Lecture 8:

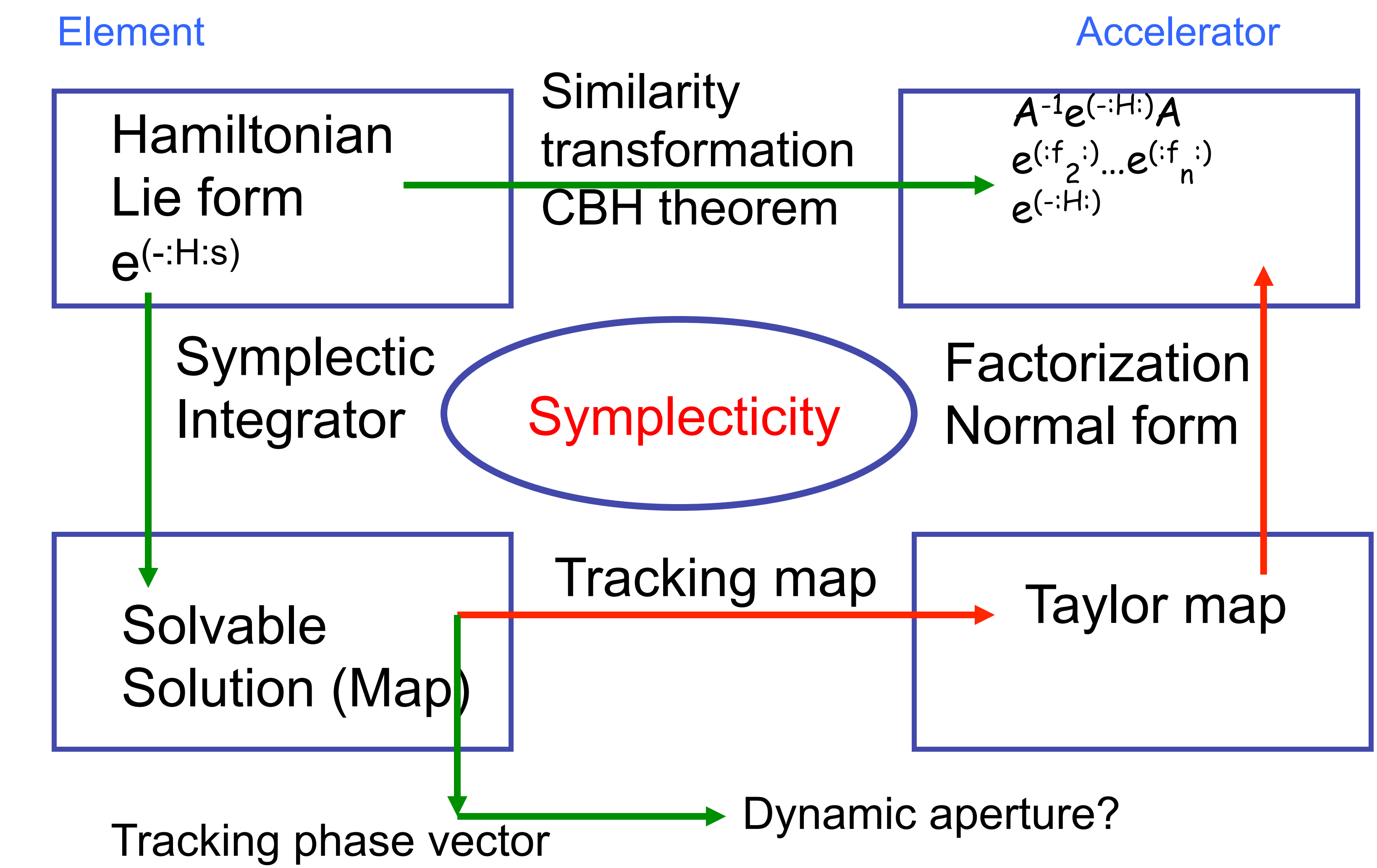
Map Analysis

Yunhai Cai
SLAC National Accelerator Laboratory

June 14, 2017

USPAS June 2017, Lisle, IL, USA

Lie Method Bases Analysis and Tracking Code



Truncated Power Series Algebra

Analytic

Given a function,

$$f(x) = \frac{1}{x + \frac{1}{x}}$$

We know that its derivative

$$f'(x) = -\frac{1 - \frac{1}{x^2}}{\left(x + \frac{1}{x}\right)^2}$$

In particular, for x=2, we have

$$f(2) = \frac{2}{5}$$

$$f'(2) = -\frac{3}{25}$$

TPSA

Rules:

$$(a_0, a_1) + (b_0, b_1) = (a_0 + b_0, a_1 + b_1)$$

$$\frac{1}{(a_0, a_1)} = \left(\frac{1}{a_0}, -\frac{a_1}{a_0^2}\right)$$

Compute:

$$\frac{1}{(2,1) + \frac{1}{(2,1)}} = \frac{1}{(2,1) + (\frac{1}{2}, -\frac{1}{4})}$$

$$= \frac{1}{(\frac{5}{2}, \frac{3}{4})} = \left(\frac{2}{5}, -\frac{3}{25}\right)$$

Result in: $f(v) = (f(a_0), f'(a_0))$

Starting: $v = (a_0, 1)$

Algebra or Rules

The rules can be derived from the rules of derivatives. But they can also be understood using the Taylor expansion,

Plus:

$$a = a_0 + a_1 x$$
$$b = b_0 + b_1 x$$
$$a + b = (a_0 + b_0) + (a_1 + b_1) x$$

Inverse:

$$a = a_0 + a_1 x$$
$$\frac{1}{a} = \frac{1}{a_0 + a_1 x} = \frac{1}{a_0 (1 + \frac{a_1}{a_0} x)} \approx \frac{1}{a_0} (1 - \frac{a_1}{a_0} x) = \frac{1}{a_0} - \frac{a_1}{a_0^2} x$$

Multiplication:

Truncation

$$ab = (a_0 + a_1 x)(b_0 + b_1 x) \approx a_0 b_0 + (a_0 b_1 + a_1 b_0) x$$

Symplectic Matrix

M is a sysmplectic matrix if it has the property that

$$\tilde{M}JM = J,$$

where J is

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

J is anti-symmetric and symplectic.

Dragt-Finn Factorization

Given a nonlinear Taylor map M, we

$$\mathcal{M}_1^{-1}\mathcal{M} = I_2$$

Here M_1 is the linear part of M. It is clear that I_2 is a second order of nonlinear map near identity. It's lowest perturbation is the second order, indicated with its subscript. Now, we would like to write I_2 as a Lie operator , namely

$$\mathcal{M}_1^{-1}\mathcal{M} = I_2 = \exp[:f_3:]$$

Once we have f_3, then we can compute the next of by

$$\mathrm{e}^{-:f_3:}\mathcal{M}_1^{-1}\mathcal{M} = I_3$$

I_3 is a third of order nonlinear map near identity. Similar process can be continued to the next order. Finally, this procedure leads to the Dragt-Finn factorization,

$$\mathcal{M} = \mathcal{M}_1\,\mathrm{e}^{:f_3:}\mathrm{e}^{:f_4:}\ldots\mathrm{e}^{:f_{n+1}:}$$

Here n is the truncation order of the Taylor map M.

