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US/AS JANUARY 2004

Summary of JB LECTURES

START WITH MICROSCOPIC PHASE SPACE DENSITY

$$\boxed{N(\underline{x},\underline{v},t) = \sum_{i=1}^{N} \delta(\underline{x} - \underline{X}_i(t))\,\delta(\underline{v} - \underline{V}_i(t))}$$ Klimontovitch Density

$\frac{\partial N}{\partial t}$ + MAXWELLS EQUATIONS $\Rightarrow$ KLIMONTIVICH EQUATION:

$$\boxed{\frac{\partial N}{\partial t} + v \cdot \nabla_x N(\underline{x},\underline{v},t) - \frac{q}{m}(E^m + v \times B^m) \cdot \nabla_v N(\underline{x},\underline{v},t) = 0}$$

$$\boxed{\text{or } \frac{dN(\underline{x},\underline{v},t)}{dt} = 0}$$

Letting $N = f + \delta f$ $\quad f = \langle N \rangle$

$E^m = E + \delta E$ $\quad E = \langle E_m \rangle$

$B^m = B + \delta B$ $\quad B = \langle B_m \rangle$

PERFORMING LOCAL AVERAGES TO OBTAIN SMOOTH & "SLIKEY" QUANTITIES:

$$\boxed{\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{x}} + \frac{d\underline{v}}{dt} \cdot \frac{\partial f}{\partial \underline{v}} = \frac{\partial f}{\partial t_c}}$$

We estimated $\dfrac{|\partial f/\partial t_c|}{|\frac{q\underline{E}}{m} \cdot \partial f/\partial \underline{v}|} \sim \dfrac{1}{16\,\lambda_D^3\, n_0} << 1$

$\lambda_D = V_{TH}/\omega_p$

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{x}} + \frac{d\underline{p}}{dt} \cdot \frac{\partial f}{\partial \underline{p}} = 0 \qquad \dot{p} = -\frac{\partial H}{\partial \underline{x}} \; ; \; \dot{x} = \frac{\partial H}{\partial \underline{p}}$$

$$\boxed{\frac{df}{dt} = 0}$$ LIOUVILLES EQUATION (INCOMPRESSIBILITY OF PHASE VOLUME)

DEFINE NORMALIZED EMITTANCES PROPORTIONAL TO $\Delta p_z \Delta z$

$\Delta p_x \Delta x$

$\Delta p_z \Delta y$

SO THAT

$$\varepsilon_{Nx}^2 = \gamma^2\beta^2\left(\langle x^2\rangle\langle x'\rangle - \langle x x'\rangle^2\right)$$

$\Rightarrow$ CONSTANT IF FORCES ARE LINEAR IN X & FILAMENTATION IS ABSENT. (LINEAR WITHOUT COUPLING TO Z, OR Y).

WE DERIVED TWO SETS OF PARTICLE EQUATION OF MOTION:

PARAXIAL EQUATION (FOR AXISYMMETRIC SYSTEMS) $\left(\frac{\partial}{\partial\theta} = 0\right)$

STARTING WITH THE LORENTZ FORCE EQUATION $\frac{d\underline{p}}{dt} = q(\underline{E} + \underline{v} \times \underline{B})$ IN CYL. COORD.

$$\underbrace{\frac{d}{dt}(\gamma m \dot{r})}_{\text{INERTIAL}} - \underbrace{\gamma m r \dot{\theta}^2}_{\text{CENTRIFUGAL}} = q\Big(\underbrace{\frac{V''}{2} r}_{\substack{E_r \text{ external} \\ (\text{DIVERGENCE OF } E = 0)}} + \underbrace{r\dot{\theta} B}_{v_\theta B_z}\Big) + q\Big(\underbrace{E_r^{self}}_{\text{SELF-FIELDS}} + \underbrace{v_z B_\theta^{self}}_{\text{SELF-FIELDS}}\Big)$$

$$\boxed{\dot{r} = \frac{dr}{dt};\ r' = \frac{dr}{ds} = \frac{\dot{r}}{\beta c}}$$

