CAVITY FUNDAMENTALS

Jean Delayen
Center for Accelerator Science
Old Dominion University
and
Thomas Jefferson National Accelerator Facility
RF Cavity

- Mode transformer (TEM→TM)

- Impedance transformer (Low Z→High Z)

- Space enclosed by conducting walls that can sustain an infinite number of resonant electromagnetic modes

- Shape is selected so that a particular mode can efficiently transfer its energy to a charged particle

- An isolated mode can be modeled by an LRC circuit
RF Cavity

Lorentz force

\[ \vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \]

An accelerating cavity needs to provide an electric field \( E \) longitudinal with the velocity of the particle.

Magnetic fields provide deflection but no acceleration.

DC electric fields can provide energies of only a few MeV.

Higher energies can be obtained only by transfer of energy from traveling waves \( \rightarrow \) resonant circuits.

Transfer of energy from a wave to a particle is efficient only is both propagate at the same velocity.
Equivalent Circuit for an rf Cavity

Simple LC circuit representing an accelerating resonator

Metamorphosis of the LC circuit into an accelerating cavity

Chain of weakly coupled pillbox cavities representing an accelerating module

Chain of coupled pendula as its mechanical analogue
Electromagnetic Modes

Electromagnetic modes satisfy Maxwell equations

\[ \left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \begin{bmatrix} \vec{E} \\ \vec{H} \end{bmatrix} = 0 \]

With the boundary conditions (assuming the walls are made of a material of low surface resistance)

- no tangential electric field \( \vec{n} \times \vec{E} = 0 \)
- no normal magnetic field \( \vec{n} \cdot \vec{H} = 0 \)
Electromagnetic Modes

Assume everything

\[ e^{-i\omega t} \]

\[
\left( \nabla^2 + \frac{\omega^2}{c^2} \right) \begin{cases} E \\ H \end{cases} = 0
\]

For a given cavity geometry, Maxwell equations have an infinite number of solutions with a sinusoidal time dependence.

For efficient acceleration, choose a cavity geometry and a mode where:

- Electric field is along particle trajectory
- Magnetic field is 0 along particle trajectory
- Velocity of the electromagnetic field is matched to particle velocity
Accelerating Field (gradient)

Voltage gained by a particle divided by a reference length

\[ E = \frac{1}{L} \int E_z(z) \cos(\omega z / \beta c) dz \]

For velocity-of-light particles

\[ L = \frac{N \lambda}{2} \]

For less-than-velocity-of-light cavities, there is no universally adopted definition of the reference length
Design Considerations

\[ \frac{H_{s,\text{max}}}{E_{\text{acc}}} \]
minimum critical field

\[ \frac{E_{s,\text{max}}}{E_{\text{acc}}} \]
minimum field emission

\[ \frac{< H_s^2 >}{E_{\text{acc}}^2} \]
minimum shunt impedance, current losses

\[ \frac{< E_s^2 >}{E_{\text{acc}}^2} \]
minimum dielectric losses

\[ \frac{U}{E_{\text{acc}}^2} \]
minimum control of microphonics

maximum voltage drop for high charge per bunch
Energy Content

Energy density in electromagnetic field:

\[ u = \frac{1}{2} \left( \varepsilon_0 E^2 + \mu_0 H^2 \right) \]

Because of the sinusoidal time dependence and the 90° phase shift, the energy oscillates back and forth between the electric and magnetic field.

Total energy content in the cavity:

\[ U = \frac{\varepsilon_0}{2} \int_V dV \, |E|^2 = \frac{\mu_0}{2} \int_V dV \, |H|^2 \]
Power Dissipation

Power dissipation per unit area

\[ \frac{dP}{da} = \frac{\mu_0 \omega \delta}{4} |H_\parallel|^2 = \frac{R_s}{2} |H_\parallel|^2 \]

Total power dissipation in the cavity walls

\[ P = \frac{R_s}{2} \int_A da |H_\parallel|^2 \]
Quality Factor

Quality Factor $Q_0$:

\[ Q_0 = \frac{\text{Energy stored in cavity}}{\text{Energy dissipated in cavity walls per radian}} = \frac{\omega_0 U}{P_{\text{diss}}} \]

\[ = \omega_0 \tau_0 = \frac{\omega_0}{\Delta \omega_0} \]

\[ Q_0 = \frac{\omega \mu_0}{R_s} \int_V dV |H|^2 \int_A da |H_\parallel|^2 \]
Geometrical Factor

