_\varepsilon^{a_{15}} + a_{16} X_d^{a_{17}} X_\varepsilon^{a_{18}} X_\gamma^{a_{19}}$$

Fitted coefficients

a_1=0.45	a_2=0.57	a_3=0.55	a_4=1.6
a_5=3	a_6=2	a_7=0.35	a_8=2.9
a_9=2.4	a_{10}=51	a_{11}=0.95	a_{12}=3
a_{13}=5.4	a_{14}=0.7	a_{15}=1.9	a_{16}=1140
a_{17}=2.2	a_{18}=2.9	a_{19}=3.2	

Comparison with Experiments

Energy	13.6 GeV
γ	26,615
ε_n	0.4 μ
σ_γ/γ	0.0001
I_p	3.4 kA
λ_u	3 cm
K	3.5
ρ	.00057
β_{av}	20 m
L_{1D}	2.4 m
η_D	.097
η_ε	.151
η_γ	.101
L_{3D}	2.9 m



Measured gain length = 2.85 m
Good agreement with MX parameterization

Summary

- 1D theory involves *N* first-order equations for *N* electrons coupled with two first-order equations for radiation field amplitude and 1st harmonic current.

Electron phase
$$\frac{d\psi_n}{dt} = 2k_u c\eta_n$$

Electron energy
$$\frac{d\eta_n}{dz} = -\frac{e}{m_0 c^2 \gamma_R} \mathrm{Re}\left\{\left[\frac{\hat{K}\tilde{E}_x}{2\gamma_R} - \frac{i\mu_0 c^2}{\omega_r} \cdot \tilde{j}_1\right]\exp(i\psi_n)\right\}$$

Radiation field amplitude
$$\frac{d\tilde{E}_x}{dz} = -\frac{\mu_0 c\hat{K}}{4\gamma_R}\tilde{j}_1$$

Harmonic current
$$\tilde{j}_1 = j_0 \frac{2}{N}\sum_{n=1}^{N}\exp(-i\psi_n)$$

- The third-order FEL equation leads to the cubic equation that has three roots corresponding to the growing, decaying and oscillating modes.

- 3D effects (diffraction, emittance, and energy spread) can be included as modifications to the 1D theory via Ming-Xie parameterization.