Phase Space Representation. Ensemble of Particles, Emittance.

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From Single Particle to a Beam

- The number of particles per bunch in most accelerators can range between 10^5 to 10^{13}.

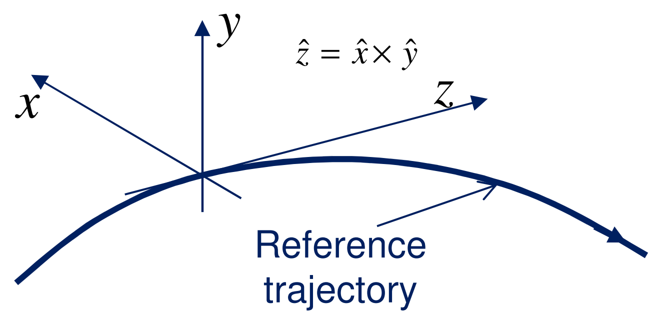
- Integrating the particle motion for such a large number of particles along accelerators with length ranging from few meters up to tens of kilometers can be a tough (impossible) task.

- Fortunately, *statistical mechanics* gives us very developed tools for representing and dealing with sets of large number of particles.

- Quite often, the statistical approach give us elegant and powerful insights on properties of the beam that could be hard to extract by approaching the problem using single particle techniques.

A Convenient Reference Frame

- **From Lecture 3:**
  - **In accelerators we are interested in studying particles along their trajectory. A natural choice is to refer all the particles relatively to a *reference trajectory* .**
- **Such a trajectory is assumed to be the solution of the Lorentz equation for the particle with the nominal parameters (reference particle).**
- **In each point of this trajectory we can define a Cartesian frame (for example) moving with the reference particle .**



- **In this frame the reference particle is always at the origin and its momentum is always parallel to the direction of the z axis.**
- **The coordinates $\{x, y, z\}$ for an arbitrary particle represent its displacement relatively to the reference particle along the three directions.**
- **In the lab frame the particle moves on the curvilinear coordinate s with speed ds/dt.**

Phase Space Representation

In relativistic classical mechanics, the motion of a single particle is totally defined when, at a given instant t, the position r and the momentum p of the particle are given together with the forces (fields) acting on the particle.

$$\bar{r}_i = x_i\,\hat{x} + y_i\,\hat{y} + z_i\,\hat{z} \qquad \bar{F} = F_x\,\hat{x} + F_y\,\hat{y} + F_z\,\hat{z} \qquad \bar{p} = p_{xi}\,\hat{x} + p_{yi}\,\hat{y} + p_{zi}\,\hat{z}$$

It is quite convenient to use the so-called *phase space* representation, a 6-D space where the i^{th} particle assumes the coordinates:

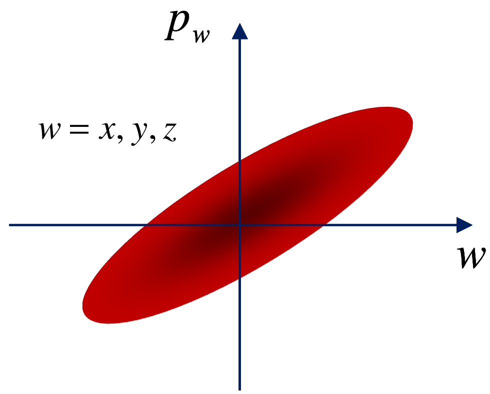
$$P_i \equiv \left\{ x_i, p_{xi}, y_i, p_{yi}, z_i, p_{zi} \right\}$$

In most accelerator physics calculations, the three planes can be considered with very good approximation as decoupled.
In this situation, it is possible and convenient to study the particle evolution independently in each of the planes:

$$\left\{ x_i, p_{xi} \right\} \qquad \left\{ y_i, p_{yi} \right\} \qquad \left\{ z_i, p_{zi} \right\}$$

Particles Systems & Ensembles

The phase space can now be used for representing particles:



The set of possible states for a system of N particles is referred as *ensemble* in statistical mechanics.

In the statistical approach, the particles lose their individuality. The properties of the whole system as a new individual entity are now studied.

The system is now fully represented by the density of particles f_{6D} and f_{2D} :

$$f_{6D}(x, p_x, y, p_y, z, p_z)\,dx\,dp_x\,dy\,dp_y\,dz\,dp_z \qquad f_{2D}(w, p_w)\,dw\,dp_w \qquad w = x, y, z$$

The above expressions indicate the number of particles contained in the elementary volume of phase space for the 6D and 2D cases respectively.

$$\int f_{6D}\,dx\,dp_x\,dy\,dp_y\,dz\,dp_z = N \qquad \int f_{2D}\,dw\,dp_w = N \qquad w = x, y, z$$

Important properties of the density functions can now be derived. Under particular circumstances, such properties allow to calculate the time evolution of the particle system without going through the integration of the motion for each single particle.