Geometrical Factor QRs ($\Omega$)
Product of the Quality Factor and the surface resistance
Independent of size and material
Depends only on shape of cavity and electromagnetic mode

$$G = QR_s = \omega \mu_0 \frac{\int_V dV \left| \mathbf{H} \right|^2}{\int_A da \left| \mathbf{H}_\parallel \right|^2} = 2\pi \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{1}{\lambda} \frac{\int_V dV \left| \mathbf{H} \right|^2}{\int_A da \left| \mathbf{H}_\parallel \right|^2} = \frac{2\pi \eta}{\lambda} \frac{\int_V dV \left| \mathbf{H} \right|^2}{\int_A da \left| \mathbf{H}_\parallel \right|^2}$$

$\eta \approx 377\Omega$ Impedance of vacuum
Shunt Impedance, $R/Q$

Shunt impedance $R_{sh}$:

$$R_{sh} \equiv \frac{V_c^2}{P_{diss}} \quad \text{in } \Omega$$

$V_c$ = accelerating voltage

Note: Sometimes the shunt impedance is defined as or quoted as impedance per unit length (ohm/m)

$$\frac{V_c^2}{2P_{diss}}$$

$R/Q$ (in $\Omega$)

$$\frac{R}{Q} = \frac{V^2}{P} \frac{P}{\omega U} = \frac{E^2}{U} \frac{L^2}{\omega}$$
Q – Geometrical Factor (Q Rs)

Q: \[ \frac{\text{Energy content}}{\text{Energy dissipated during one radian}} = \frac{U}{P} = \frac{\omega}{\tau} = \frac{\omega}{\Delta\omega} \]

Rough estimate (factor of 2) for fundamental mode:

\[ \omega = \frac{2\pi c}{\lambda} \approx \frac{2\pi}{\sqrt{\varepsilon_0 \mu_0}} \frac{1}{2L} \]

\[ U = \frac{\mu_0}{2} \int H^2 d\nu \approx \frac{\mu_0}{2} \frac{1}{2} H_0^2 \frac{\pi L^3}{6} \]

\[ P = \frac{1}{2} R_s \int H^2 dA = \frac{1}{2} R_s \frac{1}{2} H_0^2 \pi L^2 \]

\[ QR_s \sim \frac{\pi}{6} \sqrt{\frac{\mu_0}{\varepsilon_0}} = 200\Omega \]

G = QRs is size (frequency) and material independent.

It depends only on the mode geometry.

It is independent of number of cells.

For superconducting elliptical cavities \( QR_s \sim 275\Omega \)
Shunt Impedance \((R_{sh}), R_{sh} R_s, R/Q\)

\[
R_{sh} = \frac{V^2}{P} \approx \frac{E_z^2 L^2}{\frac{1}{2} R_s H_0^2 \pi L^2 \frac{1}{2}}
\]

In practice for elliptical cavities

\[R_{sh} R_s \approx 33,000 \left(\Omega^2\right) \text{ per cell}\]

\[R_{sh} / Q \approx 100\Omega \text{ per cell}\]

\(R_{sh} R_s\) and \(R_{sh} / Q\)

Independent of size (frequency) and material
Depends on mode geometry
Proportional to number of cells
Power Dissipated per Unit Length or Unit Area

\[
\frac{P}{L} \propto \frac{1}{R} \left( \frac{QR_s}{Q} \right) \frac{E^2 R_s}{\omega}
\]

For normal conductors \( R_s \propto \omega^{1/2} \)

\[
\frac{P}{L} \propto \omega^{1/2}
\]
\[
\frac{P}{A} \propto \omega
\]

For superconductors \( R_s \propto \omega^2 \)

\[
\frac{P}{L} \propto \omega^2
\]
\[
\frac{P}{A} \propto \omega^2
\]
External Coupling

- Consider a cavity connected to an rf source

- A coaxial cable carries power from an rf source to the cavity

- The strength of the input coupler is adjusted by changing the penetration of the center conductor

