Single-particle longitudinal dynamics and magnetic bunch compression

MV

revised 21-June 2015

Outline

1. **Single-particle equations of longitudinal motion through a Linac.**
   – Ultra-relativistic approximation
   – Acceleration through standing-wave structure
   – Generating an "energy-chirp"

2. **Magnetic-chicane compressors**
   – Conceptual picture
   – The 4-bend chicane compressor
   – Alternate magnetic compression layouts
   – Momentum compaction and compression factor

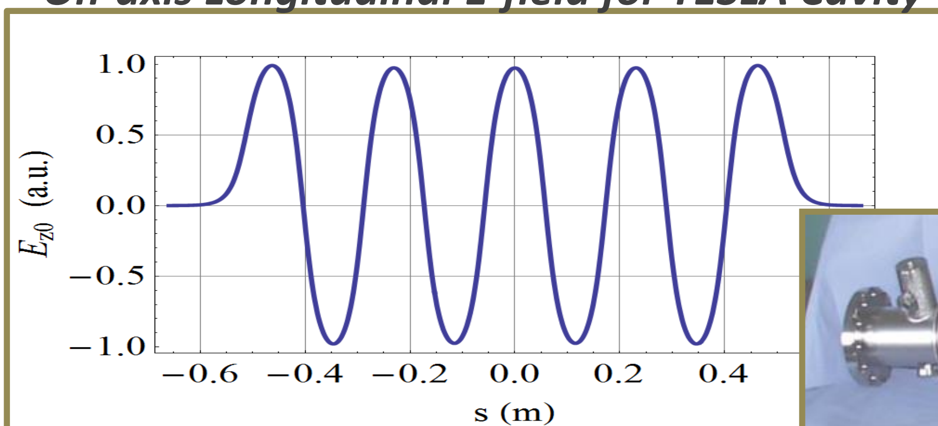
3. **The need to linearize the beam longitudinal phase-space for efficient compression**

Longitudinal dynamics: motion through an accelerating structure

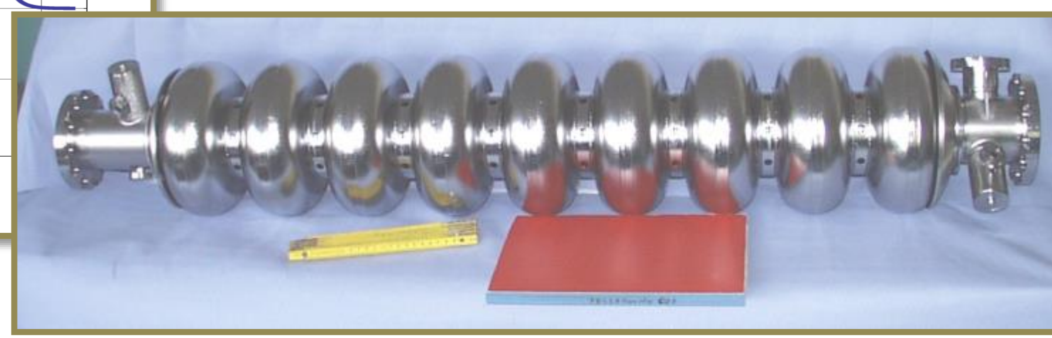
- **Neglect transverse motion for now and focus on <u>longitudinal motion</u>**
  - In first approximation the longitudinal dynamics is unaffected by the transverse motion.

- **Longitudinal dynamics through an accelerating structure.**
  - Dynamics is driven by the longitudinal component of electric field
  - Consider **standing-wave structures**

On-axis Longitudinal E-field for TESLA Cavity



1.3GHz , Super Conducting (SC) 9-Cell Tesla RF cavities are operated as standing-wave structures



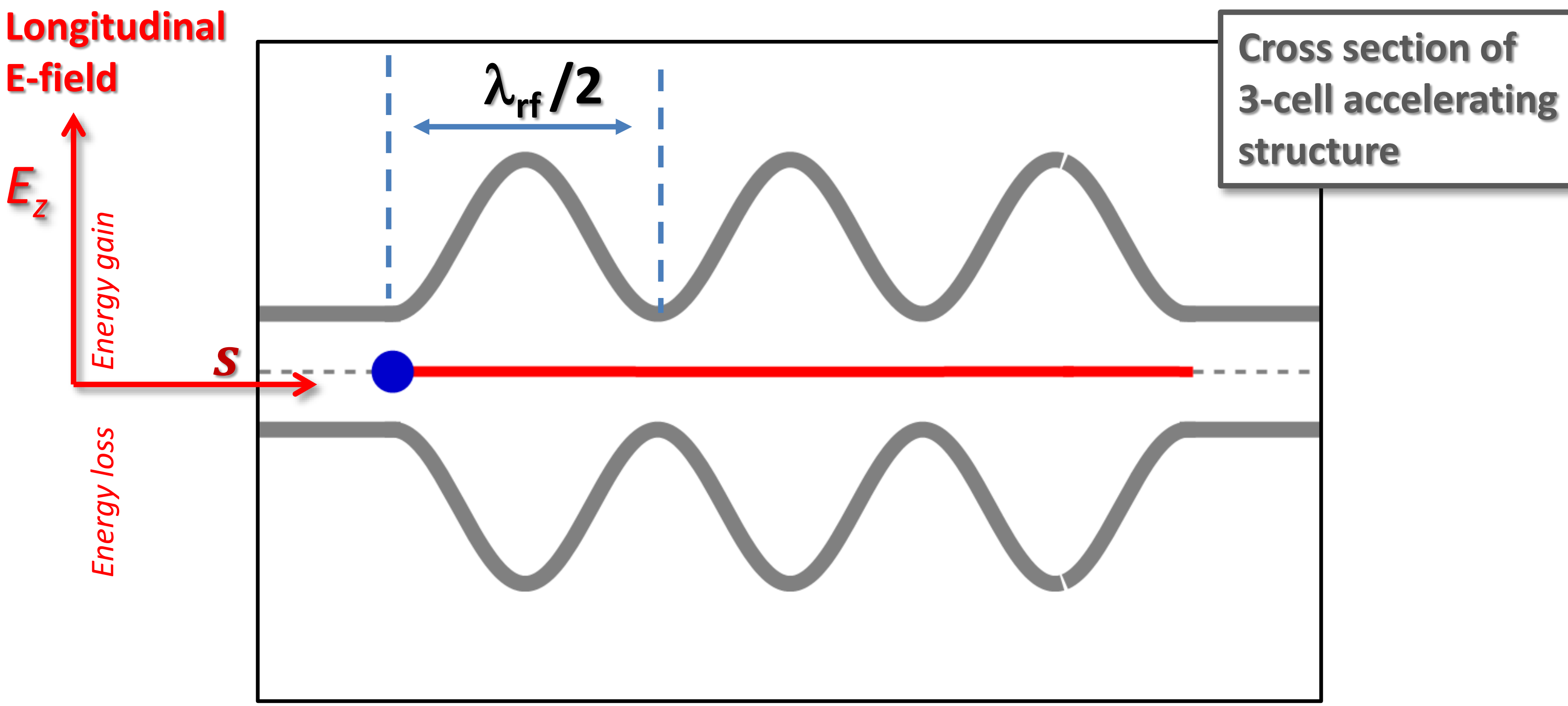
On axis (x=y=0) accelerating field: $$E_z(t,s) = E_{z0}(s)\cos(\omega_{\rm rf} t + \varphi_{\rm rf})$$

rf frequency $\omega_{\rm rf} = \frac{2\pi}{f_{\rm rf}}$

rf phase

Design structure so that as the electron moves from cell to cell it sees the same sign of E_z.

Acceleration through a standing-wave structure



$$E_z(t,s) = E_{z0}(s)\cos(\omega_{\mathrm{rf}} t + \varphi_{\mathrm{rf}})$$

- **Operating mode**
  - the electron travels through one cell in half rf period → cell length is half the rf wavelength $\lambda_{\mathrm{rf}} = c/f_{\mathrm{rf}}$
  - "π-mode" is the standard operating mode for standing-wave structures

Equations of motion for single particle dynamics

- **Neglect collective effects, dissipative effects (e.g. radiation)**
- **Canonical (Hamiltonian) formalism**
- **Define coordinate system, orbit of reference particle:**
  - Transverse coordinates $x = p_x = y = p_y = 0$ (ideally corresponding to center of magnets, accelerating structures ,etc)
  - Longitudinal coordinates for an individual particle:
  - *different options are possible depending on choice of the independent variable with respect to which we parametrize the orbit:*

Independent variable	1st canonical coordinate	2nd canonical coordinate
t , time	$s(t)$, orbit path-length	$p_s(s)$, longitudinal momentum
s , orbit path-length	$t(s)$, time of flight	$E(s)$, (total) energy

- **Coordinates for any particle in bunch can be expressed in terms of deviations from reference orbit**

Hamiltonian for charged-particle interaction with E&M field and equations of motion through rf structures

- **General Hamiltonian** with time $\boldsymbol{t}$ as the independent variable [i.e. a particle orbit solution of the canonical equations are $(\vec{x}, \vec{p}) = \overrightarrow{(x(t)}, \vec{p}(t))$]

$$H = \sqrt{m^2c^4 + c^2\left(\vec{p} - q\vec{A}(\vec{x})\right)^2} + q\phi(\vec{x})$$

$$\vec{B}(\vec{x}) = \vec{\nabla} \times \vec{A}(\vec{x}) \qquad \vec{E}(\vec{x}) = -q\vec{\nabla}\phi(\vec{x})$$

- **Specialize Hamiltonian to longitudinal motion** for on-axis electron $(q = -e)$ interacting with electric field $E_z(t,s) = E_{z0}(s)\cos(\omega_{\mathrm{rf}}t + \varphi_{\mathrm{rf}})$.
  - $\vec{s}$ direction is the same as the $\vec{z}$ direction
  - Identify scalar potential associated with field from standing wave structure $\boldsymbol{\phi} = \cos\left(\omega_{rf}t + \varphi_{rf}\right) \int_0^s E_{z0}(s')ds'$

$$H = \sqrt{m^2c^4 + p_s^2c^2} - e\phi(s,t)$$

- **Equations of motion** :

$$\boxed{\begin{aligned} \frac{dp_s}{dt} &= -\frac{\partial H}{\partial s} = -eE_{z0}(s)\cos(\omega_{\mathrm{rf}}t + \varphi_{\mathrm{rf}}) \\ \frac{ds}{dt} &= \frac{\partial H}{\partial p_s} = \frac{c^2p_s}{\sqrt{m^2c^4 + c^2{p_s}^2}} = \frac{p_s}{m\gamma} \end{aligned}}$$

Things are simpler in the ultrarelativistic approximation

- Through most of the linac the electron velocity is close enough to spead of light $\boldsymbol{v_z \sim c}$, i.e. $\gamma = \infty$

$$\boldsymbol{\frac{ds}{dt} \simeq c} \qquad \frac{dp_s}{dt} = -eE_{z0}(s)\cos(\omega_{\mathrm{rf}}t + \varphi_{\mathrm{rf}})$$

  - **Note:** The ultra-relativistic approx. does not apply in the injector. Non-relativistic motion can be exploited to do bunch compression (velocity bunching). We'll see this tomorrow
  - **Caution:** for proper modeling of other aspects of beam dynamics (e.g. space charge) γ cannot be taken to be infinite in the Linac either.