θ-component:

$$p_\theta = \gamma m r^2 \dot{\theta} + \frac{q B(z) r^2}{2} = \text{constant}$$

$$= \gamma m r^2 \beta c \theta' + \frac{q B r^2}{2} = \text{constant}$$

$$\underbrace{r''}_{\text{INERTIAL}} + \underbrace{\frac{\gamma'}{\beta^2 \gamma} r'}_{\substack{\text{ACCELERATION} \\ (\text{INERTIA})}} + \underbrace{\frac{\gamma''}{2\beta^2\gamma} r}_{\substack{E_r \\ (\text{CONVERGENCE OF FIELD LINES})}} + \underbrace{\left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r}_{v_\theta B_z \text{ CENTRIFUGAL}} - \underbrace{\left(\frac{p_\theta}{\gamma\beta m c}\right)^2 \frac{1}{r}}_{\text{CENTRIFUGAL}} - \underbrace{\frac{q}{\gamma^3 m v_z^2} \frac{\lambda(r)}{2\pi\epsilon_0 r}}_{\text{SELF FIELD}} = 0$$

STATISTICAL AVERAGE OF THIS EQUATION

$$r_b'' + \frac{\gamma'}{\beta^2\gamma} r_b' + \frac{\gamma''}{2\beta^2\gamma} r_b + \left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r_b - \frac{4\langle p_\theta\rangle^2}{(\gamma m \beta c)^2 r_b^3} - \frac{\varepsilon_r^2}{r_b^3} - \frac{Q}{r_b} = 0$$

$$\varepsilon_r^2 \equiv 4\left(\langle r^2\rangle\langle r'^2\rangle - \langle r r'\rangle^2 + \langle r^2\rangle\langle r^2\theta'^2\rangle - \langle r^2\theta'\rangle^2\right); \quad Q = \frac{q\lambda}{2\pi\epsilon_0 \gamma^3 \beta^2 m c}$$

$$= \varepsilon_x^2 - 4\langle r^2\theta'\rangle^2 \quad (\text{if } \rho = \rho(r) \text{ only})$$

EQUATION OF MOTION

RETURN TO X, Y COORDINATES

$$x'' + \frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z)x' = \frac{-q}{\gamma^3 m v_z^2}\frac{\partial \varphi}{\partial x} \pm \begin{cases} \frac{qB'}{\gamma m v_z}x & \text{for magnetic quads} \\ \frac{qE'}{\gamma m v_z^2}x & \text{for electric quads} \end{cases}$$

Let $\frac{\gamma m v_z}{q} = \frac{p}{q} \equiv [B\rho] \equiv$ RIGIDITY

$$y'' + \frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z)y' = \frac{-q}{\gamma^3 m v_z^2}\frac{\partial \varphi}{\partial y} \mp \begin{cases} \frac{B'}{[B\rho]}y & \text{magnetic} \\ \frac{qE'}{\gamma m v_z^2}y & \text{electric} \end{cases}$$

ENVELOPE EQUATION

$$r_x^2 = 4\langle x^2\rangle; \quad r_y^2 = 4\langle y^2\rangle$$

$$r_x' = \frac{4\langle xx'\rangle}{r_x}$$

$$r_x'' = \frac{4\langle xx''\rangle}{r_x} + \frac{\varepsilon_x^2}{r_x^3}; \qquad \varepsilon_x^2 = 16(\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2)$$

$$r_y'' = \frac{4\langle yy''\rangle}{r_y} + \frac{\varepsilon_y^2}{r_y^3} \qquad \varepsilon_y^2 = 16(\langle y^2\rangle\langle y'^2\rangle - \langle yy'\rangle^2)$$

for magnetic focusing:

$$r_x'' + \frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z)r_x' + \frac{4q}{\gamma^3 m v_z^2}\frac{\langle x\frac{\partial\varphi}{\partial x}\rangle}{r_x} \mp \frac{B'}{[B\rho]}r_x - \frac{\varepsilon_x^2}{r_x^3} = 0$$

$$r_y'' + \frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z)r_y' + \frac{4q}{\gamma^3 m v_z^2}\frac{\langle y\frac{\partial\varphi}{\partial y}\rangle}{r_y} \pm \frac{B'}{[B\rho]}r_x - \frac{\varepsilon_y^2}{r_y^3} = 0$$