Extraction of a First Order Lie Factor

To solve the equation,

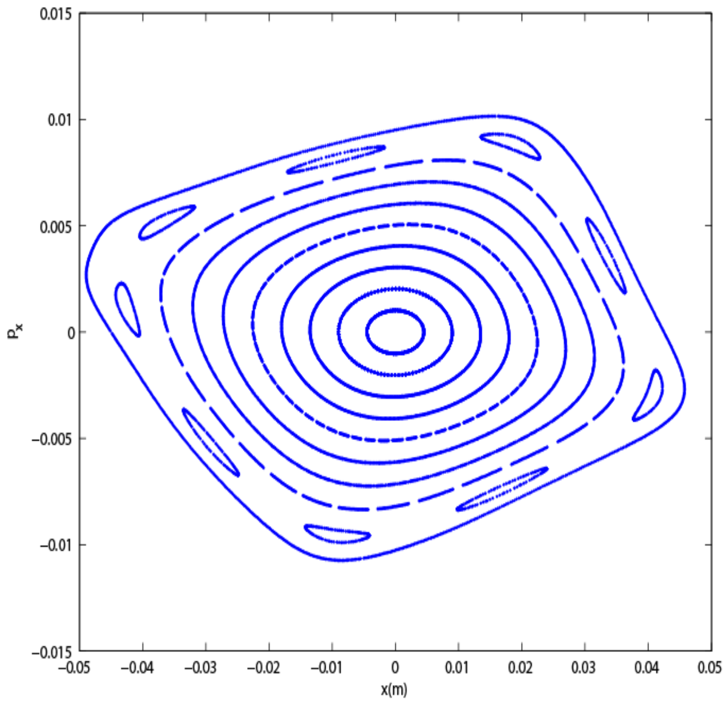
$$[f_{n+1}, z] = I_n$$

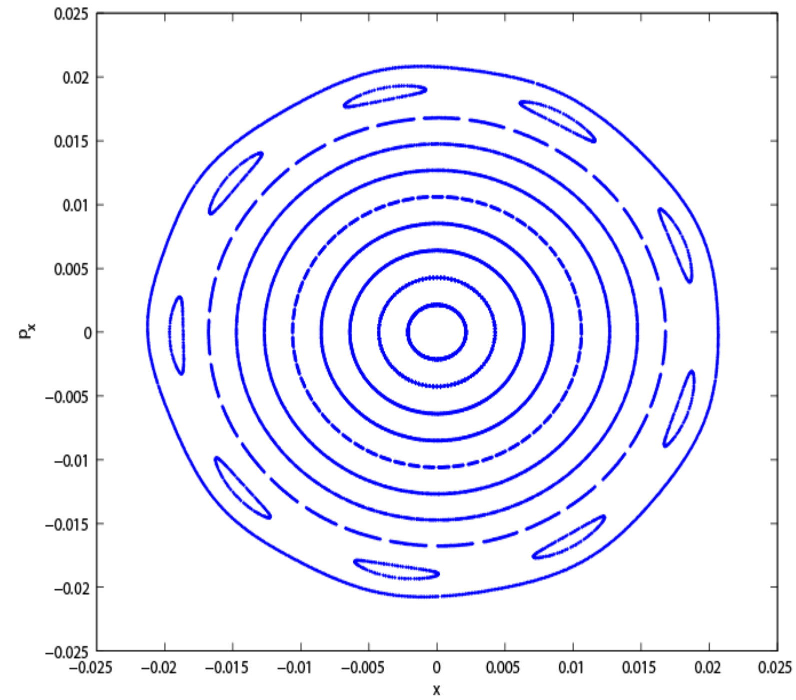
Here z is the vector in the phase space in the Poisson bracket. Its solution is given by

$$f_{n+1} = \frac{1}{n+1}\sum_{k=1}^{3}[z_{2k}(I_n)_{2k-1} - z_{2k-1}(I_n)_{2k}]$$

It is valid only if the map is symplectic.

Nonlinear Normal Form





Physical coordinates ⟶ Normalized coordinates

Transformation approximated by a 10th order Taylor map

How to Construct "Ascript"

We use eigen vectors to construct a symplectic matrix

$$U = [E_I, iE_{-I}, E_{II}, iE_{-II}, E_{III}, iE_{-III}],$$

which is symplectic and has the property that

$$U^{-1}MU = \Lambda = \mathrm{diag}(e^{i2\pi\nu_I}, e^{-i2\pi\nu_I}, e^{i2\pi\nu_{II}}, e^{-i2\pi\nu_{II}}, e^{i2\pi\nu_{III}}, e^{-i2\pi\nu_{III}})$$

"Ascript" is defined as A=UK has the property that

$$A^{-1}MA = R = K^{-1}\Lambda K$$

$$K = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i & 0 & 0 & 0 & 0 \\ -i & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -i & 0 & 0 \\ 0 & 0 & -i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -i \\ 0 & 0 & 0 & 0 & -i & 1 \end{pmatrix}$$

Further more A is symplectic and real.
Clearly, it is an extension of one dimension.

