LA-UR-14-23754

High Intensity RF Linear Accelerators

2.2 Focusing of Intense Beams

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Albuquerque, New Mexico, 23 – 27 June 2014

Content

Hamiltonian of particle motion in quadrupole focusing channel

Hamiltonian of charged particle

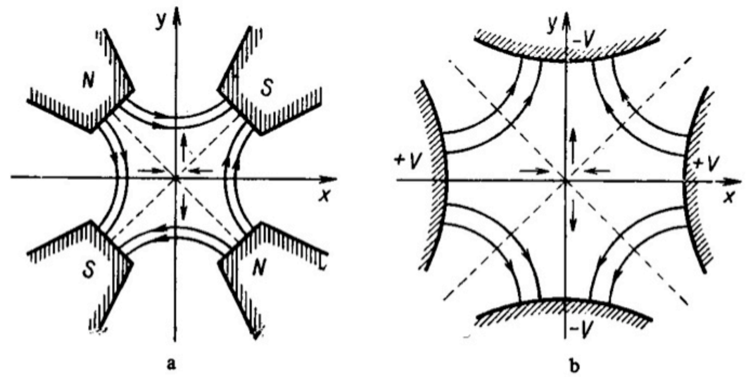
$$H = c\sqrt{m^2c^2 + (P_x - qA_x)^2 + (P_y - qA_y)^2 + (P_z - qA_z)^2} + q\,U$$

Vector potential $\vec{A} = \vec{A}_{magn} + \vec{A}_b$

is a combination of that of magnetic lenses, $\vec{A}_{magn}$, and of that of the beam, $\vec{A}_b$,

Scalar potential $U = U_{el} + U_b$

is a combination of the scalar potential of the electrostatic focusing field, U_{el}, and of the space charge potential of the beam, U_b.



(a) Magnetic quadrupole and (b) electric quadrupole.

Vector - potential of an ideal magnetic quadrupole lens with gradient G_{magn} inside the lens is given by

$$A_{z\ magn} = \frac{G_{magn}}{2}(x^2 - y^2)$$

Electrostatic quadrupole with gradient G_{el}, creates the field with electrostatic potential

$$U_{el} = -\frac{G_{el}}{2}(x^2 - y^2)$$

Transversal components of mechanical momentum are equal to that of canonical momentum, $p_x = P_x$, $p_y = P_y$, and Hamiltonian can be written as:

$$K = c\sqrt{m^2c^2 + p_x^2 + p_y^2 + (P_z - q\,A_z)^2} + qU$$

In the moving system of coordinates, particles are static, therefore, vector potential of the beam equals to zero, $\vec{A}_b' = 0$. According to Lorentz transformations, components of vector potential of the beam are converted into laboratory system of coordinates as follow

$$A_{xb} = 0\ , \qquad A_{yb} = 0\ , \qquad A_{zb} = \beta \frac{U_b}{c}$$

Total vector-potential of the structure is therefore

$$A_z = A_{z\ magn} + \frac{\beta}{c} U_b$$

Kinetic energy of the beam is typically much larger than the potential energy of focusing elements and than the potential energy of the beam. Therefore, $P_z >> qA_z$, and we can substitute canonical momentum by the mechanical momentum:

$$(P_z - qA_z)^2 \approx P_z^2 - 2\, P_z\, q\, A_z \approx p_z^2 - 2\, p_z\, q\, A_z$$

It corresponds to the case when longitudinal particle motion is not affected by the transverse motion, which is typical for beam transport.

Hamiltonian can be rewritten as

$$K = mc^2 \sqrt{(1+\frac{p_z^2}{m^2c^2}) + \frac{p_x^2+p_y^2}{m^2c^2} - \frac{2\,q\,p_z A_z}{m^2c^2}} + qU_{el} + qU_b$$

The term in brackets is close to square of reduced particle energy: $1+\frac{p_z^2}{m^2c^2} \approx \gamma^2$

Taking that term out of square root gives for Hamiltonian:

$$K = mc^2\gamma \sqrt{1 + \frac{p_x^2+p_y^2}{(\gamma m\,c)^2} - \frac{2\,q\,p_z A_z}{(\gamma m\,c)^2}} + qU_{el} + qU_b$$

After expansion of small terms $\sqrt{1+x} \approx 1 + x/2$, the Hamiltonian becomes:

$$K = mc^2\gamma + \frac{p_x^2+p_y^2}{2mc\gamma} - \frac{2\,q\,p_z\,(A_{z\ magn} + \frac{\beta}{c}U_b)}{2mc\gamma} + qU_{el} + qU_b$$

Removing the constant $mc^2\gamma$ results in the general form of Hamiltonian in a focusing channel:

$$H = \frac{p_x^2+p_y^2}{2\,m\,\gamma} + q(U_{el} - \beta c A_{z\ magn}) + q\,\frac{U_b}{\gamma^2}$$

Both U_{el} and $A_{z\ magn}$ can be a combination of that of multipole lenses of an arbitrary order.

Kapckinsky-Vladimirsky (KV) beam envelope equations

Consider now dynamics of the beam in focusing quadrupole channel including space charge forces of the beam. All particles move with the same longitudinal velocity βc, and the longitudinal space charge forces are equal to zero. Hamiltonian of particle motion in qudrupole channel with space charge is given by

$$H = \frac{p_x^2 + p_y^2}{2\, m\, \gamma} + q\frac{G(z)}{2}(x^2 - y^2) + q\,\frac{U_b}{\gamma^2}\,. \qquad (2.96)$$

Assume that transverse space charge forces are linear functions of coordinates. Correctness of this assumption will be checked later. Linear equation of motion are

$$\frac{d^2x}{dz^2} + k_x^{'}(z)x = 0\,, \qquad (2.97)$$

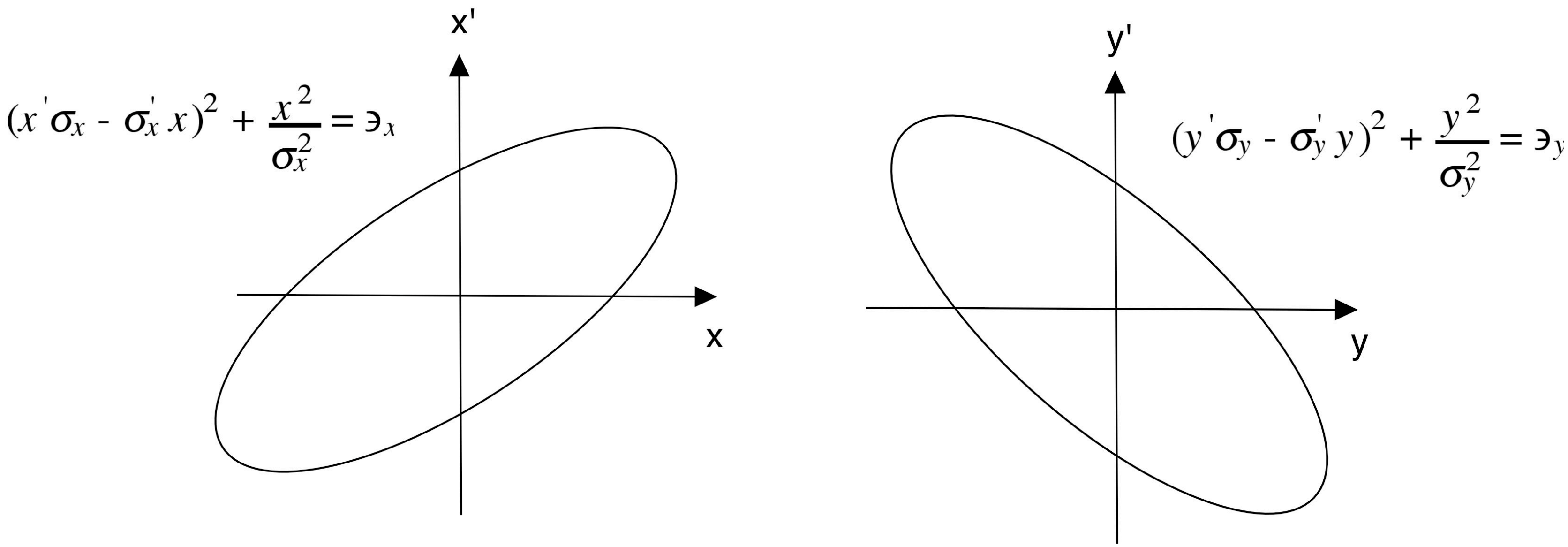
$$\frac{d^2y}{d\,z^2} + k_y^{'}(z)\,y = 0\,, \qquad (2.98)$$

where $k_x^{'}(z)$, $k_y^{'}(z)$ are modified focusing strengths including space charge. Equations of motion (2.97), (2.98) are linear, therefore, invariant of Courant-Snyder, is valid in both planes *(x, x')*, *(y, y')* for space charge regime as well.

Self-consistent solution can be obtained when distribution function is expressed as a function of integrals of motion. Due to equations of motion in linear field are uncoupled, Courant-Snyder invariants are conserved at every phase plane:

$$(x'\sigma_x - \sigma_x' x)^2 + \frac{x^2}{\sigma_x^2} = \ni_x \,, \qquad (2.99)$$

$$(y'\sigma_y - \sigma_y' y)^2 + \frac{y^2}{\sigma_y^2} = \ni_y \,. \qquad (2.100)$$



Courant-Snyder invariants.

Values of $Э_x$, $Э_y$ are areas of ellipses at phase planes (beam emittances), which are the constants of motion during beam transport. Let us express beam distribution function as a function of values $Э_x$, $Э_y$:

$$f = f_o\, \delta(Э_x + Э_y - F_o) \tag{2.101}$$

where f_o, F_o, ν are constants defined below and $\delta(\xi)$ is the Dirac delta -function:

$$\delta(\xi) = \begin{cases} \infty, & \xi = 0 \\ 0, & \xi \neq 0 \end{cases}, \tag{2.102}$$

$$\int_a^b f(\xi)\delta(\xi - X)\, d\xi = \begin{cases} 0, & X<a, \quad X>b, \\ 1/2 f(X), & X = a \;\; or \;\; X = b, \\ f(X), & a<X<b \end{cases} \tag{2.103}$$

In the selected distribution, Eq. (2.101), particles are placed at the surface of four-dimensional ellipsoid:

$$F(x, x', y, y') = (x'\sigma_x - \sigma_x' x)^2 + \frac{x^2}{\sigma_x^2} + (y'\sigma_y - \sigma_y' y)^2 + \frac{y^2}{\sigma_y^2} - F_o = 0. \tag{2.104}$$

Boundary of (x,y) Projection of Beam Distribution

Let us find *boundary of projection* of the surface $F(x,x',y,y')=0$ on the plane $(x,\ y)$. Boundary of projection of the four-dimensional surface $F(x,x',y,y')=0$ on arbitrary two-dimensional plane is obtained by equating to zero the partial derivatives of function $F(x,x',y,y')$ over the rest of variables:

$$\frac{\partial F(x,\ x',\ y,\ y')}{\partial x'}=0, \qquad \frac{\partial F(x,\ x',\ y,\ y')}{\partial y'}=0, \qquad (2.105)$$

and substitution of the solutions of equations (2.105) into equation $F(x,x',y,y')=0$. Actually, for every fixed value of x, the point at the boundary of projection corresponds to maximum possible value of y:

$$\frac{\partial y}{\partial x'}=0\ , \qquad \frac{\partial y}{\partial y'}=0\ , \qquad (2.106)$$

or, according to differentiation of implicit functions,

$$\frac{\partial y}{\partial x'}=-\frac{\dfrac{\partial F}{\partial x'}}{\dfrac{\partial F}{\partial y}}, \qquad \frac{\partial y}{\partial y'}=-\frac{\dfrac{\partial F}{\partial y'}}{\dfrac{\partial F}{\partial y}}, \qquad (2.107)$$

which coincides with Eq. (2.105).

Partial derivatives over variables x', y' in equation of four-dimensional ellipsoid are:

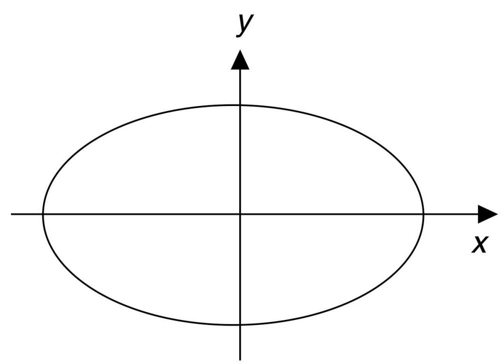
$$\frac{\partial F}{\partial x'} = 2\ (x'\sigma_x - \sigma_x' x)\sigma_x = 0\ , \qquad (2.108)$$

$$\frac{\partial F}{\partial y'} = 2\ (y'\sigma_y - \sigma_y' y)\sigma_y = 0\ . \qquad (2.109)$$

Substitution of solution of equations $\partial F/\partial x' = 0$, $\partial F/\partial y' = 0$ into equation $F(x, x', y, y') = 0$ gives the expression for the boundary of particle projection on plane (*x, y*):

$$\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} = F_o. \qquad (2.110)$$

Therefore, particles of beam distribution, Eq. (2.101), are surrounded by ellipse, Eq. (2.110),with semi-axes $R_x = \sigma_x\sqrt{F_o}$, $R_y = \sigma_y\sqrt{F_o}$ and the area of ellipse $S = \pi\,\sigma_x\,\sigma_y F_o$.



Boundary of projection of KV beam on *(x,y)*.

Space Charge Density of the Beam

Space charge density of the beam is an integral of distribution function over the rest variables x', y':

$$\rho(x,y)=f_o\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta\{(x'\sigma_x - \sigma_x' x)^2+\frac{x^2}{\sigma_x^2}+(y'\sigma_y - \sigma_y' y)^2+\frac{y^2}{\sigma_y^2} - F_o\}dx'dy'. \qquad (2.111)$$

To find particle density, Eq.(2.111), let us make substitution of the new variables, α, Ω, for old variables, x', y', according to transformation:

$$(x'\sigma_x - \sigma_x' x) = \alpha \cos\Omega , \qquad (2.112)$$

$$(y'\sigma_y - \sigma_y' y) = \alpha \sin\Omega . \qquad (2.113)$$

Inverse transformation is

$$x' = \frac{1}{\sigma_x}(\alpha \cos\Omega + x\sigma_x') , \qquad (2.114)$$

$$y' = \frac{1}{\sigma_y}(\alpha \sin\Omega + y\sigma_y') . \qquad (2.115)$$

Phase-space element is transformed according to: $dx'\,dy' = \begin{vmatrix} \frac{\partial x'}{\partial \alpha} & \frac{\partial x'}{\partial \Omega} \\ \frac{\partial y'}{\partial \alpha} & \frac{\partial y'}{\partial \Omega} \end{vmatrix} d\alpha\, d\Omega = \frac{\alpha\, d\alpha\, d\Omega}{\sigma_x\, \sigma_y}$ (2.116)

With introduced transformation, Eqs. (2.112), (2.113), the space charge density of the beam is

$$\rho(x, y) = \frac{f_o}{\sigma_x \sigma_y} \int_o^{2\pi} \int_o^{\infty} \delta(\alpha^2 + \frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - F_o)\, \alpha\, d\alpha\, d\Omega =$$

$$= \frac{\pi f_o}{\sigma_x \sigma_y} \int_o^{\infty} \delta(\alpha^2 + \frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - F_o)\, d\alpha^2 \quad (2.117)$$

Let us use one more transformation: $\alpha^2 = u$, (2.118)

$$\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - F_o = -u_o. \quad (2.119)$$

With new transformation, space charge density is $\rho(x, y) = \frac{\pi f_o}{\sigma_x \sigma_y} \int_o^{\infty} \delta(u - u_o)\, du$. (2.120)

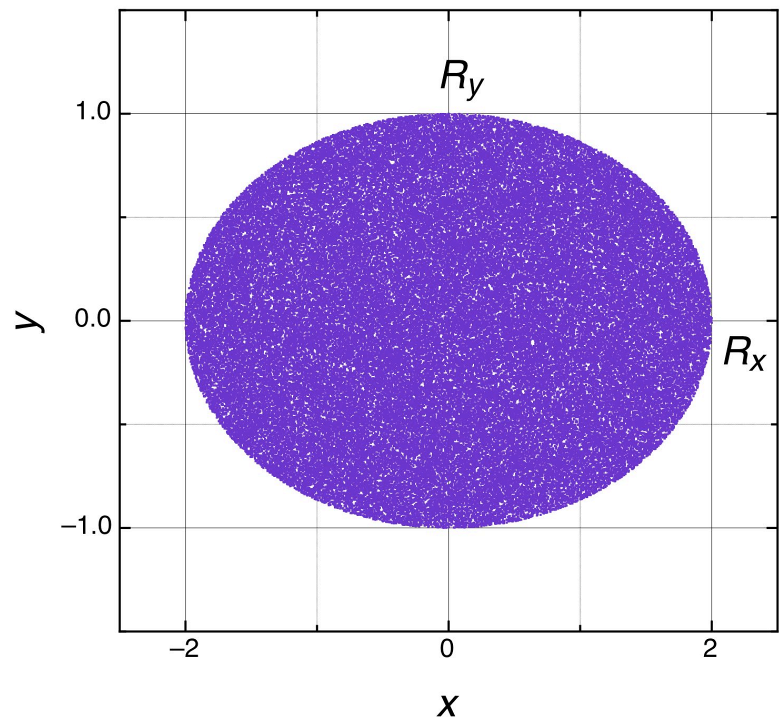
As far as the value of u_o is always positive inside the ellipse, Eq. (2.110), the integral over delta function in Eq. (2.120) is equal to unity and space charge density is equal to constant:

$$\rho(x, y) = \frac{\pi f_o}{\sigma_x \sigma_y} = \rho_o . \quad (2.121)$$

KV distribution gives projection on plane (x, y) as uniformly populated ellipse, Eq. (2.110).

Space charge density of elliptical beam with current I, semi-axis R_x, R_y, and longitudinal velocity β is

$$\rho_o = \frac{I}{\pi\beta c R_x R_y} \tag{2.122}$$



Projection of KV beam on *(x,y)*.

Projections of beam distribution on *(x,x')*

Consider particle distribution at phase plane *(x, x')*. Follow the method described above and put the following derivatives over variables *y, y'* to zero

$$\frac{\partial F(x, x', y, y')}{\partial y} = 0, \qquad \frac{\partial F(x, x', y, y')}{\partial y'} = 0 \,. \tag{2.123}$$

Substitution of the solution of Eqs. (2.123) into Eq. (2.101) gives us the boundary of particle distribution at phase plane (*x, x'*):

$$(x'\sigma_x - \sigma_x' x)^2 + \frac{x^2}{\sigma_x^2} = F_o \,, \tag{2.124}$$

which is also the ellipse. To find an area of ellipse, let us change the variables:

$$\begin{cases} \dfrac{x}{\sigma_x} = r_x \cos\theta \\ x\sigma_x' - x'\sigma_x = r_x \sin\theta \end{cases} . \tag{2.125}$$

Transformation, Eq. (2.125), in explicit form is

$$\begin{cases} x = r_x \sigma_x \cos\theta \\ x' = r_x \sigma_x' \cos\theta - \dfrac{r_x}{\sigma_x} \sin\theta \end{cases} . \tag{2.126}$$

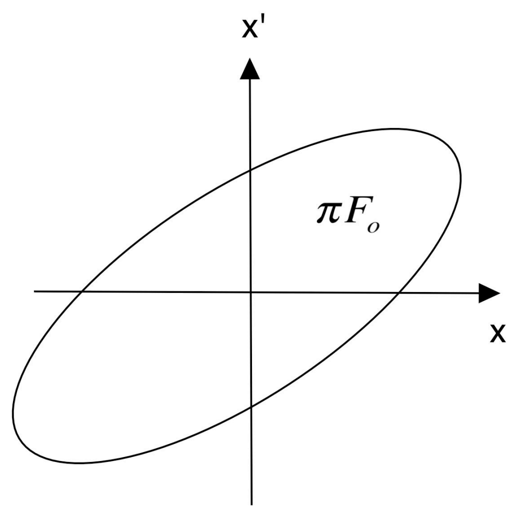
Phase space element is transformed analogously to Eq. (2.116) as

$$dx\,dx' = r_x\,dr_x\,d\theta\ . \qquad (2.127)$$

With the new variables, equation for the ellipse boundary, Eq. (2.124), is $r_x^2 = F_o$. Area of the ellipse, occupied by the particles, is:

$$S = \int_o^{2\pi}\int_o^{F_o} r_x\,dr_x\,d\theta = \pi F_o . \qquad (2.128)$$

Therefore, parameter $F_o = \text{Э}_x$ is equal to beam emittance at phase plane (x, x').



Boundary of KV beam projection on *(x,x')*.

Distribution of particles at phase plane, $\rho_x(x, x')$, is obtained via integration of distribution function, Eq. (2.101), over remaining variables *y, y'*:

$$\rho(x,x')=f_o\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta\{(x'\sigma_x-\sigma_x' x)^2+\frac{x^2}{\sigma_x^2}+(y'\sigma_y-\sigma_y' y)^2+\frac{y^2}{\sigma_y^2}-F_o\}dy\,dy'. \qquad (2.129)$$

Let us make transformation from variables *y, y'* to new variables T, ψ in Eq. (2.129):

$$(y'\sigma_y-\sigma_y' y)^2=T\cos\psi, \qquad (2.130)$$

$$\frac{y^2}{\sigma_y^2}=T\sin\psi. \qquad (2.131)$$

Phase space element $dy\,dy'$ is transformed analogously to (2.116):

$$dy\,dy'=T\,dT\,d\psi. \qquad (2.132)$$

Integration of Eq. (2.129) gives distribution in phase plane $\rho_x(x, x')=\rho_x(r_x^2)$:

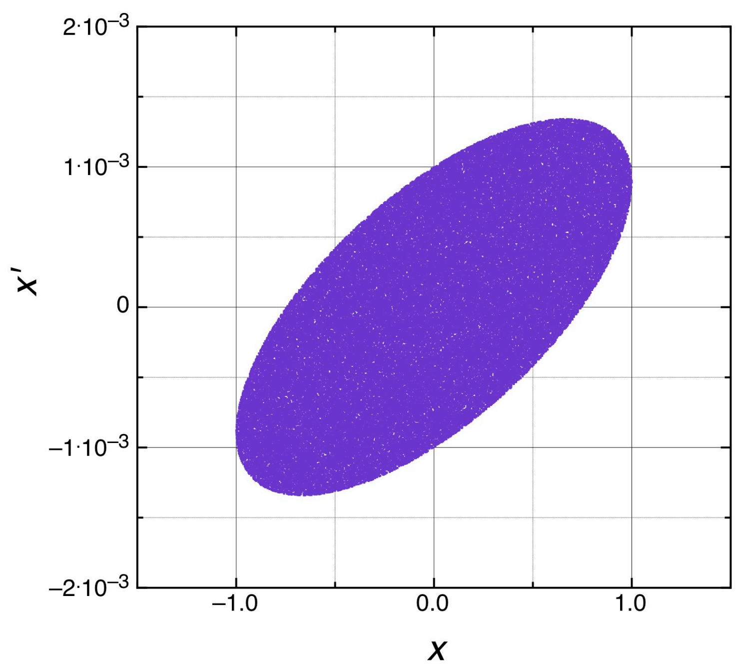
$$\rho_x(r_x^2)=\pi f_o\int_o^{\infty}\int_o^{2\pi}\delta(r_x^2+T^2-F_o)\,T\,dT\,d\psi=\pi f_o. \qquad (2.133)$$

Integral, Eq. (2.133), is evaluated in the same way as that in Eq. (2.117). Therefore, distribution of particles at phase plane (*x, x'*) is uniform inside the ellipse, Eq. (2.124).

