Single-particle longitudinal dynamics and magnetic bunch compression

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revised 21-June 2015
Outline

   - Ultra-relativistic approximation
   - Acceleration through standing-wave structure
   - Generating an “energy-chirp”

2. Magnetic-chicane compressors
   - Conceptual picture
   - The 4-bend chicane compressor
   - Alternate magnetic compression layouts
   - Momentum compaction and compression factor

3. The need to linearize the beam longitudinal phase-space for efficient compression
Longitudinal dynamics: motion through an accelerating structure

- Neglect transverse motion for now and focus on longitudinal motion
  - In first approximation the longitudinal dynamics is unaffected by the transverse motion.

- Longitudinal dynamics through an accelerating structure.
  - Dynamics is driven by the longitudinal component of electric field
  - Consider standing-wave structures

On-axis Longitudinal E-field for TESLA Cavity

\[
E_Z(t, s) = E_{Z0}(s) \cos(\omega_{\text{rf}} t + \varphi_{\text{rf}})
\]

1.3GHz, Super Conducting (SC) 9-Cell Tesla RF cavities are operated as standing-wave structures

On axis (x=y=0) accelerating field:

Design structure so that as the electron moves from cell to cell it sees the same sign of \( E_Z \).

\[
\omega_{\text{rf}} = \frac{2\pi}{f_{\text{rf}}}
\]

rf frequency

rf phase
Acceleration through a standing-wave structure

**Operating mode**
- the electron travels through one cell in half rf period → cell length is half the rf wavelength $\lambda_{rf} = c/f_{rf}$
- “$\pi$-mode” is the standard operating mode for standing-wave structures

$$E_z(t, s) = E_{z0}(s) \cos(\omega_{rf} t + \varphi_{rf})$$
Equations of motion for single particle dynamics

- Neglect collective effects, dissipative effects (e.g. radiation)
- Canonical (Hamiltonian) formalism

Define coordinate system, orbit of reference particle:
- Transverse coordinates \( x = p_x = y = p_y = 0 \) (ideally corresponding to center of magnets, accelerating structures, etc)
- Longitudinal coordinates for an individual particle:
  - different options are possible depending on choice of the independent variable with respect to which we parametrize the orbit:

<table>
<thead>
<tr>
<th>Independent variable</th>
<th>1st canonical coordinate</th>
<th>2nd canonical coordinate</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t ), time</td>
<td>( s(t) ), orbit path-length</td>
<td>( p_s(s) ), longitudinal momentum</td>
</tr>
<tr>
<td>( s ), orbit path-length</td>
<td>( t(s) ), time of flight</td>
<td>( E(s) ), (total) energy</td>
</tr>
</tbody>
</table>

- Coordinates for any particle in bunch can be expressed in terms of deviations from reference orbit
Hamiltonian for charged-particle interaction with E&M field and equations of motion through rf structures

• **General Hamiltonian** with time $t$ as the independent variable [i.e. a particle orbit solution of the canonical equations are $(\vec{x}, \vec{p}) = (\vec{x}(t), \vec{p}(t))$ ]

\[
H = \sqrt{m^2c^4 + c^2(\vec{p} - q\vec{A}(\vec{x}))^2} + q\phi(\vec{x})
\]

\[
\vec{B}(\vec{x}) = \vec{\nabla} \times \vec{A}(\vec{x})
\]

\[
\vec{E}(\vec{x}) = -q\vec{\nabla} \phi(\vec{x})
\]

• **Specialize Hamiltonian to longitudinal motion** for on-axis electron ($q = -e$) interacting with electric field $E_z(t, s) = E_{z0}(s)\cos(\omega_{rf}t + \varphi_{rf})$.
  