- When the motion is ultra-relativistic it is more convenient to **use $\boldsymbol{s}$ instead of $\boldsymbol{t}$** as the independent variable:

$$\boldsymbol{\frac{dp_s}{ds} = \frac{dp_s}{dt}\frac{dt}{ds} \simeq -\frac{e}{c}\, E_{z0}(s)\, cos(\omega_{\mathrm{rf}} t(s) + \varphi_{\mathrm{rf}})}$$

- What's the meaning? $\boldsymbol{t(s)}$ **is the arrival time** of the electron measured by an observer at longitudinal position $\boldsymbol{s}$

- Say, $s = 0$ is the entrance of accelerating structure. Identify the particle that passes there at $t = 0$ as the **reference particle**. For this particle we have $\boldsymbol{t \equiv t_r(s) = s/c}$

Specify particle coordinate w.r.t reference particle

- We specify the time of arrival of any particle in the bunch relatively to the time of arrival of the **reference (or "synchronous") particle** i.e.:

Δt(s)<0 means particle is ahead of reference particle (it arrives earlier at s)

$$\boldsymbol{\Delta t(s) = t(s) - t_r(s)}$$

- Instead of expressing separation between particles in terms of difference of time of flight we can also express it in terms of distance, i.e.

Use notation z instead of Δz for simplicity

$$\boldsymbol{z = \Delta t / c}$$

- Q: How does z (or Δt) vary as a particle travels through an accelerating structure?
  - A: In the ultra-relativistic approximation all particles are described as moving with the same velocity (= c). So the separation between particles doesn't change:

$$\frac{dz}{ds} \simeq 0$$

- Next: Find the energy change experienced by an electron travelling through an RF travelling wave structure.

Energy change through rf structure

$$\frac{dp_s}{ds} = -\frac{e}{c} E_{z0}(s)\cos(\omega_{\rm rf} t(s) + \varphi_{\rm rf}) = -\frac{e}{c} E_{z0}(s)\cos(\omega_{\rm rf}(t_r(s) + \Delta t) + \varphi_{\rm rf}) =$$

$$= -\frac{e}{c} E_{z0}(s)\cos(k_{\rm rf} s + k_{\rm rf} z + \varphi_{\rm rf})$$

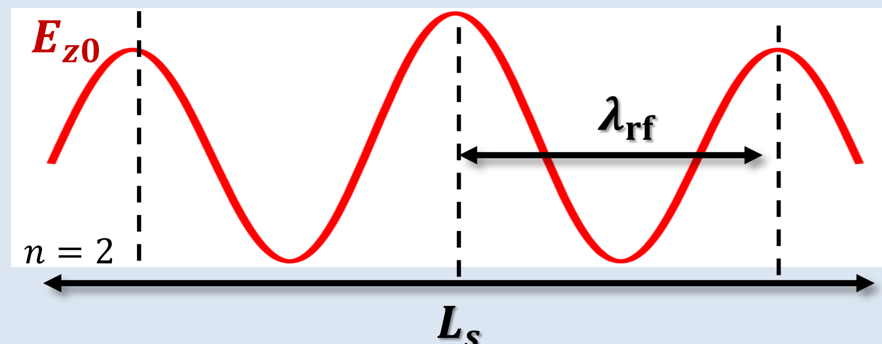
$k_{\rm rf} = \omega_{\rm rf}/c$
RF-wavenumber

- Ideally $E_{z0}(s)$ is symmetric with respect the structure midpoint
  - Expand in a cos Fourier-series, with vanishing values at boundaries. Assume idealized profile with exact symmetry w.r.t rf structure center

$$E_{z0}(s) = \sum_m E_{0m}\cos\left(m\,\frac{\pi}{L_s}s\right) \qquad m=1,3,5,\ldots$$

$$s \in [-L_s/2,\, L_s/2]$$

$$L_s = (n+1/2)\lambda_{\rm rf}$$



Energy change by electron with coordinate z:

$$\Delta E = c\int_{-L_s/2}^{L_s/2} ds\,\frac{dp_s}{ds} = e\underbrace{\left[-\sum_m E_{0m}\int_{-L_s/2}^{L_s/2}\cos\left(m\,\frac{\pi}{L_s}s\right)\cos(k_{rf}s)\,ds\right]}_{V}\cos(k_{\rm rf}z + \varphi_{\rm rf})$$

(sin term vanishes)

Acceleration Voltage

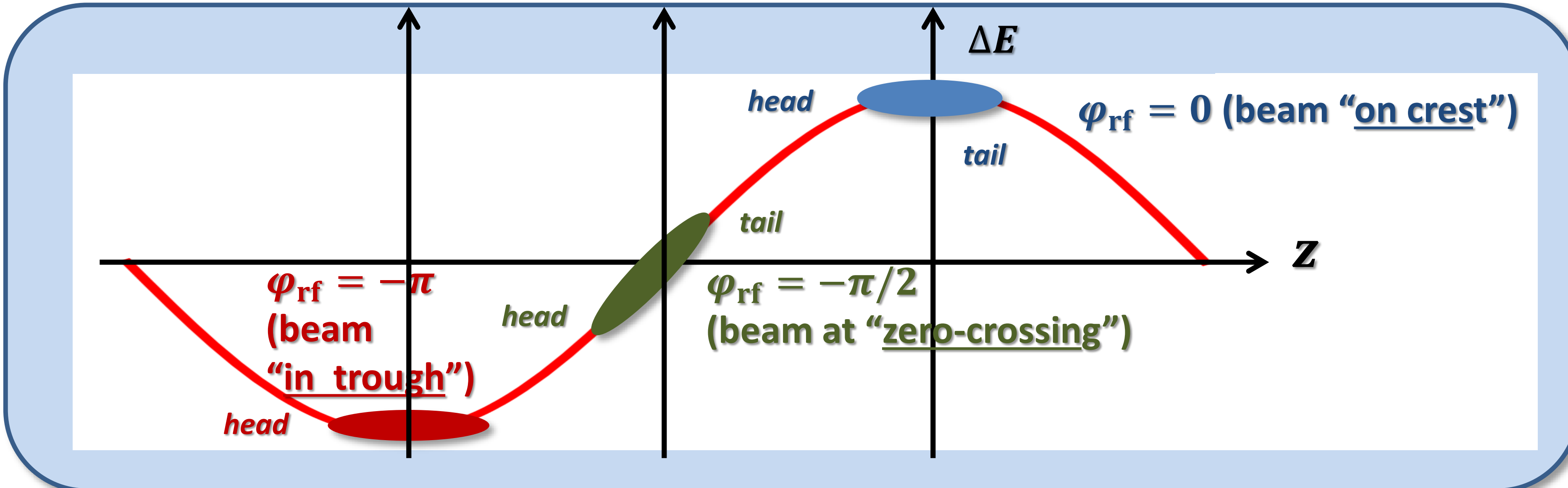
$$\mathrm{E}_a \equiv \frac{\mathbf{V}}{L_s}$$

Acceleration gradient

$$\boxed{\Delta E(z) = eV\cos(k_{\rm rf}z + \varphi_{\rm rf})}$$

A couple of remarks on formula for energy change

$$E(z) = E_i + eV\cos(k_{\mathrm{rf}}z + \varphi_{\mathrm{rf}})$$



- Acceleration of reference particle at $z = 0$

$$E = E_i + eV\cos(\varphi_{\mathrm{rf}})$$ (*"zero-phase is on crest" rf-phase convention*)

- Max. acceleration is when the cavity is operated "*on crest*" $\varphi_{\mathrm{rf}} = 0$.
  - "*Zero-field*" crossing (when there is no net acceleration) corresponds to $\varphi_{\mathrm{rf}} = \pm\pi/2$
  - "*In trough*" corresponds to $\varphi_{\mathrm{rf}} = \pm\pi$

- However, another convention is often used where rf phase is shifted by 90deg
  - In this case the "crest" corresponds to $\varphi_{\mathrm{rf}} = \pi/2$

$$E(z) = E_i + eV\sin(k_{\mathrm{rf}}z + \varphi_{\mathrm{rf}})$$

***Elegant* rf phase convention**

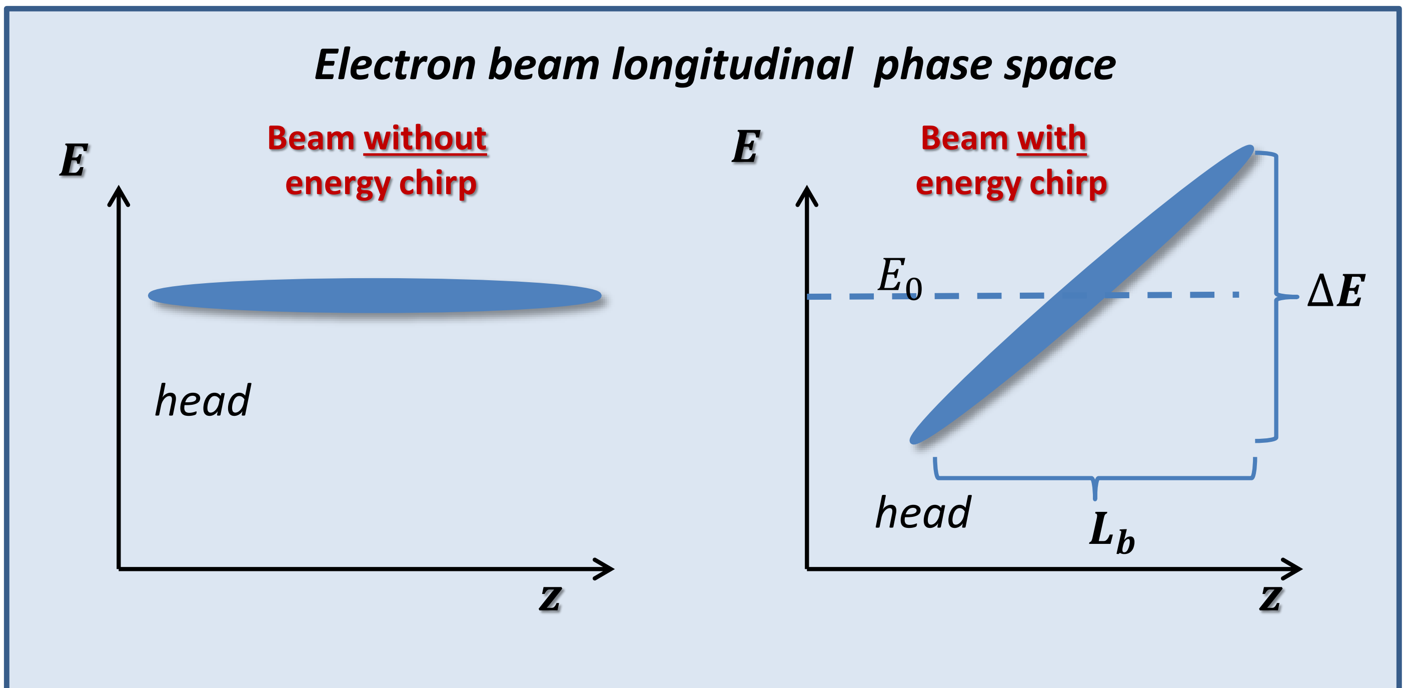
- Same formula applies to acceleration through travelling wave structures

How do we choose the rf phase ?