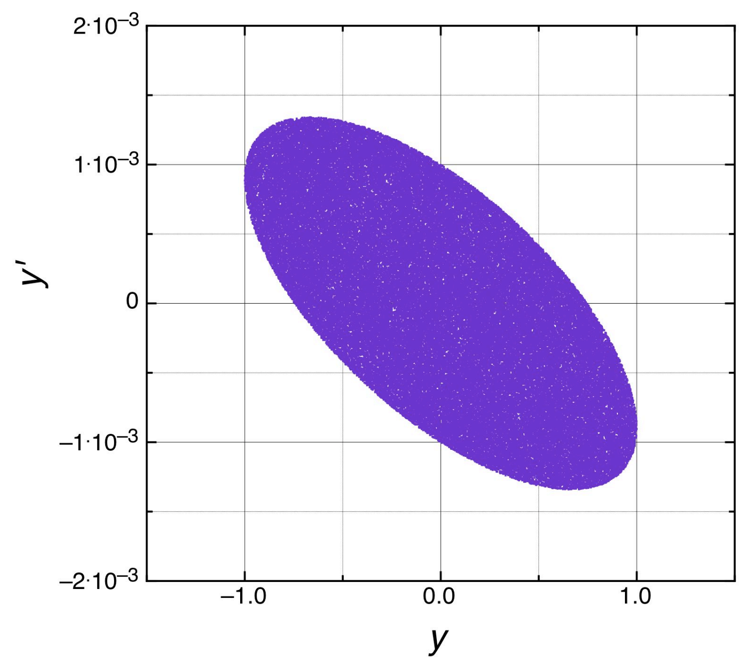
Analogously, distribution of particles in phase plane *(y, y')* is uniform inside the ellipse

$$(y' \sigma_y - \sigma_y' y)^2 + \frac{y^2}{\sigma_y^2} = F_o. \qquad (2.134)$$

Finally, KV distribution provides two-dimensional elliptical projections at every pair of phase-space coordinates with uniform particle distribution within each ellipse.



Projection of KV beam on (x, x')



Projection of KV beam on (y, y')

Potential of the beam, U_b, is to be found from Poisson's equation:

$$\frac{\partial^2 U_b}{\partial x^2} + \frac{\partial^2 U_b}{\partial y^2} = -\frac{\rho(z)}{\varepsilon_o} , \tag{2.136}$$

where space charge density

$$\rho(z) = \begin{cases} \dfrac{I}{\pi\beta c R_x R_y}, & \dfrac{x^2}{R_x^2} + \dfrac{y^2}{R_y^2} \le 1 \\ 0, & \dfrac{x^2}{R_x^2} + \dfrac{y^2}{R_y^2} \ge 1 \end{cases} . \tag{2.137}$$

Solution of Eq. (2.136) for potential of elliptical charged cylinder with current I and beam envelopes R_x, R_y is:

$$U_b(x, y, z) = -\frac{I}{4\pi\varepsilon_o \beta c R_x R_y}\left[x^2 + y^2 - \frac{R_x - R_y}{R_x + R_y}(x^2 - y^2)\right] , \tag{2.138}$$

and field components $\vec{E} = -gradU_b$ are:

$$E_x = \frac{I}{\pi\varepsilon_o \beta c R_x (R_x + R_y)} x , \tag{2.139}$$

$$E_y = \frac{I}{\pi\varepsilon_o \beta c R_y (R_x + R_y)} y . \tag{2.140}$$

Uniformly populated beam with elliptical cross section provides linear space charge forces. Therefore, initial suggestion about linearity of particle equations of motion in presence of space charge forces is correct.

Hamiltonian of particle motion within the beam with elliptical cross section is:

$$H=\frac{p_x^2+p_y^2}{2m\gamma}+q\,G(z)\frac{(x^2-y^2)}{2}-\frac{q\,I}{4\pi\varepsilon_o\gamma^2\beta c R_x R_y}\left[x^2+y^2-\frac{R_x-R_y}{R_x+R_y}(x^2-y^2)\right]. \qquad (2.141)$$

Equations of particle motion in presence of space charge forces are:

$$\frac{d^2x}{dz^2}+\left[k_x(z)-\frac{4\,I}{I_c\,\beta^3\gamma^3R_x(R_x+R_y)}\right]x=0\ , \qquad (2.142)$$

$$\frac{d^2y}{d\,z^2}+\left[k_y(z)-\frac{4\,I}{I_c\,\beta^3\gamma^3R_y(R_x+R_y)}\right]y=0\ . \qquad (2.143)$$

Characteristic current:

$$I_c=4\pi\varepsilon_o\frac{mc^3}{q}=3.13\cdot10^7\frac{A}{Z}[Ampere]$$

Eqs. (2.142), (2.143) are similar to that without space c harge forces, where instead of functions $k_x(z)$, $k_y(z)$ the modified functions $\tilde{k}_x(z)$, $\tilde{k}_y(z)$ are used:

$$\tilde{k}_x(z)=k_x(z)-\frac{4\,I}{I_c\,\beta^3\gamma^3R_x(R_x+R_y)}\ , \qquad (2.144)$$

$$\tilde{k}_y(z)=k_y(z)-\frac{4\,I}{I_c\,\beta^3\gamma^3R_y(R_x+R_y)}\ . \qquad (2.145)$$

Substitution of expressions (2.144), (2.145) instead of $k_x(z)$, $k_y(z)$ into envelope equations (2.56), (2.57) gives us the *KV envelope equations* for the beam with space charge forces:

$$\frac{d^2R_x}{dz^2} - \frac{\text{э}_x^2}{R_x^3} + k_x(z)\,R_x - \frac{4\,I}{I_c\,\beta^3\gamma^3(R_x+R_y)} = 0\ , \qquad (2.146)$$

$$\frac{d^2R_y}{dz^2} - \frac{\text{э}_y^2}{R_y^3} + k_y(z)\,R_y - \frac{4\,I}{I_c\,\beta^3\gamma^3(R_x+R_y)} = 0. \qquad (2.147)$$

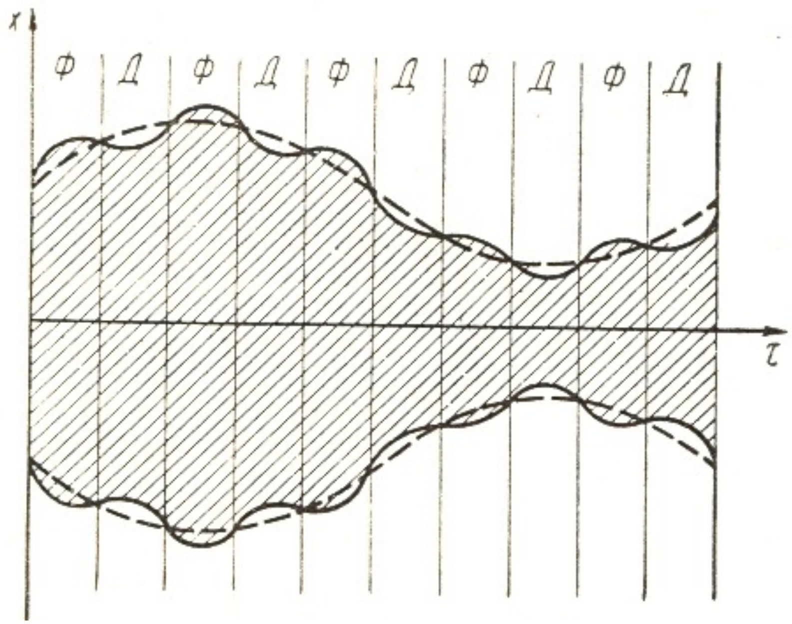
Equations (2.146), (2.147) are non-linear differential equations of the second order. They can be formally derived from Hamiltonian:

$$H = \frac{(R_x')^2}{2} + \frac{(R_y')^2}{2} + k_x(z)\frac{R_x^2}{2} + k_y(z)\frac{R_y^2}{2} + 2P^2 ln\frac{1}{R_x+R_y} + \frac{\text{э}_x^2}{2R_x^2} + \frac{\text{э}_y^2}{2R_y^2}\ , \qquad (2.148)$$

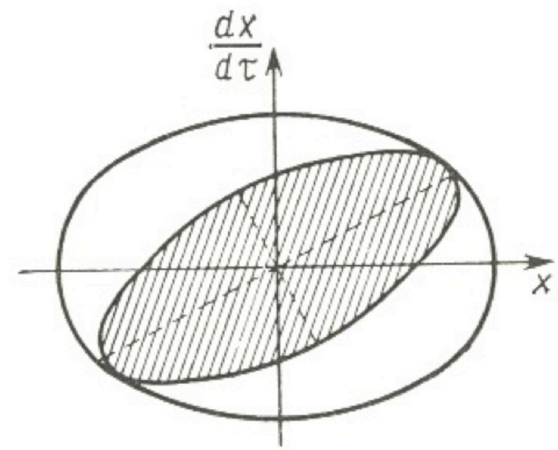
where parameter P^2 is called the generalized perveance

$$P^2 = \frac{2I}{I_c\,\beta^3\gamma^3}\ . \qquad (2.149)$$

In general case, solution of the set of envelope equations, Eqs. (2.146), (2.147) are non-periodic functions, which corresponds to envelopes of unmatched beam. However, if functions $k_x(z)$, $k_y(z)$ are periodic, there is a periodic solution of envelope equations. Envelope equations can be solved numerically at the period of structure via varying the initial conditions $R_x(0), R_x'(0), R_y(0), R_y'(0)$ unless the solution at the end of period coinsides with initial conditions $R_x(L) = R_x(0)$, $R_x'(L) = R_x'(0)$, $R_y(L) = R_y(0)$, $R_y'(L) = R_y'(0)$. Again, as in case of beam with negligible current, this beam is called the matched beam. It occupies the smallest fraction of aperture of the channel.

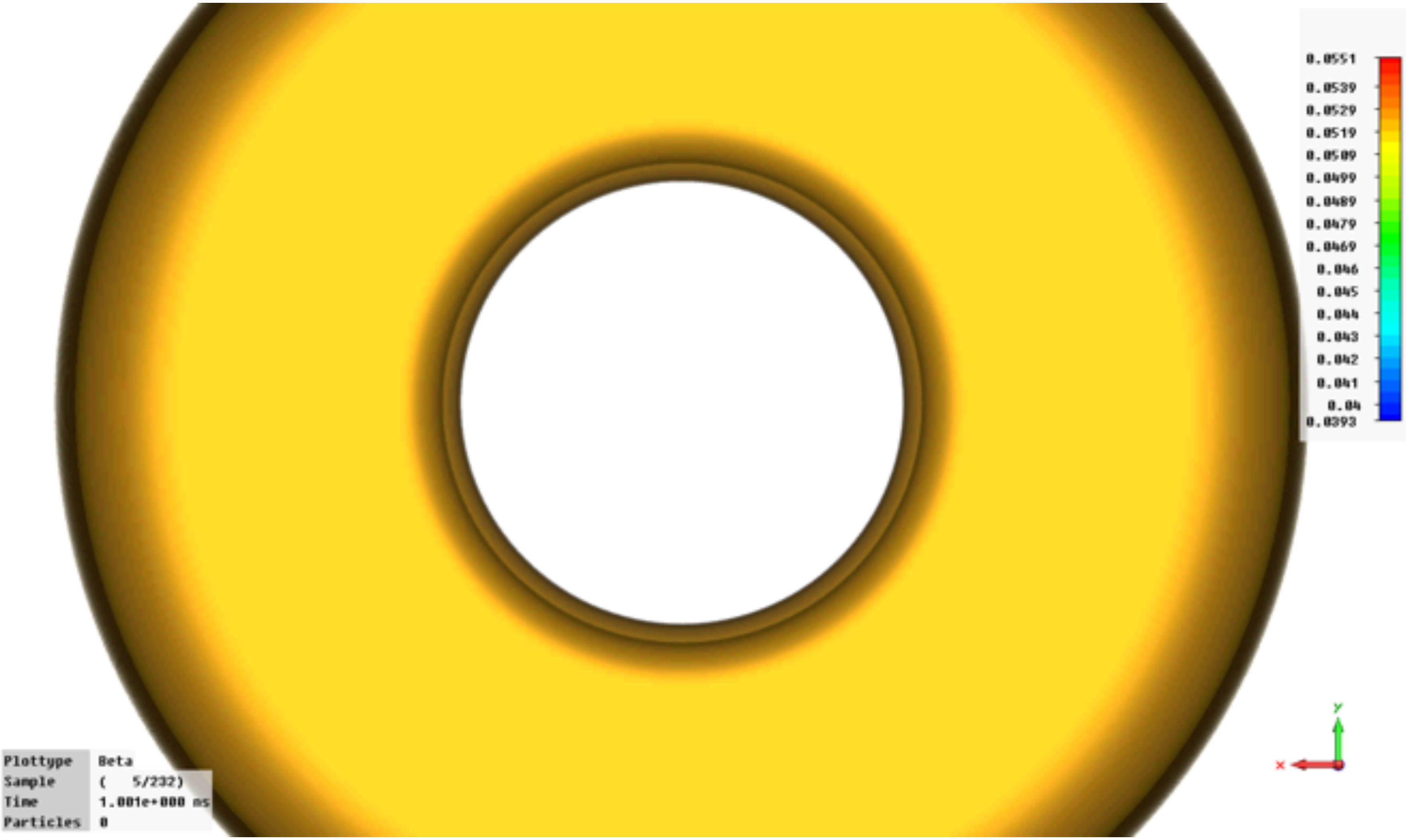


The envelope of unmatched beam in a quadrupole channel



Effective beam emittance.

Matched Beam Focusing



Matched beam in RF linear accelerator (Courtesy of Sergey Kurennoy).

Averaged Beam Envelopes

For focusing channels, where phase advance per period is small, $\mu_o/2\pi << 1$, one can use smooth approximation to beam envelopes. Analogously to particle trajectories in smoothed approximation, solution for beam envelopes can be represented as:

$$R_x(z) = \bar{R}_x(z) + \xi_x(z), \qquad (2.150)$$

$$R_y(z) = \bar{R}_y(z) + \xi_y(z), \qquad (2.151)$$

where $\bar{R}_x(z)$, $\bar{R}_y(z)$ are smoothed envelopes, and $\xi_x(z)$, $\xi_y(z)$ are small fast oscillating functions. The following approximations can be used:

$$\frac{1}{R_x^3} \approx \frac{1}{\bar{R}_x^3}\left(1 - 3\frac{\xi_x}{\bar{R}_x}\right), \qquad (2.152)$$

$$\frac{1}{R_x + R_y} \approx \frac{1}{\bar{R}_x + \bar{R}_y} - \frac{1}{(\bar{R}_x + \bar{R}_y)^2}\xi_x - \frac{1}{(\bar{R}_x + \bar{R}_y)^2}\xi_y. \qquad (2.153)$$

Eqs. (2.146), (2.147) formally can be considered as single body oscillations in the alternating-gradient field with addition of potential function describing “emittance” and “current” terms. Averaged values of that terms in the envelope equations are

$$\overline{\frac{\text{э}_x^2}{\overline{R}_x^3}\left(1 - 3\frac{\xi_x}{\overline{R}_x}\right)} = \frac{\text{э}_x^2}{\overline{R}_x^3}, \qquad \overline{\frac{\text{э}_y^2}{\overline{R}_y^3}\left(1 - 3\frac{\xi_y}{\overline{R}_y}\right)} = \frac{\text{э}_y^2}{\overline{R}_y^3} \tag{2.154}$$

$$\overline{\frac{2P^2}{\overline{R}_x + \overline{R}_y}} - \overline{\frac{2P^2}{(\overline{R}_x + \overline{R}_y)^2}\xi_x} - \overline{\frac{2P^2}{(\overline{R}_x + \overline{R}_y)^2}\xi_y} = \frac{2P^2}{\overline{R}_x + \overline{R}_y}. \tag{2.156}$$

The resulting field is a combination of effective field and potential field . Finally, envelope equations in smoothed approximation are

$$\frac{d^2\overline{R}_x}{dz^2} - \frac{\text{э}_x^2}{\overline{R}_x^3} + \left(\frac{\mu_o}{L}\right)^2\overline{R}_x - \frac{4I}{I_c\beta^3\gamma^3(\overline{R}_x + \overline{R}_y)} = 0, \tag{2.157}$$

$$\frac{d^2\overline{R}_y}{dz^2} - \frac{\text{э}_y^2}{\overline{R}_y^3} + \left(\frac{\mu_o}{L}\right)^2\overline{R}_y - \frac{4I}{I_c\beta^3\gamma^3(\overline{R}_x + \overline{R}_y)} = 0. \tag{2.158}$$

Small fast components are the same as that for single particle, because they are defined by fast oscillating functions only: $\xi_x = \xi_{\max}\sin 2\pi\frac{z}{L}$ $\quad \xi_y = -\xi_{\max}\sin 2\pi\frac{z}{L}$

Beam envelopes: $\boxed{R_x(z) = \overline{R}_x(z) + \xi_x(z)}$

$\boxed{R_y(z) = \overline{R}_y(z) + \xi_y(z)}$

Each of equations (2.157), (2.158) contain two defocusing terms: one is proportional to square of beam emittance and another one is proportional to beam current. Consider beam with the values of envelopes close to each other, $\overline{R}_x \approx \overline{R}_y = R$, and with equal emittances in both planes $э_x = э_y = \varepsilon/(\beta\gamma)$. Ratio of that two terms gives us estimation, which factor dominates in beam transport:

$$b = \frac{2}{(\beta\gamma)} \frac{I}{I_c} \frac{R^2}{\varepsilon^2} . \qquad (2.159)$$

Transport with $b >> 1$ corresponds to space-charge dominated regime, while $b << 1$ corresponds to emittance- dominated regime. The value of b is the ratio of beam brightness, $B = I/\varepsilon^2$, to normalization value of I_c / R^2. It is reasonable to call parameter b the dimensionless beam brightness. Additional factor of $2/(\beta\gamma)$ indicates that significance of the space charge forces drops with increasing of beam energy. Beam with high value of beam brightness, B, can be both in space charge dominated regime, and in emittance-dominated regime, depending on particles energy.

Acceptance of the Channel Estimated from Beam Envelopes

In the limit of negligible current, $I = 0$, KV envelope equations are decoupled. Consider matched beam, $\overline{R}_x'' = \overline{R}_y'' = 0$, with equal emittances in both planes $\text{э}_x = \text{э}_y = \text{э}$:

$$-\frac{\text{э}^2}{\overline{R}_x^3} + \left(\frac{\mu_o}{L}\right)^2 \overline{R}_x = 0, \tag{2.190}$$

$$-\frac{\text{э}^2}{\overline{R}_y^3} + \left(\frac{\mu_o}{L}\right)^2 \overline{R}_y = 0. \tag{2.191}$$

Equations (2.190), (2.191) have the common solution:

$$R_o^2 = \frac{\text{э}\, L}{\mu_o}. \tag{2.192}$$

Beam envelopes for negligible current:

$$R_x(z) = R_o + \xi_{\max} \sin 2\pi \frac{z}{L}$$

$$R_y(z) = R_o - \xi_{\max} \sin 2\pi \frac{z}{L}$$

For FD structure:

$$\boxed{\xi_{\max} = R_o \frac{4\sqrt{3}}{\pi^3} \mu_o}$$

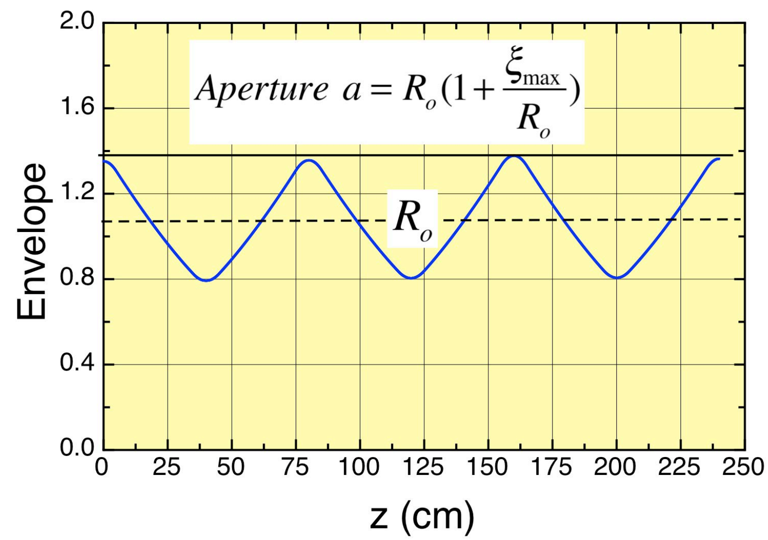
It defines the averaged beam radius in quadrupole channel for the beam with negligible space charge forces. *Acceptance of the channel*, A, is the maximum emittance of the beam, which could be transported through the channel without beam losses. In quadrupole channel, beam envelopes are oscillating functions, Eqs. (2.150), (2.151). Aperture, a, is reached by particles with $R_o + \xi_{max} = a$. From Eq. (2.192), acceptance of the channel in smooth approximation is

$$A = \frac{a^2 \mu_o}{L\,(1+\delta_{max})^2} \quad \text{, where} \quad \boxed{\delta_{max} = \frac{\xi_{max}}{R_o}} \qquad (2.193)$$

The value of δ_{max} in FD channel was estimated as $\delta_{max} = \frac{4\sqrt{3}}{\pi^3}\mu_o$,

Acceptance: $\boxed{A = \frac{a^2 \mu_o}{L(1+0.447\mu_o)}}$

Normalized acceptance of the channels: $\varepsilon_{ch} = \beta\gamma A$



<u>Compare with FODO acceptance obtained from matrix method:</u>

$$\boxed{A = \frac{a^2}{L}\frac{\sin\mu_o}{(1+\sin\frac{\mu_o}{2})}}$$

For a matched beam, the maximum and minimum beam envelopes are $R_{max} \approx R_o(1+\delta_{max})$ $\quad R_{min} \approx R_o(1-\delta_{max})$

Beam Radius in Space-Charge Dominated Regime

When space charge forces are not negligible, smoothed KV equations for matched beam, $\bar{R}_x'' = \bar{R}_y'' = 0$, are

$$-\frac{\epsilon^2}{\bar{R}_x^3} + \left(\frac{\mu_o}{L}\right)^2 \bar{R}_x - \frac{4\,I}{I_c\,\beta^3\gamma^3(\bar{R}_x + \bar{R}_y)} = 0\ , \qquad (2.196)$$

$$-\frac{\epsilon^2}{\bar{R}_y^3} + \left(\frac{\mu_o}{L}\right)^2 \bar{R}_y - \frac{4\,I}{I_c\,\beta^3\gamma^3(\bar{R}_x + \bar{R}_y)} = 0. \qquad (2.197)$$

Eqs. (2.196), (2.197), have common solution $\bar{R}_x = \bar{R}_y = R$ defined by:

$$-\frac{\epsilon^2}{R^3} + \left(\frac{\mu_o}{L}\right)^2 R - \frac{2\,I}{I_c\,\beta^3\gamma^3 R} = 0. \qquad (2.198)$$

Combination of Eq. (2.192) and Eq. (2.198) gives:

$$R - \frac{R_o^4}{R^3} - \frac{2\, I\, R_o^4}{I_c\, \beta^3 \gamma^3 R\, \text{э}^2} = 0 \ . \qquad (2.199)$$

From the last equation, the averaged beam radius in space – charge regime is expressed via beam radius with negligible space charge forces as

$$R = R_o \sqrt{b_o + \sqrt{1 + b_o^2}} \ , \qquad (2.200)$$

where b_o is the space charge parameter:

$$b_o = \frac{1}{(\beta\gamma)} \frac{I}{I_c} \frac{R_o^2}{\varepsilon^2} \ . \qquad (2.201)$$

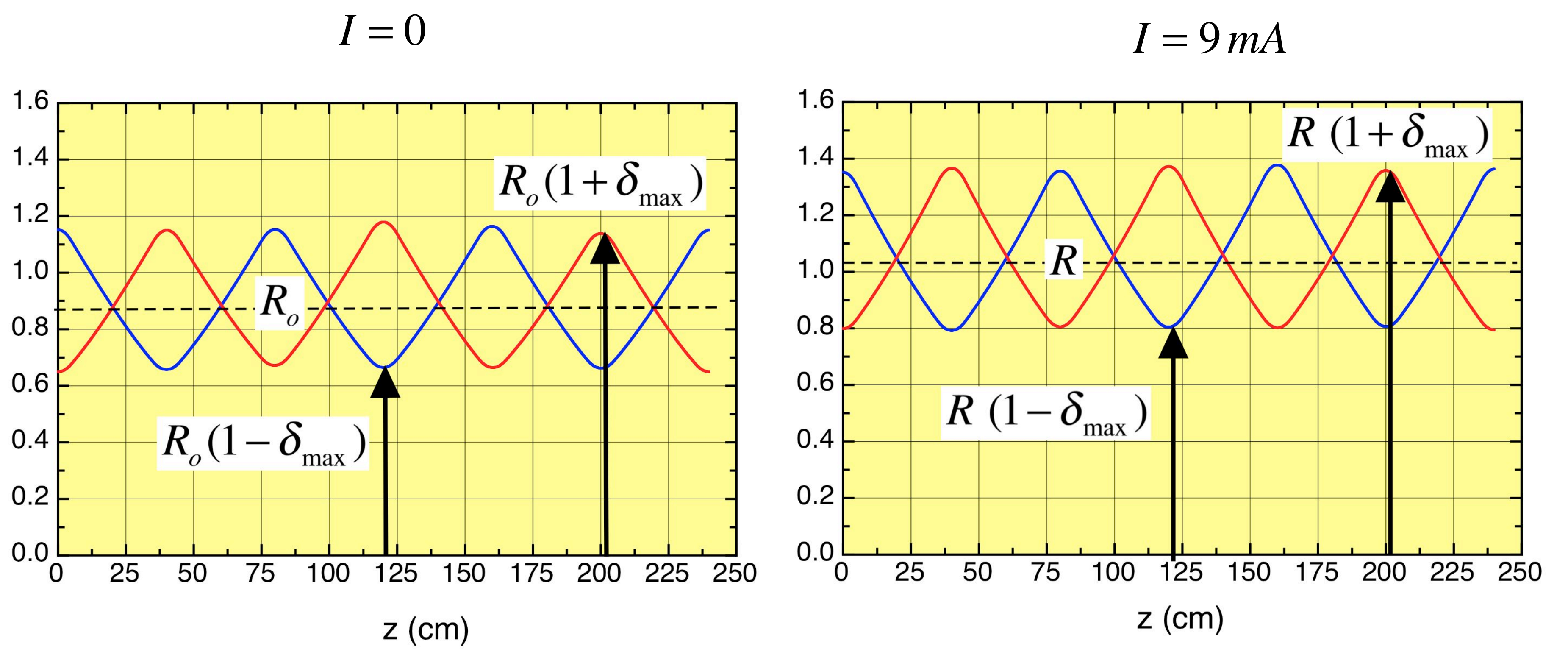
Equation (2.201) indicates that matched beam radius increases with beam current.