Hamiltonian Systems

A system of variables q (generalized position) and p (generalized momentum) is Hamiltonian when exists a function $H(q, p, t)$ that allows to describe the evolution of the system by:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \qquad \begin{aligned} \bar{q} &\equiv \{q_1, q_2, ..., q_N,\} \\ \bar{p} &\equiv \{p_1, p_2, ..., p_N,\} \end{aligned}$$

The function H is called Hamiltonian and q and p are referred as canonical conjugate variables.

In the particular case that q are the usual spatial coordinates $\{x, y, z\}$ and p their conjugate momentum components $\{p_x, p_y, p_z\}$, H coincides with the total energy of the system:

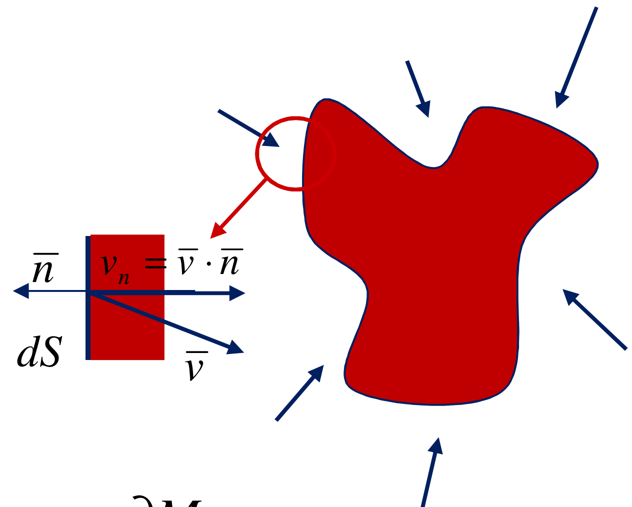
$$H = U + T = \text{Potential Energy} + \text{Kinetic Energy}$$

Non-Hamiltonian Forces:

- **Stochastic processes (collisions, quantum emission, diffusion, …)**
- **Inelastic processes (ionization, fusion, fission, annihilation, …)**
- **Dissipative forces (viscosity, friction, …)**

The Continuity Equation

If there is a flow of matter going inside a given volume, then the density inside the volume must increase in order to conserve the mass.



By indicating the density by ρ :

$$\rho(x, y, z, t)dx\,dy\,dz \equiv mass\,in\,the\,volume \quad dV = dx\,dy\,dz$$

$$dm = -\rho\, v_n dt\, dS = -\rho\,\bar{v}\cdot\bar{n}\, dS\, dt \;\Rightarrow\; \frac{dm}{dt} = -\rho\,\bar{v}\cdot\bar{n}\, dS \;\Rightarrow\; \frac{\partial M}{\partial t} = -\int_S \rho\,\bar{v}\cdot\bar{n}\, dS$$

But it is also true that:

$$M = \int_V \rho\, dV \;\Rightarrow\; \frac{\partial}{\partial t}\int_V \rho\, dV = -\int_S \rho\,\bar{v}\cdot\bar{n}\, dS. \quad \textbf{But} \quad \int_S \bar{F}\cdot\bar{n}\, dS = \int_V \nabla\cdot\bar{F}\, dV$$

$$\Rightarrow \int_S \rho\,\bar{v}\cdot\bar{n}\, dS = \int_V \nabla\cdot\rho\,\bar{v}\, dV \;\Rightarrow\; \frac{\partial}{\partial t}\int_V \rho\, dV = -\int_V \nabla\cdot\rho\,\bar{v}\, dV$$

$$\Rightarrow \boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\,\bar{v} = 0}$$

This expression, known as the continuity equation, is a consequence of the mass conservation law

The Liouville Theorem

Let us now use the continuity equation with our phase space. For simplicity we will use a 2D distribution, but the same exact results apply to the more general 6D case.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \bar{v} = 0 \qquad \text{Let} \quad \rho \equiv f_{2D}(x, p_x) \ \text{ and } \ \bar{v} \equiv \{\dot{x}, \dot{p}_x\}$$

$$\frac{\partial f_{2D}}{\partial t} + \nabla \cdot f_{2D}\bar{v} = \frac{\partial f_{2D}}{\partial t} + \frac{\partial(\dot{x} f_{2D})}{\partial x} + \frac{\partial(\dot{p}_x f_{2D})}{\partial p_x} = \frac{\partial f_{2D}}{\partial t} + \frac{\partial f_{2D}}{\partial x}\dot{x} + \frac{\partial f_{2D}}{\partial p_x}\dot{p}_x + f_{2D}\frac{\partial \dot{p}_x}{\partial p_x} + f_{2D}\frac{\partial \dot{x}}{\partial x}$$