- There is a fixed output coupler, the transmitted power probe, which picks up power transmitted through the cavity. This is usually very weakly coupled
Consider the rf cavity after the rf is turned off. Stored energy $U$ satisfies the equation:

$$\frac{dU}{dt} = -P_{\text{tot}}$$

Total power being lost, $P_{\text{tot}}$, is:

$$P_{\text{tot}} = P_{\text{diss}} + P_e + P_t$$

$P_e$ is the power leaking back out the input coupler. $P_t$ is the power coming out the transmitted power coupler. Typically $P_t$ is very small $\Rightarrow P_{\text{tot}} \approx P_{\text{diss}} + P_e$

Recall

$$Q_0 \equiv \frac{\omega_0 U}{P_{\text{diss}}}$$

Similarly define a “loaded” quality factor $Q_L$:

$$Q_L \equiv \frac{\omega_0 U}{P_{\text{tot}}}$$

Now

$$\frac{dU}{dt} = -\frac{\omega_0 U}{Q_L} \Rightarrow U = U_0 e^{-\frac{\omega_0 t}{Q_L}}$$

$\therefore$ energy in the cavity decays exponentially with time constant:

$$\tau_L = \frac{Q_L}{\omega_0}$$
Cavity with External Coupling

Equation

\[
\frac{P_{\text{tot}}}{\omega_0 U} = \frac{P_{\text{diss}}}{\omega_0 U} + \frac{P_e}{\omega_0 U}
\]

suggests that we can assign a quality factor to each loss mechanism, such that

\[
\frac{1}{Q_L} = \frac{1}{Q_0} + \frac{1}{Q_e}
\]

where, by definition,

\[
Q_e \equiv \frac{\omega_0 U}{P_e}
\]

Typical values for CEBAF 7-cell cavities: \(Q_0=1\times10^{10}, Q_e \approx Q_L=2\times10^7\).
Cavity with External Coupling

• Define “coupling parameter”: \( \beta \equiv \frac{Q_0}{Q_e} \)

\[
\frac{1}{Q_L} = \frac{1 + \beta}{Q_0}
\]

\( \beta \) is equal to: \( \beta = \frac{P_e}{P_{\text{diss}}} \)

• It tells us how strongly the couplers interact with the cavity. Large \( \beta \) implies that the power leaking out of the coupler is large compared to the power dissipated in the cavity walls.
Several Loss Mechanisms

\[ P = \sum P_i \]

- wall losses
- power absorbed by beam
- coupling to outside world

Associate \( Q \) will each loss mechanism

\[ Q_i = \omega \frac{U}{P_i} \]

(index 0 is reserved for wall losses)

Loaded \( Q \): \( Q_L \)

\[ \frac{1}{Q_L} = \frac{\sum P_i}{\omega U} = \sum \frac{1}{Q_i} \]

Coupling coefficient:

\[ \beta_i = \frac{Q_0}{Q_i} = \frac{P_i}{P_0} \]

\[ Q_L = \frac{Q_0}{1 + \sum \beta_i} \]
Another Simple Model: Coaxial Half-wave Resonator

\[ 2b \]

\[ 2a \]

\[ L \]
Coaxial Half-wave Resonator

Capacitance per unit length

\[ C = \frac{2\pi \varepsilon_0}{\ln\left(\frac{b}{a}\right)} = \frac{2\pi \varepsilon_0}{\ln\left(\frac{1}{\rho_0}\right)} \]

Inductance per unit length

\[ L = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{r_0}\right) = \frac{\mu_0}{2\pi} \ln\left(\frac{1}{\rho_0}\right) \]
Coaxial Half-wave Resonator

Center conductor voltage

\[ V(z) = V_0 \cos \left( \frac{2\pi}{\lambda} z \right) \]

Center conductor current

\[ I(z) = I_0 \sin \left( \frac{2\pi}{\lambda} z \right) \]

Line impedance

\[ Z_0 = \frac{V_0}{I_0} = \frac{\eta}{2\pi} \ln \left( \frac{1}{\rho_0} \right), \quad \eta = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 377\Omega \]
Coaxial Half-wave Resonator

Peak Electric Field

d: coaxial cylinders

\( V_p \): Voltage on center conductor
Outer conductor at ground
\( E_p \): Peak field on center conductor
Coaxial Half-wave Resonator