In averaging method, small function ξ is defined by fast oscillating term only. Function $\delta_{\max} = \xi_{\max} / R$ does not depend on beam current and beam emittance, therefore

$$R_{\max} \approx R\,(1 + \delta_{\max})$$

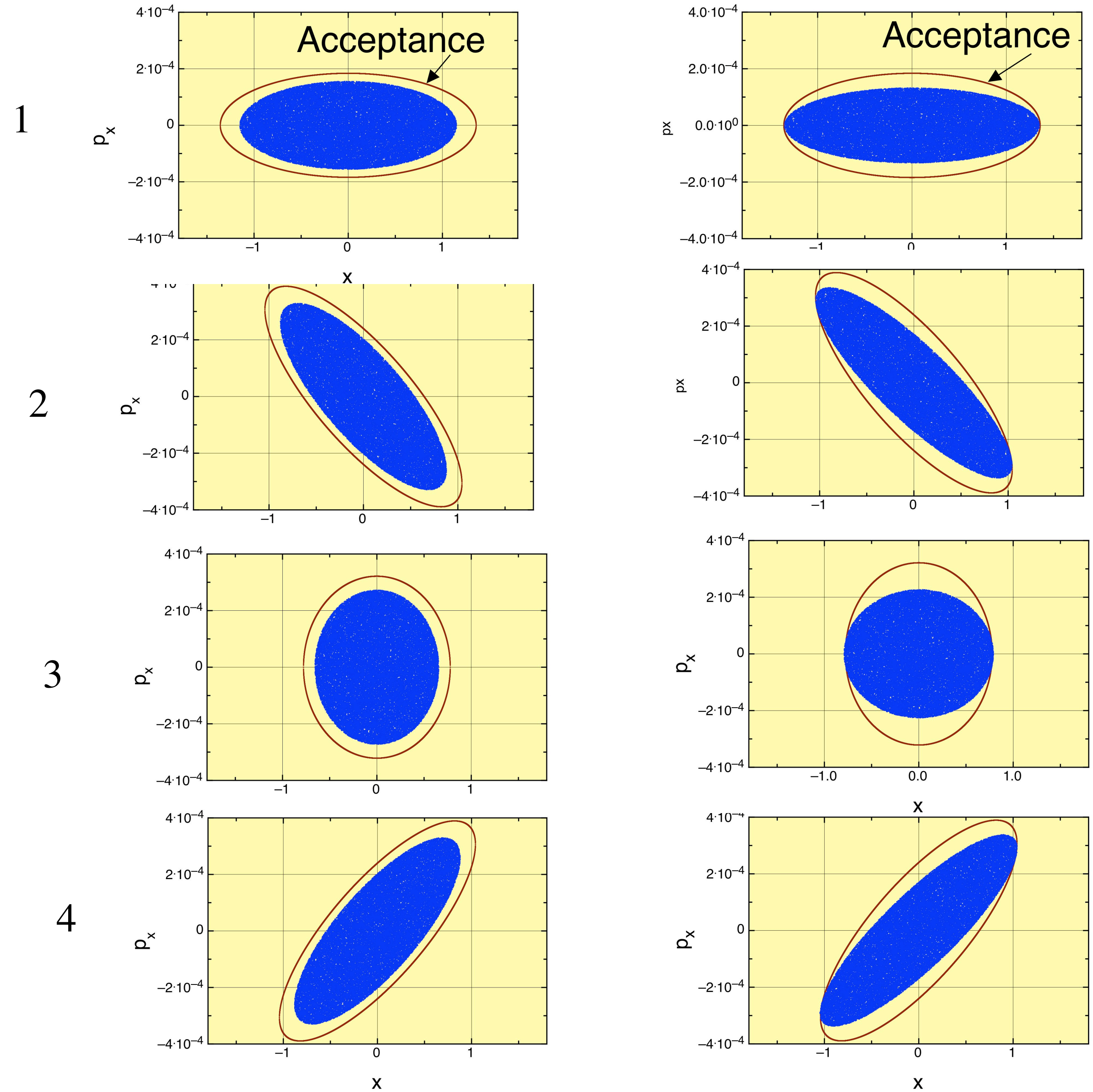
$$R_{\min} \approx R\,(1 - \delta_{\max})$$

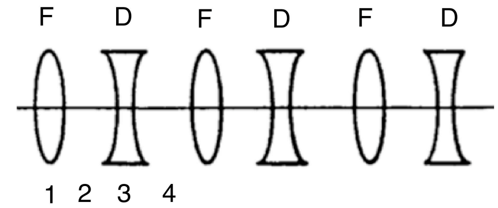
Matched Beam Envelopes Versus Beam Current



Envelopes of the proton beam with energy of E = 150 keV, and normalized emittance of ε = 0.1788 π cm mrad, propagating in a quadrupole FODO channel with magnetic lenses of length of D =10 cm, field gradient of G_m = 1.6 Tesla/m, and period of the structure of L = 80 cm.

Matched beam: I = 0 I > 0





Transverse Oscillation Frequency in Space-Charge Dominated Regime

Eqs. (2.142), (2.143) define particle trajectory in quadrupole channel in presence of space charge field of the uniformly populated beam with elliptical cross-section. Taking $\bar{R}_x \approx \bar{R}_y = R$, equation for single particle trajectory in smoothed approximation is

$$\frac{d^2X}{dz^2} + [(\frac{\mu_o}{L})^2 - \frac{2\,I}{I_c\,\beta^3\gamma^3R^2}]\,X = 0\,, \tag{2.202}$$

and similar for y - direction. Expression (2.202) can be re-written as

$$\frac{d^2X}{dz^2} + (\frac{\mu}{L})^2 X = 0\,, \tag{2.203}$$

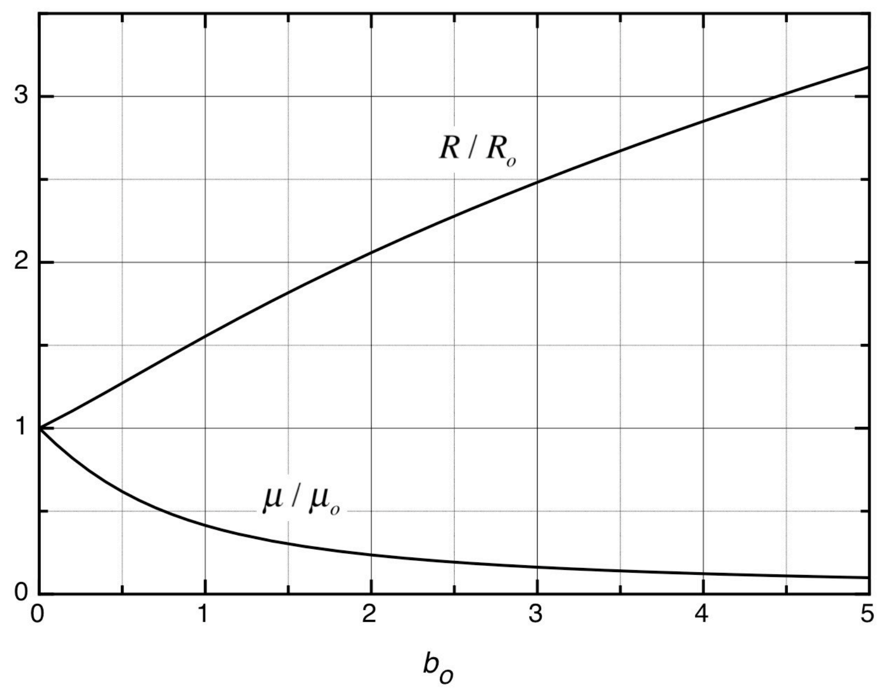
where μ is the averaged betatron frequency in presence of space charge forces, which is also called the depressed betaron tune:

$$\mu^2 = \mu_o^2 - \frac{2I}{I_c(\beta\gamma)^3}(\frac{L}{R})^2\,. \tag{2.204}$$

Eq. (2.204) indicates that space charge forces result in depression of transverse oscillations. Substitution of expression for beam radius in space charge dominated regime, Eq. (2.200), and expression for space charge parameter, Eq. (2.201), into Eq. (2.204) gives for μ

$$\mu = \mu_o(\sqrt{1 + b_o^2} - b_o)\,. \tag{2.205}$$

Transverse oscillation frequency drops with increase of beam current, but remains non-zero. Therefore, *beam stability can be provided at any value of beam current*. However, increase of beam current requires increase of aperture of the channel, and *stability of transverse oscillations can be provided at arbitrary high value of beam current, but in the channel with infinitely large aperture*.



Averaged beam radius and transverse oscillation frequency as functions of space charge parameter b_o.

Connection between μ, μ_o, b

Let us rewrite equation (2.204):

$$\mu_o^2 = \mu^2 + \frac{2I}{I_c(\beta\gamma)^3}(\frac{L}{R})^2$$

$$\mu_o^2 = \mu^2[1 + \frac{2I}{I_c(\beta\gamma)^3\mu^2}(\frac{L}{R})^2]$$

Substitute beam emittance:

$$Э = \frac{\mu R^2}{L}$$

$$\mu_o^2 = \mu^2[1 + \frac{2I}{I_c\beta\gamma}(\frac{R}{\varepsilon})^2]$$

Connection between phase advance per period μ_o, μ, and dimensionless space charge parameter *b*

$$\mu_o^2 = \mu^2(1 + b)$$

$$\mu^2 = \frac{\mu_o^2}{1+b}$$

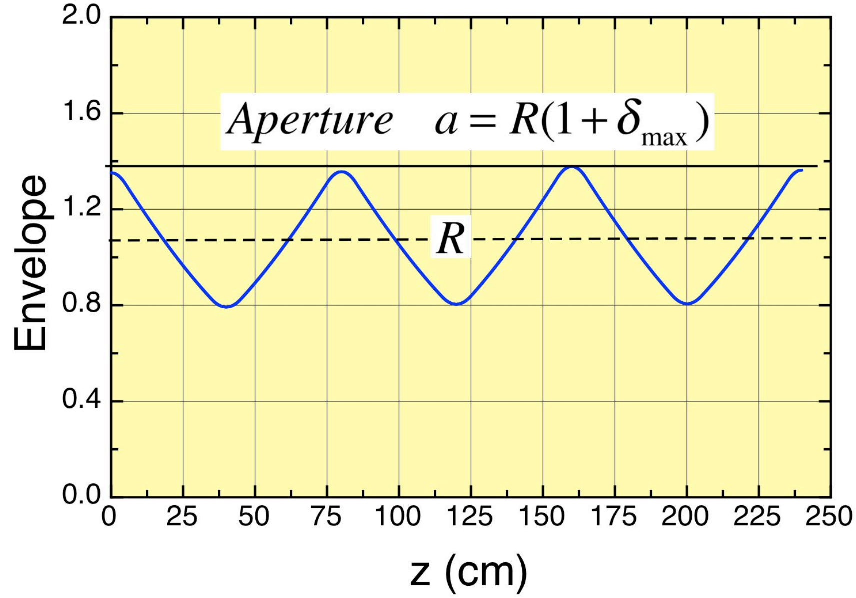
$$b = \frac{\mu_o^2}{\mu^2} - 1$$

э

Beam Current Limit

Maximum beam current corresponds to the beam, which fills in all available aperture, $a = R(1 + \delta_{max})$, or, taking into account Eqs. (2.192), (2.200):

$$a = \sqrt{\frac{э\, L}{\mu_o}} \sqrt{b_o + \sqrt{1 + b_o^2}}\ (1 + \delta_{max}) \cdot \qquad (2.206)$$



For $b_o = 0$, equation (2.206) describes the beam with maximum possible emittance in the channel, equal to acceptance of the channel, $э = A$:

$$a = \sqrt{\frac{A\ L}{\mu_o}}\ (1 + \delta_{max}) \cdot \qquad (2.207)$$

Ratio of last two equations gives us the relationship between acceptanceof the channel and the maximum emittance of the beam with non-zero current, which fills in all aperture of the channel:

$$э = \frac{A}{b_o + \sqrt{1 + b_o^2}}, \qquad \text{or} \qquad э = A(\sqrt{1 + b_o^2} - b_o)\,. \qquad (2.208)$$

Substitution of the expression for space charge parameter b_o, Eq. (2.201), into Eq. (2.208) gives for maximum transported beam current:

$$\boxed{I_{\max} = \frac{I_c}{2} (\beta\gamma)^3 A \frac{\mu_o}{L} (1 - \frac{э^2}{A^2})}\,. \qquad (2.209)$$

Approximation of the value of limited beam current by Eq. (2.209) becomes better with increase of beam current, because in this case the transverse oscillation frequency $\mu / 2\pi << 1$ drops and smooth approximation is improved.

Rms Beam Envelopes (F.Sacherer, P.Lapostolle, PAC 1971)

Rms envelope equations

$$\frac{d^2\tilde{X}}{dz^2} - \frac{э_x^2}{\tilde{X}^3} + k_x(z)\,\tilde{X} - \frac{4\,I}{I_c\,\beta^3\gamma^3(\tilde{X}+\tilde{Y})} = 0$$

$$\frac{d^2\tilde{Y}}{dz^2} - \frac{э_y^2}{\tilde{Y}^3} + k_y(z)\,\tilde{Y} - \frac{4\,I}{I_c\,\beta^3\gamma^3(\tilde{X}+\tilde{Y})} = 0$$

2-rms beam envelopes

$$R_x = \tilde{X} = 2\sqrt{<x^2>}$$

$$R_y = \tilde{Y} = 2\sqrt{<y^2>}$$

4-rms beam emittances

$$э_x = 4\sqrt{<x^2><x'^2> - <xx'>^2}$$

$$э_y = 4\sqrt{<y^2><y'^2> - <yy'>^2}$$

Beam Drift in Free Space

Important case is propagation of the beam in the area without any external fields. Consider transport of a round beam $R_x = R_y = R$ in drift space, described by envelope equation

$$\frac{d^2R}{dz^2} - \frac{\epsilon^2}{R^3} - \frac{P^2}{R} = 0 . \qquad \text{(D-1)}$$

Equation (D-1) has the first integral:

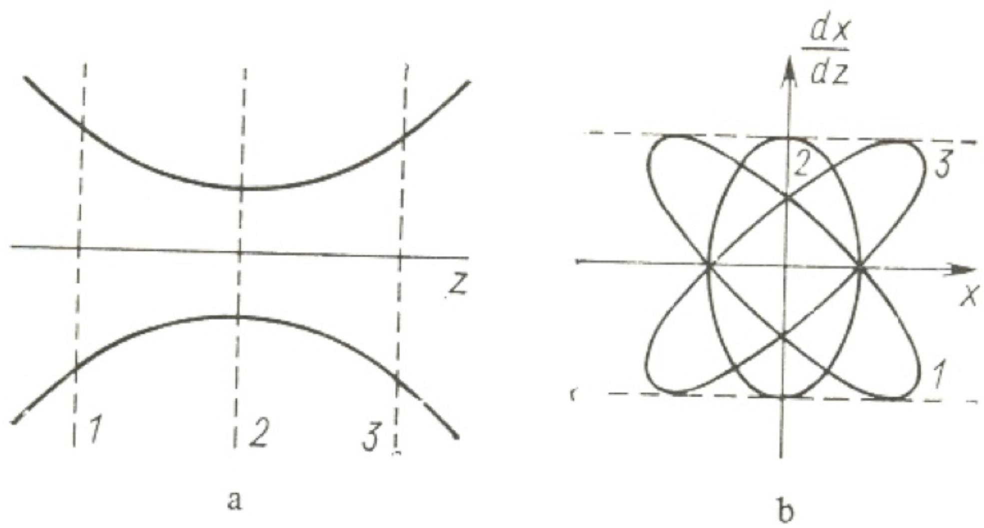
$$\left(\frac{dR}{dz}\right)^2 = \left(\frac{dR}{dz}\right)_o^2 + \left(\frac{\epsilon}{R_o}\right)^2\left(1 - \frac{R_o^2}{R^2}\right) + P^2 \ln\left(\frac{R}{R_o}\right)^2 \qquad \text{(D-2)}$$

which determines divergence of the beam as a function of initial beam parameters, beam current, and beam emittance. Eq. (D-2) can be further integrated to determine distance, where beam with initial radius of R_o and initial divergence $R_o^{'}$ is evolved up the radius R

$$z = \frac{R_o^2}{2\epsilon}\int_1^{(\frac{R}{R_o})^2} \frac{ds}{\sqrt{[1+(\frac{R_o R_o^{'}}{\epsilon})^2]s + (\frac{P R_o}{\epsilon})^2 s \ln s - 1}} \qquad \text{(D-3)}$$

Eq. (D-3) can be integrated in case of negligible current, *P = 0:*

$$\frac{R}{R_o} = \sqrt{\left(1 + \frac{R_o^{'}}{R_o} z\right)^2 + \left(\frac{\epsilon}{R_o^2}\right)^2 z^2} \qquad \text{(D-4)}$$



Drift of the beam with finite value of phase space (a) beam envelope, (b) phase space deformation.

Drift of Space-Charge Dominated Beam

Another case is drift of the beam with negligible beam emittance, but non-zero beam current. Eq. (D-2) has the form

$$\left(\frac{dR}{dz}\right)^2 = \left(\frac{dR}{dz}\right)_o^2 + P^2 \ln\left(\frac{R}{R_o}\right)^2 \qquad \text{(D-5)}$$

To determine expansion of the beam from waist point, let us put initial beam divergence $R_o' = 0$, then Eq. (D-5) becomes

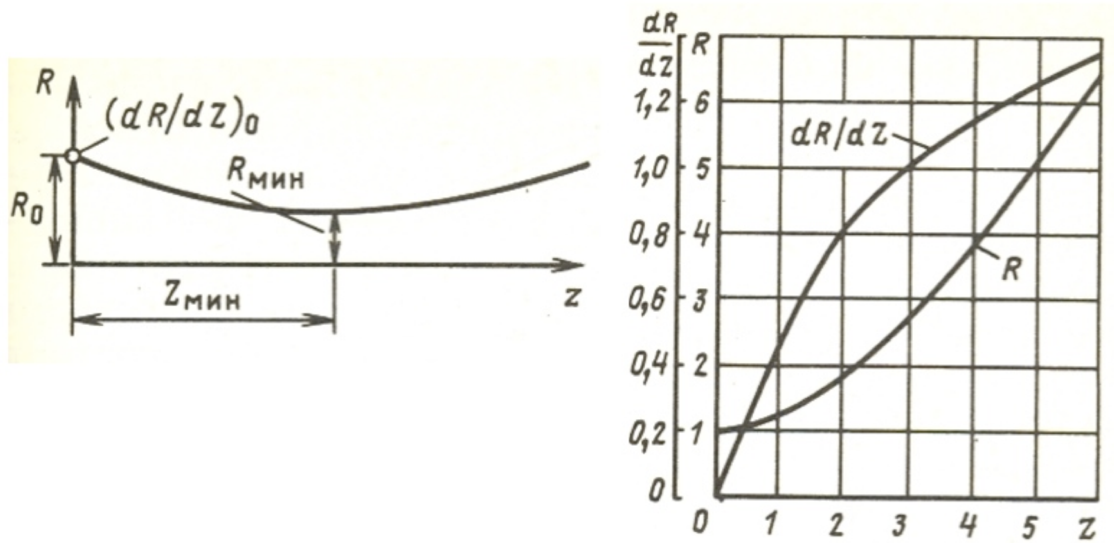
$$\left(\frac{dR}{dz}\right)^2 = P^2 \ln\left(\frac{R}{R_w}\right)^2 \qquad \text{(D-6)}$$

Eq. (D-6) has an approximate solution

$$\boxed{\frac{R}{R_w} \approx 1 + 0.25Z^2 - 0.017Z^3} \qquad \text{(D-7)}$$

$$Z = 2\frac{z}{R_w}\sqrt{\frac{I}{I_c(\beta\gamma)^3}} \qquad \text{(D-8)}$$

where z is counted from the waist point. Eq. (D-7) gives good results for for $0 < Z < 3.2$ and $1 < R/R_w < 3$.



Envelope of an axial-symmetric beam in drift space (Molokovsky, Sushkov, 2005).