- For maximum acceleration, the cavities should be operated on crest …

Q: **Why do we ever want to operate the cavities off-crest?**
A: *To control the beam "energy chirp", i.e. the correlation between a particle position* z *within the bunch and its energy* E

- – The ability to put an energy chirp on a beam is needed to do bunch compression through a magnetic chicane (see following slides)



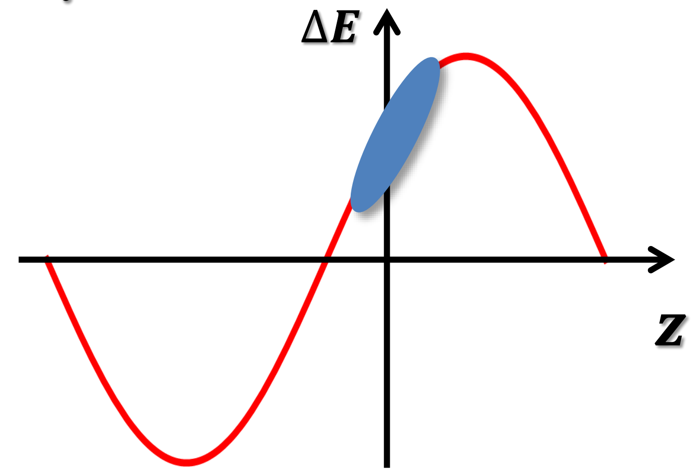
Definition of *linear (relative) energy chirp*

$$h_1 = \frac{\Delta E}{E_0 L_b}$$

Linear chirp from an rf structure operated off crest

Taylor expand through first order in z:

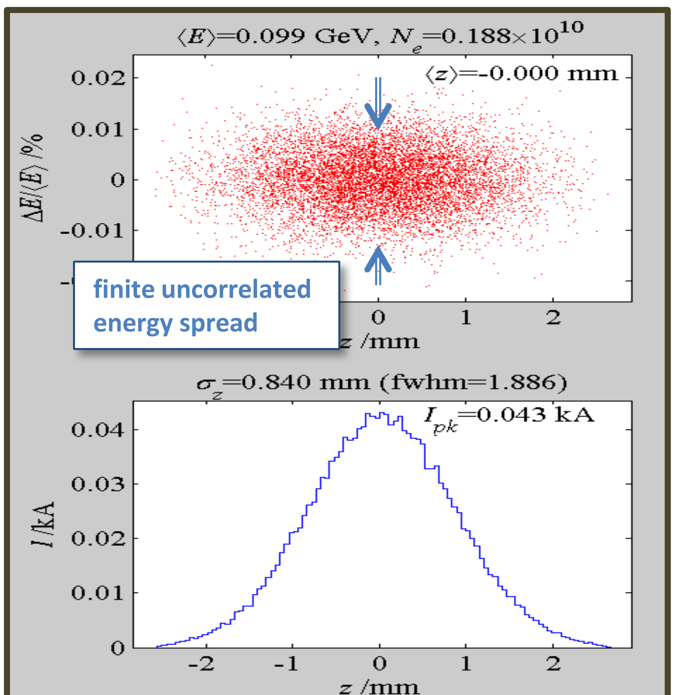
$$E(z) = E_i + eV\cos(k_{\mathrm{rf}}z + \varphi_{\mathrm{rf}}) \approx$$
$$E_i + eV\cos\varphi_{\mathrm{rf}} - eVk_{\mathrm{rf}}\,z\sin\varphi_{\mathrm{rf}} + O(z)^2$$



Linear chirp
(exit of structure)

$$\boldsymbol{h_1 = \frac{1}{E(0)}\frac{dE(0)}{dz} = -\frac{eVk_{\mathrm{rf}}\sin\varphi_{\mathrm{rf}}}{(E_i + eV\cos\varphi_{\mathrm{rf}})}}$$
$$\simeq \frac{E(L_b/2) - E(-L_b/2)}{E(0)\,L_b}$$

Beam @ entrance of structure



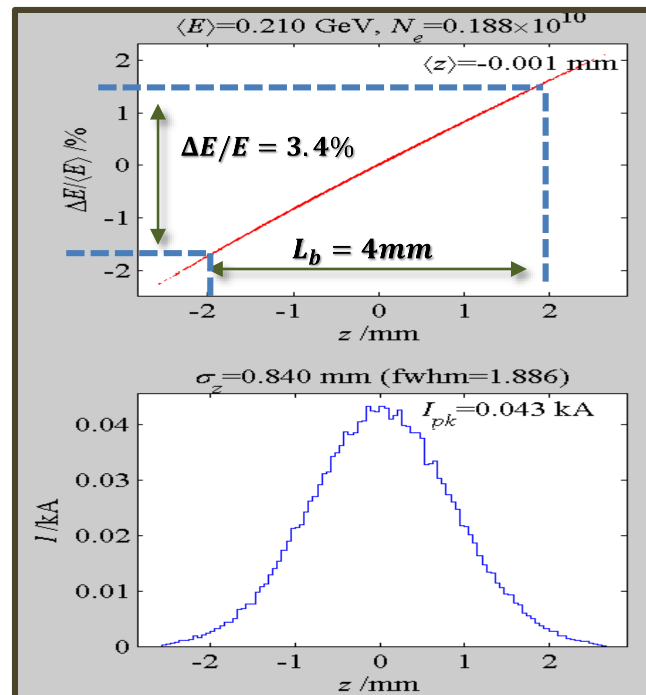
Example of off-crest acceleration:

f_{rf} =1.3 GHz
$\lambda_{\mathrm{rf}} = 23\ cm$

$V_0 = 129\ MV$
$\varphi_{\mathrm{rf}} = -30.3^o$

$$\boldsymbol{h_1 \sim \frac{0.034}{0.004m} = 8.5m^{-1}}$$

Beam @ exit of structure



How can we compress an ultra-relativistic beam?

- Longitudinal density (peak current) of bunches out of injector is typically too low (10s A) for efficient lasing (we need 100s A, at least). We need to compress the bunch.

- To compress the bunch we need to be able to change the electrons' longitudinal coordinate z

- We have problem: equation of motion of ultra-relativistic electron (trough an accelerating structure or transport line):

Relative longitudinal position of particles in the bunch does not change (the beam is 'frozen').

$$\boxed{\frac{dz}{ds} \simeq 0}$$

We need to make the electrons interact with something so that the ***electrons can slip with respect to each other*** *in some controllable way*

Solution: bring in a magnetic field

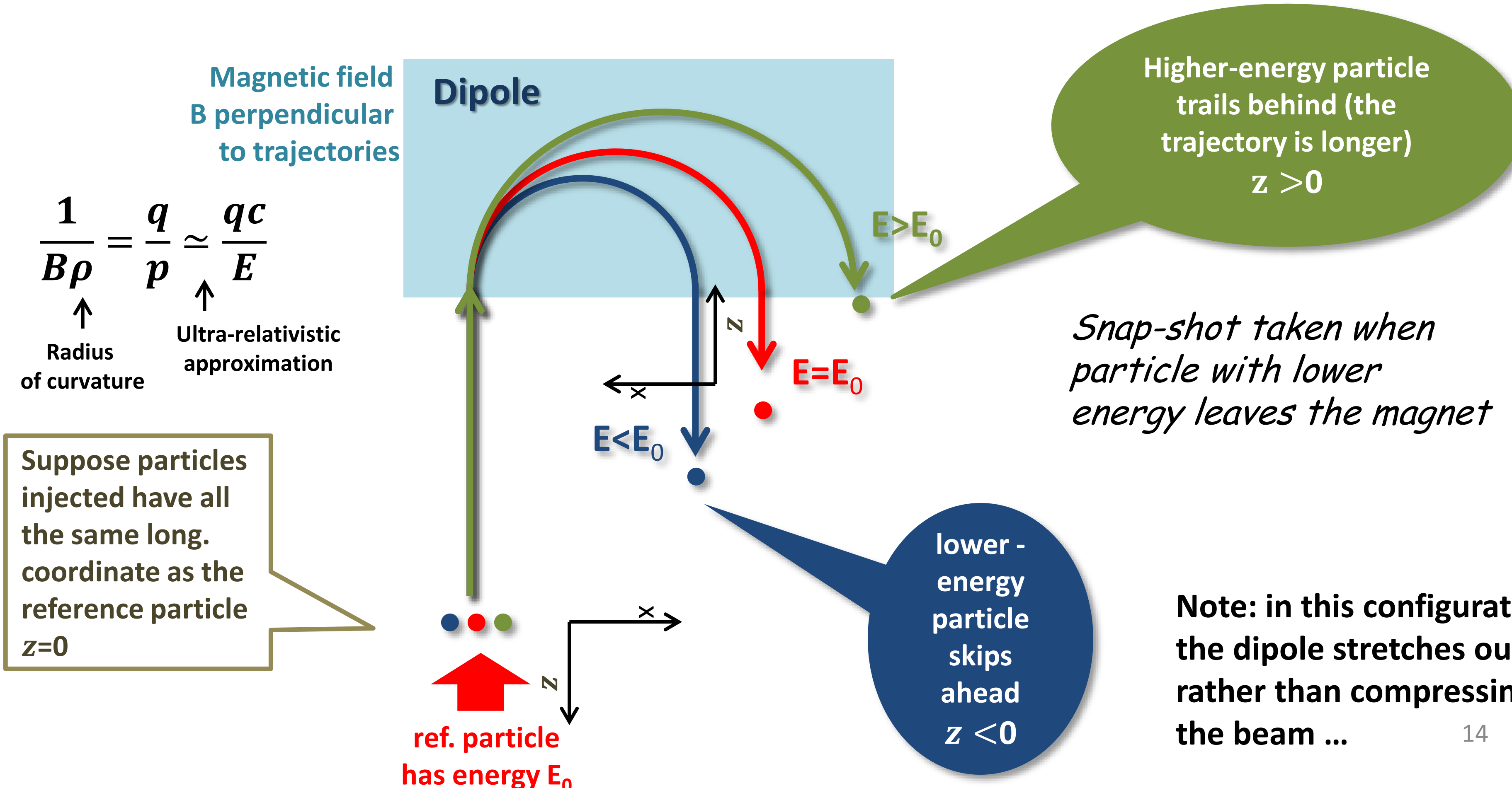
Coming up with a concept for a bunch compressor I