(for electric focusing $\frac{B'}{[B\rho]} \to \frac{qE'}{\gamma m v_z^2}$)

SPACE CHARGE TERM WITH ELLIPTICAL SYMMETRY

NOW DEFOCUSING IN ONE DIRECTION AND FOCUSING IN THE OTHER $\Rightarrow$ RADIAL SYMMETRY SHOULD BE REPLACED BY ELLIPTICAL SYMMETRY:

$$\rho = \rho\left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2}\right)$$

CAN BE SHOWN THAT

$$\left\langle x \frac{\partial \phi}{\partial x} \right\rangle = \frac{-\lambda}{4\pi\varepsilon_0} \frac{r_x}{r_x + r_y}$$

$$\left\langle y \frac{\partial \phi}{\partial y} \right\rangle = \frac{-\lambda}{4\pi\varepsilon_0} \frac{r_y}{r_x + r_y}$$

USE $\phi(x,y) = \dfrac{-r_x r_y}{4\varepsilon_0} \displaystyle\int_0^\infty \frac{\hat{\rho}(\chi)\, ds}{\sqrt{r_x^2+s}\,\sqrt{r_y^2+s}}$ to prove, where $\hat{\rho}(\chi) = \dfrac{d\rho}{d\chi}$

$$\rho(x,y) = \hat{\rho}(\chi)\big|_{s=0}$$

$$\chi = \frac{x^2}{r_x^2+s} + \frac{y^2}{r_y^2+s}$$

DEFINING $Q = \dfrac{2\lambda}{4\pi\varepsilon_0 \gamma^3 m v_z^2}$

$$\Rightarrow \boxed{\begin{aligned} & r_x'' + \frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z) r_x' - \frac{2Q}{r_x + r_y} \mp \frac{B'}{[B\rho]} r_x - \frac{\varepsilon_x^2}{r_x^3} = 0 \\ & r_y'' + \frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z) r_y' - \frac{2Q}{r_x + r_y} \pm \frac{B'}{[B\rho]} r_y - \frac{\varepsilon_y^2}{r_y^3} \end{aligned}}$$

(for Electric Focusing $\dfrac{B'}{[B\rho]} \to \dfrac{qE'}{\gamma m v_z^2}$).

ENVELOPE EQUATIONS DERIVED

RMS (ENVELOPES DEFINED IN TERMS OF RMS QUANTITIES; EMITTANCE NOT GUARANTEED TO BE CONSERVED).

1. PARAXIAL r_b ; $\rho(r)$
2. ELLIPTICAL r_x, r_y ; $\rho\left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2}\right)$
3. LONGITUDINAL r_z FOR $E_z = \frac{-q}{4\pi\epsilon_0}\frac{\partial\lambda}{\partial z} \propto z$; $[\lambda \propto (1 - 4z^2/r_z^2)]$
4. ELLIPSOIDAL of BUNCHES $r_\perp, r_z$ (ALSO r_x, r_y, r_z c.f. Wangler, section 9.9).
   (ρ CONSTANT)
5. ELLIPTICAL WITH IMAGES r_x, r_y $\rho\left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2}\right)$
6. LARMOR FRAME (CONTINUOS FOCUSING) $\tilde{r}_x, \tilde{r}_y$ $\rho(r)$

KINETIC ENVELOPE EQUATIONS (CONSTRAINT EQUATION ON A PARTICULAR DISTRIBUTION FUNCTION; EMITTANCE CONSERVED)

1. KV FOR ELLIPTICAL UNIFORM BEAMS $f(x, x', y, y')$
   [IDENTICAL TO #2 ABOVE]
2. NEUFFER DISTRIBUTION FOR 1D $f(z, z')$
   [IDENTICAL TO #3 ABOVE]