Maximum Beam Current Transported Through the Tube

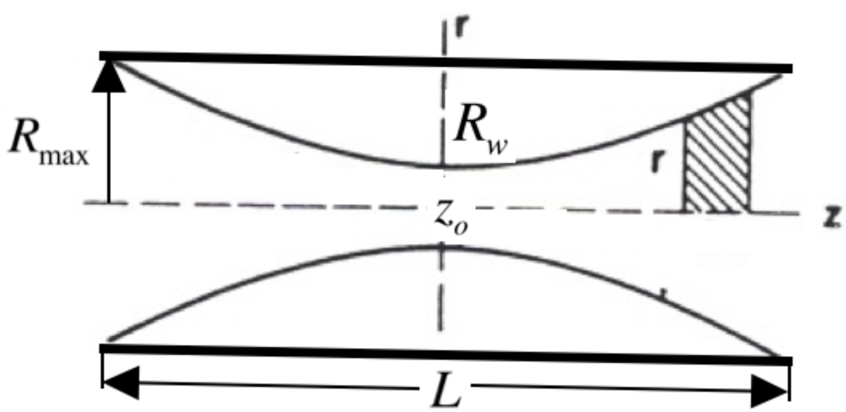
In practical applications, it is important to know the maximum beam current which can be transported through the tube of length L and radius R_{max}. From symmetry point, it is clear that beam should have a waist size $R = R_w$ and zero divergence in the middle of the tube $z = z_o$. Thus, equation (D-6) can be integrated in this case to determine beam expansion from minimal size $R = R_w$ to max size of $R = R_{max}$:

$$\frac{1}{\bar{R}_{\max}}\int_1^{\bar{R}_{\max}}\frac{d\bar{R}}{\sqrt{\ln\bar{R}}} = \sqrt{2}P\frac{(z-z_o)}{R_{\max}}. \qquad \bar{R} = R/R_w \qquad \text{(D-9)}$$

The left hand side of Eq. (D-9) has a maximum value of 1.082 for $\bar{R}_{\max} = R_{max}/R_w = 2.35$. The maximum radius is achieved at $z - z_o = L/2$, which in turn yields $P_{\max}L/(\sqrt{2}R_{\max}) = 1.082$. From this expression, the maximum transported current through the tube is

$$I_{\lim} = 1.17 I_c(\beta\gamma)^3(\frac{R_{\max}}{L})^2. \qquad \text{(D-10)}$$

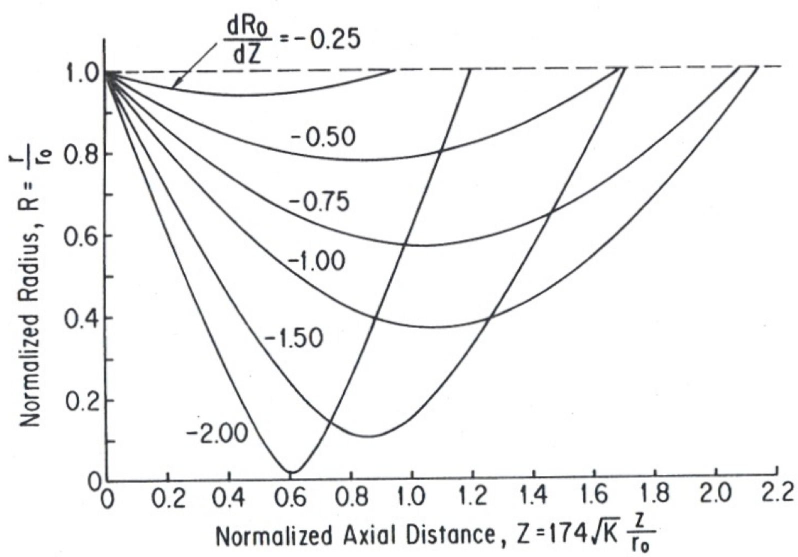
Required beam slope at the entrance of the tube can be determined from Eq. (D-6):



$$\frac{dR}{dz} = \sqrt{\frac{4I_{\lim}}{I_c(\beta\gamma)^3}\ln(\frac{R_{\max}}{R_w})} \approx 2\frac{R_{\max}}{L} \qquad \text{(D-11)}$$

On maximum current transported through the tube

Optimization of Beam Drift Space



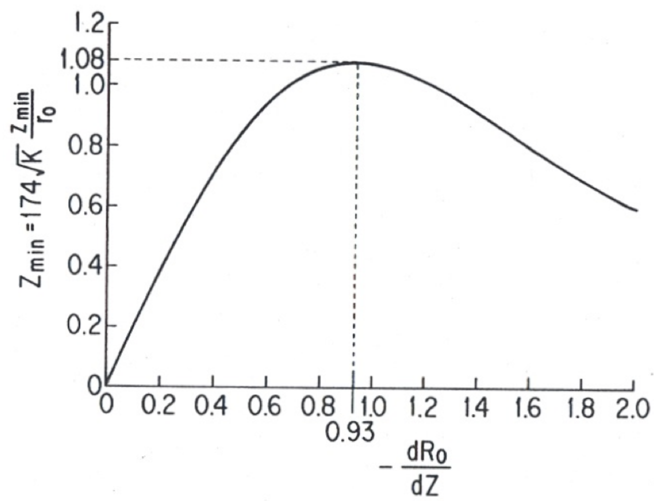


Figure 11: The position of the minimum beam radius as a function of dR_0/dZ. From A.S. Gilmour, Jr.[5]

Beam radius at waist point, $R = R_w$, can be determined from Eq. (D-5) as a function of beam radius R_o and initial beam convergence $R_o^{'}$ assuming in waist point $dR/dz = 0$:

$$R_w = R_o \exp[-(\frac{R_o^{'}}{\sqrt{2P}})^2] \quad \text{(D-12)}$$

To determine distance, where the beam reaches it's waist, let us rewrite Eq. (D-5) including notations, Eq. (D-8):

$$Z = \int_1^{\bar{R}} \frac{d\bar{R}}{\sqrt{\ln \bar{R} + (d\bar{R}_o / dZ)^2}} \quad \text{(D-13)}$$

Using substitution $u = \sqrt{\ln \bar{R} + (d\bar{R}_o / dZ)^2}$, Eq. (D-15) for waist point, where $\bar{R} = 1$, is reduced to

$$Z_{\min} = 2\, e^{-(d\bar{R}_o/dZ)^2} \int_o^{|d\bar{R}_o/dZ|} \exp(u^2)\, du \quad \text{(D-14)}$$

Envelope Instability

Averaging procedure (smooth approximation) was based on assumption that solution of envelope equations are stable

Envelope Equations

$$\frac{d^2 R_x}{dz^2} - \frac{\epsilon_x^2}{R_x^3} + k(z) R_x - \frac{2P^2}{(R_x + R_y)} = 0$$

$$\frac{d^2 R_y}{dz^2} - \frac{\epsilon_y^2}{R_y^3} - k(z) R_y - \frac{2P^2}{(R_x + R_y)} = 0$$

Let us represent solution as a combination of periodic solutions $\tilde{R}_x(z),\ \tilde{R}_y(z)$ and deviations from that $\xi_x(z),\ \xi_y(z)$

$$R_x(z) = \tilde{R}_x(z) + \xi_x(z)$$

$$R_y(z) = \tilde{R}_y(z) + \xi_y(z)$$

Equations for deviations from periodic solution:

$$\xi_x'' + \xi_x a_1(z) + \xi_y a_o(z) = 0$$

$$\xi_y'' + \xi_y a_2(z) + \xi_x a_o(z) = 0$$

$$a_o(z) = \frac{2P^2}{(\tilde{R}_x + \tilde{R}_y)^2} \qquad a_1(z) = k(z) + 3\frac{\epsilon_x^2}{\tilde{R}_x^4} + \frac{2P^2}{(\tilde{R}_x + \tilde{R}_y)^2} \qquad a_2(z) = -k(z) + 3\frac{\epsilon_y^2}{\tilde{R}_y^4} + \frac{2P^2}{(\tilde{R}_x + \tilde{R}_y)^2}$$

Envelope Oscillations Modes

In smooth approximation $\tilde{R}_x = \tilde{R}_y = \bar{R}$ and equations for deviations from periodic solution, where coefficients

$$\xi_x'' + \xi_x a_1 + \xi_y a_o = 0$$

$$\xi_y'' + \xi_y a_1 + \xi_x a_o = 0$$

$$a_o = \frac{P^2}{2\bar{R}^2} \qquad a_1 = a_2 = \frac{\mu_o^2}{L^2} + 3\frac{э_x^2}{\bar{R}^4} + \frac{P^2}{2\bar{R}^2}$$

Taking into account expression for phase advances (depressed and undepressed), as well as expression for unnormalized beam emittance, we get equations for oscillations of two envelope modes

$$\mu^2 = \mu_o^2 - P^2\left(\frac{L}{R}\right)^2$$

$$э = \frac{\mu R^2}{L}$$

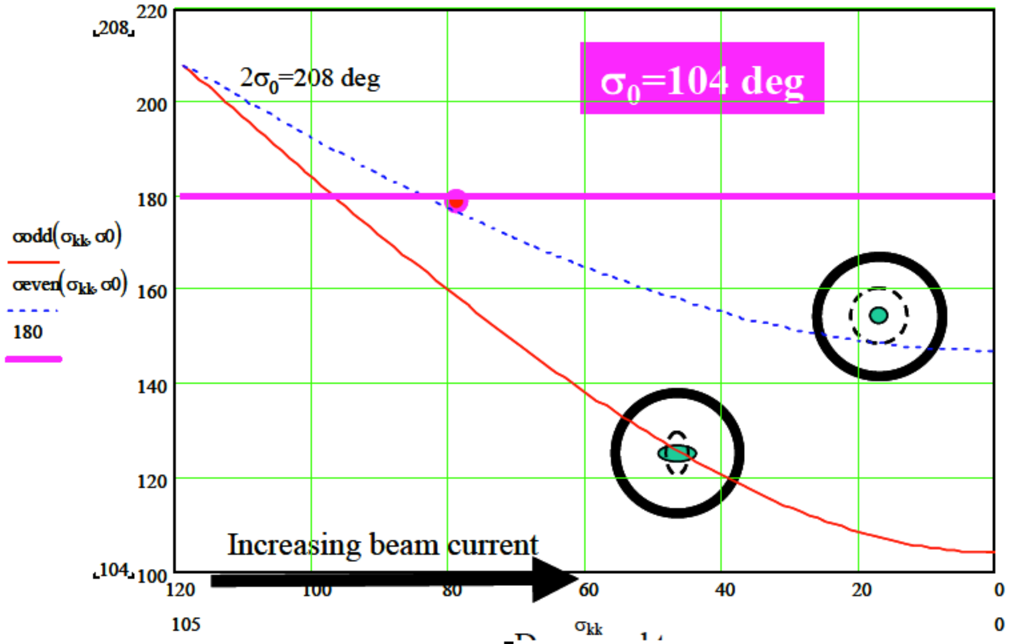
Symmetric envelope mode

$$(\xi_x + \xi_y)'' + \frac{\sigma_{even}^2}{L^2}(\xi_x + \xi_y) = 0 \qquad \sigma_{even} = \sqrt{2(\mu_o^2 + \mu^2)}$$

Anti-symmetric envelope mode

$$(\xi_x - \xi_y)'' + \frac{\sigma_{odd}^2}{L^2}(\xi_x - \xi_y) = 0 \qquad \sigma_{odd} = \sqrt{\mu_o^2 + 3\mu^2}$$

Envelope Instability: $\sigma_{even} = 180^o$ or $\sigma_{odd} = 180^o$

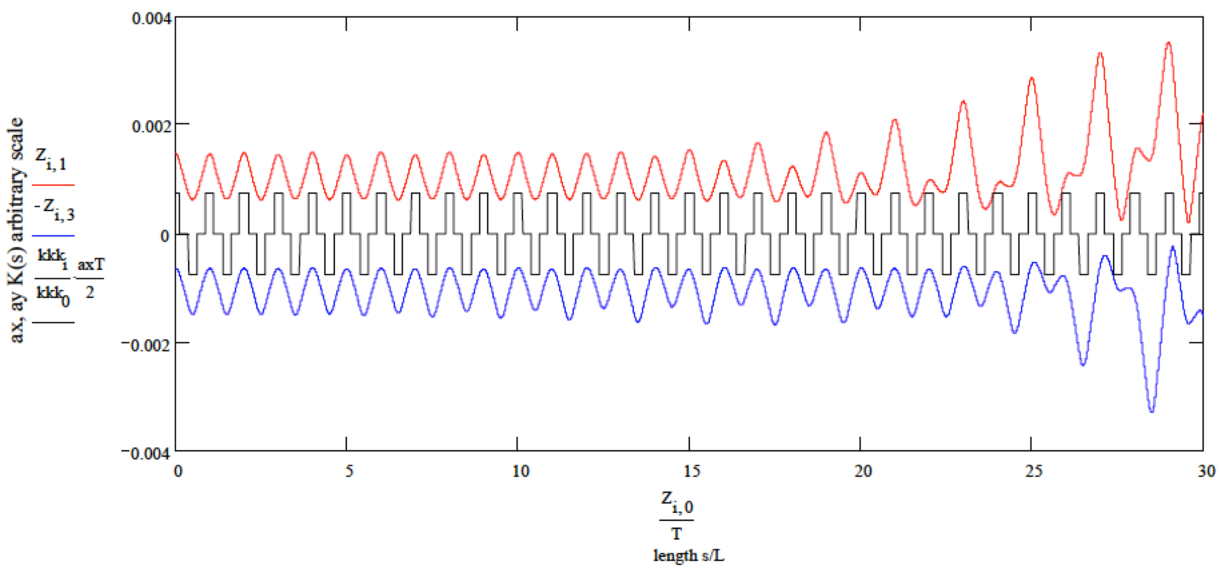


$$\sigma_{even} = 2\mu_o\sqrt{\frac{1}{2}+\frac{1}{2}(\frac{\mu}{\mu_o})^2} < 2\mu_o$$

$$\sigma_{odd} = 2\mu_o\sqrt{\frac{1}{4}+\frac{3}{4}(\frac{\mu}{\mu_o})^2} < 2\mu_o$$

No instability for $\mu_o < 90^o$

$\sigma 0 \cdot \frac{180}{\pi} = 104.07954$ $\mu \cdot \frac{180}{\pi} = 69.62981$ $\sqrt{2\cdot(\sigma 0^2+\mu^2)}\cdot\frac{180}{\pi} = 177.09241$ $\sqrt{\sigma 0^2 + 3\cdot\mu^2}\cdot\frac{180}{\pi} = 159.30311$



Beam envelope instability for $\mu_o = 104^o$, $\mu = 70^o$ (A.Pisent, CERN-2005-004, p.198).

Multipole KV Beam Instability Modes (I.Hofmann, L.Laslett, L.Smith, 1983, I.Hofmann , 1998)

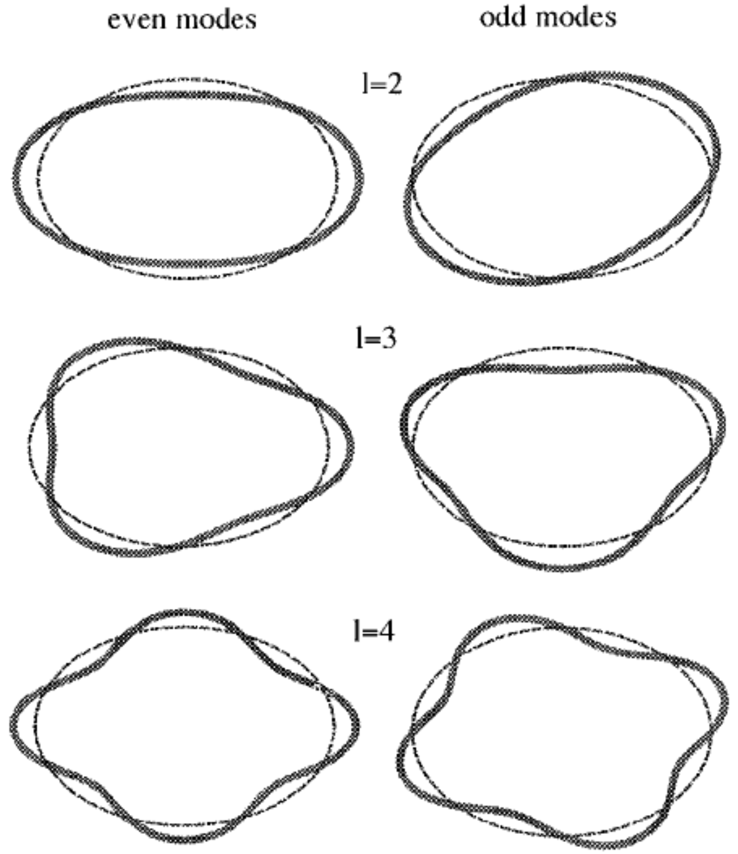


FIG. 1. Beam cross sections for second, third and fourth order even and odd modes (schematic, with x horizontal and y vertical coordinates).

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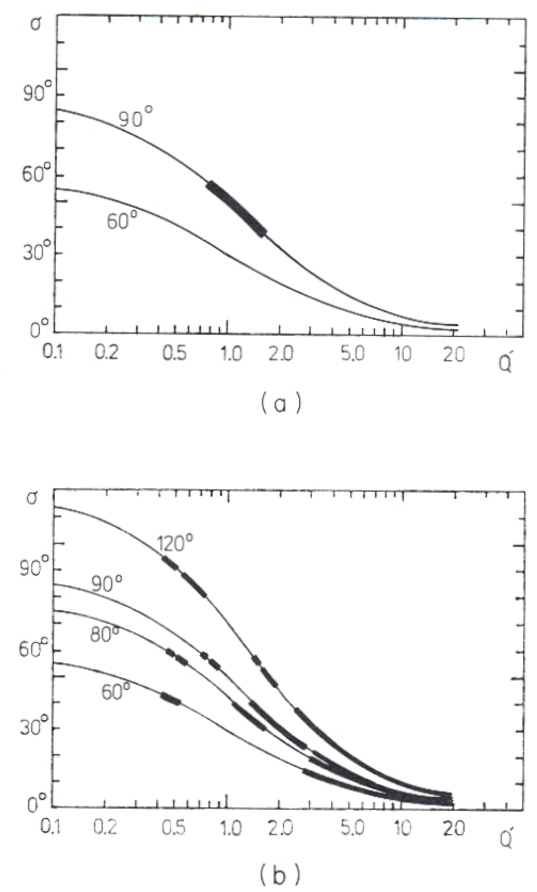
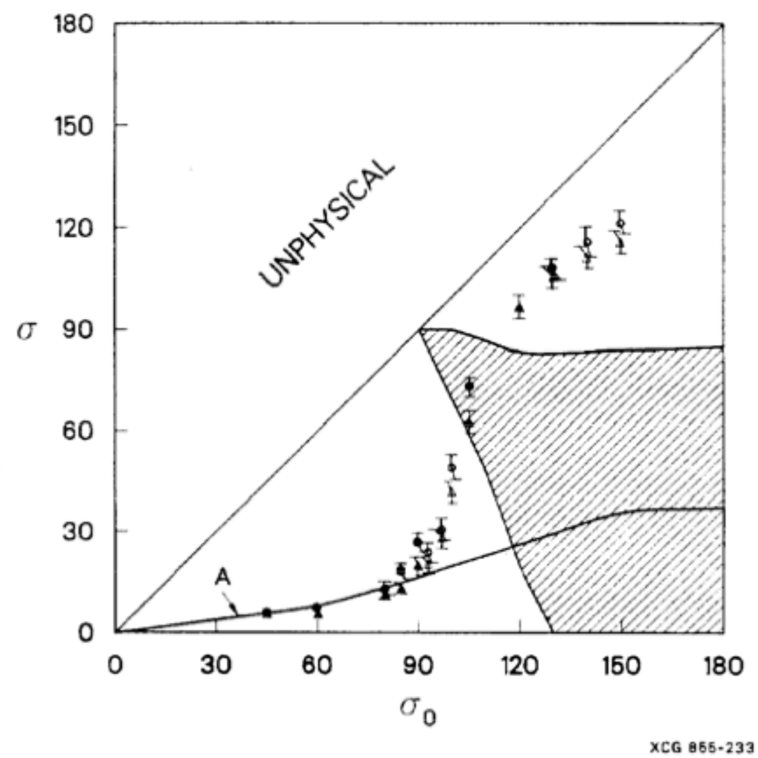


FIG. 19. Instability bands in the phase advance σ for a FODO channel ($\eta = \frac{1}{2}$) and different σ_0: (a) "third-order" modes and (b) "fourth-order" modes.

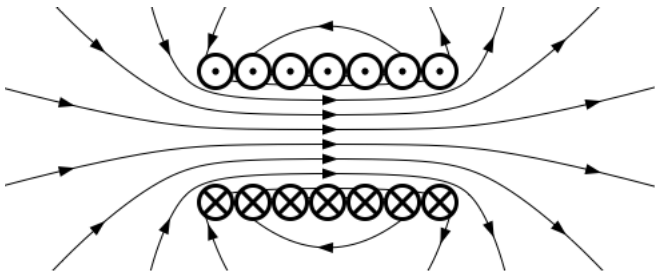
Experiments on Stability of Transport Beam at LBNL (1985) and University of Maryland (1995)

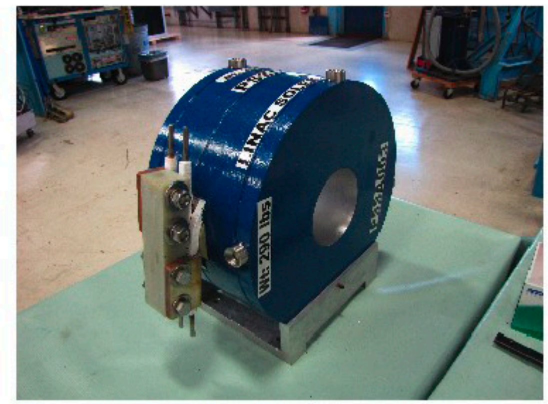
single beam transport channel was constructed at Lawrence Berkeley National Laboratory using 82 electrostatic quadrupole lenses in a FODO configuration, using a cesium beam, as part of the heavy-ion inertial-fusion program. Systematic experiments were conducted by Tiefenback and Keefe, [40] where the beam was matched in both transverse planes, and both σ_0 and σ/σ_0 were varied. The envelope instability predicted by K–V periodic-focusing beam-transport theory for a phase advance per period of $\sigma_0 > 90^\circ$ led to major beam degradation with beam loss. No instability modes predicted by K–V theory, below the $\sigma_0 = 90^\circ$ envelope instability were observed. Similarly, in a systematic experimental study carried out in a solenoid focusing lattice at University of Maryland, [41] the envelope instability was also observed with major beam loss. This was investigated systematically by varying σ/σ_0 and changing σ_0 from below to above 90°. Below 90°, no other instability predicted by the theory, including the third-order (sextupole) mode for $\sigma_0 > 60^\circ$, was found. The conclusion is that for real beams in periodic-focusing channels, the envelope instability predicted by theory for a phase advance per period of $\sigma_0 > 90^\circ$ is the only instability of this theory that leads to emittance growth.



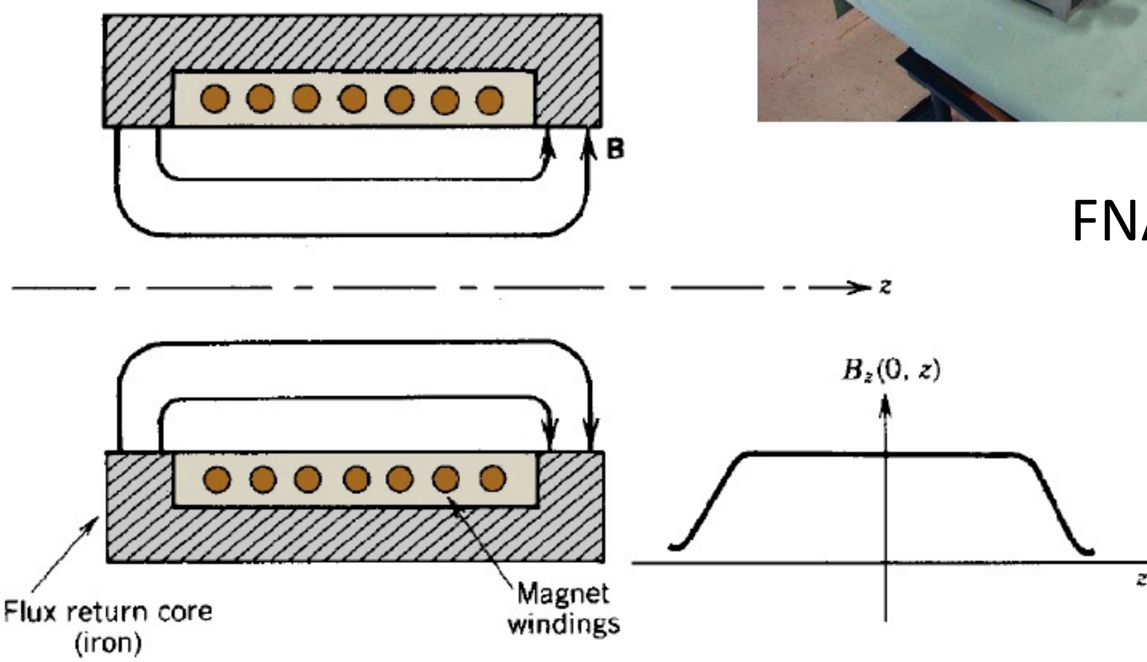
g. 4. Plotted are calculated σ values for stable and apparently stable beams for various σ_0. Filled-in symbols represent beams with the same current and emittance at the beginning and end of the lattice. Hollow symbols mark σ values derived from beams reproducing ϵ and current over at least the last 10 periods, as illustrated in Fig. 3 for σ_0 =100°. Circles mark σ values derived using full beam distribution RMS emittance. Triangles mark calculations using central 95% current of the phase space distribution. The shaded region marks the calculated instability of the envelope equations. Curve A marks the region of equivalent σ attainable at injection with our limited source emittance.