- **Basic observation: particles with different energy in magnetic field follow different trajectories (e.g. a spectrometer)**
  - A spectrometer exploits the particle separation in the transverse direction (x). We are interested in the fact that this is associated with different path-lengths. Meaning: a magnetic field also introduces a separation longitudinally

$$\frac{1}{B\rho} = \frac{q}{p} \simeq \frac{qc}{E}$$

Radius of curvature

Ultra-relativistic approximation

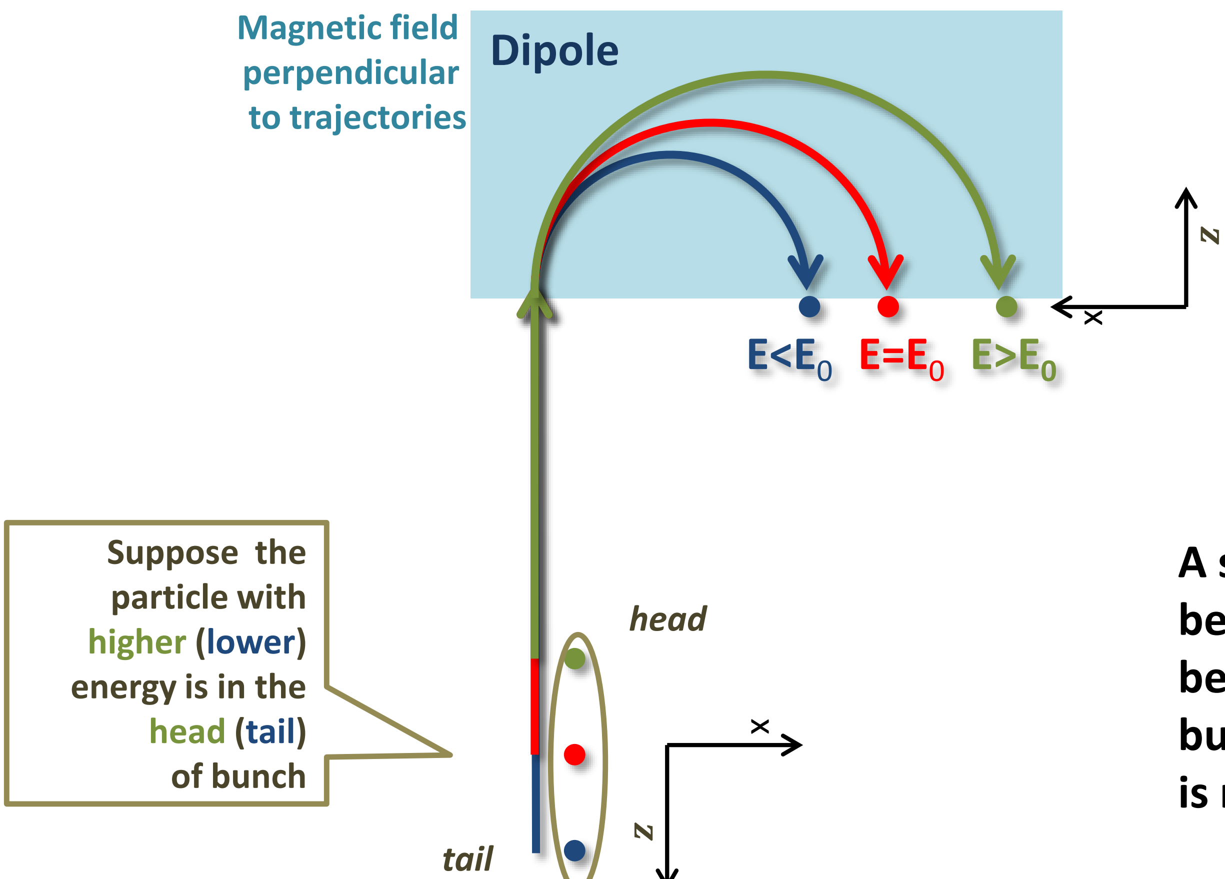


Snap-shot taken when particle with lower energy leaves the magnet

Note: in this configuration the dipole stretches out rather than compressing the beam …

Coming up with a design for a compressor II

- By introducing a properly defined correlation between E and z we can control the differential path-length among portions of the beam and effectively compress



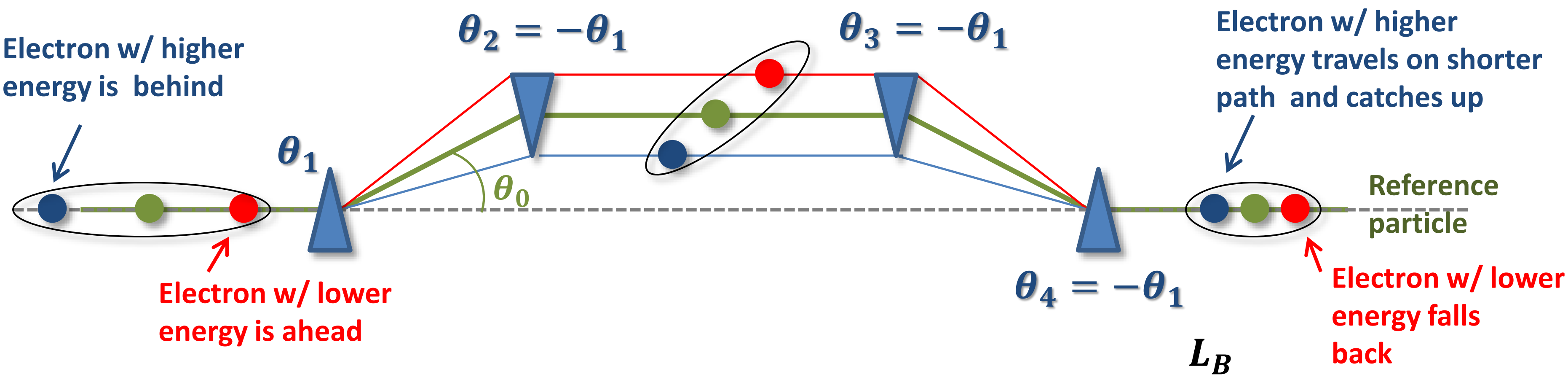
A single 180 deg bending magnet could in principle be used as a compressor but dispersion in the x-direction is not good for us …

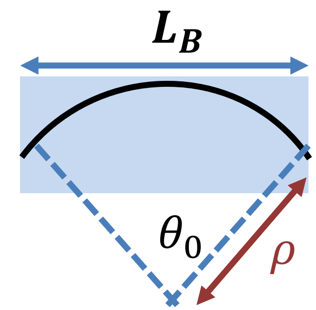
From concept to realization of a practical compressor

- **The spectrometer example tells us that we can use dipoles (magnetic field) and particle energy/position correlation within bunch correlation to compress**
  - Happily we know how to create an E/z correlation: Accelerate off-crest!
  - A single magnet in principle would work (not in practice…)

- **Problem: find a combination of dipoles that satisfies the following requirements:**
  - The system should be an overall **achromat** *(after we are done with compression electrons with different energy should not spread out horizontally)*
  - **Vanishing overall bend angle** *(After compression the beam "keeps going straight", unless we are designing a different kind of machine e.g. an ERL)*
  - **Modest bend angle** for each dipole *(short magnets; and synchrotron radiation emitted does not perturb the beam too much – more on this later)*

The most popular bunch compressor: four-dipole, C-shape chicane





- Bend angle for on-momentum (reference) particle:

$$\theta_0 \simeq \frac{L_B}{\rho} = \frac{eB}{p_0} L_B$$

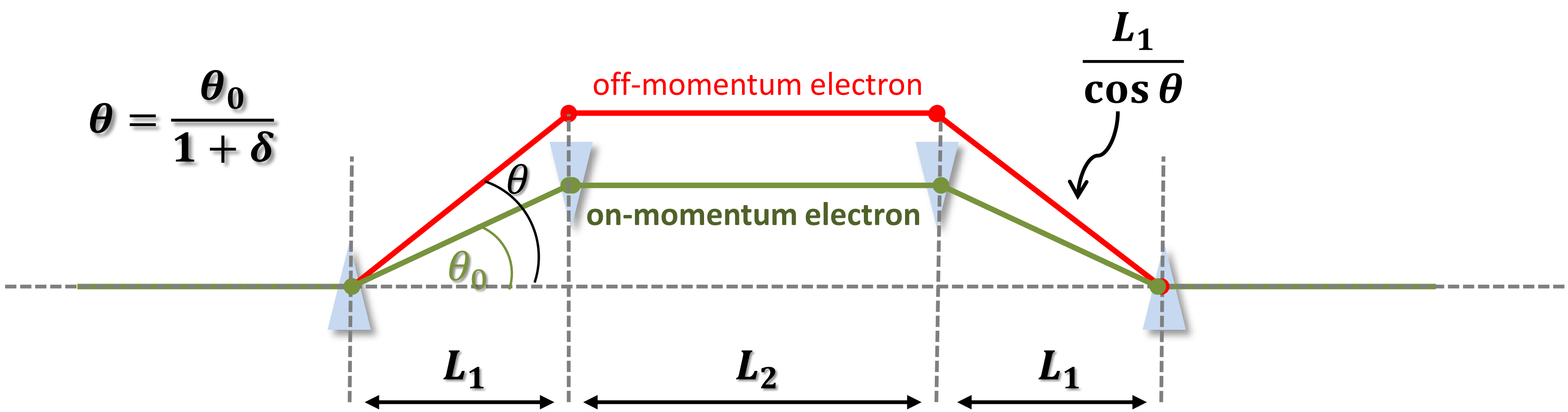
- Bend angle for a particle off momentum

$$\theta = \frac{\theta_0}{1+\delta} \qquad \delta \equiv \frac{\Delta p}{p_0} \simeq \frac{\Delta E}{E_0} \text{ (ultra-relativistic approx.)}$$

- The system is an achromat by design (barring magnet errors/imperfections)

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = 0 \quad (any\ particle\ momentum)$$

1^{st}-order calculation of path-length dependence on δ



- Thin lens approximation for the dipoles (finite bend angle resulting from infinitesimally short dipole and infinitely large magnetic field): $\theta = \frac{L_B \to 0}{R_B \to 0} = \boldsymbol{finite}$
- **Path-length of off-momentum electron:** $s = \dfrac{2L_1}{\cos\theta} + L_2$
- **Path-length of on-momentum (reference-particle) electron:** $s_0 = \dfrac{2L_1}{\cos\theta_0} + L_2$

Path-length difference:

Aside on the "R"-matrix

- Electron coordinate in 6D phase space $\vec{x} = (x, x', y, y', z, \delta)$
- Linear dynamics from point s_0 to point s_1: $\vec{x}_1 = R(s_0 \to s_1)\vec{x}_0$

$R(s_0 \to s_1)$

$\vec{x}_0$

$\vec{x}_1 = R\vec{x}_0$

Electron coordinate vector at entrance of beamline

s_0 s_1

Electron coordinate vector at exit of beamline

- Most general form of transfer-matrix in Linac section containing horizontal bends (in the absence of x/y coupling and acceleration)

$$
\boldsymbol{R} = \begin{bmatrix} R_{11} & R_{12} & 0 & 0 & 0 & R_{16} \\ R_{21} & R_{22} & 0 & 0 & 0 & R_{26} \\ 0 & 0 & R_{33} & R_{34} & 0 & 0 \\ 0 & 0 & R_{43} & R_{44} & 0 & 0 \\ R_{51} & R_{52} & 0 & 0 & 1 & R_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

By design, through a chicane:

$R_{16} = R_{26} = R_{51} = R_{52} = 0$

"R_{56}" for a 4-bend chicane

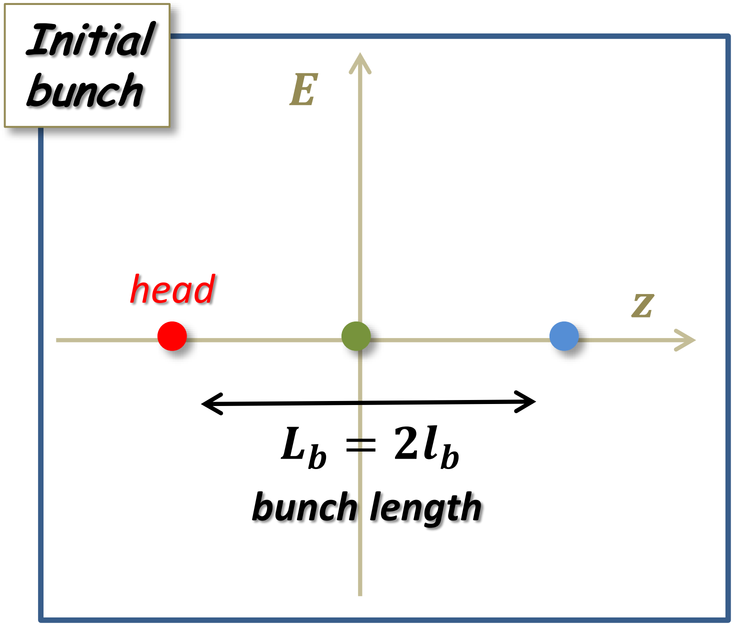
- **Longitudinal slippage?** $\boldsymbol{z_1 = z_0 + R_{51}x_0 + R_{52}x_0' + R_{56}\delta_0}$
  - or $\Delta z = z_1 - z_0 = R_{51}x_0 + R_{52}x_0' + R_{56}\delta_0$
- **What is $\boldsymbol{R_{56}}$ for a chicane?** ($\boldsymbol{R_{51} = R_{52} = 0}$, by design)
  - From previous slides: $\Delta s = -2L_1\theta_0^2\delta_0$
  - $\Delta t = \Delta s/c$. Recall we defined "z" as scaled time $\Delta z = c\Delta t$ therefore $\Delta z = \Delta s$
  - $\boldsymbol{R_{56} = -2L_1\theta_0^2}$

Negative sign: a higher-energy particle follows a shorter path and ends up ahead ($z < 0$, according to our sign convention).