  – $\vec{s}$ direction is the same as the $\vec{z}$ direction
  
  – Identify scalar potential associated with field from standing wave structure

\[
\phi = \cos(\omega_{rf}t + \varphi_{rf}) \int_0^s E_{z0}(s')ds'
\]

\[
H = \sqrt{m^2c^4 + p_s^2c^2 - e\phi(s, t)}
\]

• **Equations of motion** :

\[
\frac{dp_s}{dt} = -\frac{\partial H}{\partial s} = -eE_{z0}(s)\cos(\omega_{rf}t + \varphi_{rf})
\]

\[
\frac{ds}{dt} = \frac{\partial H}{\partial \vec{p}_s} = \frac{c^2p_s}{\sqrt{m^2c^4 + c^2p_s^2}} = \frac{p_s}{m\gamma}
\]
Things are simpler in the ultrarelativistic approximation

- Through most of the linac the electron velocity is close enough to speed of light \( v_z \sim c \), i.e. \( \gamma = \infty \)

\[
\frac{ds}{dt} \simeq c
\]

\[
\frac{dp_s}{dt} = -e E_{z0}(s) \cos(\omega_{rf} t + \varphi_{rf})
\]

- **Note:** The ultra-relativistic approx. does not apply in the injector. Non-relativistic motion can be exploited to do bunch compression (velocity bunching). We'll see this tomorrow.

- **Caution:** for proper modeling of other aspects of beam dynamics (e.g. space charge) \( \gamma \) cannot be taken to be infinite in the Linac either.

- When the motion is ultra-relativistic it is more convenient to use \( s \) instead of \( t \) as the independent variable:

\[
\frac{dp_s}{ds} = \frac{dp_s}{dt} \frac{dt}{ds} \simeq -\frac{e}{c} E_{z0}(s) \cos(\omega_{rf} t(s) + \varphi_{rf})
\]

- What's the meaning? \( t(s) \) is the arrival time of the electron measured by an observer at longitudinal position \( s \)

- Say, \( s = 0 \) is the entrance of accelerating structure. Identify the particle that passes there at \( t = 0 \) as the reference particle. For this particle we have \( t \equiv t_r(s) = s/c \)
Specify particle coordinate w.r.t reference particle

- We specify the time of arrival of any particle in the bunch relatively to the time of arrival of the reference (or “synchronous”) particle i.e.:

\[ \Delta t(s) = t(s) - t_r(s) \]

\( \Delta t(s) < 0 \) means particle is ahead of reference particle (it arrives earlier at \( s \))

- Instead of expressing separation between particles in terms of difference of time of flight we can also express it in terms of distance, i.e.

\[ z = \Delta t/c \]

- Q: How does \( z \) (or \( \Delta t \)) vary as a particle travels through an accelerating structure?
  - A: In the ultra-relativistic approximation all particles are described as moving with the same velocity \( (\approx c) \). So the separation between particles doesn't change:

\[ \frac{dz}{ds} \approx 0 \]

- Next: Find the energy change experienced by an electron travelling through an RF travelling wave structure.
Energy change through rf structure

\[
\frac{dE}{ds} = -\frac{e}{c} E_{z0}(s) \cos(\omega_{rf} t(s) + \varphi_{rf}) = -\frac{e}{c} E_{z0}(s) \cos(\omega_{rf} (t_r(s) + \Delta t) + \varphi_{rf}) = -\frac{e}{c} E_{z0}(s) \cos(k_{rf} s + k_{rf} z + \varphi_{rf})
\]

- Ideally \( E_{z0}(s) \) is symmetric with respect the structure midpoint
  - Expand in a \( \cos \) Fourier-series, with vanishing values at boundaries. Assume idealized profile with exact symmetry w.r.t rf structure center

\[
E_{z0}(s) = \sum_m E_{0m} \cos(m \frac{\pi}{L_s} s) \quad m=1,3,5,...
\]

\[ s \in [-L_s/2, L_s/2] \]

\[ L_s = (n + 1/2) \lambda_{rf} \]