Solenoid Focusing





FNAL Solenoids



Solenoidal magnetic lens (Humphries, 1999).

Dynamics in Axial-Symmetric Magnetic Field

Equation of radial particle motion:

$$\ddot{r} + \frac{qE_z\beta_z}{mc\gamma}\dot{r} - \frac{P_\theta^2}{m^2\gamma^2 r^3} + r\left(\frac{qB_z}{2m\gamma}\right)^2 - \frac{qE_r}{m\gamma} = 0$$

An area of special interest in beam dynamics is an axially-symmetric static field, $E_\theta = 0$, $B_\theta = 0$, which is common in beam transport. In this case, all partial derivatives over the azimuth angle are equal to zero, $\partial/\partial\theta = 0$, and the canonical angular momentum is a constant of motion:

$$P_\theta = m\gamma\, r^2 \frac{d\theta}{dt} + r\, qA_\theta = const\,. \qquad (1.87)$$

The angular component of the vector – potential is given by

$$A_\theta = \frac{\Psi}{2\pi r}\,, \qquad (1.88)$$

where Ψ is the magnetic flux

$$\Psi = \int_o^r B_z\, 2\pi\, r'\, dr'\,. \qquad (1.89)$$

Substitution of Eq. (1.88) into Eq. (1.87) gives:

$$r^2 \frac{d\theta}{dt} + q\frac{\Psi}{2\pi m\gamma} = const. \quad (1.90)$$

If we denote the initial conditions as $\dot{\theta}_o$, r_o, Ψ_o, Eq. (1.90) can be rewritten as

$$\boxed{r^2\dot{\theta} - r_o^2\dot{\theta}_o = -\frac{q}{2\pi m\gamma}(\Psi - \Psi_o)}, \quad (1.91)$$

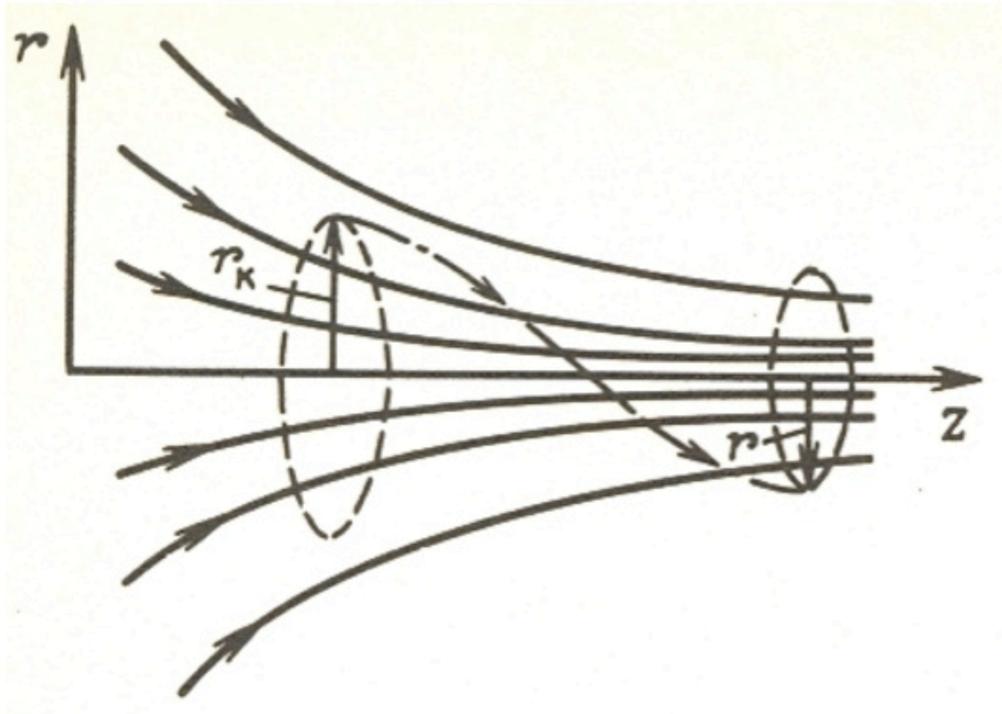
which is known as Busch's theorem. It states that change in angular momentum of a particle in a static magnetic field is defined by the change in magnetic flux comprised by the particle trajectory.

Busch's theorem can be represented as

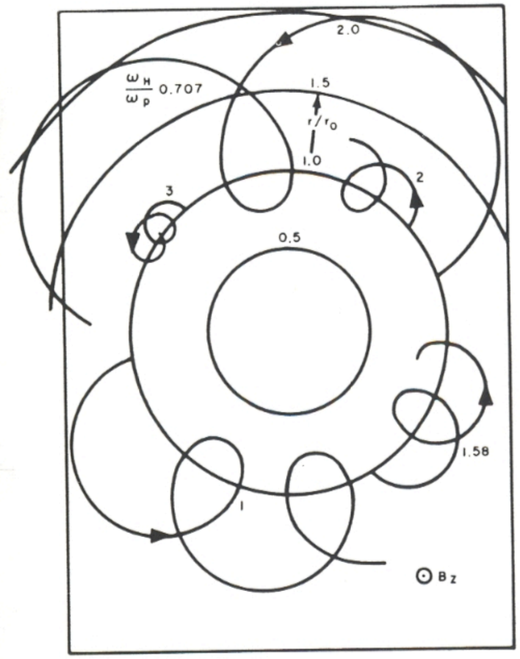
$$\dot{\theta} = \frac{P_\theta}{m\gamma\, r^2} - \omega_L , \quad (1.93)$$

where ω_L is the Larmor frequency of particle oscillations in a longitudinal magnetic field

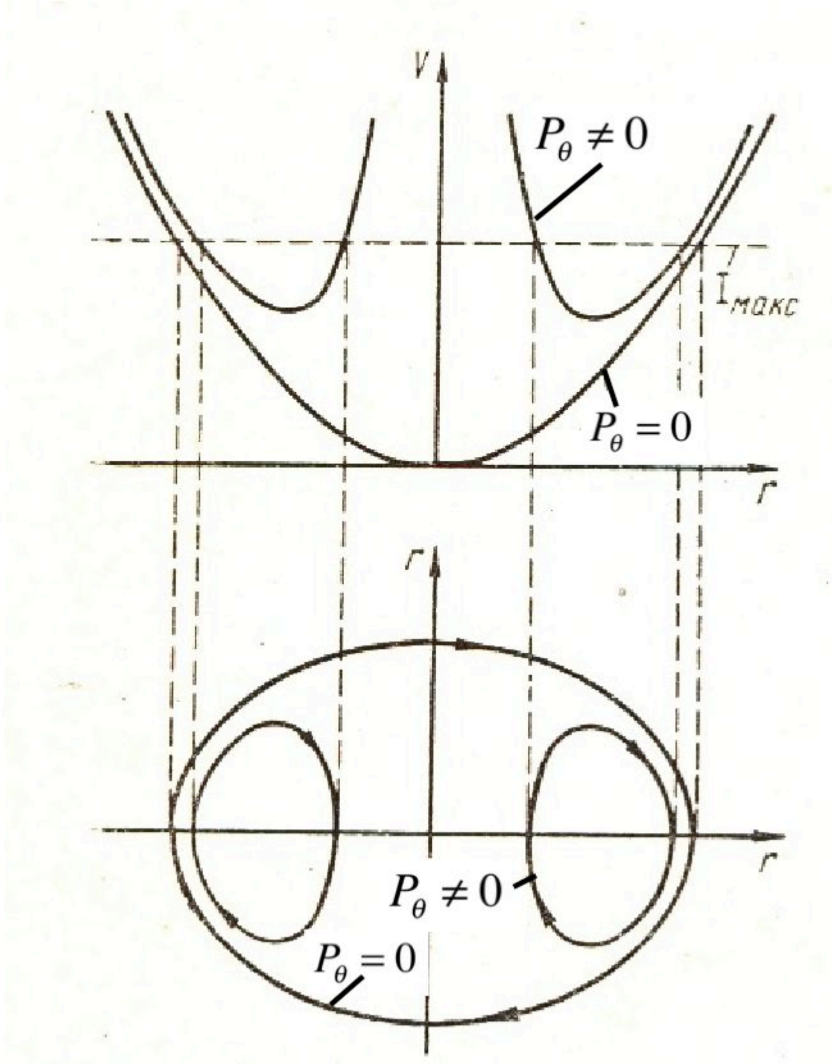
$$\omega_L = \frac{q\,B}{2m\gamma} . \quad (1.94)$$



On Busch's theorem for particle in axial-symmetric magnetic field.



Typical particle trajectories in magnetic field with beam space charge (from G. Brewer, 1967).



Phase space trajectories in magnetic field.

KV Equations in Longitudinal Magnetic Field

Consider the beam propagating in a focusing channel with longitudinal magnetic field $B_z = B(z)$. This field can be created by solenoids or permanent magnets. Like in quadrupole channel, we assume that all particles have the same value of longitudinal velocity β, which is not affected by variation of magnetic field. Vector potential has only azimuthal field component:

$$A_{\theta\ magn} = \frac{1}{2\pi r}\int_o^r B\, 2\pi r'\, dr' = \frac{B\, r}{2}\ . \qquad (2.210)$$

Components of vector potential in Cartesian coordinates are:

$$A_{x\ magn} = - A_{\theta\ magn}\ sin\theta = - B\frac{y}{2}, \qquad (2.211)$$

$$A_{y\ magn} = A_{\theta\ magn}\ cos\theta = B\frac{x}{2}\ . \qquad (2.212)$$

Hamiltonian of particle motion in presence of longitudinal magnetic field is given by

$$K = c\sqrt{m^2c^2 + (P_x + qB\frac{y}{2})^2 + (P_y - qB\frac{x}{2})^2 + (P_z - q\beta\frac{U_b}{c})^2} + qU_b. \qquad (2.213)$$

Taking into account that, $P_z >> q\,\beta U_b/c$ and repeating all derivations, resulted in Eq. (2.27), the Hamiltonian becomes

$$H = \frac{(P_x + qB\frac{y}{2})^2}{2m\gamma} + \frac{(P_y - qB\frac{x}{2})^2}{2m\gamma} + \frac{q\,U_b}{\gamma^2}\ . \qquad (2.214)$$

In longitudinal magnetic field, the canonical - conjugate variables are position and canonical momentum (x, P_x), (y, P_y), where

$$P_x = p_x - qB\frac{y}{2}, \qquad (2.215)$$

$$P_y = p_y + qB\frac{x}{2}. \qquad (2.216)$$

Emittances of the beam have to be defined at the phase planes of canonical variables (x, P_x), (y, P_y), in contrast with quadrupole channel, where canonical variables are (x, p_x), (y, p_y).

Hamiltonian, Eq. (2.214), contains cross term $(xP_y - yP_x)$. Equations of motion in longitudinal magnetic field are coupled: equation in x -direction depends on P_y and that in y -direction depend on P_x. To avoid coupling, let us make a canonical transformation to new variables $\hat{x}$, $\hat{P}_x$, $\hat{y}$, $\hat{P}_y$ according to generating function

$$F_2(x,\hat{P}_x,y,\hat{P}_y,t) = (x\hat{P}_x + y\hat{P}_y)\cos\theta(z) + (x\hat{P}_y - y\hat{P}_x)\sin\theta(z), \qquad \theta(z) = \int_{z_o}^{z}\omega_L(z)dz \qquad (2.217)$$

where $\omega_L(z) = \dfrac{qB_z(z)}{2m\gamma}$ is the Larmor frequency. Transformation from old variables to new variables are given by

$$\hat{x} = x\cos\theta - y\sin\theta, \qquad (2.218)$$

$$\hat{y} = x\sin\theta + y\cos\theta, \qquad (2.219)$$

$$\hat{P}_x = P_x\cos\theta - P_y\sin\theta, \qquad (2.220)$$

$$\hat{P}_y = P_y\cos + P_x\sin\theta. \qquad (2.221)$$

New Hamiltonian, $\widehat{H} = H + \dfrac{\partial F_2}{\partial t}$, is given by

$$\widehat{H} = \frac{\widehat{P}_x^2 + \widehat{P}_y^2}{2m\gamma} + m\gamma\omega_L^2 \frac{(\widehat{x}^2 + \widehat{y}^2)}{2} + q\frac{U_b}{\gamma^2} . \tag{2.222}$$

Hamiltonian, Eq. (2.222), is similar to that for quadrupole channel, Eq. (2.96). Analysis resulted in KV envelope equations, can be applied here as well. Because of the axial symmetry of the beam propagating in magnetic field, there will be only one envelope equation instead of two in quadrupole channel. Repeating the same derivations, which resulted in Eqs. (2.146), (2.147), we can obtain KV envelope equation for round beam in Larmor frame:

$$\widehat{R}'' - \frac{\widehat{\ni}^2}{\widehat{R}^3} + k(z)\widehat{R} - \frac{2I}{I_c \beta^3 \gamma^3 \widehat{R}} = 0 \ , \tag{2.223}$$

where

$$k(z) = \left(\frac{q B(z)}{2mc\,\beta\gamma}\right)^2 \tag{2.224}$$

In KV distribution, particles occupy surface of four-dimensional ellipsoid:

$$F(\hat{x}, \hat{x}', \hat{y}, \hat{y}') = \gamma_o \hat{x}^2 + 2\alpha_o \hat{x}\hat{x}' + \beta_o \hat{x}'^2 + \gamma_o \hat{y}^2 + 2\alpha_o \hat{y}\hat{y}' + \beta_o \hat{y}'^2 - F_o = 0 \, . \qquad (2.225)$$

Here parameters β_o and γ_o are ellipse parameters, not the particle velocity and energy. Projections of the distribution at every phase plane are uniformly populated ellipses:

$$\gamma_o \hat{x}^2 + 2\,\alpha_o \hat{x}\hat{x}' + \beta_o \hat{x}'^2 = \hat{Э} \qquad (2.226)$$

$$\gamma_o \hat{y}^2 + 2\,\alpha_o \hat{y}\hat{y}' + \beta_o \hat{y}'^2 = \hat{Э} \qquad (2.227)$$

where

$$\hat{x}' = \frac{\hat{P}_x}{m\gamma\beta_z c} \qquad (2.228)$$

$$\hat{y}' = \frac{\hat{P}_y}{m\gamma\beta_z c} \qquad (2.229)$$

Substitution of Eqs. (2.218) - (2.221) into Eq. (2.225) gives for the boundary of the four-dimensional ellipsoid occupied by the beam in laboratory frame:

$$F(x, x', y, y') = \gamma_o x^2 + 2\alpha_o x x' + \beta_o x'^2 + \gamma_o y^2 + 2\alpha_o y y' + \beta_o y'^2 - F_o = 0 \qquad (2.230)$$

Boundaries of projections of the four-dimensional beam ellipsoid and of their projections at phase planes are the same both in laboratory frame, and in Larmor frame. From Eqs. (2.218) - (2.221), transformation of phase space elements and area element in real space are

$$d\hat{x}\, d\widehat{P_x} = dx\, dP_x \,, \qquad (2.231)$$

$$d\hat{y}\, d\widehat{P_y} = dy\, dP_y \,, \qquad (2.232)$$

$$d\hat{x}\, d\hat{y} = dx\, dy. \qquad (2.233)$$

Therefore, distribution of particles within projections in both frames are also the same, and uniformly populated ellipses in Larmor frame remain the uniformly populated in laboratory frame. Finally, beam emittance and beam radius are the same in both frames, $э = \hat{э}$, $R = \hat{R}$. Therefore, we can write KV envelope equation in the laboratory frame as well:

$$R'' - \frac{э^2}{R^3} + k(z)\, R - \frac{2I}{I_c \beta^3 \gamma^3 R} = 0 \,. \qquad (2.234)$$

Beam Equilibrium in Magnetic Field

Important case is the beam transport in a constant magnetic field $B(z) = B$, which is a uniform focusing structure. Matched beam corresponds to transport with constant envelope, $R'' = 0$:

$$-\frac{\text{э}^2}{R_e^3} + \left(\frac{qB}{2mc\,\beta\gamma}\right)^2 R_e - \frac{2I}{I_c\,\beta^3\gamma^3 R_e} = 0\,. \qquad (2.235)$$

where R_e is the equilibrium beam radius. Acceptance of the channel, A, and normalized acceptance, ε_{ch}, are obtained from Eq. (2.235) taking the value of beam current $I = 0$, and equilibrium beam radius equal to aperture of the channel, $R_e = a$:

$$A = \omega_L \frac{a^2}{\beta c}\,, \qquad (2.236)$$

$$\varepsilon_{ch} = \frac{qB\,a^2}{2mc}\,. \qquad (2.237)$$

Let us note, that normalize acceptance of the channel with constant longitudinal magnetic field is energy - independent. In the equilibrium, beam envelope does not perform any oscillations and beam occupies the smallest possible area. From Eq. (2.235), the required magnetic field to keep in equilibrium the beam with radius R_e, emittance э, and current I, is

$$B = \frac{2mc\,\beta\gamma}{q R_e}\sqrt{\left(\frac{\text{э}}{R_e}\right)^2 + \frac{2I}{I_c\,\beta^3\gamma^3}}\,. \qquad (2.238)$$

Maximum Transported Beam Current in Uniform Magnetic Field

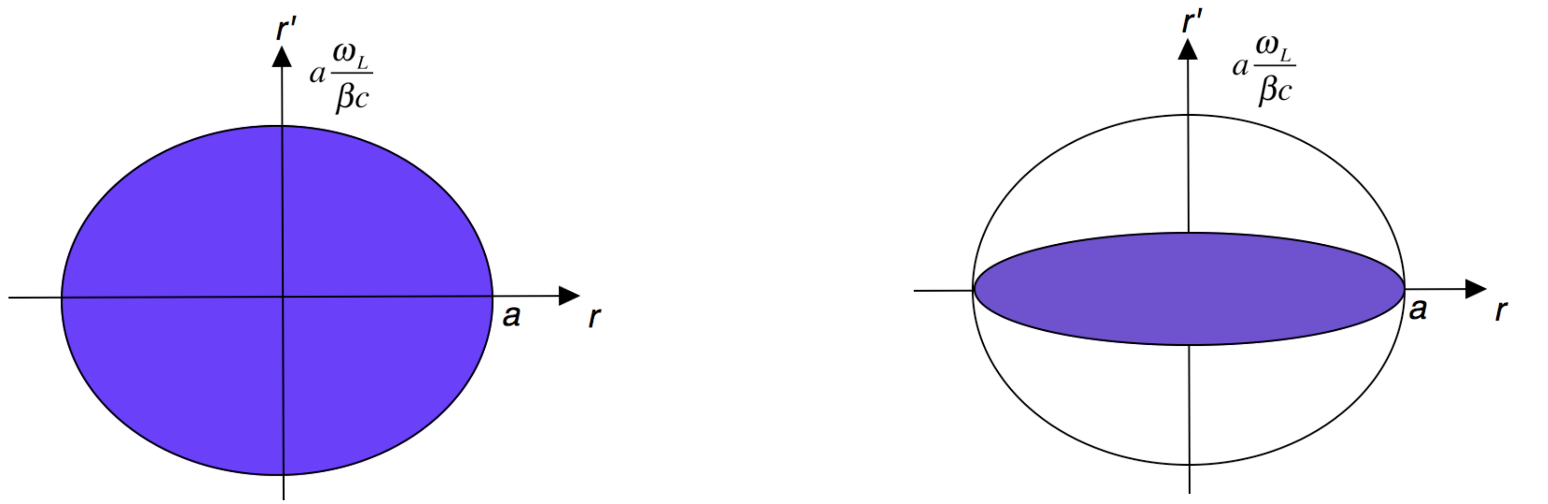
Taking $R_e = a$, and expressing explicitly the value of beam current from the last equation gives for maximum transported beam current:

$$I_{max} = \frac{I_c}{2} (\beta\gamma) \left(\frac{qBa}{2mc}\right)^2 \left(1 - \frac{\text{э}^2}{A^2}\right). \qquad (2.239)$$

Equation (2.239) can be re-written as

$$I_{max} = \frac{I_c}{2} (\beta\gamma) \left(\frac{\varepsilon_{ch}}{a}\right)^2 \left(1 - \frac{\varepsilon^2}{\varepsilon_{ch}^2}\right) . \qquad (2.240)$$

$I = 0$, $\text{э} = A$ $\qquad\qquad$ $I \neq 0$, $\text{э} < A$



Matched beam in uniform magnetic field for zero current mode, and for space charge dominated mode.

Brillouin Flow

Important specific case is the equilibrium of the beam with negligible emittance э = 0, which is called the Brillouin flow:

$$BR_e = 2\sqrt{2}\,\frac{mc}{q}\sqrt{\frac{I}{\beta\gamma\, I_c}}\ . \qquad (2.241)$$

As far as beam with zero emittance cannot be achieved when particle source is inserted in magnetic field, Brillouin flow is realized for the beam born outside magnetic field. If particles are born with zero beam emittance, the transverse mechanical momentum of all particles at the source are equal to zero. Due to conservation of azimuthal canonical particle momentum, all particles obtain azimuthal rotation after entering magnetic field

$$p_\theta = -\, q\frac{B_z\, r}{2}\ , \qquad \text{or} \qquad \dot{\theta} = -\,\omega_L\ . \qquad (2.242)$$

Oscillations Around Equilibrium Radius

Realistic beams usually are not in equilibrium with focusing magnetic field. Consider small deviation of beam radius from equilibrium condition, $R = R_e + \xi$, where $\xi << R_e$. In this case

$$\frac{1}{R} \approx \frac{1}{R_e}(1 - \frac{\xi}{R_e}), \tag{2.243}$$

$$\frac{1}{R^3} \approx \frac{1}{R_e^3}(1 - 3\frac{\xi}{R_e}) . \tag{2.244}$$

Then, envelope equation becomes

$$\frac{d^2\xi}{dz^2} - \frac{\epsilon^2}{R_e^3}(1 - 3\frac{\xi}{R_e}) + (\frac{\omega_L}{\beta c})^2 (R_e + \xi) - \frac{2I}{I_c \beta^3 \gamma^3 R_e}(1 - \frac{\xi}{R_e}) = 0. \tag{2.245}$$

Taking into account equilibrium condition, Eq (2.235), the equation for small deviation of the beam from equilibrium is

$$\frac{d^2\xi}{dz^2} + 3\frac{\epsilon^2}{R_e^4}\xi + (\frac{\omega_L}{\beta c})^2 \xi + \frac{2I}{I_c \beta^3 \gamma^3 R_e^2}\xi = 0 . \tag{2.246}$$

Beam equilibrium condition, Eq. (2.235), can be written as

$$\frac{\epsilon^2}{R_e^4} = (\frac{\omega_L}{\beta c})^2 \frac{1}{1 + b}. \tag{2.247}$$

where b is the dimensionless beam brightness, Eq. (2.159)

$$b = \frac{2}{(\beta\gamma)^3} \frac{I}{I_c} \frac{R_e^2}{\epsilon^2} . \tag{2.248}$$

Last term in Eq. (2.246) can be also expressed through parameter b:

$$\frac{2I}{I_c \beta^3 \gamma^3 R_e^2} \xi = \frac{э^2}{R_e^4} b \xi . \qquad (2.249)$$

Substitution of Eqs. (2.247), (2.249) into Eq. (2.246) gives for small derivation:

$$\frac{d^2 \xi}{dz^2} + 2 \left(\frac{\omega_L}{\beta c}\right)^2 \left(\frac{2+b}{1+b}\right) \xi = 0. \qquad (2.250)$$

Solution of Eq. (2.250) can be written as

$$\xi = \xi_o \cos\left(\sqrt{2\left(\frac{2+b}{1+b}\right)} \frac{\omega_L}{\beta c} z + \Psi_o\right). \qquad (2.251)$$

From Eq. (2.251) it follows that in emittance-dominated regime, $b \to 0$, envelope oscillates with double Larmor frequency:

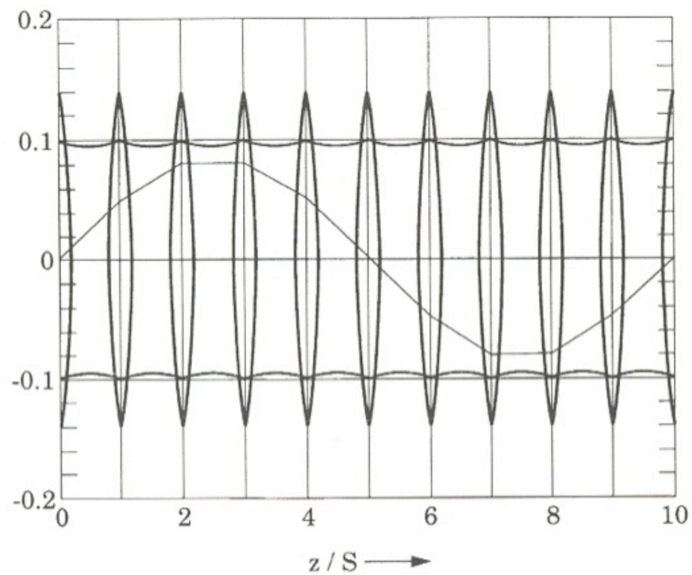
$$\xi = \xi_o \cos\left(2 \frac{\omega_L}{\beta c} z + \Psi_o\right) , \qquad (2.252)$$

while in space-charge dominated regime, $b \to \infty$, frequency of oscillation is $\sqrt{2}$ smaller:

$$\xi = \xi_o \cos\left(\sqrt{2} \frac{\omega_L}{\beta c} z + \Psi_o\right). \qquad (2.253)$$

Beam Transport in Periodic Structure of Axial-Symmetric Lenses

Periodic axial-symmetric magnetic field is often used in focusing of particle beams. Most existing ion Low Energy Beam Transport lines are based on solenoid focusing. Modern accelerator projects utilize superconducting solenoids in combination with superconducting accelerating cavities for acceleration of high-intensity particle beams.