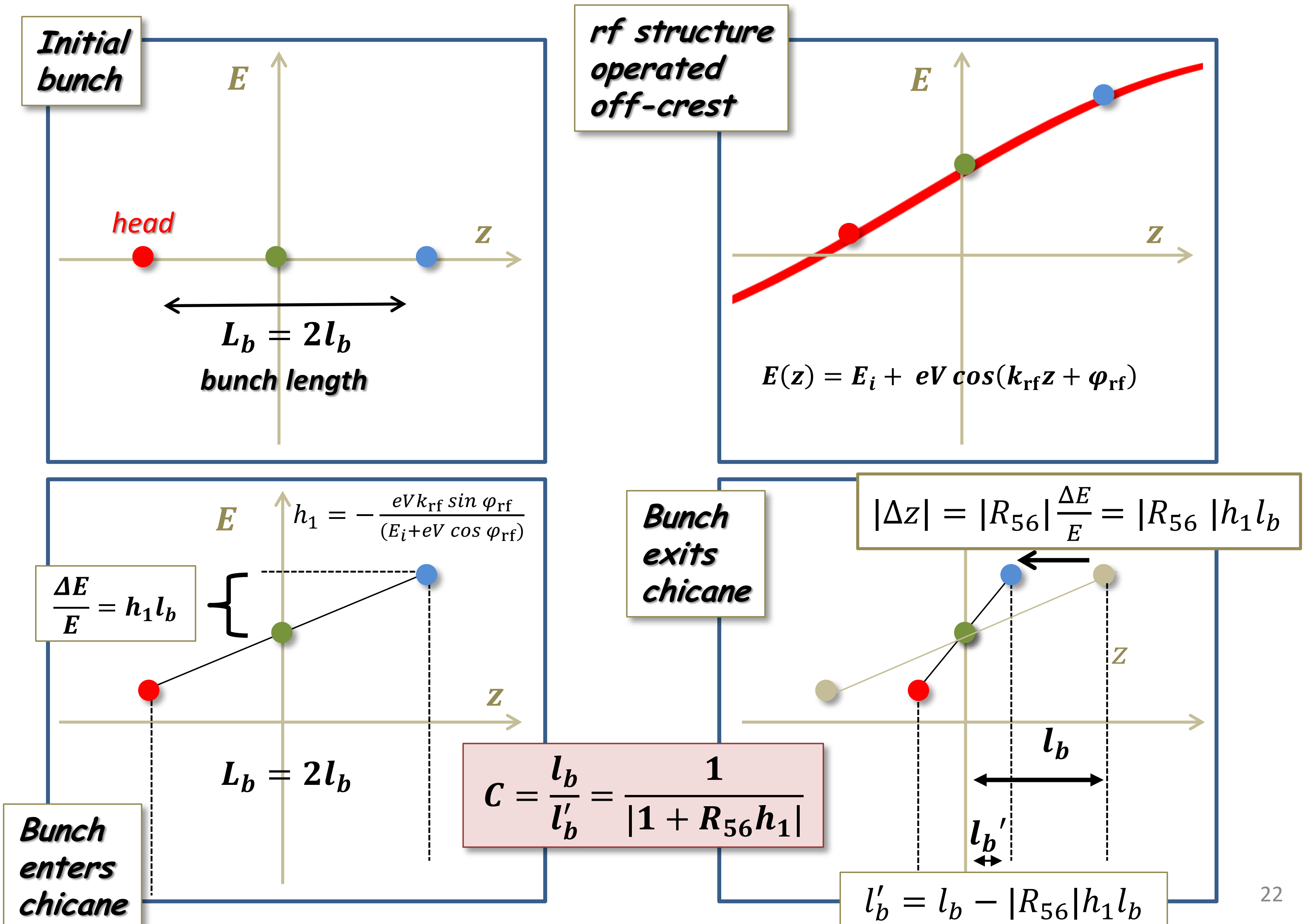
- **Beyond linear dynamics:**
- $X_i = R_{ij}x_j + T_{ijk}x_jx_k + \ \ldots$ where $x_i = (\vec{x})_i$
- **Longitudinal slippage:** $\boldsymbol{z_1 = z_0 + R_{56}\delta_0 + T_{566}\delta_0^2 + \ \ldots}$

We'll get back to this later

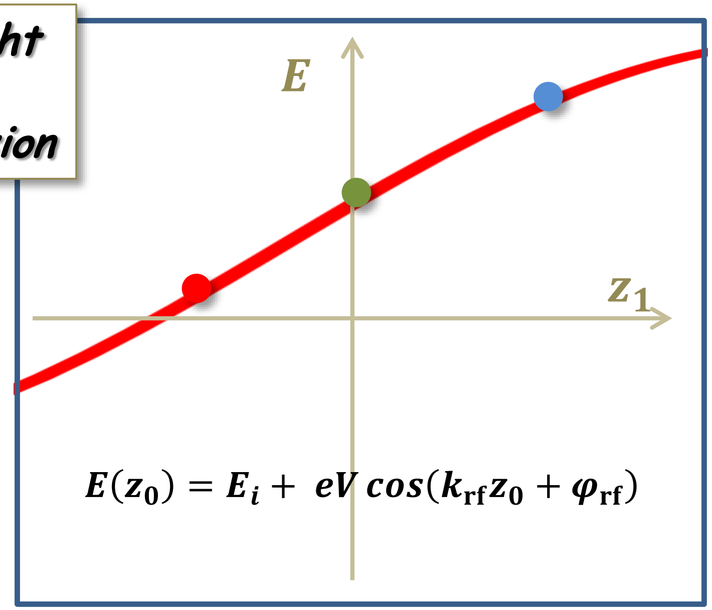
Expression for compression factor



Expression for compression factor



A more formal definition of compression factor



Think of beam as a line in
E/z phase space (negligible slice energy spread)

- Action through the compressor:

$$z_1 = z_0 + R_{56}\boldsymbol{\delta_0} = z_0 + R_{56}\frac{\boldsymbol{E(z_0) - E_{BC}}}{\boldsymbol{E_{BC}}}$$

- Differentiate:

$$\Delta z_1 = \Delta z_0 + R_{56}\frac{\mathbf{1}}{\boldsymbol{E_{BC}}}\frac{\boldsymbol{dE(z_0)}}{\boldsymbol{dz_0}}\Delta z_0$$

$$\Delta z_1 = \Delta z_0\left(1 + R_{56}\frac{\mathbf{1}}{\boldsymbol{E_{BC}}}\frac{\boldsymbol{dE(z_0)}}{\boldsymbol{dz_0}}\right)$$

$$\Delta z_1 = \Delta z_0(1 + R_{56}\boldsymbol{h_1}) \equiv \Delta z_0/C$$

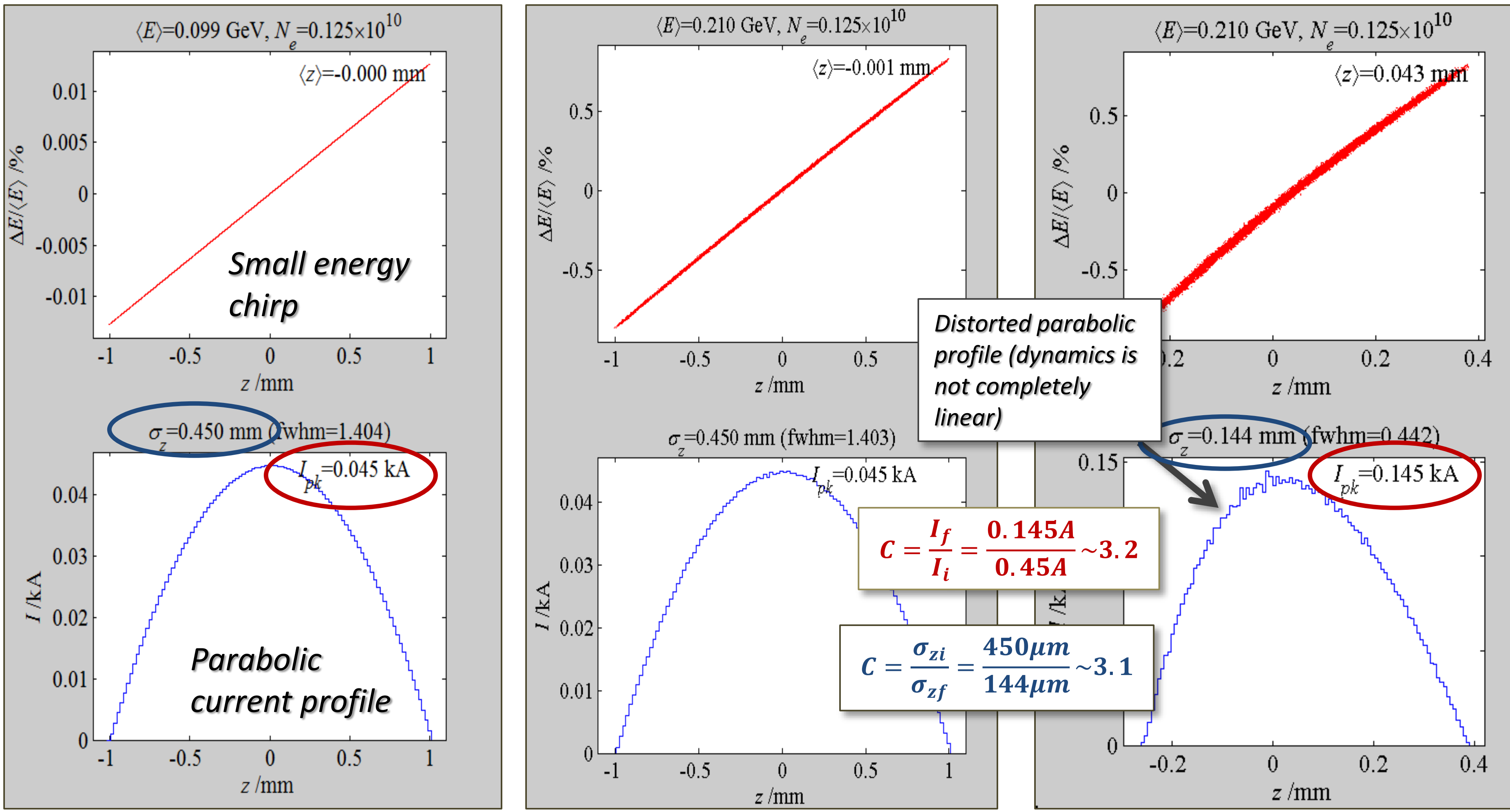
$$|\Delta z_1| \equiv |\Delta z_0|/C$$

$$\boxed{\frac{\mathbf{1}}{\boldsymbol{C}} \equiv \left|\frac{\boldsymbol{dz_1(z_0)}}{\boldsymbol{dz_0}}\right|}$$

If $E(z_0)$ - the energy chirp - is nonlinear then C depends on z (compression will vary along bunch). Generally, we refer to $C(z = 0)$ as the nominal compression factor.