Energy change by electron with coordinate \( z \):

\[
\Delta E = c \int_{-L_s/2}^{L_s/2} ds \frac{dE}{ds} = e \left[ -\sum_m E_{0m} \int_{-L_s/2}^{L_s/2} \cos(m \frac{\pi}{L_s} s) \cos(k_{rf} s) ds \right] \cos(k_{rf} z + \varphi_{rf})
\]

\[
\Delta E(z) = eV \cos(k_{rf} z + \varphi_{rf})
\]

\( k_{rf} = \omega_{rf}/c \)

RF-wavenumber

\[ E_{a} \equiv \frac{V}{L_s} \]

Acceleration Voltage

Acceleration gradient
A couple of remarks on formula for energy change

\[ E(z) = E_i + eV \cos(k_{rf}z + \varphi_{rf}) \]

- Acceleration of reference particle at \( z = 0 \)

\[ E = E_i + eV \cos(\varphi_{rf}) \]  \hspace{1cm} (“zero-phase is on crest” rf-phase convention)

- Max. acceleration is when the cavity is operated "on crest" \( \varphi_{rf} = 0 \).
  - "Zero-field" crossing (when there is no net acceleration) corresponds to \( \varphi_{rf} = \pm \pi/2 \)
  - "In trough" corresponds to \( \varphi_{rf} = \pm \pi \)

- However, another convention is often used where rf phase is shifted by 90deg
  - In this case the "crest" corresponds to \( \varphi_{rf} = \pi/2 \)

\[ E(z) = E_i + eV \sin(k_{rf}z + \varphi_{rf}) \]

- Same formula applies to acceleration through travelling wave structures
How do we choose the rf phase?

- For maximum acceleration, the cavities should be operated on crest ...

Q: Why do we ever want to operate the cavities off-crest?
A: To control the beam "energy chirp", i.e. the correlation between a particle position $z$ within the bunch and its energy $E$

- The ability to put an energy chirp on a beam is needed to do bunch compression through a magnetic chicane (see following slides)

Electron beam longitudinal phase space

Beam without energy chirp

Beam with energy chirp

Definition of linear (relative) energy chirp

$h_1 = \frac{\Delta E}{E_0 L_b}$
Linear chirp from an rf structure operated off crest

Taylor expand through first order in $z$:

$$E(z) = E_i + eV \cos(k_{rf}z + \varphi_{rf}) \approx E_i + eV \cos \varphi_{rf} - eV k_{rf} z \sin \varphi_{rf} + O(z)^2$$

Linear chirp (exit of structure)

$$h_1 = \frac{1}{E(0)} \frac{dE(0)}{dz} = -\frac{eV k_{rf} \sin \varphi_{rf}}{(E_i + eV \cos \varphi_{rf})} \approx \frac{E(L_b/2) - E(-L_b/2)}{E(0) L_b}$$

Example of off-crest acceleration:

$$f_{rf} = 1.3 \text{ GHz}$$

$$\lambda_{rf} = 23 \text{ cm}$$

$$V_0 = 129 \text{ MV}$$

$$\varphi_{rf} = -30.3^\circ$$

$$h_1 \sim \frac{0.034}{0.004m} = 8.5 m^{-1}$$
How can we compress an ultra-relativistic beam?

• Longitudinal density (peak current) of bunches out of injector is typically too low (10s A) for efficient lasing (we need 100s A, at least). We need to compress the bunch.

• To compress the bunch we need to be able to change the electrons’ longitudinal coordinate $z$.

• We have problem: equation of motion of ultra-relativistic electron (through an accelerating structure or transport line):

$$\frac{dz}{ds} \approx 0$$

Relative longitudinal position of particles in the bunch does not change (the beam is ‘frozen’).

We need to make the electrons interact with something so that the electrons can slip with respect to each other in some controllable way.