But if our system is Hamiltonian ➡ $f_{2D}\frac{\partial \dot{x}}{\partial x} + f_{2D}\frac{\partial \dot{p}_x}{\partial p_x} = f_{2D}\frac{\partial^2 H}{\partial x \partial p_x} - f_{2D}\frac{\partial^2 H}{\partial x \partial p_x} = 0$

➡ $$\frac{\partial f_{2D}}{\partial t} + \nabla \cdot f_{2D}\bar{v} = \frac{\partial f_{2D}}{\partial t} + \frac{\partial f_{2D}}{\partial x}\dot{x} + \frac{\partial f_{2D}}{\partial p_x}\dot{p}_x = \frac{d f_{2D}}{d t}$$

$$\frac{d f_{2D}}{d t} = 0$$

Liouville Theorem: The phase space density for a Hamiltonian system is an invariant of the motion. Or equivalently, the phase space volume occupied by the system is conserved.

Decoupling the Problem: the Longitudinal Phase Space

- **Quite often in accelerators the phase space planes are weakly coupled. In particular, we can treat the longitudinal plane independently from the transverse one in the large majority of the cases.**
- **The effects of the weak coupling can be then investigated as a perturbation of the uncoupled solution.**
- **In the longitudinal plane we apply electric fields for accelerating the particles and changing their energy.**
- **It becomes natural to use energy as one of the longitudinal plane variable together with its canonical conjugate time.**
- **In accelerator physics, the *relative energy variation* δ and the *relative time 'distance'* τ with respect to a reference particle are often used:**

$$\delta = \frac{E - E_0}{E_0} \qquad \tau = t - t_0$$

- **According to Liouville, in the presence of Hamiltonian forces, the area occupied by the beam in the longitudinal phase space is conserved.**
- **The longitudinal phase space case has been already addressed in the longitudinal dynamics lecture, we will now concentrate in …**

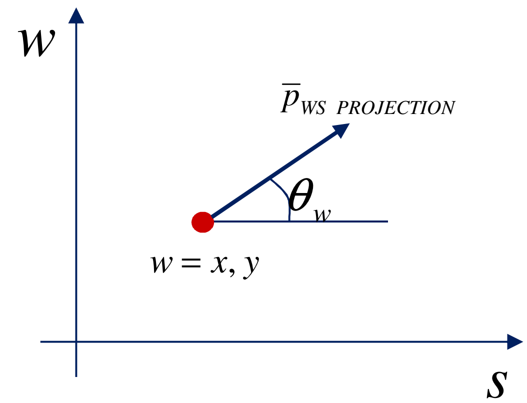
The Transverse Phase Space

For the transverse planes $\{x, p_x\}$ and $\{y, p_y\}$, it is usually used a modified phase space where the momentum components are replaced by:

$$p_{xi} \rightarrow x' = \frac{dx}{ds} \qquad\qquad p_{yi} \rightarrow y' = \frac{dy}{ds}$$

The physical meaning of the new variables:

$$x' = \frac{dx}{ds} = \tan\theta_x \qquad\qquad y' = \frac{dy}{ds} = \tan\theta_y$$



The relation between this new variables and the momentum (when $B_z = 0$) is:

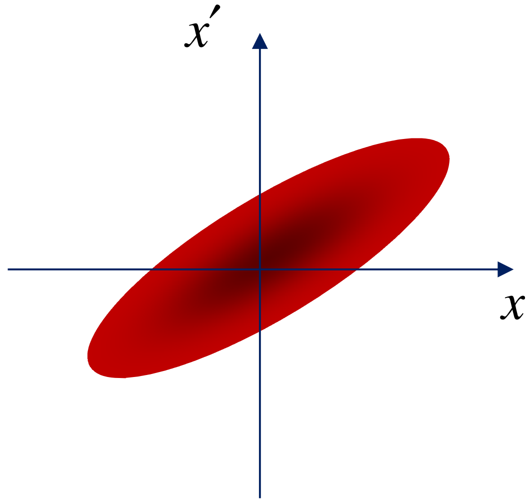
$$p_x = \gamma m_0 \frac{dx}{dt} = \gamma m_0 v_s \frac{dx}{ds} = \gamma \beta m_0 c x'$$

$$p_y = \gamma \beta m_0 c y'$$

$$where \quad \beta = \frac{v_s}{c} \quad and \quad \gamma = \left(1 - \beta^2\right)^{-1/2}$$

Note that x and p_x are canonical conjugate variables while x and x' are not, unless there is no acceleration (γ and β constant)

Definition of Emittance



We will consider the decoupled case and use the $\{w, w'\}$ plane where w can be either x or y.