Example of macroparticle simulation: off-crest acceleration + compression

(Idealized) beam out of the injector E=100MeV	Beam accelerated off-crest to E=210MeV	Beam @ exit of compressor

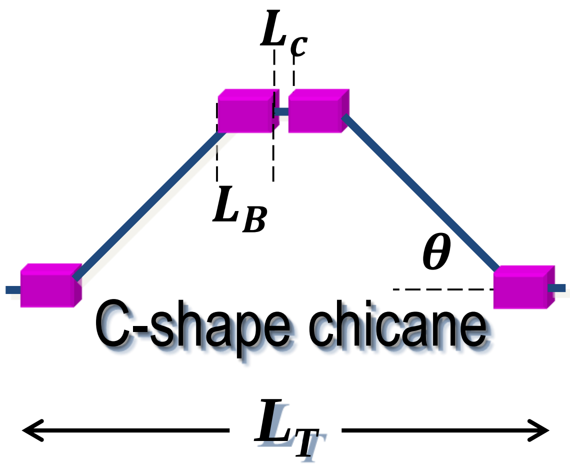


$$C = \frac{I_f}{I_i} = \frac{0.145A}{0.45A} \sim 3.2$$

$$C = \frac{\sigma_{zi}}{\sigma_{zf}} = \frac{450\mu m}{144\mu m} \sim 3.1$$

Various options for bunch compressor design

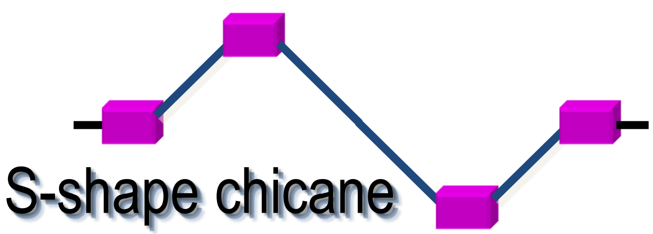
Bunch head < 0



FLASH
LCLS
FERMI
X-FEL
SACLA

$$R_{56} \approx -2\theta^2\left(\frac{L_T}{2} - \frac{4}{3}L_B - \frac{\Delta L_c}{2}\right) < 0$$

simple, achromatic



FLASH
X-FEL

$$R_{56} \approx -2\theta^2\left(\frac{L_T}{2} - \frac{4}{3}L_B\right) < 0$$

achromatic,

SLC arcs
NLC BC2

$$R_{56} \approx \frac{\theta_T^2 L_T}{4N_c^2 \sin^2(\mu_x/2)} > 0$$

reverse sign

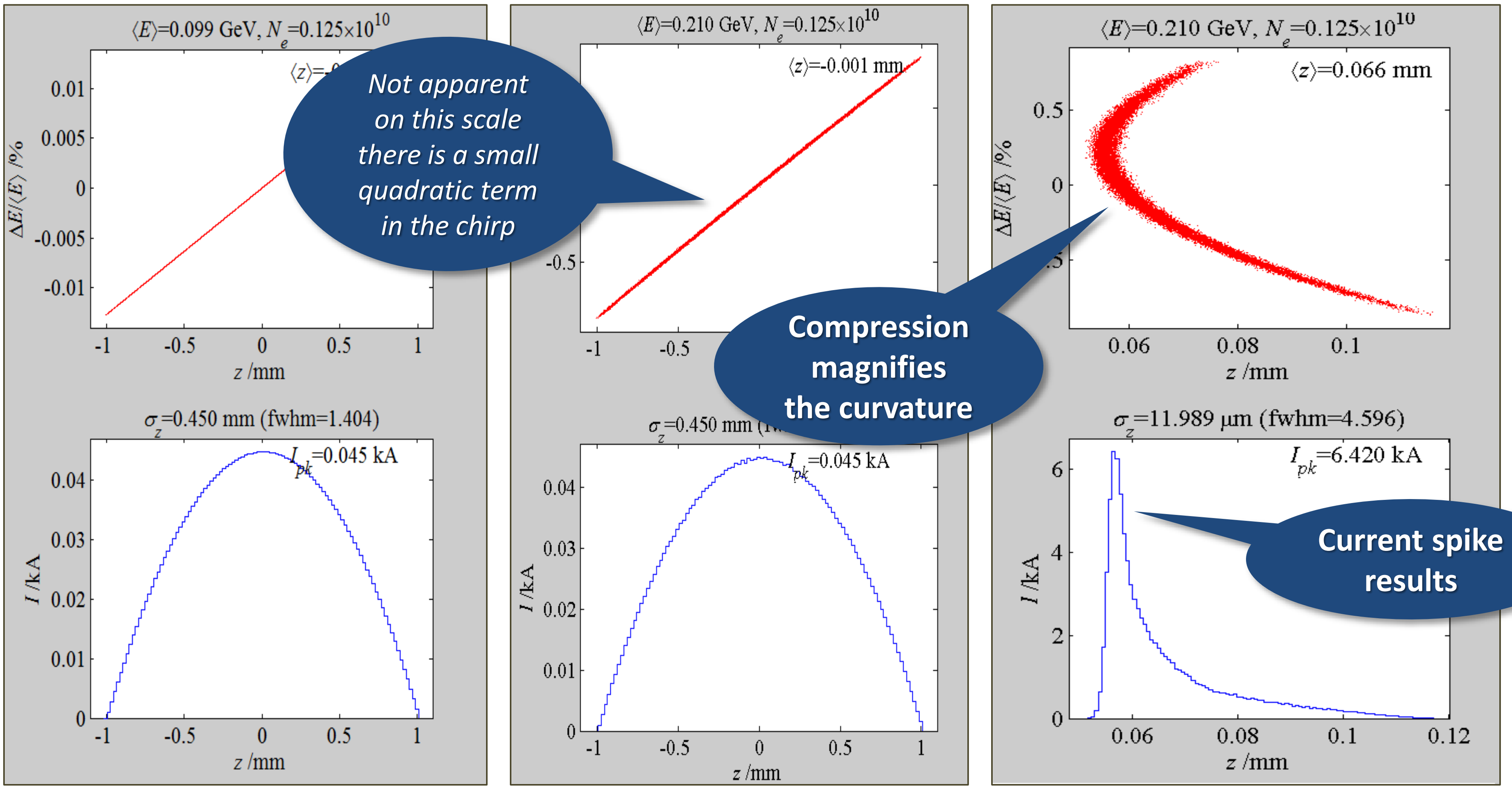
Formulas valid in the small-angle approx. Courtesy of P. Emma

Cranking up compression …

(Idealized) beam out of the injector E=100MeV

Beam accelerated off-crest to E=210MeV

Beam @ exit of compressor



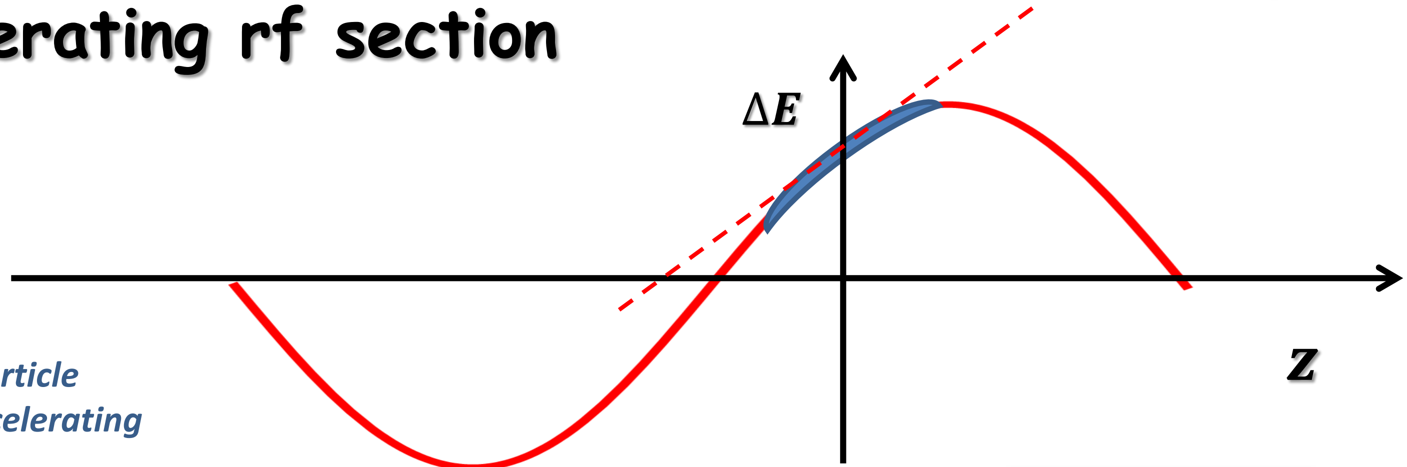
Non-linearities in the rf waveform compromise beam quality after compression

- **Spiky current profiles are generally not desirable**
  - *We like high peak current, but if the beam is very spiky only a small fraction of the beam may end up having sufficiently high current*
  - Spiky currents are associated with large energy spread (not good)
  - *If we do external-laser seeding in rgw FEL we like to have a bunch core where the current is about uniform*
  - rf and other wakefield effects are magnified by presence of spikes and will make 'spikiness' even worse when bunch is further compressed.

- **Is there a way to fix this?**
  - *One can deal with the problem by reducing compression (not good).*
  - Choosing a small k_{rf} for the accelerating structure (not practical; generally, choice of rf frequency is determined by other considerations)

- **Effective solution was proposed by D. Dowell, *et al.* in the ~90's**
  - Compensate the dominant (quadratic) nonlinearity by use of harmonic cavities

Analysis of rf waveform nonlinearities through accelerating rf section



Energy of particle at exit of accelerating structure

$$E_I = E_i + eV\cos(kz+\varphi) \simeq E_i + \boxed{eV\cos\varphi} - k\,zeV_0\sin\varphi \boxed{- e\frac{Vk^2}{2}z^2\cos\varphi} + O(z^3)$$

0-order term >0 (acceleration)

Quadratic term <0

- How can we compensate the quadratic term?
  - Idea: pass beam through a second rf section (with different rf wavenumber)

0-order term <0

$$E_{II} = E_I + eV_H\cos(k_H z + \varphi_H) \simeq E_I + \boxed{eV_H\cos\varphi_H} - k_H z\,eV_H\sin\varphi_H \boxed{- e\frac{V_H k_H^2}{2}z^2\cos\varphi_H} + O(z^3)$$

$$-\frac{Vk^2}{2}\cos\varphi + \frac{V_H k_H^2}{2} = 0 \quad \Rightarrow \quad V_H = \frac{k^2}{k_H^2}V\cos\varphi$$

To cancel quadratic curvature from accelerating structure this term should be >0; i.e. $\cos\phi_H < 0$, say ($\cos\varphi = -1$). This structure is decelerating