Particle trajectory and matched beam envelope in a periodic thin lens array (Reiser, 1994).

Linear Transfer Matrix of Solenoid (S.Y. Lee, Accelerator Physics, (1999) p.180)

2. *Linear transfer Matrix of a Solenoid:* The particle equation of motion in an ideal solenoidal field is

$$x'' + 2gz' + g'z = 0, \qquad z'' - 2gx' - g'x = 0,$$

where the solenoidal field strength is $g = \frac{eB_{\parallel}(s)}{2p}$.

(a) Show that the coupled equation of motion becomes

$$y'' - j2gy' - jg'y = 0,$$

where $y = x + jz$, and j is the complex imaginary number.

(b) Transforming coordinates into rotating frame with

$$\bar{y} = ye^{-j\theta(s)}, \quad \text{where} \quad \theta = \int_0^s g ds,$$

show that the system is decoupled, and the equation of motion becomes

$$\bar{y}'' + g^2\bar{y} = 0.$$

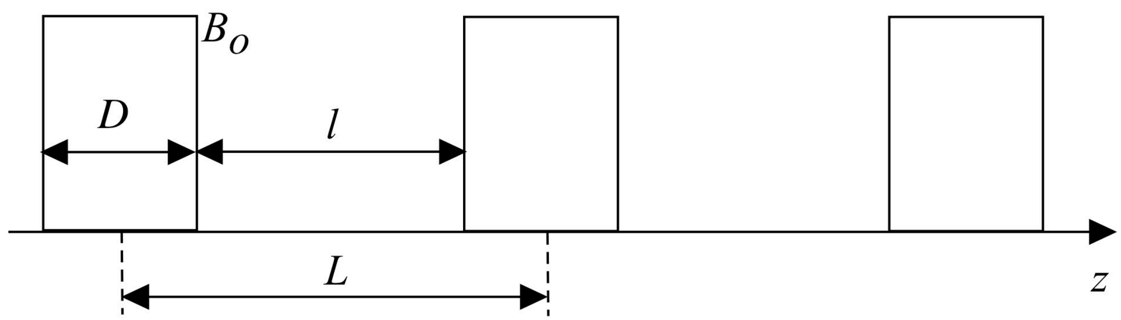
Thus both horizontal and vertical planes are focused by the solenoid.

(c) Show that the transfer matrix in the rotating frame is

$$\tilde{M} = \begin{pmatrix} \cos\theta & \frac{1}{g}\sin\theta & 0 & 0 \\ -g\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \frac{1}{g}\sin\theta \\ 0 & 0 & -g\sin\theta & \cos\theta \end{pmatrix},$$

where $\theta = gs$.[89]

Matrix Method for Periodic Structure of Axial-Symmetric Lenses



Periodic structure of focusing solenoids.

The transformation matrix in a rotating frame through a period of the structure between centers of solenoids

$$\begin{pmatrix} \cos\frac{\theta}{2} & \frac{D}{\theta}\sin\frac{\theta}{2} \\ -\frac{\theta}{D}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} & \frac{D}{\theta}\sin\frac{\theta}{2} \\ -\frac{\theta}{D}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \cos\theta - \frac{l}{2D}\theta\sin\theta & \frac{D}{\theta}\sin\theta + l\cos^2\frac{\theta}{2} \\ -\frac{\theta}{D}\sin\theta + l(\frac{\theta}{D})^2\sin^2\frac{\theta}{2} & \cos\theta - \frac{l}{2D}\theta\sin\theta \end{pmatrix}$$

Rotational angle of particle trajectory in a solenoid

$$\theta = \frac{qB_oD}{2mc\beta\gamma}$$

Matched Beam with *I* = 0 in Periodic Focusing Structure

From the matrices, the value of betatron tune shift per period, μ_o, is determined by $\cos\mu_o = \cos\theta - \theta\sin\theta(L-D)/(2D)$. Adopting the expansions $\cos\xi = 1-\xi^2/2+\xi^4/24$ and $\sin\xi = \xi - \xi^3/6$, the value of betatron tune shift per period reads:

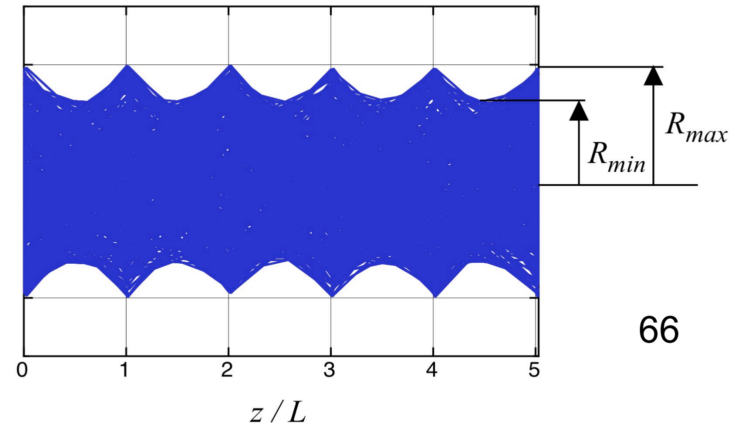
$$\mu_o = \theta\sqrt{\frac{L}{D}}\sqrt{1-\frac{\theta^2}{6}[1-\frac{1}{2}(\frac{D}{L}+\frac{L}{D})]}\,. \qquad (1.4)$$

Thus, the maximum and minimum values of the beta-function $\beta_{max/min} = m_{12}/\sin\mu_o$ in the channel are given by:

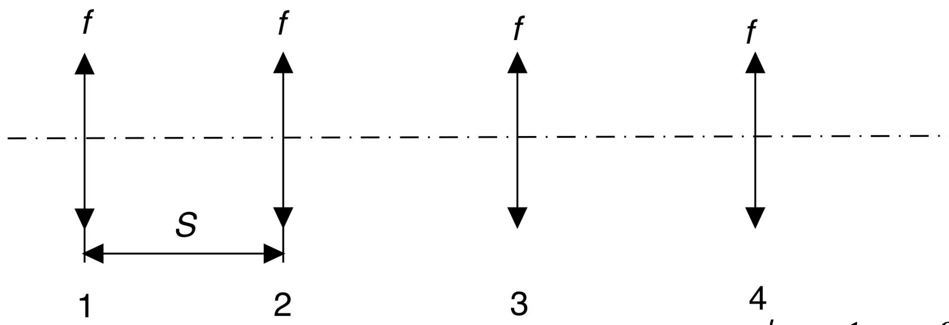
$$\beta_{min} = \frac{(L-D)\cos\theta - \frac{(L-D)^2\theta}{4D}\sin\theta + D\frac{\sin\theta}{\theta}}{\sin\mu_o} \qquad \beta_{max} = \frac{L\cos^2\frac{\theta}{2}[1-\frac{D}{L}(1-\frac{\tan\theta/2}{(\theta/2)})]}{\sin\mu_o} \qquad (1.5)$$

Eqs. (1.5) determine the maximum $R_{max} = \sqrt{\beta_{max}\ni}$ and minimum $R_{min} = \sqrt{\beta_{min}\ni}$ matched envelope of the beam with unnormalized emittance, $\ni$, and negligible beam current, *I* = 0. Acceptance of the channel with aperture radius, *a*, is given by $A = a^2/\beta_{max}$:

$$A = \frac{a^2\sin\mu_o}{L\,[1-\frac{D}{L}(1-\frac{\tan\theta/2}{(\theta/2)})]\cos^2\frac{\theta}{2}}\,. \qquad (1.6)$$



Thin Lens Analysis of Periodic Focusing Circle Lenses



Focal length of thin solenoid lens

$$f = \frac{D}{\theta^2} = \frac{4}{D}\left(\frac{mc\beta\gamma}{qB_o}\right)^2$$

Transformation matrix between lens centers:

$$M_2 M_S M_1 = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{S}{2f} & S \\ -\frac{1}{f}+\frac{S}{4f^2} & 1-\frac{S}{2f} \end{pmatrix}$$

Phase advance per period

$$\cos\mu_o \approx 1-\frac{\mu_o^2}{2} = \frac{m_{11}+m_{22}}{2} = 1-\frac{S}{2f}$$

$$\mu_o = \sqrt{\frac{S}{f}}$$

Max value of beta-function

$$\beta_{\max} = \frac{m_{12}}{\sin\mu_o}$$

$$\beta_{\max} = \frac{S}{\sin\mu_o}$$

Transformation matrix between drift centers

$$M_{\frac{S}{2}} M_f M_{\frac{S}{2}} = \begin{pmatrix} 1 & \frac{S}{2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{S}{2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{S}{2f} & \frac{S}{2}(2-\frac{S}{2f}) \\ -\frac{1}{f} & 1-\frac{S}{2f} \end{pmatrix}$$

Min value of beta-function

$$\beta_{\min} = \frac{m_{12}}{\sin\mu_o} \qquad \beta_{\min} = \frac{S}{\sin\mu_o}\left(1-\frac{S}{4f}\right)$$

$$\beta_{\min} = \frac{S}{\sin\mu_o}\left(1-\frac{\mu_o^2}{4}\right)$$

Acceptance and Stability Criteria

Acceptance of the channel $A = \dfrac{a^2}{\beta_{\max}}$ $\boxed{A = \dfrac{a^2}{S}\sin\mu_o}$

Maximum acceptance $\boxed{A_{\max} = \dfrac{a^2}{S}}$ $\mu_o = \dfrac{\pi}{2}$ $\cos\mu_o = 0$ $\boxed{S = 2f}$

Single particle stability criteria: $|\cos\mu_o| \leq 1$ $0 \leq S \leq 4f$ $\boxed{\dfrac{f}{S} \geq \dfrac{1}{4}}$

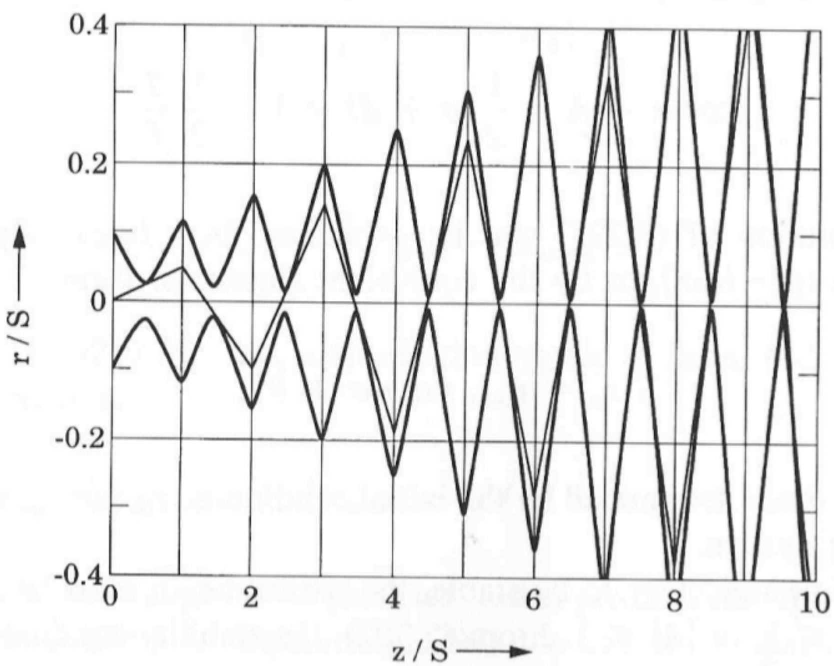


Figure 3.25. Particle trajectory and beam envelope in a periodic thin-lens array with focal length $f = 0.246S$, slightly below the stability threshold ($f = 0.25S$). The particle motion is unstable in this case.

Matching of the Beam with Negligible Current

Max beam radius

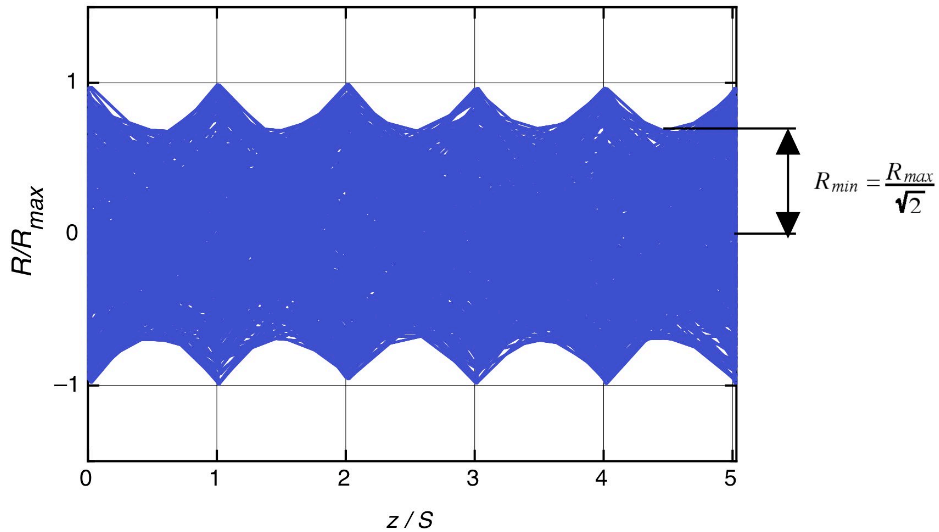
$$R_{\max} = \sqrt{\beta_{\max}\epsilon} \qquad R_{\max} = \sqrt{\frac{\epsilon S}{\sin \mu_o}}$$

$$R_{\max} = \sqrt{\epsilon S}\,\left[\frac{4(\frac{f}{S})^2}{4(\frac{f}{S})-1}\right]^{1/4}$$

Min beam radius

$$R_{\min} = \sqrt{\beta_{\min}\epsilon}$$

$$R_{\min} = R_{\max}\sqrt{1-\frac{S}{4f}}$$



For max acceptance $S = 2f$

$$\frac{R_{\max}}{R_{\min}} = \sqrt{2}$$

Matched beam with zero current in periodic structure of axial-symmetric lenses.

Dynamics of Space-Charge Dominated Beam in Periodic Solenoid Structure

Envelope Equation

$$\frac{d^2R}{dt^2} + \omega_L^2 R - \frac{\text{э}^2(\beta c)^2}{R^3} - \frac{2I\,c^2}{I_c R\beta\gamma^3} = 0$$

Fourier Expansion of Magnetics Field

$$B^2(z) = B_o^2\left[\frac{D}{L} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{\pi n D}{L}\right)\cos\left(\frac{2\pi n z}{L}\right)\right]$$

Envelope Equation with Expansion of Magnetic Field

$$\frac{d^2R}{dt^2} = -\frac{R}{2\pi}\left(\frac{qB_o}{m\gamma}\right)^2\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{\pi n D}{L}\right)\cos\left(\frac{2\pi n\beta c t}{L}\right) - \frac{RD}{4L}\left(\frac{qB_o}{m\gamma}\right)^2 + \frac{\text{э}^2(\beta c)^2}{R^3} + \frac{2I\,c^2}{I_c R\beta\gamma^3}$$

Envelope equation describes oscillations in combination of fast oscillating field and slow varying field:

$$\ddot{R} = \sum_{n=1}^{\infty} F_n(R)\cos(\omega_n t) - \frac{\partial U(R)}{\partial R}, \tag{3.4}$$

where

$$F_n(R) = -\frac{R}{2\pi n}\left(\frac{qB_o}{m\gamma}\right)^2 \sin\left(\pi n\frac{D}{L}\right), \qquad \omega_n = \frac{2\pi n\beta c}{L}, \tag{3.5}$$

$$\frac{\partial U(R)}{\partial R} = \frac{RD}{4L}\left(\frac{qB_o}{m\gamma}\right)^2 - \frac{\text{э}^2(\beta c)^2}{R^3} - \frac{2I\,c^2}{I_c R\,\beta\gamma^3} \tag{3.6}$$

According to the averaging method, such motion can be approximated by combination of slow variable $R_{aver}(t)$ and small amplitude fast oscillations $\xi(t)$:

$$R(t) = R_{aver}(t) + \xi(t) \tag{3.7}$$

Averaging method gives the same value for betatron tune shift as matrix method. Eq. (3.4) gives the equation for slow envelope variable

$$\frac{d^2 R_{aver}}{dz^2} - \frac{э^2}{R_{aver}^3} + \frac{\mu_o^2}{L^2} R_{aver} - \frac{P^2}{R_{aver}} = 0 \tag{3.18}$$

Fast oscillation component of the beam envelope is determined by

$$\xi(z) \approx -\frac{q}{m\gamma}\frac{F_1(R_{aver})}{\omega_1^2}\cos\omega_1 t = R_{aver}\frac{\theta^2}{2\pi^3}(\frac{L}{D})^2 \sin(\pi\frac{D}{L})\cos(2\pi\frac{z}{L}) \tag{3.20}$$

Finally, solution of envelope equation can be expressed as

$$R(z) = R_{aver}(z)(1 + \vartheta_{\max}\cos 2\pi\frac{z}{L}), \qquad \upsilon_{\max} = \frac{\theta^2}{2\pi^3}(\frac{L}{D})^2 \sin(\pi\frac{D}{L}) \tag{3.21}$$

Matched beam corresponds to constant value of average beam envelope $R_{aver}(z) = \bar{R}_{aver}$ and can be determined from envelope equation assuming $R_{aver}''(z) = 0$:

$$\bar{R}_{aver} = \bar{R}_{aver}(0)\sqrt{b_o + \sqrt{1 + b_o^2}} \tag{3.22}$$

where $\bar{R}_{aver}(0)$ is the matched average beam size with negligible space charge,

$$\bar{R}_{aver}(0) = \sqrt{\frac{э\, L}{\mu_o}} \tag{3.23}$$

and b_o is the space charge parameter:

$$b_o = \frac{1}{(\beta\gamma)^3}\frac{I}{I_c}(\frac{\bar{R}_{aver}(0)}{э})^2 \tag{3.24}$$

The minimum and maximum matched beam envelope in presence of space charge forces are given by:

$$R_{max/min} = \bar{R}_{aver}(1 \pm \vartheta_{max}), \tag{3.25}$$

Maximum beam current is achieved when maximum beam size is equal to aperture of the channel $R_{max} = a$, which is determined from Eqs. (3.22) - (3.25) as

$$a = \sqrt{\frac{э L}{\mu_o}}\sqrt{b_o + \sqrt{1+b_o^2}}\,(1+\upsilon_{max}) \tag{3.26}$$

For negligible beam intensity, $b_o = 0$, Eq. (3.26) determines the beam with maximum possible emittance (acceptance of the channel) approximated by envelope equation $э = A_{env}$:

$$a = \sqrt{\frac{A_{env} L}{\mu_o}}(1+\upsilon_{max}) \tag{3.27}$$

Aapproximation to acceptance of the channel

$$A_{env} = \frac{a^2 \mu_o}{L(1+\vartheta_{max})^2} \quad , \tag{3.28}$$

The maximum beam current is:

$$I_{max} = \frac{I_c}{2}\frac{\mu_o}{L}A_{env}(\beta\gamma)^3[1 - (\frac{э}{A_{env}})^2]. \tag{3.29}$$

Applicability of Smooth Approximation to Beam Envelope

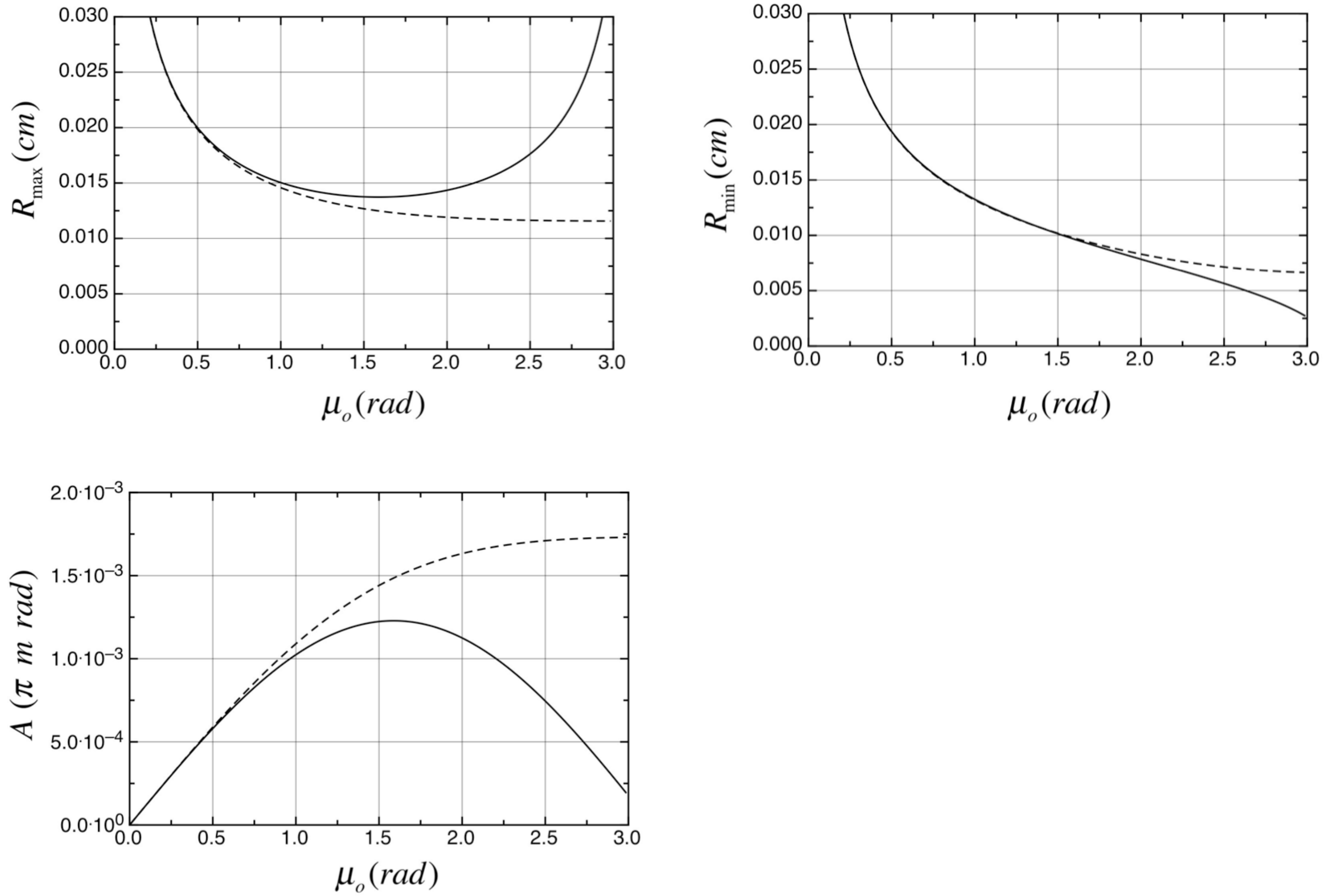


Fig. 3. Minimum and maximum beam sizes in periodic solenoid structure with $D/L = 0.034$: (sol line) solution from matrix analysis, (dotted line) smooth approximation to beam envelope.