Solution: bring in a magnetic field
Coming up with a concept for a bunch compressor I

- Basic observation: particles with different energy in magnetic field follow different trajectories (e.g. a spectrometer)
  - A spectrometer exploits the particle separation in the transverse direction (x). We are interested in the fact that this is associated with different path-lengths. Meaning: a magnetic field also introduces a separation longitudinally.

\[
\frac{1}{B \rho} = \frac{q}{p} \approx \frac{qc}{E}
\]

Suppose particles injected have all the same long. coordinate as the reference particle \(z=0\)

\[
\text{Radius of curvature}
\]

\[
\text{Ultra-relativistic approximation}
\]

Dipole

Magnetic field \(B\) perpendicular to trajectories

Higher-energy particle trails behind (the trajectory is longer) \(z > 0\)

Snap-shot taken when particle with lower energy leaves the magnet

Lower-energy particle skips ahead \(z < 0\)

Note: in this configuration the dipole stretches out rather than compressing the beam ...

\[
\text{Ref. particle has energy } E_0
\]
Coming up with a design for a compressor II

- By introducing a properly defined correlation between $E$ and $z$ we can control the differential path-length among portions of the beam and effectively compress

\[
E = E_0 \\
E < E_0 \\
E > E_0
\]

Suppose the particle with higher (lower) energy is in the head (tail) of bunch

A single 180 deg bending magnet could in principle be used as a compressor but dispersion in the $x$-direction is not good for us ...
From concept to realization of a practical compressor

• The spectrometer example tells us that we can use dipoles (magnetic field) and particle energy/position correlation within bunch correlation to compress
  – Happily we know how to create an E/z correlation: Accelerate off-crest!
  – A single magnet in principle would work (not in practice...)

• Problem: find a combination of dipoles that satisfies the following requirements:
  – The system should be an overall achromat (after we are done with compression electrons with different energy should not spread out horizontally)
  – Vanishing overall bend angle (After compression the beam “keeps going straight”, unless we are designing a different kind of machine e.g. an ERL)
  – Modest bend angle for each dipole (short magnets; and synchrotron radiation emitted does not perturb the beam too much - more on this later)
The most popular bunch compressor: four-dipole, C-shape chicane

- Bend angle for on-momentum (reference) particle:
  \[ \theta_0 \approx \frac{L_B}{\rho} = \frac{eB}{p_0}L_B \]

- Bend angle for a particle off momentum
  \[ \theta = \frac{\theta_0}{1 + \delta} \quad \delta \equiv \frac{\Delta p}{p_0} \approx \frac{\Delta E}{E_0} \] (ultra-relativistic approx.)

- The system is an achromat by design (barring magnet errors/imperfections)
  \[ \theta_1 + \theta_2 + \theta_3 + \theta_4 = 0 \] (any particle momentum)
1st-order calculation of path-length dependence on $\delta$

$$\theta = \frac{\theta_0}{1 + \delta}$$

- Thin lens approximation for the dipoles (finite bend angle resulting from infinitesimally short dipole and infinitely large magnetic field): $\theta = \frac{L_B}{R_B} \rightarrow 0 = \text{finite}$

- Path-length of off-momentum electron: $S = \frac{2L_1}{\cos \theta} + L_2$

- Path-length of on-momentum (reference-particle) electron: $S_0 = \frac{2L_1}{\cos \theta_0} + L_2$

Path-length difference:
Aside on the “R”-matrix

- Electron coordinate in 6D phase space $\vec{x} = (x, x', y, y', z, \delta)$
- Linear dynamics from point $s_0$ to point $s_1$: $\vec{x}_1 = R(s_0 \rightarrow s_1)\vec{x}_0$

$$\vec{x}_0 \xrightarrow{\text{beamline}} \vec{x}_1 = R\vec{x}_0$$

- Most general form of transfer-matrix in Linac section containing horizontal bends (in the absence of x/y coupling and acceleration)

$$R = \begin{bmatrix}
R_{11} & R_{12} & 0 & 0 & 0 & R_{16} \\
R_{21} & R_{22} & 0 & 0 & 0 & R_{26} \\
0 & 0 & R_{33} & R_{34} & 0 & 0 \\
0 & 0 & R_{43} & R_{44} & 0 & 0 \\
R_{51} & R_{52} & 0 & 0 & 1 & R_{56} \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