CURRENT LIMITS

SOLENOIDS:

$$Q_{max} = \left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r_b^2 \qquad \omega_c = \frac{qB(s)}{m}$$

EINZEL LENSES:

$$Q_{max} = \frac{3\pi^2}{8}\left(\frac{q\phi_0}{mv_0^2}\right)^2\left(\frac{r_b}{L}\right)^2$$

$\pm\phi_0$ = potential of each electrode
L = distance between electrodes

QUADRUPOLES:

$$Q_{max} = F(\eta)\,\eta^2 k^2 L^2 r_b^2$$

$$k = \begin{cases} \dfrac{B'}{[B\rho]} & \text{magnetic} \\[2ex] \dfrac{qE'}{\gamma m v_z^2} & \text{electric} \end{cases} \qquad F(\eta) = \begin{cases} \dfrac{1}{4} & \text{THIN LENS, WEAK } \sigma_0 \\[2ex] \dfrac{2}{\pi^2}\left(\dfrac{\sin\left[\frac{\eta\pi}{2}\right]}{\eta\pi/2}\right)^2 & \text{FOURIER EXPANSION} \\[2ex] \dfrac{1}{4}\left(1-\dfrac{2\eta}{3}\right) & \text{THICK LENS (LEE PESSENDEN LASLETT)} \end{cases}$$

$\sigma_0 \cong \eta L^2 k$ (THIN LENS, SMALL σ_0) (USE σ_0 TO ELIMINATE L)

$\cos\sigma_0 \cong 1 - \dfrac{\eta^2 k^2 L^4}{2}\left[1 - \dfrac{2\eta}{3}\right]$ (THICK LENS, LEE, PESSENDEN, LASLETT)

Let $\theta = \sqrt{k}\,\eta L$:

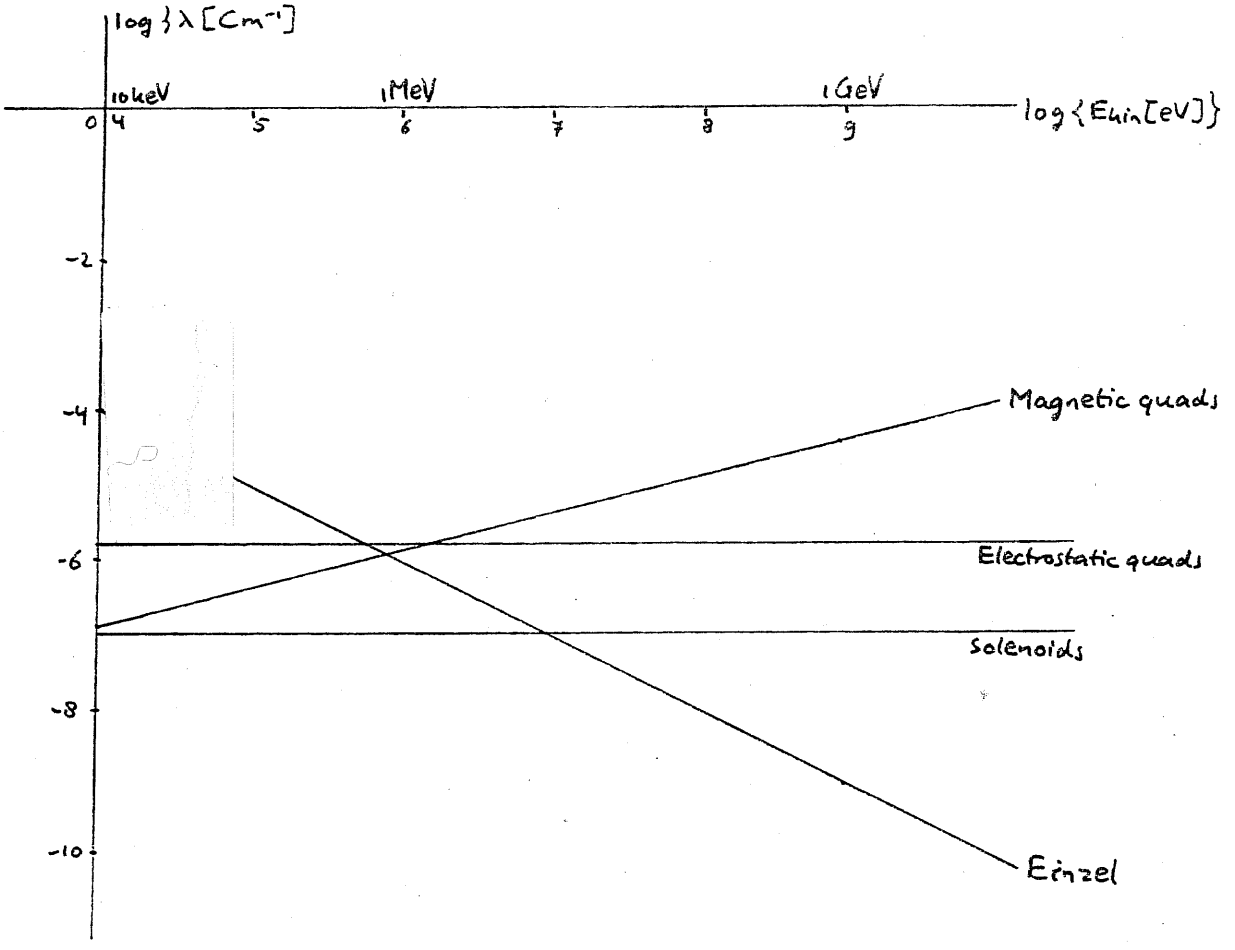
$$\cos\sigma_0 = \cos\theta\cosh\theta + \frac{(1-\eta)}{\eta}\theta(\cos\theta\cosh\theta - \sin\theta\cosh\theta) - \frac{1}{2}\frac{(1-\eta)^2}{\eta^2}\theta^2\sin\theta\sinh\theta$$