Smooth Approximation in Space-Charge Dominated Regime

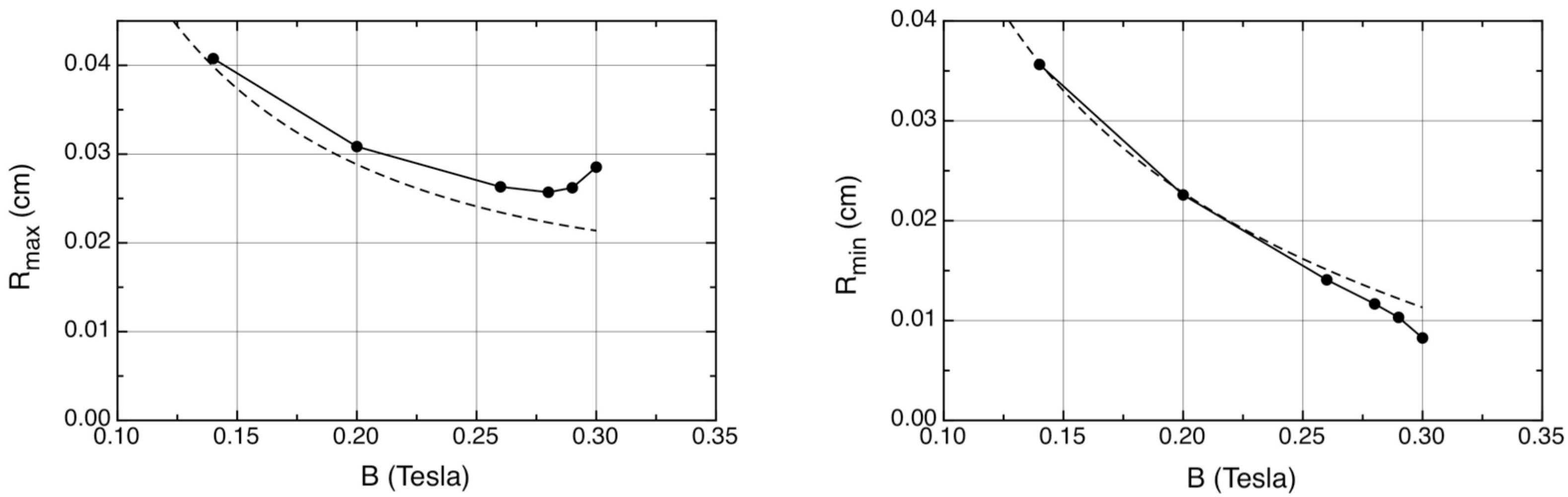
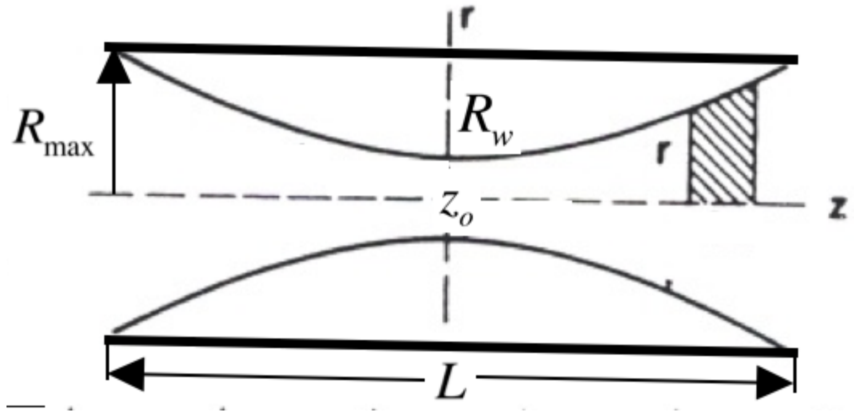


Fig. 4. Minimum and maximum beam sizes in periodic solenoid structure with D/L = 0.034 f space-charge dominated proton beam with energy of W = 35 keV, beam current I = 3.5 mA aı beam emittance э $= 9.262\ \pi\ cm\ mrad$: (solid line) exact solution of envelope equation, Eq. (3.1 (dotted line) smooth approximation to beam envelope, Eq. (3.25).

Maximum Transported Beam Current



On maximum current transported through the tube

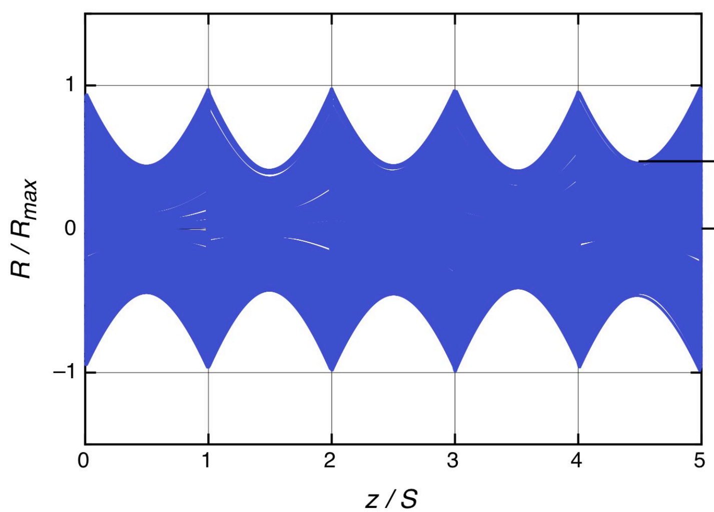
Maximum beam current

$$I_{lim} = 1.17 I_c (\beta\gamma)^3 (\frac{R_{max}}{L})^2$$

Beam slope after lens

$$\frac{dR}{dz} = \sqrt{\frac{4 I_{lim}}{I_c (\beta\gamma)^3} \ln(\frac{R_{max}}{R_w})} \approx 2 \frac{R_{max}}{L}$$

Required focal length $f \approx \frac{L}{4}$



$$\frac{R_{min}}{R_{max}} = \frac{1}{2.35}$$

Matched beam with maximum current in periodic structure of axial-symmetric lenses.

Stationary beam equilibrium in linear focusing channel

In general case, the Hamiltonian is not a constant of motion, because potentials can depend on time, $\vec{A} = \vec{A}(t)$, $U = U(t)$. Note that even if the potentials of the external field, $\vec{A}_{ext}$, U_{ext}, are time-independent, the beam field potentials, $\vec{A}_b$, U_b, might still depend on time, and the Hamiltonian remains time-dependent. If an additional condition of matching the beam with the channel (where the beam distribution remains stationary) is applied, explicit dependence on time disappears from the beam potentials. In this case, the Hamiltonian becomes time-independent, and therefore, is an integral of motion. The Hamiltonian, can then be used to find the unknown distribution function of the beam via the expression $f = f(H)$ and the subsequent solution of equation for space charge potential (Kapchinsky, 1985).

Hamiltonian corresponding to the motion in averaged linear focusing field is given by

$$H = \frac{p_x^2 + p_y^2}{2\,m\,\gamma} + \frac{m\gamma\Omega_r^2}{2}(x^2 + y^2) + q\frac{U_b}{\gamma^2}, \qquad (4.26)$$

where Ω_r is the frequency of smoothed particle oscillations. If the beam is matched with the continuous channel, space charge potential U_b is constant, and Hamiltonian is a constant of motion.

Let us transform Hamiltonian, Eq. (4.26), to another one, multiplying Eq. (4.26) by a constant:

$$K = \frac{L^2}{m\gamma\,(\beta c)^2} H \tag{4.39}$$

It corresponds to changing of independent time variable t for dimensionless time $\tau = t\beta c / L$. New Hamiltonian is given by

$$K = \frac{\dot{x}^2 + \dot{y}^2}{2} + \frac{\mu_o^2}{2}(x^2 + y^2) + \frac{q\,L^2\,U_b}{m\,c^2\,\gamma^3\beta^2}, \tag{4.40}$$

where $\dot{x} = dx / d\tau$, $\dot{y} = dy / d\tau$. Let us use particle radius $R^2 = x^2 + y^2$ and total transverse momentum $P^2 = \dot{x}^2 + \dot{y}^2$, where

$$\dot{x} = P\cos\theta, \; \dot{y} = P\sin\theta\,. \tag{4.41}$$

Hamiltonian, Eq. (4.40), is now

$$K = \frac{P^2}{2} + \frac{\mu_o^2}{2} R^2 + \frac{q\,L^2\,U_b}{m\,c^2\,\gamma^3\beta^2} \tag{4.42}$$

Consider the following distribution:

$$f = \begin{cases} f_o, & K \le K_o \\ 0, & K > K_o \end{cases} . \quad (4.43)$$

According to transformation, Eq. (4.41), space charge density of the beam is expressed as

$$\rho(R) = 2\pi\, q f_o \int_o^{P_{max}(R)} P dP = \pi\, q f_o\, P_{max}^2(R). \quad (4.44)$$

For each value of R, the maximum value of transverse momentum P_{max} *(R)* is achieved for $K = K_o$. From Eq. (4.40)

$$P_{max}^2(R) = 2K_o - \mu_o^2 R^2 - \frac{2\, q L^2 U_b}{m c^2 \gamma^3 \beta^2} . \quad (4.45)$$

Therefore, space charge density, Eq. (4.44), is

$$\rho(R) = \pi\, q f_o \left(2K_o - \mu_o^2 R^2 - \frac{2\, q L^2 U_b}{m c^2 \gamma^3 \beta^2}\right) . \quad (4.46)$$

Poisson's equation for unknown space charge potential of the beam U_b is

$$\frac{1}{R}\frac{d}{dR}\left(R \frac{dU_b}{dR}\right) = -\frac{\pi\, q f_o}{\varepsilon_o}\left(2K_o - \mu_o^2 R^2 - \frac{2\, q L^2 U_b}{m c^2 \gamma^3 \beta^2}\right) . \quad (4.47)$$

Let us introduce notation:

$$R_o = \frac{\varepsilon_o m c^2 \beta^2 \gamma^3}{2\pi q^2 f_o L^2}, \qquad s = \frac{R}{R_o}, \qquad (4.48)$$

Then, Poisson's equation, Eq. (4.47) is

$$\frac{1}{s}\frac{d}{ds}(s\frac{dU_b}{ds}) - U_b = \frac{m c^2 \beta^2 \gamma^3}{q L^2}(\frac{\mu_o^2 s^2 R_o^2}{2} - K_o). \qquad (4.49)$$

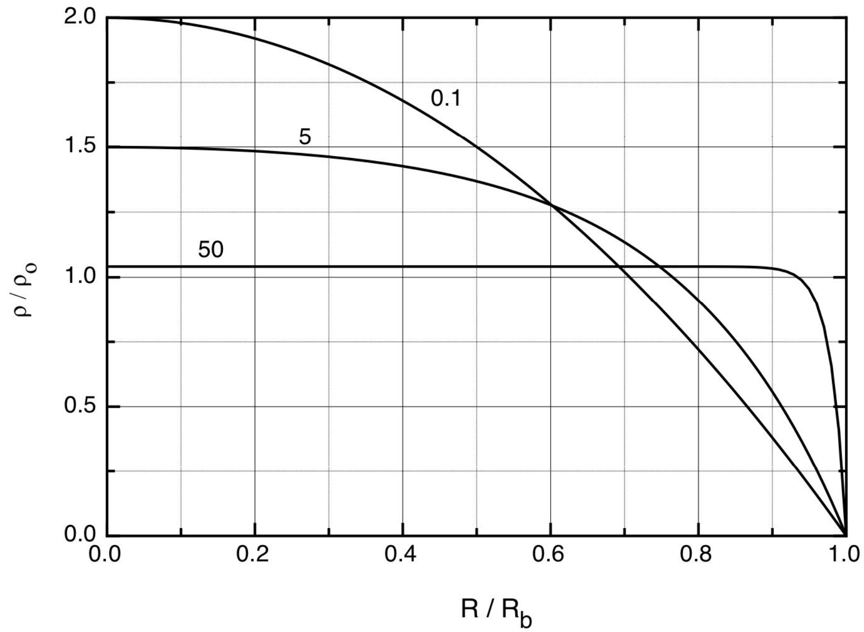
Solution of differential equation (4.49) is a combination of general solution of the homogeneous equation $U_b^{(u)} = C_o I_o(s)$ and of a particular solution of non-homogeneous equation $U_b^{(n)} = C_1 s^2 + C_2$

$$U_b = \frac{m c^2 \beta^2 \gamma^3}{q L^2}[(2\mu_o^2 R_o^2 - K_o)(I_o(s) - 1) - \frac{\mu_o^2 s^2 R_o^2}{2}]. \qquad (4.57)$$

Space charge density profile

$$\rho(s_b \frac{R}{R_b}) = \frac{\rho_o}{[1 - \frac{2I_1(s_b)}{s_b I_o(s_b)}]}[1 - \frac{I_o(s_b \frac{R}{R_b})}{I_o(s_b)}], \qquad (4.74)$$

where the following notation is used: $s_b = \frac{R_b}{R_o}$ (4.60)

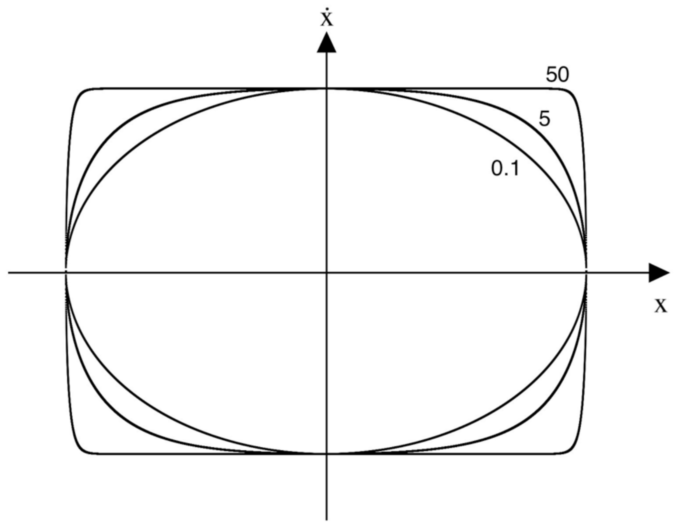


Density profile, Eq. (4.74), for different values of parameter s_b.

Projection of the volume at the phase plane $(x, \dot{x})$:

$$\frac{s_b^2}{4\mu_o^2 R_b^2}\dot{x}^2 + \frac{1}{I_o(s_b)} I_o(s_b \frac{x}{R_b}) = 1 \quad . \qquad (4.65)$$

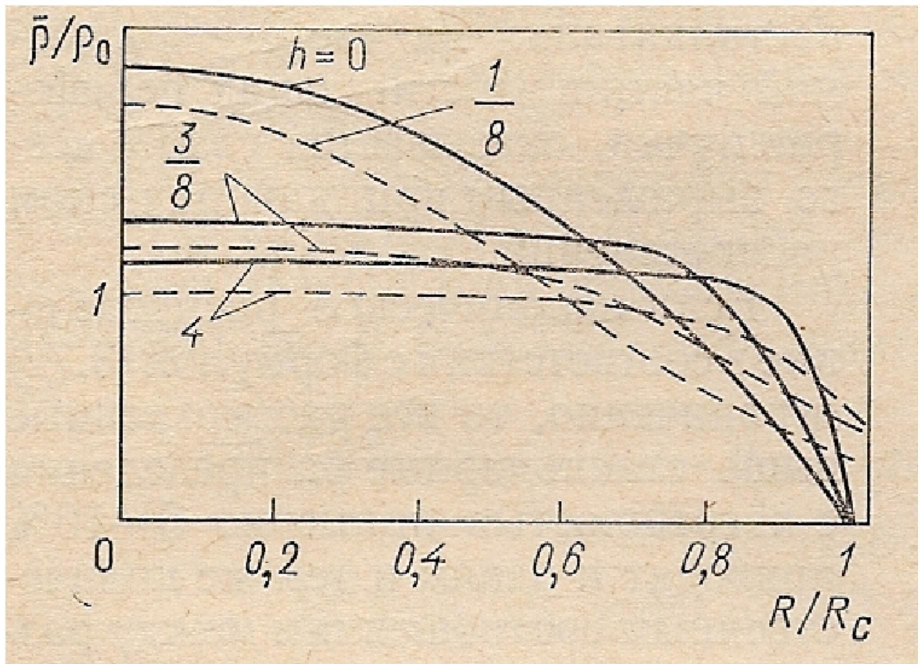
Eq. (4.65) describes the boundary of phase space of the beam at the plane $(x, \dot{x})$.



Boundary phase space trajectories of particles, Eq. (4.65), for different values of parameter s_b.

Similar results can be obtained for another distribution function

$$f = f_o \exp(-\frac{H}{H_o})$$



Space charge density for different distributions:

(solid) $f = f_o,\ H \le H_o$

(dotted) $f = f_o \exp(-H / H_o)$

Performed anlysis shows, that for small values of space charge forces, particle phase space trajectories are close to elliptical, and beam profile density is essentially nonlinear. With increase of space charge forces, boundary particle trajectories become more close to rectangular, and density beam profile becomes more uniform. In space charge dominated regime, stationary beam profile tend to be uniform, and space charge field of the beam compensates for external field.

Non-Uniform Beam Equilibrium

Non-uniform beam in general case is intrinsically mismatched with linear focusing channel. Meanwhile, it is possible to find matched solution for non-uniform beam without emittance growth, if we refuse from linearity of focusing field.

Assume that the beam is propagating in continuously focusing channel, and is matched with the channel. Hence, the Hamiltonian is a constant of motion:

$$H = \frac{p_x^2 + p_y^2}{2m\gamma} + q\,U(x,y) = const\,. \qquad (4.3)$$

The total potential of the structure is a combination of the external focusing potential, U_{ext}, and the space charge potential U_b of the beam, $U = U_{ext} + U_b \gamma^{-2}$. The time-independent distribution function of a matched beam obeys Vlasov's equation, where the partial derivative of the distribution function over time is equal to zero due to assumption of a matched beam:

$$\frac{1}{m\gamma}\left(\frac{\partial f}{\partial x} p_x + \frac{\partial f}{\partial y} p_y\right) - q\left(\frac{\partial f}{\partial p_x}\frac{\partial U}{\partial x} + \frac{\partial f}{\partial p_y}\frac{\partial U}{\partial y}\right) = 0\,. \qquad (4.4)$$

Eq. (4.4) can be solved to find the total potential of the structure, U, as a function of beam distribution function $f(x, p_x, y, p_y)$. The distribution function of the beam is supposed to be given. Therefore, the self - potential of the beam U_b is also a known function derived from Poisson's equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U_b}{\partial r}\right) = -\frac{\rho(r)}{\varepsilon_o}, \tag{4.5}$$

Combining solutions of Vlasov's equation for total potential of the structure, U, and space charge potential of the beam, obtained from Poisson's equation, U_b, the external potential of the focusing structure can be found

$$U_{ext} = U - \frac{U_b}{\gamma^2}. \tag{4.6}$$

The solution of this problem is unique for every specific particle distribution.

Consider a z - uniform beam with Gaussian distribution function in four - dimensional phase space:

$$f = f_o \, exp\left(-2 \frac{x^2 + y^2}{R^2} - 2 \frac{p_x^2 + p_y^2}{p_o^2} \right). \tag{4.7}$$

This distribution has an elliptical phase space projection at every phase plane with normalized root-mean-square beam emittance:

$$\varepsilon = \frac{4}{m\,c} \sqrt{<x^2><p_x{}^2> - <xp_x>^2} = R \, \frac{p_o}{m\,c} \, . \tag{4.8}$$

Substituting the distribution function, Eq. (4.7), into Vlasov's equation yields an expression for the total unknown potential of the structure:

$$\frac{mc^2}{q} \frac{1}{\gamma} (x p_x + y p_y) = \frac{R^4}{\varepsilon^2} \left(p_x \frac{\partial U}{\partial x} + p_y \frac{\partial U}{\partial y} \right). \tag{4.9}$$

Vlasov's equation can be separated into two independent parts for x- and y- coordinates respectively:

$$\frac{\partial U}{\partial x} = \frac{mc^2 \varepsilon^2}{\gamma \, q R^4} \, x \, , \qquad \frac{\partial U}{\partial y} = \frac{mc^2 \varepsilon^2}{\gamma \, q R^4} \, y \, . \tag{4.10}$$

Combining solutions of Eq. (4.10), the total potential of the structure is a quadratic function of coordinates which creates linear focusing field E_{tot}:

$$U(x,y) = \frac{mc^2}{q}\frac{1}{\gamma}\frac{\varepsilon^2}{R^4}\left(\frac{x^2+y^2}{2}\right), \qquad (4.11)$$

$$E_{tot} = -\frac{mc^2}{q}\frac{1}{\gamma}\frac{\varepsilon^2}{R^4}\, r\ . \qquad (4.12)$$

The appearance of quadratic terms in the total potential of the structure is quite clear because phase space projections of the beam have elliptical shape which is conserved in linear field. The space charge field of the beam, E_b, is calculated from Poisson's equation using a known space charge density function of the beam ρ_b:

$$\rho_b = \rho_o\ exp\left(-2\frac{r^2}{R^2}\right), \qquad (4.13)$$

$$E_b = -\frac{\partial U_b}{\partial r} = \frac{I}{2\pi\,\varepsilon_o\,\beta\, c}\frac{1}{r}\left[1 - exp\left(-2\frac{r^2}{R^2}\right)\right], \qquad (4.14)$$

where $\rho_o = 2I/(\pi c\beta R^2)$ is the space charge density at the axis.

Subtraction of the space charge field from the total field of the structure gives the expression for the external focusing field of the structure which is required for conservation of the beam:

$$E_{ext} = -\frac{mc^2}{qR\gamma}\left[\frac{\varepsilon^2 r}{R^3} + 2\frac{I}{I_c\beta\gamma}\frac{R}{r}\left(1 - exp\left(-2\frac{r^2}{R^2}\right)\right)\right], \quad (4.15)$$

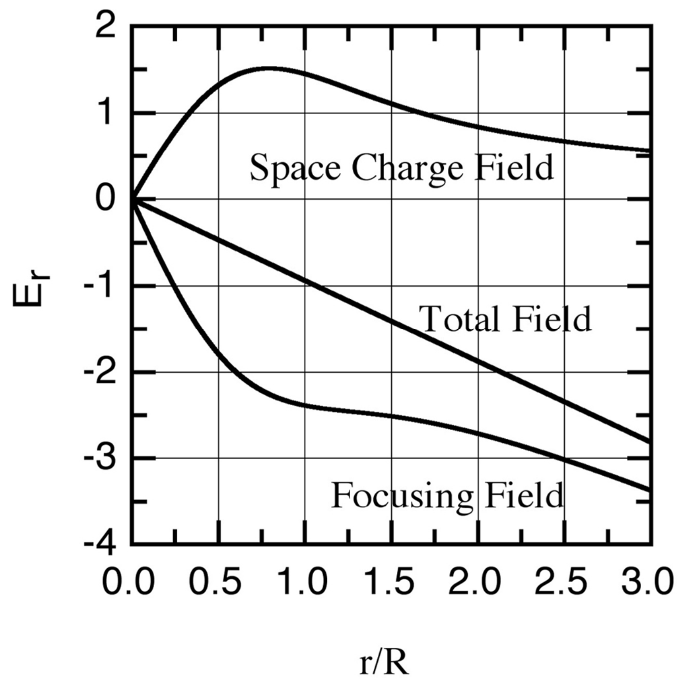
The relevant potential of the focusing field is given by the expression:

$$U_{ext}(r) = \frac{mc^2}{q\gamma}\left[\left(\frac{\varepsilon^2}{2R^4} + \frac{2I}{I_c\beta\gamma R^2}\right)r^2 + \frac{2I}{I_c\beta\gamma}\left(-\frac{r^4}{2R^4} + \frac{2}{9}\frac{r^6}{R^6} + \ldots + \frac{(-1)^{k+1}2^k \; r^{2k}}{2k\; k!\, R^{2k}}\right)\right]. \quad (4.16)$$

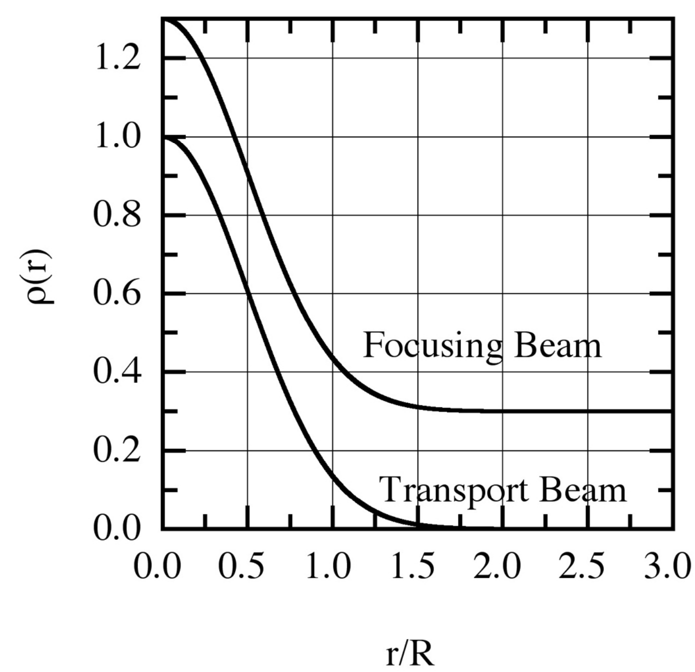
External potential of the structure, E q. (4.16), consists of two parts: quadratic (which produces linear focusing) and the part with higher order terms which describe nonlinear focusing. The linear part depends on the values of beam emittance and on the beam current, while the nonlinear part depends on beam current only. This means that the external field has t o compensate the nonlinearity of self-field of th e beam and produ ce required linear focusing of th e beam to kee p the elliptical beam phase space distribution.