By design, through a chicane:
$$R_{16} = R_{26} = R_{51} = R_{52} = 0$$
"$R_{56}$" for a 4-bend chicane

- **Longitudinal slippage?**
  \[ z_1 = z_0 + R_{51}x_0 + R_{52}x'_0 + R_{56}\delta_0 \]
  - or $\Delta z = z_1 - z_0 = R_{51}x_0 + R_{52}x'_0 + R_{56}\delta_0$

- **What is $R_{56}$ for a chicane?** ($R_{51} = R_{52} = 0$, by design)
  - From previous slides: $\Delta s = -2L_1\theta^2_0\delta_0$
  - $\Delta t = \Delta s/c$. Recall we defined "z" as scaled time $\Delta z = c\Delta t$ therefore $\Delta z = \Delta s$
  - $R_{56} = -2L_1\theta^2_0$

  **Negative sign: a higher-energy particle follows a shorter path and ends up ahead ($z < 0$, according to our sign convention).**

- **Beyond linear dynamics:**
  - $X_i = R_{ij}x_j + T_{ijk}x_jx_k + \ldots$ where $x_i = (\vec{x})_i$

- **Longitudinal slippage:**
  \[ z_1 = z_0 + R_{56}\delta_0 + T_{566}\delta^2_0 + \ldots \]

  **We’ll get back to this later**
Expression for compression factor

Initial bunch

$E$

$z$

$L_b = 2l_b$

bunch length

head
Expression for compression factor

Initial bunch

\[ L_b = 2l_b \]

\[ h_1 = -\frac{eV_k \sin \varphi_{rf}}{(E_i + eV \cos \varphi_{rf})} \]

rf structure operated off-crest

\[ E(z) = E_i + eV \cos(k_{rf} z + \varphi_{rf}) \]

|ΔZ| = |R_{56}| \frac{ΔE}{E} = |R_{56}| h_1 l_b

Bunch enters chicane

\[ l'_{b} = l_b - |R_{56}| h_1 l_b \]

Bunch exits chicane

\[ C = \frac{l_b}{l'_{b}} = \frac{1}{|1 + R_{56} h_1|} \]
A more formal definition of compression factor

• **Action through the compressor:**
  \[ z_1 = z_0 + R_{56} \delta_0 = z_0 + R_{56} \frac{E(z_0) - E_{BC}}{E_{BC}} \]

• **Differentiate:**
  \[ \Delta z_1 = \Delta z_0 + R_{56} \frac{1}{E_{BC}} \frac{dE(z_0)}{dz_0} \Delta z_0 \]
  \[ \Delta z_1 = \Delta z_0 \left( 1 + R_{56} \frac{1}{E_{BC}} \frac{dE(z_0)}{dz_0} \right) \]
  \[ \Delta z_1 = \Delta z_0 (1 + R_{56} h_1) \equiv \Delta z_0 / C \]
  \[ |\Delta z_1| \equiv |\Delta z_0| / C \]

If \( E(z_0) \) - the energy chirp - is nonlinear then \( C \) depends on \( z \) (compression will vary along bunch). Generally, we refer to \( C(z = 0) \) as the nominal compression factor.

Think of beam as a line in \( E/z \) phase space (negligible slice energy spread)
Example of macroparticle simulation: off-crest acceleration + compression

(Idealized) beam out of the injector
E=100MeV

Beam accelerated off-crest to E=210MeV

Beam @ exit of compressor

Small energy chirp

Parabolic current profile

Distorted parabolic profile (dynamics is not completely linear)

$C = \frac{I_f}{I_i} = \frac{0.145 A}{0.045 A} \approx 3.2$

$C = \frac{\sigma_{zi}}{\sigma_{zf}} = \frac{450 \mu m}{144 \mu m} \approx 3.1$
Various options for bunch compressor design