Solenoids: $\log\{\lambda[Cm^{-1}]\} = -7.0$

Electrostatic quads: $\log\{\lambda[Cm^{-1}]\} = -5.8$

Magnetic quads: $\log\{\lambda[Cm^{-1}]\} = -8.9 + \frac{1}{2}\log\{E_{kin}[eV]\}$

Einzel lenses: $\log\{\lambda[Cm^{-1}]\} = 0.01 - \log\{E_{kin}[eV]\}$



ESTIMATING SPOT SIZE

$$r_x'' + \frac{(\gamma_b \beta_b)'}{\gamma_b \beta_b} r_x + K_x r_x - \frac{2Q}{r_x + r_y} - \frac{\epsilon_x^2}{r_x^3} = 0$$

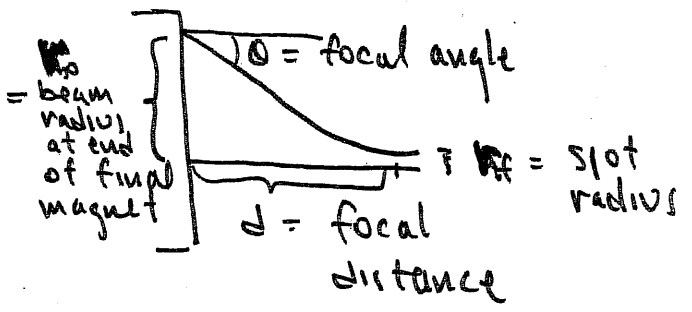
$$r_y'' + \frac{(\gamma_b \beta_b)'}{\gamma_b \beta_b} r_y + K_y r_y - \frac{2Q}{r_x + r_y} - \frac{\epsilon_y^2}{r_y^3} = 0$$

<u>IN CHAMBER</u>: NO EXTERNAL FOCUSING, NO ACCELERATION AND BEAM IS OFTEN CIRCULAR (BY DESIGN)

$$\Rightarrow \quad K_x = K_y = (\gamma_b \beta_b)' = 0 \qquad \& \quad r_x = r_y = r_b$$