We define as **emittance** the phase space area occupied by the system of particles, divided by π

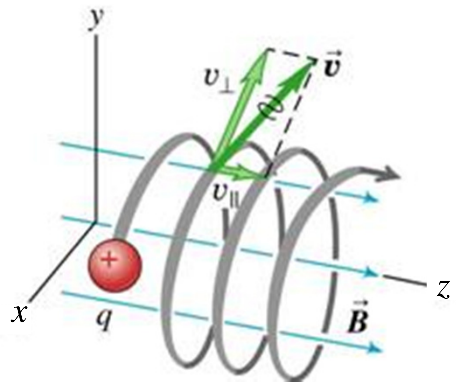
$$\varepsilon_w = \frac{A_{ww'}}{\pi} \qquad w = x, y$$

As we previously shown, x' and y' are conjugate to x and y when $B_z = 0$ and in absence of acceleration. In this case, we can immediately apply the Liouville theorem and state that for such a system the **emittance is an invariant of the motion**.

This specific case is actually extremely important.
In fact, for most of the elements in a beam transferline, such as dipoles, quadrupoles, sextupoles, …, the above conditions apply and the emittance is conserved.

Emittance in the Presence of B_z

• When the B_z component of the magnetic field is present (solenoidal lenses for example), the transverse planes become coupled and the phase space area occupied by the system in each of the transverse planes is not conserved anymore.

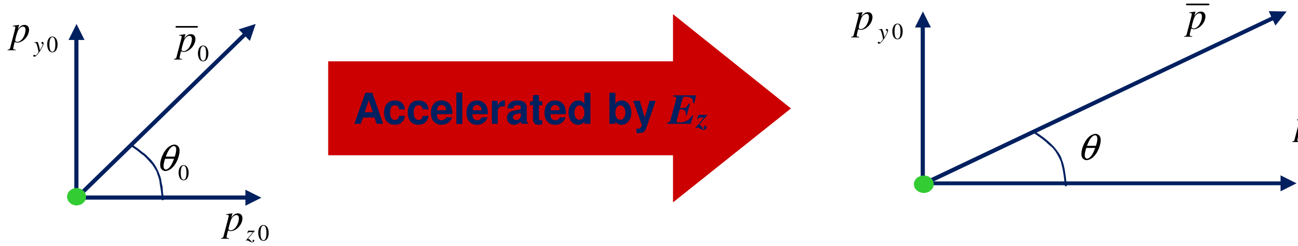


• Anyway in this situation, the Liouville theorem still applies to the 4D transverse phase space where the ipervolume occupied by our system is still a motion invariant.

• Actually, if we rotate the spatial reference frame around the z axis by the *Larmor frequency* $\omega_L = qB_z / 2\gamma m_0$, then the transverse planes become decoupled again and the phase space area in each of the planes is conserved again.

Emittance in the Presence of Acceleration

When the particles in a beam undergo to acceleration, β and γ change and the variables x and x' are not canonical anymore. Liouville theorem does not apply and the emittance is not conserved.



$$p_z = \sqrt{\frac{T^2 + 2Tm_0c^2}{T_0^2 + 2T_0m_0c^2}}\, p_{z0}$$

$$T \equiv kinetic \;\; energy$$

$$y_0' = \tan\theta_0 = \frac{p_{y0}}{p_{z0}} = \frac{p_{y0}}{\beta_0\gamma_0 m_0 c} \qquad y' = \tan\theta = \frac{p_y}{p_z} = \frac{p_{y0}}{\beta\gamma m_0 c} \qquad \frac{y'}{y_0'} = \frac{\beta_0\gamma_0}{\beta\gamma}$$

It can be shown that in this case $\dfrac{\varepsilon_y}{\varepsilon_{y0}} = \dfrac{y'}{y_0'}$

$$\boxed{\beta\gamma\varepsilon_y = \beta_0\gamma_0\varepsilon_{y0}}$$

The last expression tells us that the quantity $\beta\gamma\varepsilon$ is a system invariant during acceleration. By defining the normalized emittance:

$$\boxed{\varepsilon_{nw} = \beta\gamma\varepsilon_w \qquad w = x, y}$$

We can say that the normalized emittance is conserved during acceleration.

In other words, the acceleration couples the longitudinal plane with the transverse ones: the 6D emittance is still conserved but the transverse ones are not.