Peak magnetic field

\[ \frac{V_p}{b} = \left\{ \begin{array}{c} \eta \\ H \\ c \\ \frac{1}{\rho_0} \end{array} \right\} \begin{array}{c} \text{m, A/m} \\ \text{m, T} \\ \text{cm, G} \end{array} \]

\[ V_p : \] Voltage across loading capacitance

\[ B \approx 9 \text{ mT at 1 MV/m} \]
Power dissipation (ignore losses in the shorting plate)

\[
P = V_p^2 \frac{\pi}{4} \frac{R_s}{\eta^2} b \frac{1 + 1}{\ln^2 \rho_0} \\
P \propto \frac{R_s}{\eta^2} E^2 \beta \lambda^2
\]
Energy content

\[ U = V_p^2 \frac{\pi \varepsilon_0}{4} \lambda \frac{1}{\ln \left( \frac{1}{\rho_0} \right)} \]

\[ U \propto \varepsilon_0 E^2 \beta^2 \lambda^3 \]
Geometrical factor

\[ G = QR_s = 2\pi \eta \frac{b \ln(1/\rho_0)}{\lambda (1+1/\rho_0)} \]

\[ G \propto \eta \beta \]
Coaxial Half-wave Resonator

Shunt impedance \( \left( \frac{4V_p^2}{P} \right) \)

\[
R_{sh} = \frac{\eta^2}{R_s} \frac{16}{\pi} \frac{b}{\lambda} \frac{\ln^2 \rho_0}{1 + 1/\rho_0}
\]

\[ R_{sh} \propto \eta^2 \beta \]
Coaxial Half-wave Resonator

\[ \frac{R_{sh}}{Q} = \frac{8}{\pi^2} \eta \ln\left(\frac{1}{\rho_0}\right) \]

\[ \frac{R_{sh}}{Q} \propto \eta \]
Some Real Geometries ($\lambda/4$)
Some Real Geometries ($\lambda/4$)
\frac{\lambda}{4} Resonant Lines
$\lambda/2$ Resonant Lines
λ/2 Resonant Lines – Single-Spoke
\( \lambda/2 \) Resonant Lines – Double and Triple-Spoke
\(\lambda/2\) Resonant Lines – Multi-Spoke
1300 MHz 9-cell
Pill Box Cavity

Hollow right cylindrical enclosure

Operated in the TM$_{010}$ mode

\[ H_z = 0 \]

\[
\frac{\partial^2 E_z}{\partial^2 r} + \frac{1}{r} \frac{\partial E_z}{\partial r} = \frac{1}{c^2} \frac{\partial^2 E_z}{\partial^2 t} \quad \omega_0 = \frac{2.405c}{R}
\]

\[
E_z(r, z, t) = E_0 J_0 \left(2.405 \frac{r}{R}\right) e^{-i\omega_0 t}
\]

\[
H_\phi(r, z, t) = -i \frac{E_0}{\mu_0 c} J_1 \left(2.405 \frac{r}{R}\right) e^{-i\omega_0 t}
\]
Modes in Pill Box Cavity

- **$\text{TM}_{010}$**
  - Electric field is purely longitudinal
  - Electric and magnetic fields have no angular dependence
  - Frequency depends only on radius, independent on length

- **$\text{TM}_{0mn}$**
  - Monopoles modes that can couple to the beam and exchange energy

- **$\text{TM}_{1mn}$**
  - Dipole modes that can deflect the beam

- **TE modes**
  - No longitudinal E field
  - Cannot couple to the beam
TM Modes in a Pill Box Cavity

\[
\frac{E_r}{E_0} = -\frac{n\pi}{x_{lm}} \frac{R}{L} J''_l(x_{lm} \frac{r}{R}) \sin \left( n\pi \frac{z}{L} \right) \cos l\varphi \\
\frac{E_\varphi}{E_0} = \frac{\ln \pi}{x_{lm}^2} \frac{R^2}{rL} J_1(x_{lm} \frac{r}{R}) \sin \left( n\pi \frac{z}{L} \right) \sin l\varphi \\
\frac{E_z}{E_0} = J_1(x_{lm} \frac{r}{R}) \sin \left( n\pi \frac{z}{L} \right) \cos l\varphi \\
\frac{H_r}{E_0} = -i\omega \varepsilon \frac{l}{x_{lm}^2} \frac{R^2}{r} J_1(x_{lm} \frac{r}{R}) \cos \left( n\pi \frac{z}{L} \right) \sin l\varphi \\
\frac{H_\varphi}{E_0} = -i\omega \varepsilon \frac{R}{x_{lm}} J'_l(x_{lm} \frac{r}{R}) \cos \left( n\pi \frac{z}{L} \right) \cos l\varphi \\
\frac{H_z}{E_0} = 0
\]