$\Rightarrow$ ENVELOPE EQUATION IS:

$$r_b'' = \frac{Q}{r_b} + \frac{\epsilon^2}{r_b^3}$$



MULTIPLYING BY r_b' & INTEGRATING $\Rightarrow$

$$\frac{r_{bf}'^2}{2} - \frac{r_{bo}'^2}{2} = Q \ln \frac{r_{bf}}{r_{bo}} + \frac{\epsilon^2}{2 r_{bo}^2} - \frac{\epsilon^2}{2 r_{bf}^2}$$

<u>NOW</u> $r_{bo}' \cong \theta$ $\qquad r_{bf}$ = spot radius $\qquad r_{bf} << r_{bo}$

$r_{bf}' = 0 \qquad r_{bo} \cong d\theta$

$$\Rightarrow \boxed{\theta^2 \cong 2Q \ln\left(\frac{\theta d}{r_{bf}}\right) + \frac{\epsilon^2}{r_{bf}^2}}$$

WHEN $Q = 0$

$$r_{spot}^2 = \frac{\epsilon^2}{\theta^2} + r_{\text{CHROMATIC ABERRATION}}^2 + \ldots$$

$$r_{\text{chromatic}}^2 = \alpha^2 d^2 \left(\frac{\Delta p}{p}\right)^2 \theta^2$$

$\alpha \cong 6$ (system dependent)

LONGITUDINAL DYNAMICS SUMMARY

1 D VLASOV EQUATION

$$\frac{\partial \tilde{f}}{\partial s} + z' \frac{\partial f}{\partial z} + z'' \frac{\partial \tilde{f}}{\partial z'} = 0$$

$$E_z = \frac{-q}{4\pi\epsilon_0} \frac{\partial \lambda}{\partial z}$$ "g-factor" model

$$z'' = \frac{q E_z}{m v_0^2} \; ; \qquad \frac{\partial^2 \phi}{\partial z^2} + \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r}\right) = -\frac{\rho}{\epsilon_0}$$

CHILD-LANGMUIR IN 1-D DIODE

LEADS TO FLUID EQUATIONS

$$\frac{\partial \lambda}{\partial s} + \frac{\partial}{\partial z}(\lambda \bar{z}') = 0$$

$$\frac{\partial \bar{z}'}{\partial s} + \bar{z}' \frac{\partial \bar{z}'}{\partial z} + \frac{1}{\lambda} \frac{\partial}{\partial z}(\lambda \Delta z'^2) = \frac{q E_z}{m v_0^2}$$

qE

1D E_z: $\Rightarrow$ CHILD LANGMUIR SOLUTION $\leftarrow$ NON-LINEAR SOLUTION TO FLUID EQUATIONS

g-factor: $\Rightarrow$ SPACE-CHARGE WAVES

$\hookrightarrow$ LONGITUDINAL OR REFLEXIVE WALL INSTABILITY (IF $E_1 = z^{\nu} I_1$)

$\Rightarrow$ SPACE-CHARGE RAREFACTION WAVES $\leftarrow$ NON-LINEAR SOLUTION TO FLUID EQNS.

$\Rightarrow$ PARABOLIC BUNCH COMPRESSION $\leftarrow$ NON-LINEAR SOLUTION TO FLUID EQUATIONS

$\Rightarrow$ "EAK" FIELDS

VLASOV EQUATION ALSO $\Rightarrow$ ENVELOPE EQUATION

$$\frac{d^2 r_z}{ds^2} = \frac{\epsilon_z^2}{r_z^3} + \frac{3}{2} \frac{q Q_c}{4\pi\epsilon_0 m v^2} \frac{1}{r_z^2} - K(r) r_z$$