Required potential distributio n can be crea ted by introducing inside the transport channel an opposite charged cloud of particles (plasma lens) with the space charge density:

$$\rho_{ext} = \rho_o \, exp(-2\frac{r^2}{R^2}) + \frac{I_c \, \varepsilon^2}{2\pi c \, R^4}. \qquad (4.17)$$

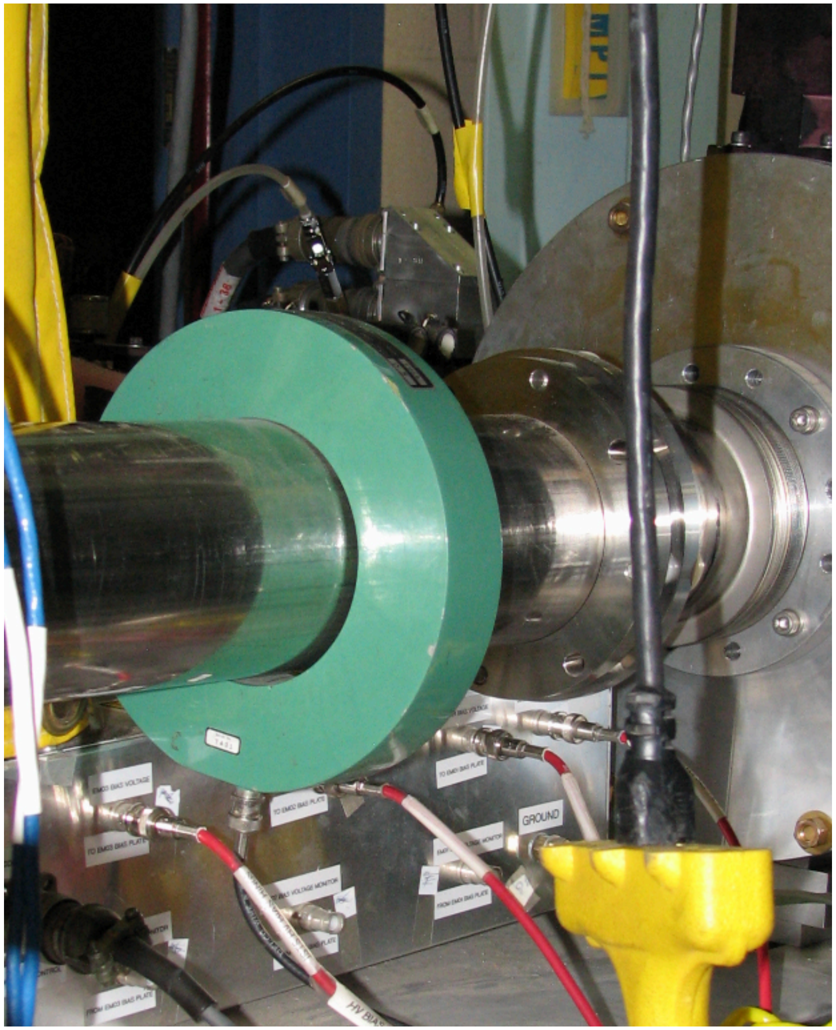


Total field of the structure E_{tot}, required external focusing field E_{ext}, and space-charge field of the Gaussian beam E_b.



Charged particle density of the transported beam with Gaussian distribution, and of the external focusing beam

Beam Current Measurement



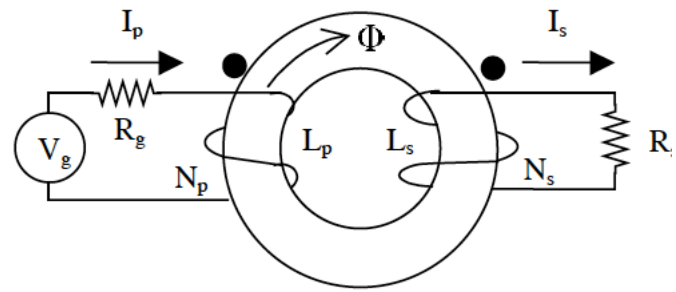


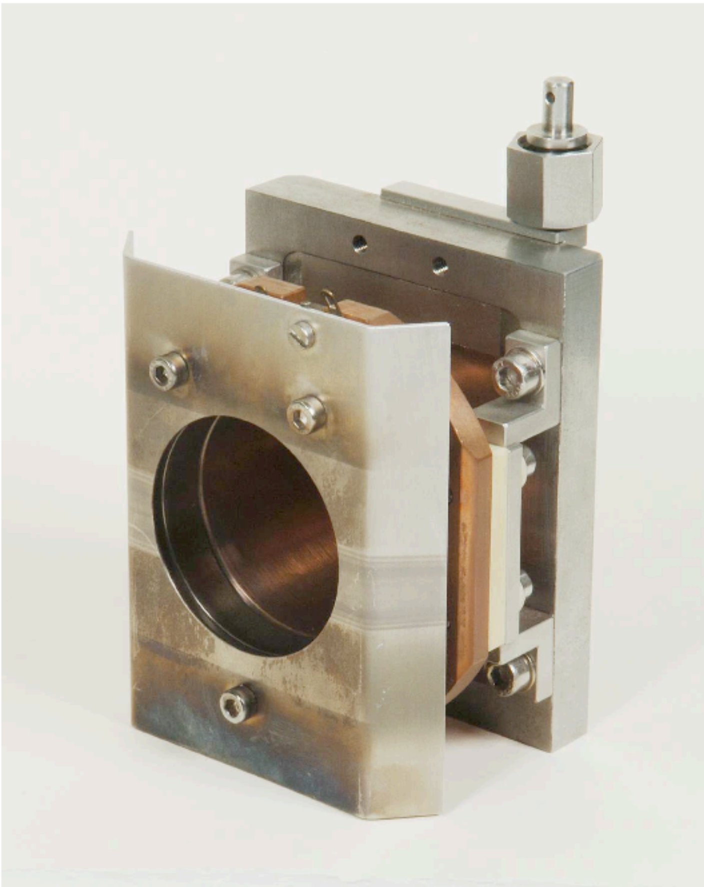
FIGURE 6. Classical transformer circuit.

Torus radii	$r_i = 70$ mm, $r_o = 90$ mm
Torus thickness	$l = 16$ mm
Torus material	Vitrovac 6025: $(CoFe)_{70\%}(MoSiB)_{30\%}$
Torus permeability	$\mu_r \simeq 10^5$ for $f < 100$ kHz, $\mu_r \propto 1/f$ above
Number of windings	10
Sensitivity	4 V/A at $R = 50\ \Omega$, 10^4 V/A with amplifier
Resolution for $S/N = 1$	40 μA_{rms} for full bandwidth
$\tau_{droop} = L/R$	0.2 ms
$\tau_{rise} = \sqrt{L_S C_S}$	1 ns
Bandwidth	2 kHz to 300 MHz

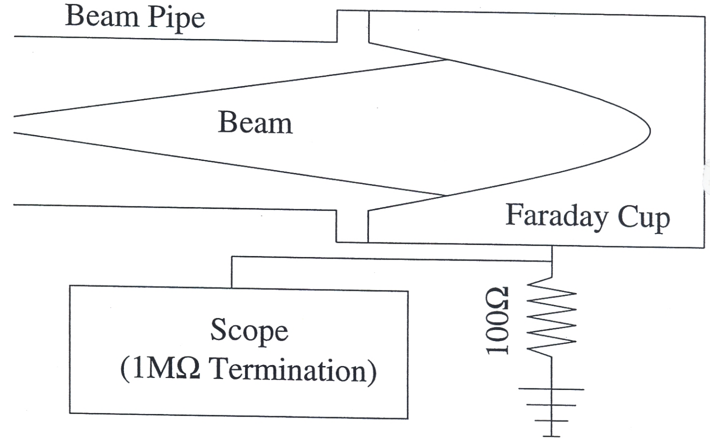
Table 2.1: Some basic specification of the GSI passive transformer.

LANL beam current monitor

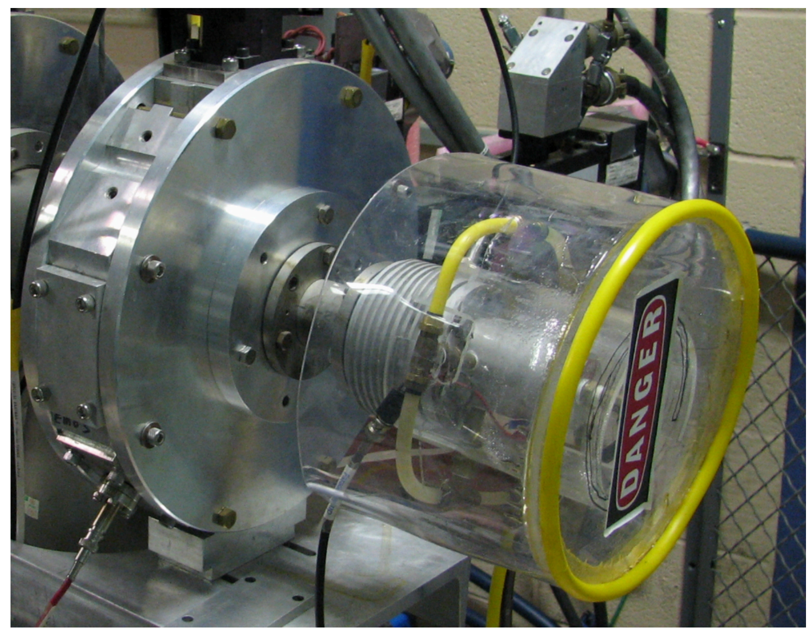
Faraday Cups



Used as a beam stop for low energy beam and as a fast current monitor.

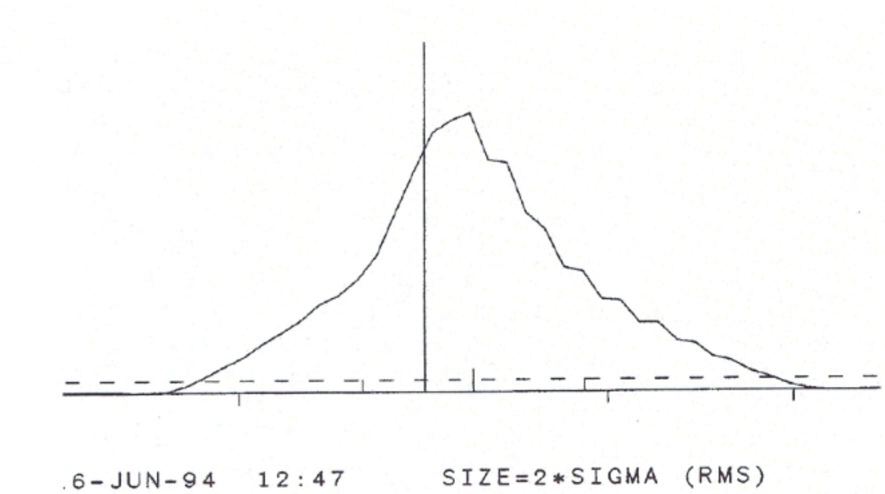


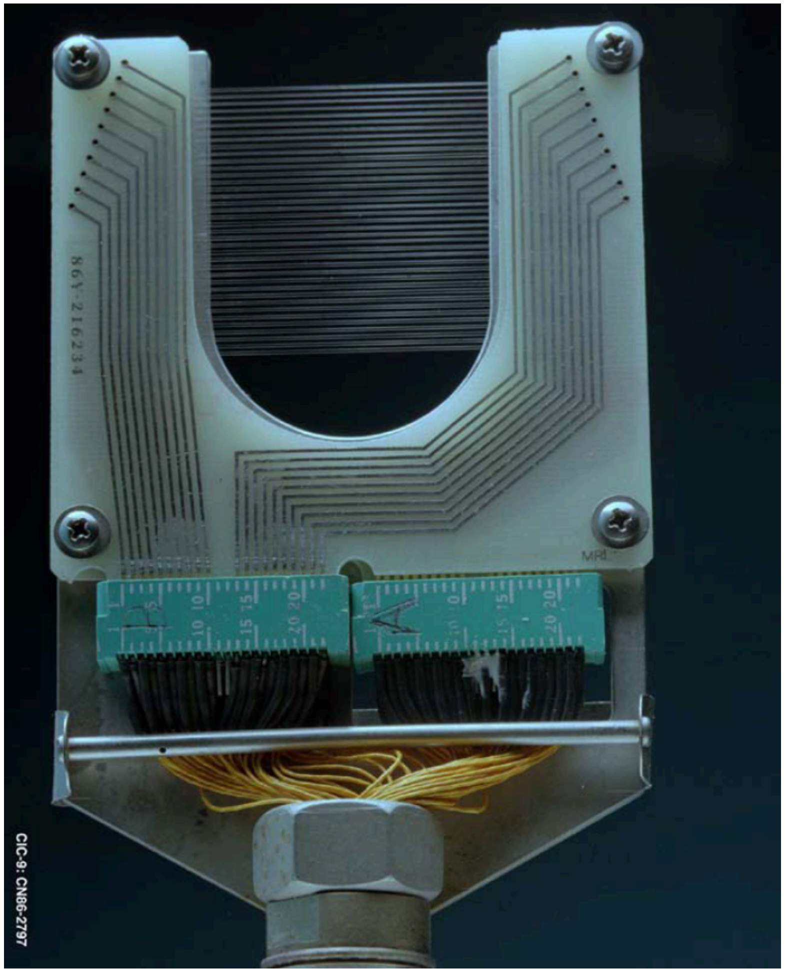
$I_{beam} = V(\text{volts})/100\,\Omega$



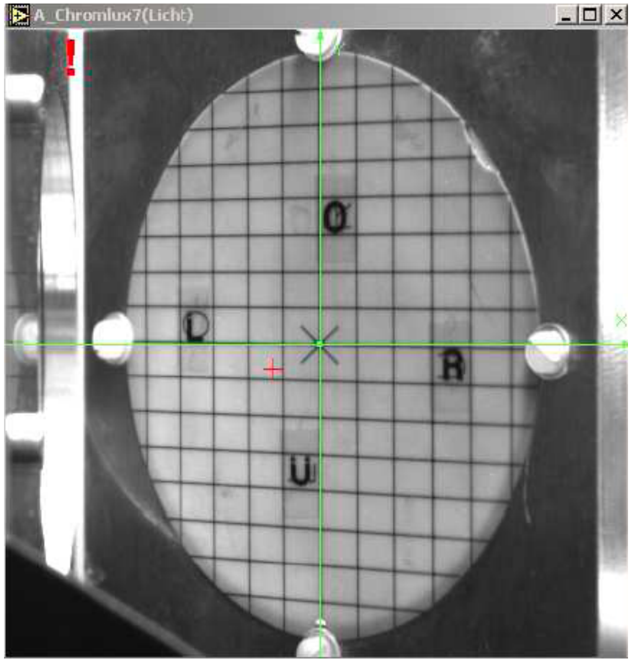
Harps (Profile Monitors)

- 1.3 mil carbon wires
- 76 wires
- 20 mil spacing
- Soldered on to g-10 board
- 1.5” aperture

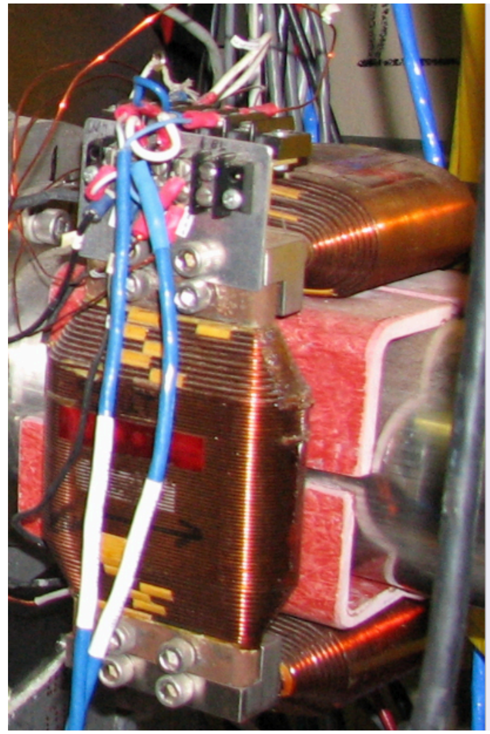




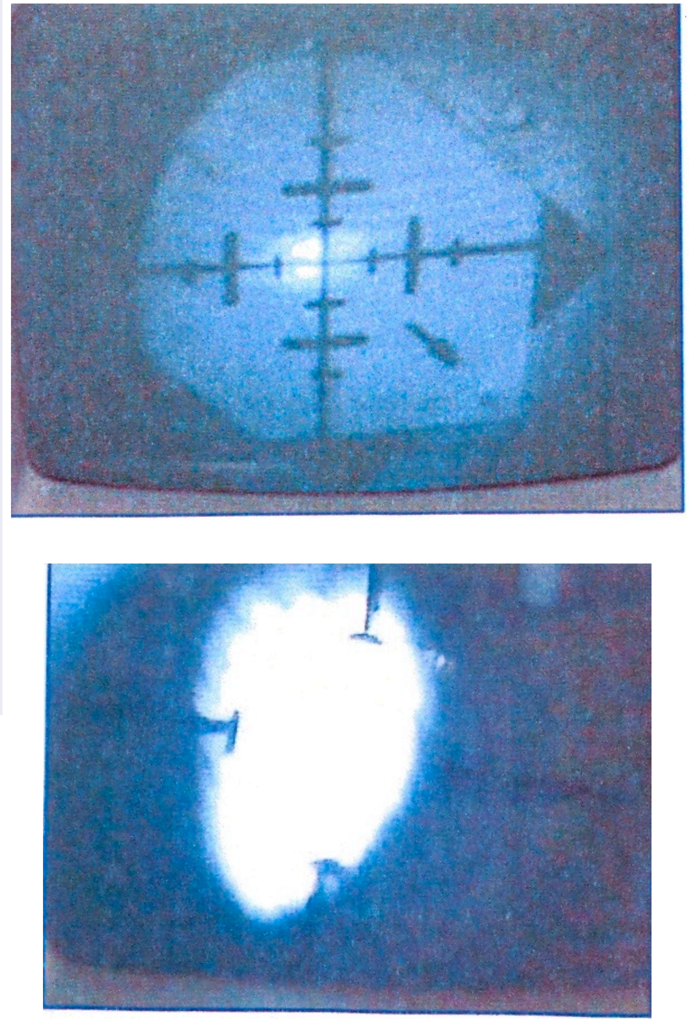
Scintillation Screens and Steering Magnets



View of a Chromolux screen with a camera. The screen is illuminated by an external light. The lines have a separation of 5mm (P.Forck, 2011).

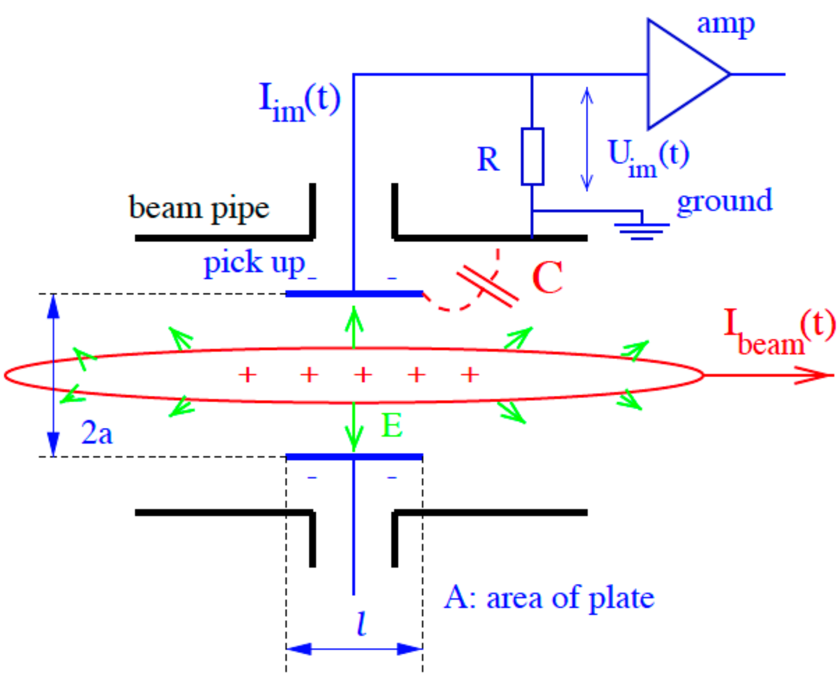


Steering magnet

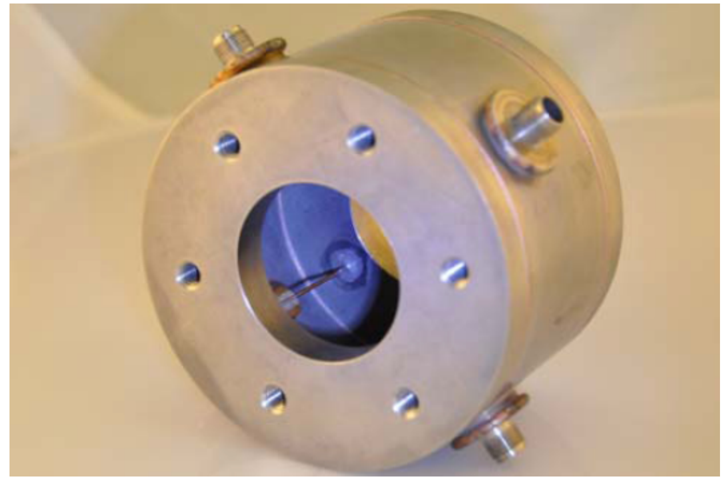


LANSCE phosphor screens illuminated by 800 MeV proton beam

Beam Position Monitors



Scheme of pick-up electrode
(P.Forck, 2011).



LANSCE BPM

Parameter	Value
Frequency of Measurement	201.25 MHz
System Response Time	50 ns
Averaging Window for System Resolution Specifications	100 µs
Position Resolution (% of radius, RMS)	0.46% (0.1mm)
Position Accuracy (% of radius)	±4.6
Position Range (% of inner electrode radius)	±60
Phase Resolution (RMS)	0.25°
Phase Linearity	±2°
Beam Current Resolution (RMS)	0.05 mA
Beam Current Accuracy	N/A
Beam Current Range	0.9 to 21 mA
Timing Uncertainty	±50 ns