\[
\omega_{lmn} = c \sqrt{\left( \frac{x_{lm}}{R} \right)^2 + \left( \frac{\pi n}{L} \right)^2}
\]

\(x_{lm}\) is the mth root of \(J_l(x)\)

\[\text{Jefferson Lab} \]

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**TM$_{010}$ Mode in a Pill Box Cavity**

\[ E_r = E_\varphi = 0 \]
\[ E_z = E_0 J_0 \left( \frac{x_{01} r}{R} \right) \]

\[ H_r = H_z = 0 \]
\[ H_\varphi = -i \omega \varepsilon E_0 \frac{R}{x_{01}} J_1 \left( \frac{x_{01} r}{R} \right) \]

\[ \omega = x_{01} \frac{c}{R} \]
\[ x_{01} = 2.405 \]

\[ R = \frac{x_{01}}{2\pi} \lambda = 0.383\lambda \]
TM_{010} Mode in a Pill Box Cavity

Energy content

\[ U = \epsilon_0 E_0^2 \frac{\pi}{2} J_1^2(x_{01})LR^2 \]

Power dissipation

\[ P = E_0^2 \frac{R_s}{\eta^2} \pi J_1^2(x_{01})(R + L)R \]

Geometrical factor

\[ G = \eta \frac{x_{01} L}{2 (R + L)} \]

\[ x_{01} = 2.40483 \]
\[ J_1(x_{01}) = 0.51915 \]
TM010 Mode in a Pill Box Cavity

Energy Gain

\[ \Delta W = E_0 \frac{\lambda}{\pi} \sin \frac{\pi L}{\lambda} \]

Gradient

\[ E_{acc} = \frac{\Delta W}{\lambda/2} = E_0 \frac{2}{\pi} \sin \frac{\pi L}{\lambda} \]

Shunt impedance

\[ R_{sh} = \frac{\eta^2}{R_s} \frac{1}{\pi^3 J_1^2(x_{01})} \frac{\lambda^2}{R(R + L)} \sin^2 \left( \frac{\pi L}{\lambda} \right) \]
Real Cavities

Beam tubes reduce the electric field on axis

- Gradient decreases
- Peak fields increase
- R/Q decreases
Real Cavities

TM010

$f = 1323$ MHz
Single Cell Cavities

- Electric field high at iris

- Magnetic field high at equator

![Diagram of single cell cavities](image)
Coupling between cells

Symmetry plane for the H field

Symmetry plane for the E field which is an additional solution

The normalized difference between these frequencies is a measure of the energy flow via the coupling region

\[ k_{cc} = \frac{\omega_\pi - \omega_0}{\omega_\pi + \omega_0} \left/ 2 \right. \]
Multi-Cell Cavities

\[ k = \frac{C}{C_k} \quad \text{and} \quad C_b = \frac{C_k}{2} \]

Mode frequencies:

\[ \frac{\omega_m^2}{\omega_0^2} = 1 + 2k \left( 1 - \cos \frac{\pi m}{n} \right) \]

\[ \frac{\omega_n - \omega_{n-1}}{\omega_0} \approx k \left( 1 - \cos \frac{\pi}{n} \right) \approx \frac{k}{2} \left( \frac{\pi}{n} \right)^2 \]

Voltages in cells:

\[ V_j^m = \sin \left( \pi m \frac{2j-1}{2n} \right) \]
Pass-Band Modes Frequencies

\[ \omega_0 (1 + 4k)^{1/2} \]

9-cell cavity
Cell Excitations in Pass-Band Modes

9 Cell, Mode 1

9 Cell, Mode 2

9 Cell, Mode 3

9 Cell, Mode 4

9 Cell, Mode 5

9 Cell, Mode 6

9 Cell, Mode 7

9 Cell, Mode 8

9 Cell, Mode 9