KINETIC SOLUTION TO VLASOV EQUATION & SATISFYING RMS ENVELOPE EQUATION IS NEUFELD DISTRIBUTION

$$f(z, z') = \frac{3N}{2\pi\epsilon_z} \sqrt{1 - \frac{z^2}{r_z^2} - \frac{r_z^2 (z' - r_z' z / r_z)^2}{\epsilon_z^2}}$$

INSTABILITIES

1. LONGITUDINAL (RESISTIVE WALL) INSTABILITY

   (FLUID INSTABILITY)

2. ELECTRON – ION INSTABILITY

   (CENTROID INSTABILITY)

STEVE TALKED ABOUT:

3. ENVELOPE INSTABILITIES

STEVE WILL TALK TODAY ABOUT:

4. KINETIC INSTABILITIES
   (DISTRIBUTION FUNCTION DEPENDENT)

5. SINGLE PARTICLE RESONANT INSTABILITIES
   - HALO
   - RING RESONANCES

HALO:

CORE TEST PARTICLE MODEL:

$$x'' = \begin{cases} -\left[k_{p0}^2 - \frac{Q}{r_b^2}\right]x & \text{for } r < r_b \\ -\left[k_{p0}^2 - \frac{Q}{r^2}\right]x & \text{for } r > r_b \end{cases}$$

$$r_b = r_{b0} + \delta r_b \cos(k_B s + \phi)$$

Gluckstern's phase-amplitude analysis:

$$x'' + \overbrace{\left[k_{p0}^2 - \frac{Q}{r_{b0}^2}\right]}^{k_p^2} x = f(x)$$

Non linear + forcing part

$x = A\sin\psi \quad x' = k_p A\cos\psi$ ← PHASE/AMPLITUDE

$\psi = k_p s + \alpha$ If $f = 0$ A & α would be constant

$\Rightarrow A' = \frac{1}{k_p r_{b0}} f\cos\psi \quad \alpha' = \frac{-1}{k_p r_{b0} A} f\sin\psi$

DEFINE RESONANT PHASE $\Psi_r = 2\psi - k_B s$

AVERAGE OVER ALL NON-RESONANT FREQUENCIES

$$A_r' = \frac{1}{k_p r_{b0}} \int_{-\pi}^{\pi} f\cos\psi \frac{d\psi}{2\pi}; \quad \alpha_r' = \frac{-1}{k_p A_r} \int_{-\pi}^{\pi} \frac{d\psi}{2\pi} f\sin\psi$$

$\rightarrow A_r', \Psi_r' \rightarrow \omega', \Psi_r' \rightarrow H(\omega, \Psi_r) \rightarrow$ GIVE RESONANT PARTICLE TRAJECTORY & SEPARATRIX