The Real Beam Case: r.m.s Emittance

For the case of a real beam composed by N particles, we start calculating the second order statistical moments of its phase space distribution:

$$\left\langle x^2 \right\rangle = \frac{\sum_{n=1}^{N} x_n^2}{N} \cong \frac{\int x^2 f_{2D}(x,x')\,dx\,dx'}{\int f_{2D}(x,x')\,dx\,dx'}$$

$$\left\langle x'^2 \right\rangle = \frac{\sum_{n=1}^{N} x_n'^2}{N} \cong \frac{\int x'^2 f_{2D}(x,x')\,dx\,dx'}{\int f_{2D}(x,x')\,dx\,dx'}$$

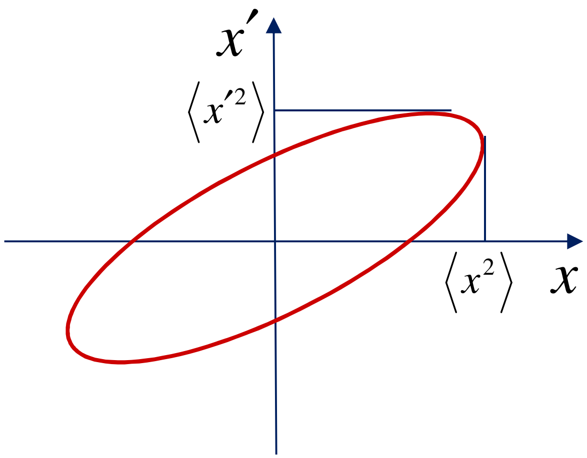
$$\left\langle x\,x' \right\rangle = \frac{\sum_{n=1}^{N} x_n x_n'}{N} \cong \frac{\int x\,x'\; f_{2D}(x,x')\,dx\,dx'}{\int f_{2D}(x,x')\,dx\,dx'}$$

We then define the rms emittance as the quantity:

$$\varepsilon_{rms} = \sqrt{\left\langle x^2 \right\rangle \left\langle x'^2 \right\rangle - \left\langle x\,x' \right\rangle^2}$$

This is equivalent to associate to the real beam an *equivalent ellipse* in the phase space with area $\pi\varepsilon_{rms}$ and equation:

$$\frac{\left\langle x'^2 \right\rangle}{\varepsilon_{rms}} x^2 + \frac{\left\langle x^2 \right\rangle}{\varepsilon_{rms}} x'^2 - 2\frac{\left\langle x\,x' \right\rangle}{\varepsilon_{rms}} x\,x' = \varepsilon_{rms}$$



Nonlinear Forces and Filamentation

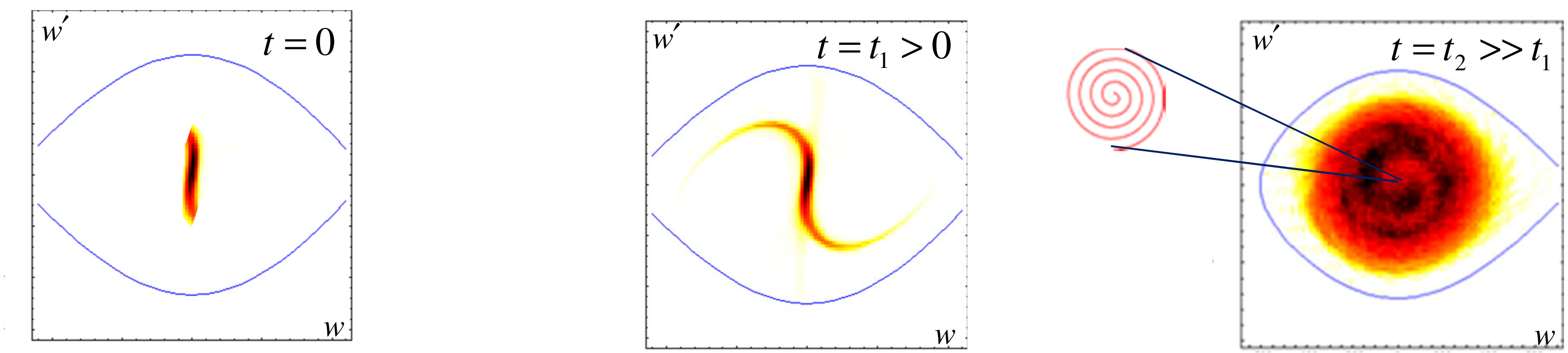
• In the case of a Hamiltonian system, as a consequence of the Liouville Theorem the emittance is conserved

• This is true even when the forces acting on the system are nonlinear (space charge, nonlinear magnetic and/or electric fields, …)

• This is not true in the case of the rms emittance.

•In the presence of nonlinear forces the rms emittance is not conserved

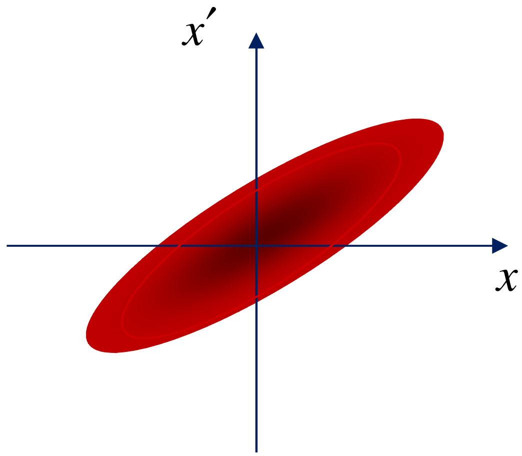
• Example: *filamentation*. Particles with different phase space coordinates, because of the nonlinear forces, move with different phase space velocity



• The emittance according to Liouville is still conserved.