**C-shape chicane**

- FLASH
- LCLS
- FERMI
- X-FEL
- SACLA

Bunch head < 0

\[ R_{56} \approx -2\theta^2 \left( \frac{L_T}{2} - \frac{4}{3} L_B - \frac{\Delta L_c}{2} \right) \]

simple, achromatic

\[ R_{56} \approx -2\theta^2 \left( \frac{L_T}{2} - \frac{4}{3} L_B \right) \]

achromatic

**S-shape chicane**

- FLASH
- X-FEL

**Arc (e.g. FODO-cell)**

- SLC arcs
- NLC BC2

\[ R_{56} \approx \frac{\theta^2 L_T}{4N_c^2 \sin^2(\mu_c/2)} \]

reverse sign

Formulas valid in the small-angle approx. Courtesy of P. Emma
Cranking up compression ...

(Idealized) beam out of the injector
E=100MeV

Beam accelerated off-crest to E=210MeV

Beam @ exit of compressor

Not apparent on this scale there is a small quadratic term in the chirp

Compression magnifies the curvature

Current spike results

\[ \langle E \rangle = 0.099 \text{ GeV}, \quad N_e = 0.125 \times 10^{10} \]

\[ \langle z \rangle = 0.001 \text{ mm} \]

\[ \sigma_z = 0.450 \text{ mm (fwhm=1.404)} \]

\[ \langle E \rangle = 0.210 \text{ GeV}, \quad N_e = 0.125 \times 10^{10} \]

\[ \langle z \rangle = 0.066 \text{ mm} \]

\[ \sigma_z = 11.989 \mu\text{m (fwhm=4.596)} \]

\[ I_{pk} = 0.045 \text{ kA} \]

\[ I_{pk} = 0.045 \text{ kA} \]

\[ I_{pk} = 6.420 \text{ kA} \]
Non-linearities in the rf waveform compromise beam quality after compression

- **Spiky current profiles are generally not desirable**
  - *We like high peak current, but if the beam is very spiky only a small fraction of the beam may end up having sufficiently high current*
  - Spiky currents are associated with large energy spread (not good)
  - *If we do external-laser seeding in rgw FEL we like to have a bunch core where the current is about uniform*
  - *rf and other wakefield effects are magnified by presence of spikes and will make 'spikiness' even worse when bunch is further compressed.*

- **Is there a way to fix this?**
  - *One can deal with the problem by reducing compression (not good).*
  - Choosing a small $k_{rf}$ for the accelerating structure (not practical; generally, choice of rf frequency is determined by other considerations)

- **Effective solution was proposed by D. Dowell, et al. in the ~90's**
  - *Compensate the dominant (quadratic) nonlinearity by use of harmonic cavities*
Analysis of rf waveform nonlinearities through accelerating rf section

\[ E_I = E_i + eV \cos(kz + \varphi) \approx E_i + eV \cos \varphi - kzeV_0 \sin \varphi - e \frac{Vk^2}{2} z^2 \cos \varphi + O(z^3) \]

Energy of particle at exit of accelerating structure

- How can we compensate the quadratic term?
  - Idea: pass beam through a second rf section (with different rf wavenumber)

\[ E_{II} = E_I + eV_H \cos(k_H z + \varphi_H) \approx E_I + eV_H \cos \varphi_H - k_H z eV_H \sin \varphi - e \frac{VHk_H^2}{2} z^2 \cos \varphi_H + O(z^3) \]

Q: How can we win? (i.e. compensate 2\textsuperscript{nd} order term and still have overall acceleration?)
A: Choose \( k_H > k \)
3rd-order Harmonic Linearizer at FLASH (3.9GHz)

- Operationally linearizer rf frequency is best chosen to be a harmonic number of rf frequency of accelerating structures (FLASH uses 1.3GHz SC accelerating structures)

Installation of cryomodule w./ linearizer

Time-resolved measurements of longitudinal phase space
Formula for setting of linearizer revisited: Life is always more complicated...