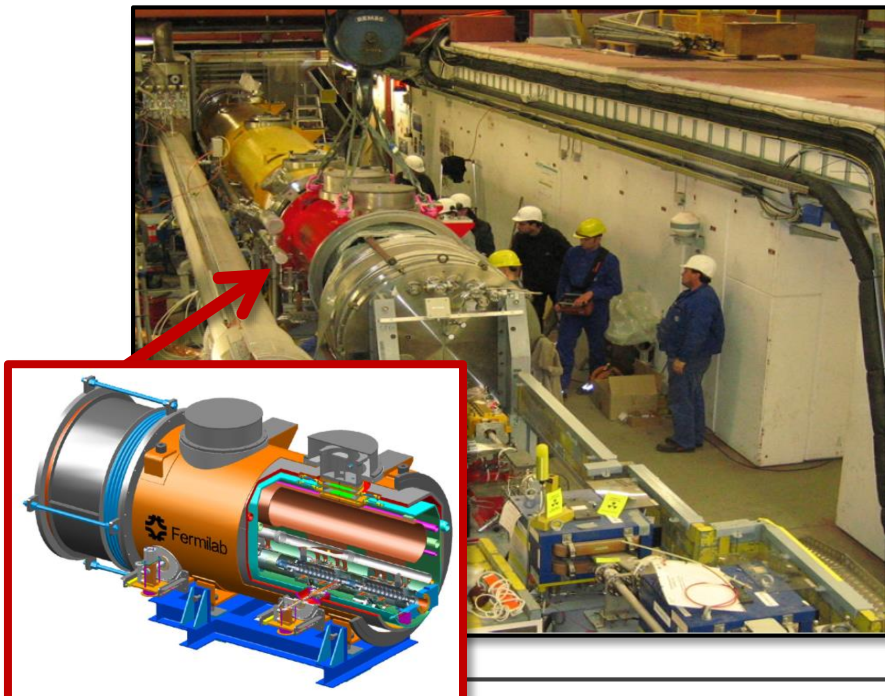
Q: *How can we win? (i.e. compensate 2nd order term and still have overall acceleration?)*

A: *Choose* $k_H > k$

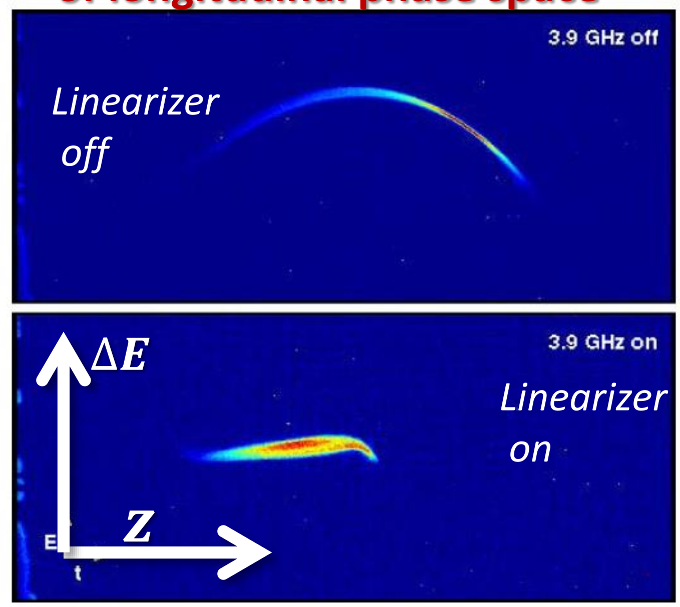
3rd –order Harmonic Linearizer at FLASH (3.9GHz)

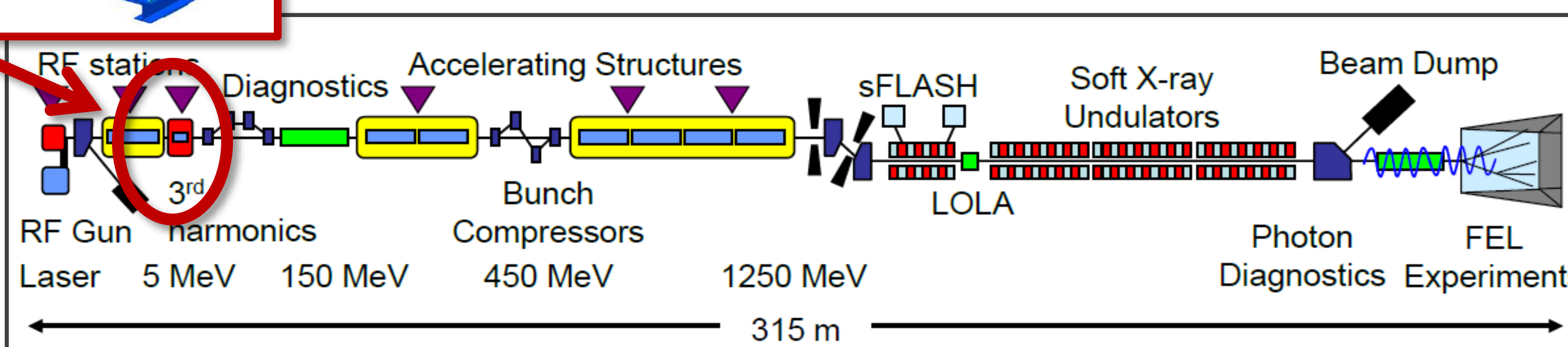
- Operationally linearizer rf frequency is best chosen to be a harmonic number of rf frequency of accelerating structures (FLASH uses 1.3GHz SC accelerating structures)

Installation of cryomodule w./ linearizer



Time-resolved measurements of longitudinal phase space





Formula for setting of linearizer revisited: Life is always more complicated…

- Nonlinear momentum compaction in chicane is usually non-negligible and has to be compensated too

$$z_1 = z_0 + R_{56}\delta_0 + \boldsymbol{T_{566}\delta_0^2}$$

For standard chicanes (Homework Exercise)

$$T_{566} \simeq -\frac{3}{2}R_{56} > 0$$

- Modified setting of harmonic cavity when accounting for the 2nd order term T_{566} in momentum compaction (*Homework Exercise*):

$$eV_H = \frac{k^2}{k_H^2 - k^2}\left\{E_{BC}\left[1 + \frac{2}{k^2}\frac{T_{566}}{|R_{56}|^3}\left(1 - \frac{1}{C}\right)^3\right] - E_i\right\}$$

Energy of beam entering Linac section

Beam energy @ compressor

(minimizing V_H favors doing compression at low energy)

Compression factor

$$C = \frac{1}{|1 + R_{56}h_1|}$$

- Formula valid for $\phi_H = -180^o$ and one-stage (single chicane) compression
- If **multiple compressors** are present, V_H setting varies somewhat but typically not too much (after first BC the bunch, is shorter and less vulnerable to rf nonlinearities)
- Further small adjustments may be needed to account for collective effects (rf wakefields, CSR).
- Alternate method to linearize: sextupole magnets within magnetic compressor (works well in arc-shaped compressors, not so well in chicanes where relatively small dispersion tends to requires too-strong sextuple magnets)

Summary highlights from this morning.

- Energy change by particle travelling through rf structure (ultra-relativistic approx.)

$$\boldsymbol{E(z) = E_i + eV\,cos(k_{\mathrm{rf}}z + \varphi_{\mathrm{rf}})}$$

- Linear chirp acquired by beam when rf structure is operated off-crest

$$\boldsymbol{h_1 = -\frac{eV\,k_{\mathrm{rf}}\,sin\,\varphi_{\mathrm{rf}}}{(E_i+eV\,cos\,\varphi_{\mathrm{rf}})}}$$

- Compression factor through beamline with finite momentum compaction R_{56}

$$\boldsymbol{C = \frac{1}{|1 + R_{56}h_1|}}$$

- Momentum compaction for 4-bend *C*-shape chicane (thin lens approximation $L_B \ll L_1$):

$$\boldsymbol{R_{56} = -2L_1\theta_0^2}$$

- Setting of harmonic cavity linearizer:

$$\boldsymbol{eV_H = \frac{k^2}{k_H^2 - k^2}\left\{E_{BC}\left[1 + \frac{2}{k^2}\frac{T_{566}}{|R_{56}|^3}\left(1 - \frac{1}{C}\right)^3\right] - E_i\right\}}$$

Summary highlights

- Undulator radiation /FEL resonance equation; undulator parameter"

$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2}\right) \qquad K = \frac{eB_0\lambda_u}{2\pi mc} \simeq 0.934\lambda_u[cm]B[T]$$

- FEL ρ (Pierce) parameter, 1D Theory FEL gainlength

$$\boxed{\rho = \frac{1}{4}\left[\frac{1}{\pi^2}\frac{I}{I_A}\frac{\lambda_u^2}{\gamma^3\sigma_x^2}(K \times [JJ])^2\right]^{1/3}} \qquad \boxed{L_g \sim \frac{1}{4\pi\sqrt{3}}\frac{\lambda_u}{\rho}}$$

- Requirements for beam relative energy spread and transverse rms emittance

$$\boxed{\sigma_\delta < \rho} \qquad \boxed{\varepsilon_\perp \lesssim \frac{\lambda}{4\pi}}$$

- E-beam brightness

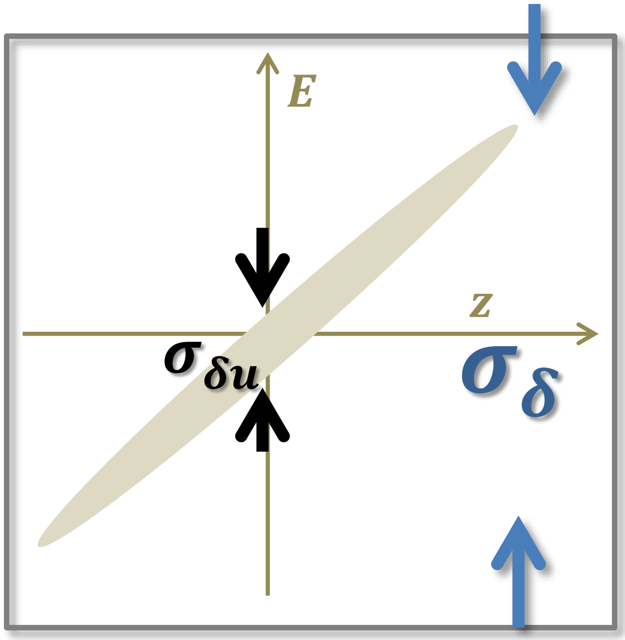
$$B_6 = \frac{N}{\varepsilon_{nx}\varepsilon_{ny}\varepsilon_{nz}} \qquad B_5 = \frac{I}{\varepsilon_{nx}\varepsilon_{ny}} \qquad B_4 = \frac{Q}{\varepsilon_{nx}\varepsilon_{ny}}$$

- Emittance growth due to angular kick perturbation

$$\boxed{\frac{\Delta\varepsilon_x}{\varepsilon_{x0}} \simeq \frac{\beta_x}{2\varepsilon_{x0}}\langle\Delta x'^2\rangle}$$

Supplemental material

Correlated vs. uncorrelated energy spread

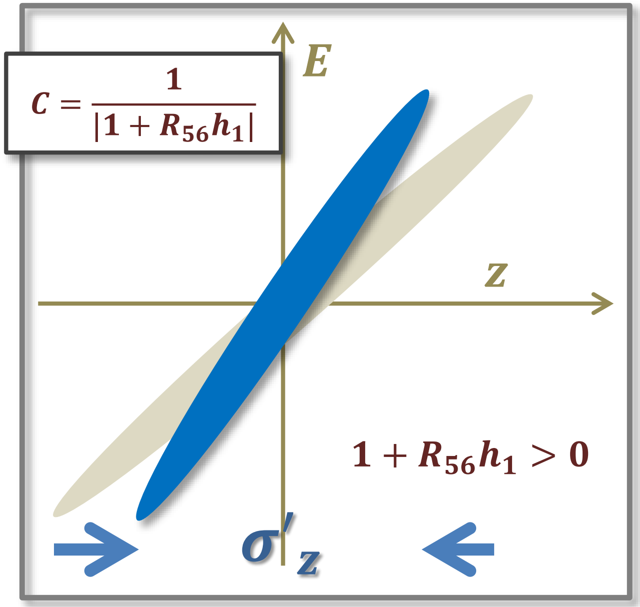


So far we have assumed model beams with negligible uncorrelated $\boldsymbol{\sigma_{\delta u}}$ (or 'slice') energy spread.