But the rms emittance calculated at later times increases.

The Twiss Parameters



We saw that a beam with arbitrary phase space distribution can be represented by an equivalent ellipse with area equal to the rms emittance divided by π and with equation:

$$\frac{\langle w'^2\rangle}{\varepsilon_{w\,rms}} w^2 + \frac{\langle w^2\rangle}{\varepsilon_w} w'^2 - 2\frac{\langle w\,w'\rangle}{\varepsilon_w} w\,w' = \varepsilon_w \qquad w = x, y$$

A convenient representation for this ellipse, often used in accelerator physics, is the one by the so-called *Twiss Parameters* β_T, γ_T and α_T :

$$\beta_{Tw} w'^2 + \gamma_{Tw} w^2 + 2\alpha_{Tw} w w' = \varepsilon_w \qquad w = x, y$$

$$\text{with} \quad \beta_{Tw}\,\gamma_{Tw} - \alpha_{Tw}^2 = 1$$

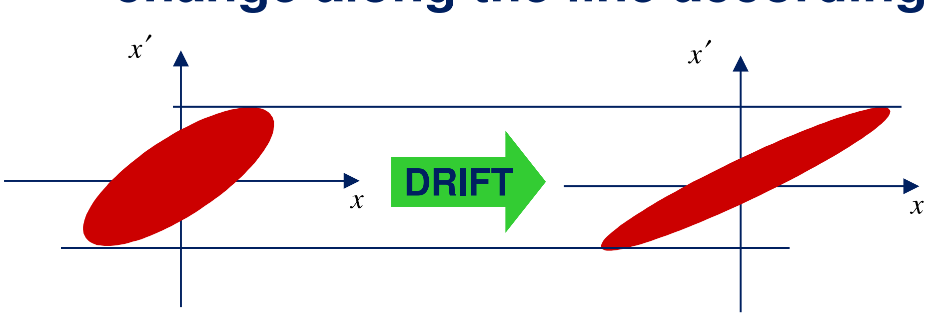
In this representation, the beam status at a given moment is totally defined when the emittance and two of the Twiss parameters are known.

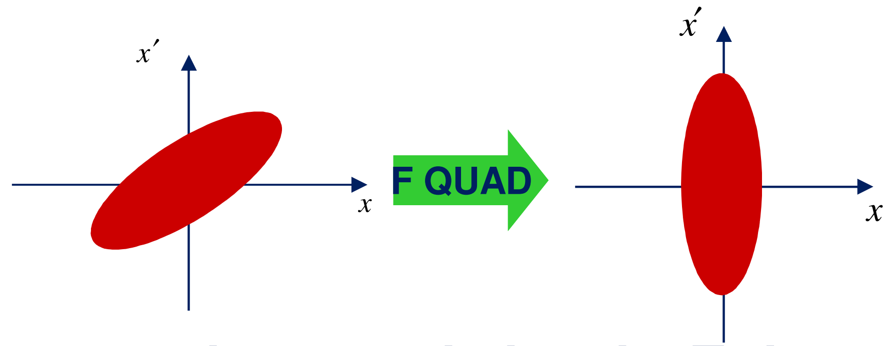
By comparing the two ellipse equations, we can derive:

$$\langle w^2\rangle = \beta_{Tw}\varepsilon_w \qquad \langle w'^2\rangle = \gamma_{Tw}\varepsilon_w \qquad \langle w\,w'\rangle = -\alpha_{Tw}\varepsilon_w \qquad w = x, y$$

Propagating the Twiss Parameters

When the beam propagates along the beamline, the eccentricity and the orientation of the equivalent ellipse change while the area remains constant (Liouville theorem). In other words, the Twiss parameters change along the line according to the action of the line elements.