SUMMARY OF ELECTRON, GAS, PRESSURE, & SCATTERING EFFECTS

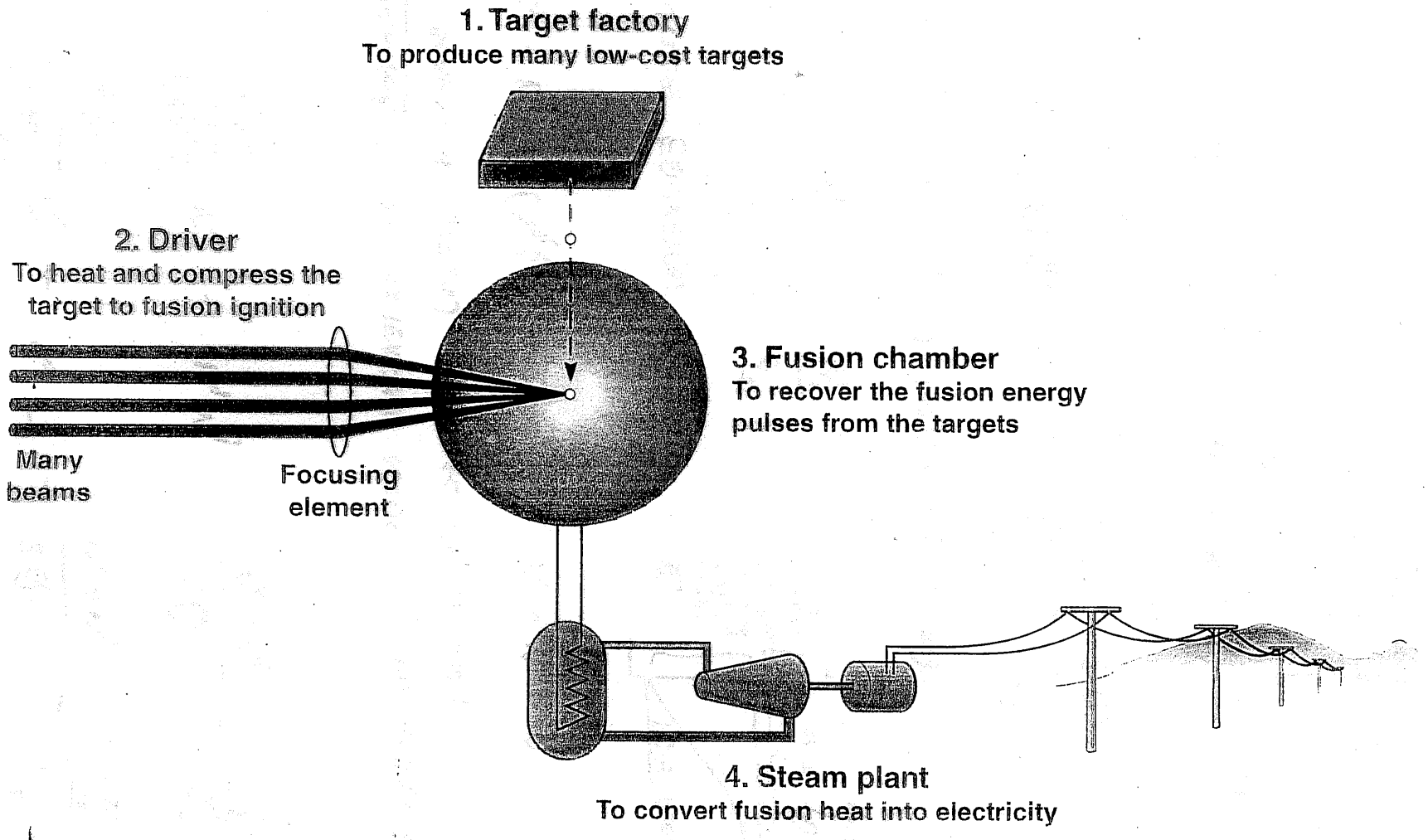
1. COULOMB COLLISIONS WITHIN BEAM CAN TRANSFER ENERGY FROM ⊥ TO ∥ AND PROVIDE LOWER LIMIT ON $T_{\parallel}$, HIGHER THAN FROM ACCELERATIVE COOLING.

2. COULOMB INTERACTIONS WITH RESIDUAL GAS NUCLEI PROVIDE A SOURCE OF EMITTANCE GROWTH (BUT NOT IMPORTANT FOR HIGHER MASS AND LONGER RESIDENCE TIMES).

3. PRESSURE INSTABILITY FROM DESORPTION OF RESIDUAL GAS BY STRIKED BEAM IONS HITTING WALL OR BEAM-IONIZED RESIDUAL GAS ATOMS, FORCED TO WALL BY E-field of BEAM. LIMITS CURRENT IN RINGS OR HIGH REP RATE LINAC.

4. ELECTRONS CAN CASCADE AND REACH A "QUASI" EQUILIBRIUM POPULATION OF SIMILAR LINE CHARGE TO THE ION BEAM. ELECTRON-ION TWO STREAM INSTABILITY IS UNSTABLE, AND CAN LEAD TO TRANSVERSE INSTABILITY, SIMILAR TO WHAT IS OBSERVED IN SOME PROTON RINGS.

Inertial Fusion Energy (IFE) power plants of the future will consist of four parts



A power-plant driver would fire about five targets per second to produce as much electricity as today's 1000-megawatt power plant

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