- Nonlinear momentum compaction in chicane is usually non-negligible and has to be compensated too

\[ z_1 = z_0 + R_{56} \delta_0 + T_{566} \delta_0^2 \]

- Modified setting of harmonic cavity when accounting for the 2\textsuperscript{nd} order term \( T_{566} \) in momentum compaction (Homework Exercise):

\[
e V_H = \frac{k^2}{k^2_H - k^2} \left\{ E_{BC} \left[ 1 + \frac{2}{k^2} \frac{T_{566}}{|R_{56}|^3} \left( 1 - \frac{1}{C} \right)^3 \right] - E_i \right\}
\]

- Formula valid for \( \phi_H = -180^\circ \) and one-stage (single chicane) compression
- If multiple compressors are present, \( V_H \) setting varies somewhat but typically not too much (after first BC the bunch is shorter and less vulnerable to rf nonlinearities)
- Further small adjustments may be needed to account for collective effects (rf wakefields, CSR).
- Alternate method to linearize: sextupole magnets within magnetic compressor (works well in arc-shaped compressors, not so well in chicanes where relatively small dispersion tends to requires too-strong sextuple magnets)

For standard chicanes (Homework Exercise)

\[ T_{566} \approx -\frac{3}{2} R_{56} > 0 \]

Energy of beam entering Linac section

Beam energy @ compressor

(minimizing \( V_H \) favors doing compression at low energy)

Compression factor

\[ C = \frac{1}{|1 + R_{56} h_1|} \]
Summary highlights from this morning.

- Energy change by particle travelling through rf structure (ultra-relativistic approx.)
  \[ E(z) = E_i + eV \cos(k_{rf}z + \phi_{rf}) \]

- Linear chirp acquired by beam when rf structure is operated off-crest
  \[ h_1 = -\frac{eV k_{rf} \sin \phi_{rf}}{(E_i + eV \cos \phi_{rf})} \]

- Compression factor through beamline with finite momentum compaction \( R_{56} \)
  \[ C = \frac{1}{|1 + R_{56} h_1|} \]

- Momentum compaction for 4-bend C-shape chicane (thin lens approximation \( L_B \ll L_1 \)):
  \[ R_{56} = -2L_1 \theta_0^2 \]

- Setting of harmonic cavity linearizer:
  \[ eV_H = \frac{k^2}{k_{H}^2 - k^2} \left\{ E_{BC} \left[ 1 + \frac{2}{k^2} \frac{T_{566}}{|R_{56}|^3} \left( 1 - \frac{1}{C} \right)^3 \right] - E_i \right\} \]
Summary highlights

• Undulator radiation /FEL resonance equation; undulator parameter"

\[ \lambda = \frac{\lambda_u}{2\gamma^2} \left( 1 + \frac{K^2}{2} \right) \]

\[ K = \frac{eB_0\lambda_u}{2\pi mc} \approx 0.934\lambda_u [cm] B[T] \]

• FEL \( \rho \) (Pierce) parameter, 1D Theory FEL gainlength

\[ \rho = \frac{1}{4} \left[ \frac{1}{\pi^2} \frac{I}{L_A} \frac{\lambda_u^2}{\gamma^3 \sigma_x^2} (K \times [JJ])^2 \right] \]

\[ L_g \sim \frac{1}{4\pi \sqrt{3}} \frac{\lambda_u}{\rho} \]

• Requirements for beam relative energy spread and transverse rms emittance

\[ \sigma_\delta < \rho \]

\[ \varepsilon_\perp \lesssim \frac{\lambda}{4\pi} \]

• E-beam brightness

\[ B_6 = \frac{N}{\varepsilon_{nx}\varepsilon_{ny}\varepsilon_{nz}} \]

\[ B_5 = \frac{I}{\varepsilon_{nx}\varepsilon_{ny}} \]

\[ B_4 = \frac{Q}{\varepsilon_{nx}\varepsilon_{ny}} \]

• Emittance growth due to angular kick perturbation

\[ \frac{\Delta \varepsilon_x}{\varepsilon_{x0}} \approx \frac{\beta_x}{2\varepsilon_{x0}} \langle \Delta x'^2 \rangle \]
Supplemental material
Correlated vs. uncorrelated energy spread

So far we have assumed model beams with negligible uncorrelated $\sigma_{\delta u}$ (or ‘slice’) energy spread.