The single particle matrix formalism can now be extended to the Twiss parameters. For example for a drift of length L in the horizontal plane:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} \Rightarrow \begin{array}{c} x = x_0 + L x'_0 \\ x' = x'_0 \end{array}$$

$$\Rightarrow \begin{array}{c} \langle x^2 \rangle = \langle (x_0 + L x'_0)^2 \rangle = \langle x_0^2 \rangle + L^2 \langle x_0'^2 \rangle + 2L \langle x_0 x'_0 \rangle \\ \langle x'^2 \rangle = \langle x_0'^2 \rangle \\ \langle x x' \rangle = \langle (x_0 + L x'_0) x'_0 \rangle = L \langle x_0'^2 \rangle + \langle x_0 x'_0 \rangle \end{array}$$

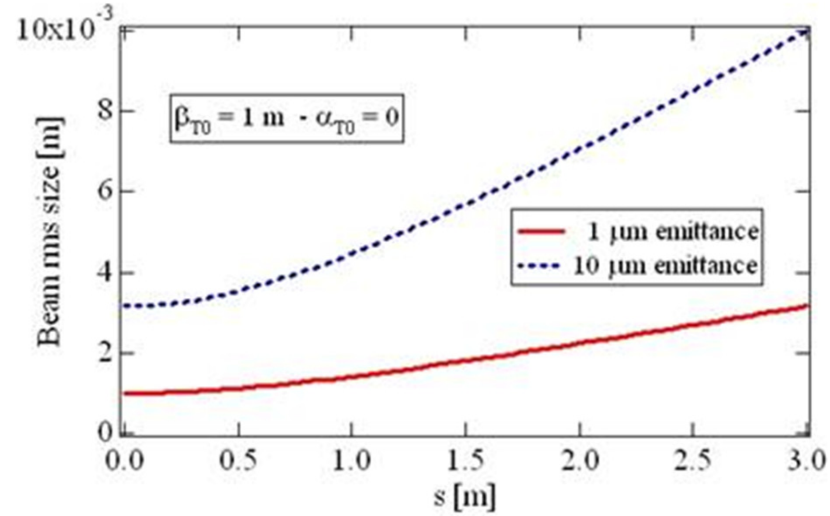
$$\Rightarrow \begin{array}{c} \beta_T \cancel{\varepsilon} = \beta_{T0} \cancel{\varepsilon} + L^2 \gamma_{T0} \cancel{\varepsilon} - 2L \alpha_{T0} \cancel{\varepsilon} \\ \gamma_T \cancel{\varepsilon} = \gamma_{T0} \cancel{\varepsilon} \\ -\alpha_T \cancel{\varepsilon} = L \gamma_{T0} \cancel{\varepsilon} - \alpha_{T0} \cancel{\varepsilon} \end{array}$$

$$\Rightarrow \begin{pmatrix} \beta_T \\ \gamma_T \\ \alpha_T \end{pmatrix} = \begin{pmatrix} 1 & L^2 & -2L \\ 0 & 1 & 0 \\ 0 & -L & 1 \end{pmatrix} \begin{pmatrix} \beta_{T0} \\ \gamma_{T0} \\ \alpha_{T0} \end{pmatrix}$$

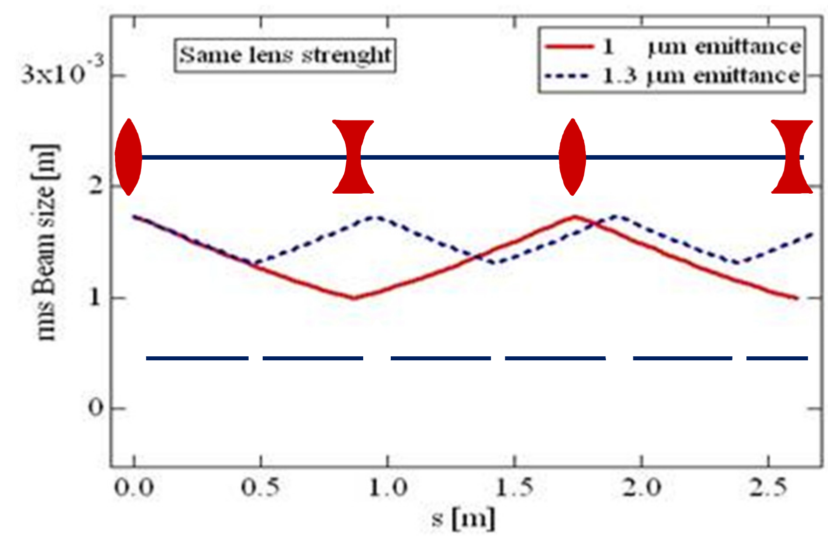
Getting Familiar with the Emittance

A couple of examples:

Propagation of beams with different emittance through a drift space

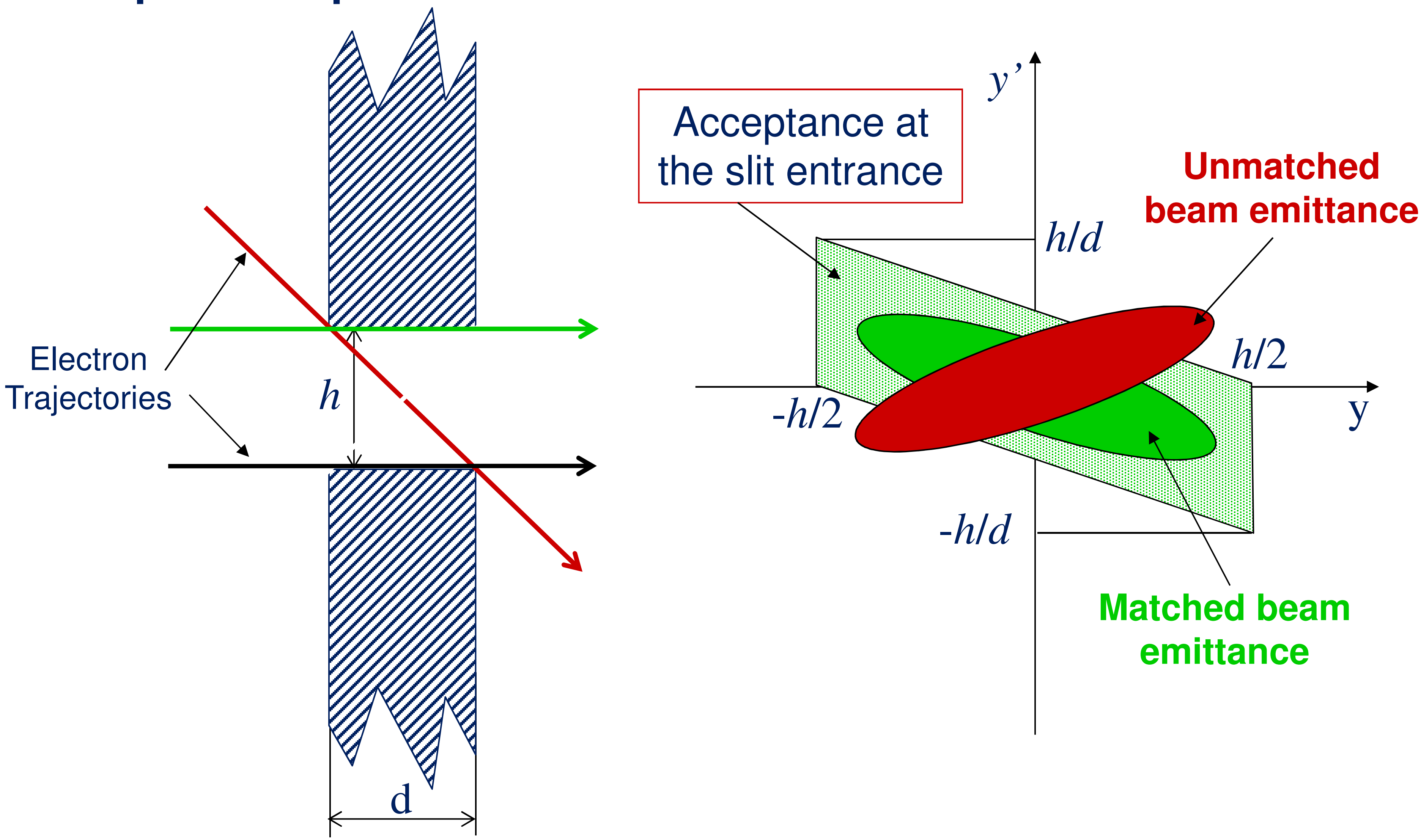


Propagation of beams with different emittance through a FODO lattice



The Concept of Acceptance

Example: Acceptance of a slit



The Emittance Role In Accelerator Applications

The emittance is a motion invariant in Hamiltonian systems: transferlines, linear accelerators (6D-emittance), heavy particles rings, ….
It is an equilibrium quantity (as it will be shown later) defined by the lattice and the synchrotron radiation emission in electron and positron rings.

It is an important quantity that plays a fundamental role in most accelerator systems:

- **Electron microscopes:** High resolution requires low emittances
- **Free electron lasers (FEL):** performance of the FEL and size of the radiating undulator strongly depends on emittance. The smaller the better.
- **Synchrotron light sources:** smaller emittances gives higher brightness
- **Colliders:** higher emittances give higher luminosity (in beam-beam limited regime)
- …

L6 Homework

1) Demonstrate the validity of the relation $\varepsilon/\varepsilon_0 = y'/y'_0$ in slide 13, where ε and ε_0 are the vertical emittances after and before acceleration by a "thin" cavity (assume that the energy in rest mass units goes from γ_0 to γ), and y' and y'_0 are the particle divergences after and before acceleration.
Tip: use the definition of rms emittance in slide 14.

2) Calculate the Twiss parameter transport matrix for both planes of a focusing quadrupole in the thin lens approximation.
(similarly to what done in slide 17 for the case of a drift)