A finite $\boldsymbol{\sigma_{\delta u}}$ limits the minimum bunch length that can be achieved (see next slide)

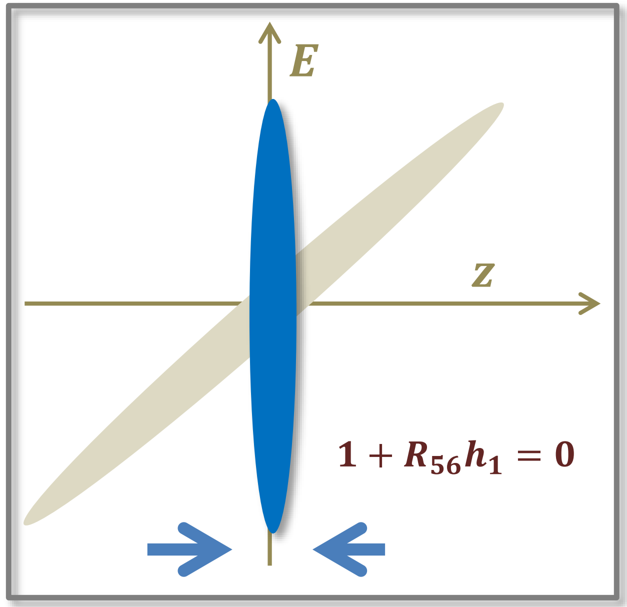
Modes of compression;

under-compression



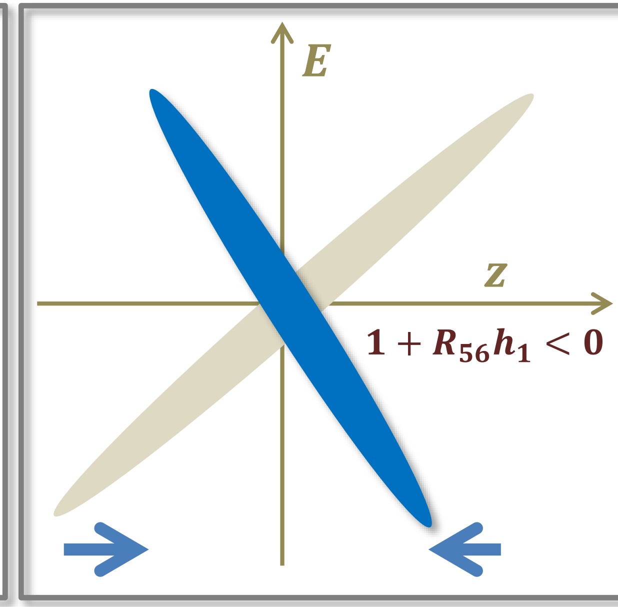
More prevalent mode of compression

max. compression



Min. bunch length determined by uncorrelated energy spread

over-compression



Sign of energy chirp is reversed

Projected energy spread before Compression… $\sigma_\delta = \sqrt{(h_1\sigma_z)^2 + \sigma_{\delta u}^2}$ …same after compression

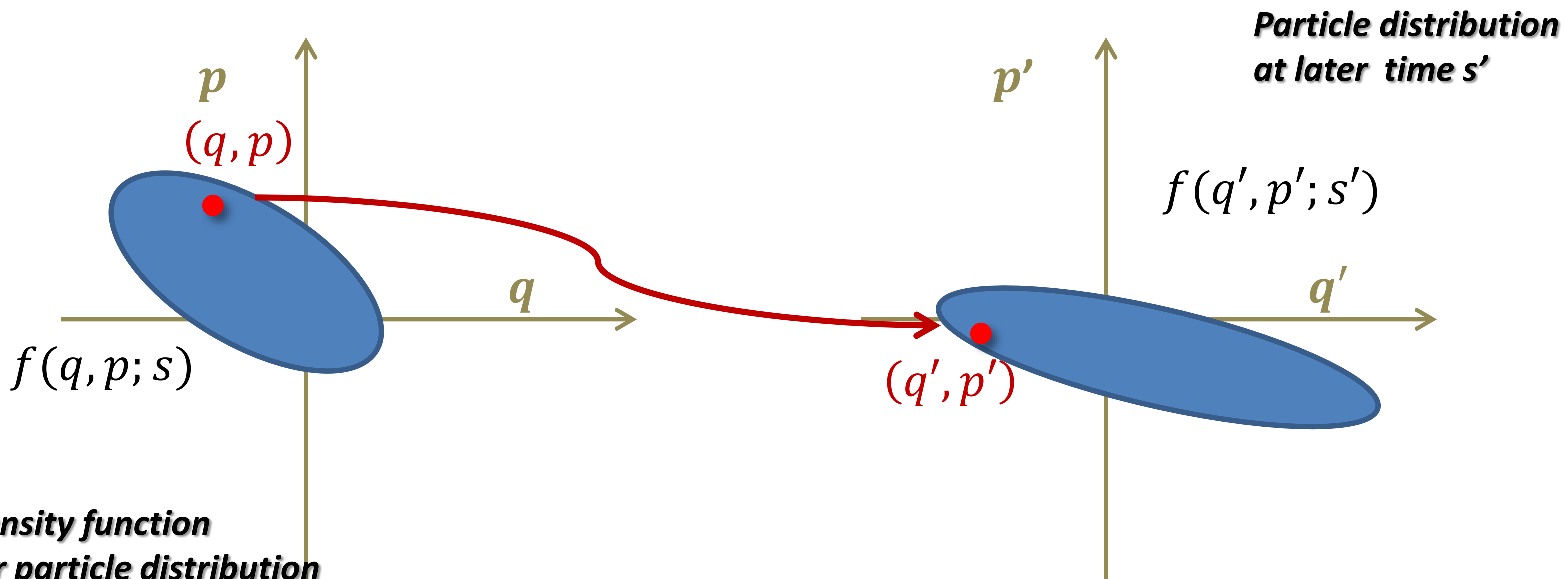
Initial uncorrelated energy spread

Slice energy spread after compression $\sigma'_{\delta u} = \frac{\sigma_z}{\sigma'_z}\sigma_{\delta u} \sim C\sigma_{\delta u}$

Bunch length after compression $\sigma'_z = \sqrt{\sigma_z^2/C^2 + (R_{56}\sigma_{\delta u})^2}$

In next few slides:
How to work out these expressions

How do particle distributions evolve in phase space?



density function
for particle distribution
at present 'time' s

$$dN = f(q,p;s)dqdp$$

The particle dynamics is described by a map from space (q,p) to space (q',p'):

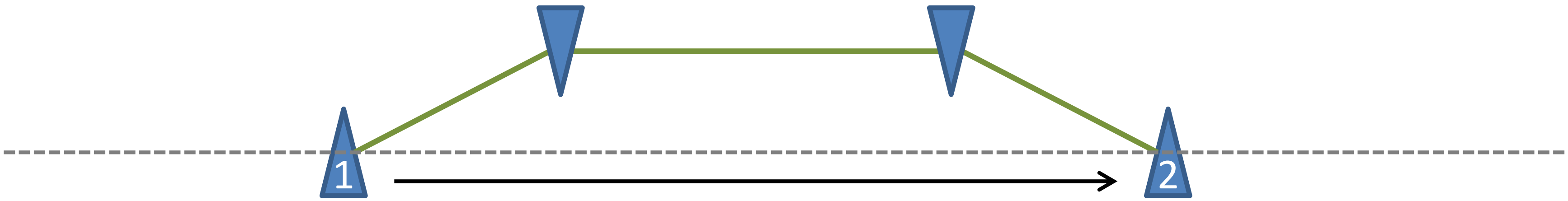
$$\begin{cases} \boldsymbol{q'} = \boldsymbol{q'}(\boldsymbol{q},\boldsymbol{p}) \\ \boldsymbol{p'} = \boldsymbol{p'}(\boldsymbol{q},\boldsymbol{p}) \end{cases}$$

- For Hamiltonian systems (volume preserving)

$$\boldsymbol{f}(\boldsymbol{q'},\boldsymbol{p'};\boldsymbol{s'}) = \boldsymbol{f}(\boldsymbol{q},\boldsymbol{p};\boldsymbol{s})$$

$$\boxed{f(q',p';s') = f(\,q(q'p'),p(q',p');s)}$$

Evolution of beam distribution through compressor (longitudinal phase space) I



Coordinates: (z_1, δ_1) (z_2, δ_2)

Beam density: $f(z_1, \delta_1; s_1)$ $f(z_2, \delta_2; s_2)$

$$\delta_1 = \frac{\Delta E}{E_{BC}} = \frac{E - E_{BC}}{E_{BC}}$$

- Assume linear approximation

$$\begin{cases} z_2 = z_1 + R_{56}\delta_1 \\ \delta_2 = \delta_1 \end{cases}$$

(Particle energy doesn't change)

- Assume gaussian model of beam distribution

$$f(z_1, \delta_1) = \frac{N}{2\pi\sigma_{z1}\sigma_{\delta 1}} \exp\left(-\frac{z_1^2}{2\sigma_{z1}^2} - \frac{(\delta_1 - h_1 z_1)^2}{2\sigma_{\delta 1}^2}\right)$$

Evolution of beam distribution through compressor II

$$\begin{cases} z_2 = z_1 + R_{56}\delta_1 \\ \delta_2 = \delta_1 \end{cases} \xrightarrow{\text{Invert}} \begin{cases} z_1 = z_2 - R_{56}\delta_1 \\ \delta_1 = \delta_2 \end{cases}$$

$$\boldsymbol{f(z_2, \delta_2; s_2) = f(\, z_1(z_2, \delta_2), \delta_1(z_2, \delta_2);\ s_1\,)} \quad =$$

$$\frac{\boldsymbol{N}}{\boldsymbol{2\pi\sigma_{z1}\sigma_{\delta 1}}}\mathbf{exp}\Big(-\frac{\boldsymbol{(z_2 - R_{56}\delta_2)^2}}{\boldsymbol{2\sigma_{z1}^2}} - \frac{\boldsymbol{[\delta_2 - h_1(z_2 - R_{56}\delta_2)]^2}}{\boldsymbol{2\sigma_{\delta 1}^2}}\Big)$$

- Calculate rms bunch length and energy spread
  - Homework exercise

$$\sigma_{z2}^2 = \langle (z_2 \ - \langle\, z_2 \rangle)^2 \rangle$$

$$\sigma_{\delta 2}^2 = \langle (\delta_2 \ - \langle \delta_2 \rangle)^2 \rangle$$

Where: $\langle\,\rangle \equiv \int dz_2 d\delta_2 f(z_2, \delta_2)$