A finite $\sigma_{\delta u}$ limits the minimum bunch length that can be achieved (see next slide)
Modes of compression:

**under-compression**

More prevalent mode of compression

More prevalent mode of compression

Min. bunch length determined by uncorrelated energy spread

Sign of energy chirp is reversed

Projected energy spread before Compression...

\[ \sigma_{\delta} = \sqrt{(h_1 \sigma_z)^2 + \sigma_{\delta \nu}^2} \]

...same after compression

Initial uncorrelated energy spread

Slice energy spread after compression

\[ \sigma'_{\delta \nu} = \frac{\sigma_z}{\sigma'_{z}} \sigma_{\delta \nu} \sim C \sigma_{\delta \nu} \]

Bunch length after compression

\[ \sigma'_{z} = \sqrt{\frac{\sigma_z^2}{C^2} + (R_{56} \sigma_{\delta \nu})^2} \]

In next few slides:
How to work out these expressions.
How do particle distributions evolve in phase space?

The particle dynamics is described by a map from space \((q, p)\) to space \((q', p')\):

\[
\begin{align*}
q' &= q'(q, p) \\
p' &= p'(q, p)
\end{align*}
\]

- For Hamiltonian systems (volume preserving)

\[
f(q', p'; s') = f(q, p; s) = f(q(q'p'), p(q', p'); s)
\]
Evolution of beam distribution through compressor (longitudinal phase space) I

Coordinates: \((z_1, \delta_1)\)

Beam density: \(f(z_1, \delta_1; s_1)\)

\[
\delta_1 = \frac{\Delta E}{E_{BC}} = \frac{E - E_{BC}}{E_{BC}}
\]

- Assume linear approximation

\[
\begin{cases}
  z_2 = z_1 + R_{56} \delta_1 \\
  \delta_2 = \delta_1
\end{cases}
\]

\((\text{Particle energy doesn’t change})\)

- Assume gaussian model of beam distribution

\[
f(z_1, \delta_1) = \frac{N}{2\pi\sigma_{z1}\sigma_{\delta1}} \exp\left(-\frac{z_1^2}{2\sigma_{z1}^2} - \frac{(\delta_1 - h_1 z_1)^2}{2\sigma_{\delta1}^2}\right)
\]
Evolution of beam distribution through compressor II

\[
\begin{aligned}
  z_2 &= z_1 + R_{56}\delta_1 \\
  \delta_2 &= \delta_1
\end{aligned}
\]

Invert

\[
\begin{aligned}
  z_1 &= z_2 - R_{56}\delta_1 \\
  \delta_1 &= \delta_2
\end{aligned}
\]

\[
f(z_2, \delta_2; s_2) = f(z_1(z_2, \delta_2), \delta_1(z_2, \delta_2); s_1) =
\]

\[
\frac{N}{2\pi\sigma_{z1}\sigma_{\delta1}} \exp\left(-\frac{(z_2 - R_{56}\delta_2)^2}{2\sigma_{z1}^2} - \frac{[\delta_2 - h_1(z_2 - R_{56}\delta_2)]^2}{2\sigma_{\delta1}^2}\right)
\]

- Calculate rms bunch length and energy spread
  - Homework exercise

\[
\sigma_{z2}^2 = \langle (z_2 - \langle z_2 \rangle)^2 \rangle
\]

\[
\sigma_{\delta2}^2 = \langle (\delta_2 - \langle \delta_2 \rangle)^2 \rangle
\]

Where: \( \langle \rangle \equiv \int dz_2 d\delta_2 f(z_2, \delta_2) \)