Solution of "Ascript"

Explicitly, "ascript" can be written

$$A = \sqrt{2}[\mathrm{Re}\,E_I, \mathrm{Im}\,E_I, \mathrm{Re}\,E_{II}, \mathrm{Im}\,E_{II}, \mathrm{Re}\,E_{III}, \mathrm{Im}\,E_{III}],$$

The eigen vectors are normalized as

$$\tilde{E}^*_{I,II,III} J E_{I,II,III} = i,$$
$$\tilde{E}^*_{-I,-II,-III} J E_{-I,-II,-III} = -i$$

How to get "ascript" directly from the one-turn matrix? Given "ascript", we have $U=AK^{-1}$, which we should use in our map analysis. How about propagation of U? $A_2=M_{1\text{->}2}*A_1$ leads to $U_2=M_{1\text{->}2}*U_1$. But that implies we need to write force in complex, That is rather "dangerous". Therefore, we should use the complex coordinates only in the analysis.

Nonlinear Normal Form in eigen bases

Here we switch to map notation. The operator on the left acts first. M is a nonlinear Taylor map, trunked at order n. Let's make a following transformation

$$\Lambda^{-1}\mathcal{U}\,\mathcal{M}\,\mathcal{U}^{-1} = I_2$$

It is clear that I_2 is a nonlinear map near identity. It's lowest perturbation is the second order, indicated with its subscript. Now, we would like to make a similarity transformation in the next order of perturbation, namely

$$\Lambda^{-1}e^{:f_3(\vec{u}):}\mathcal{U}\,\mathcal{M}\,\mathcal{U}^{-1}e^{-:f_3(\vec{u}):} = I_3$$

$$\Lambda^{-1}e^{:f_3(\vec{u}):}\Lambda\Lambda^{-1}\mathcal{U}\,\mathcal{M}\,\mathcal{U}^{-1}e^{-:f_3(\vec{u}):} = I_3$$

$$\Lambda^{-1}e^{:f_3(\vec{u}):}\Lambda I_2 e^{-:f_3(\vec{u}):} = I_3$$

$$e^{:f_3(\Lambda^{-1}\vec{u}):}I_2 e^{-:f_3(\vec{u}):} = I_3$$

Here we inserted an identity map after $e^{:f3:}$ and used the previous equation. In The last step, we performed a similarity transformation on the Lie operator.

Nonlinear Normal Form at third-order

We could rewrite this equation as

$$e^{:f_3(\Lambda^{-1}\vec{u}):}e^{:\bar{f}_3(\vec{u}):}e^{:-f_3(\vec{u}):} = \bar{I}_3$$

$$e^{:f_3(\Lambda^{-1}\vec{u})+\bar{f}_3(\vec{u})-f_3(\vec{u}):} = \bar{\bar{I}}_3$$

where {\bar f}_3 is Lie operator generates I_2 and {\bar I}_3 is another map near the identity map third-order perturbation. Since L is diagonal matrix We could easily solve f_3 in terms of {\bar f}_3. That is reason why we start with the complex base at linear transformation. The solution is

$$f_3(\vec{u}) - f_3(\Lambda^{-1}\vec{u}) = \bar{f}_3(\vec{u})$$

Once f_3 is calculated, we can compute I_3 using

$$e^{:f_3(\Lambda^{-1}\vec{u}):}I_2 e^{-:f_3(\vec{u}):} = I_3$$

Note that there are two similarity transformations to be used to simplify the calculation. Clearly, f_3 becomes large near the resonance.

Nonlinear Normal Form

Fourth-Order and Tune Shifts

In fourth-order,

$$e^{:h_4(u\bar{u}):}\Lambda^{-1}e^{:f_4(\vec{u}):}e^{:f_3(\vec{u}):}\mathcal{U}\,\mathcal{M}\,\mathcal{U}^{-1}e^{-:f_3(\vec{u}):}e^{-:f_4(\vec{u}):}=I_4$$

Here we could like to absorb the third-order terms in I_3 to f_4 and H_4, which foes not have any dependence on the phase of the complex coordinates. Once again it is much easy to obtain h_4 in a complex coordinate. Note, L and h_4 commute.

$$e^{:h_4(u\bar{u}):}\Lambda^{-1}e^{:f_4(\vec{u}):}e^{:f_3(\vec{u}):}\mathcal{U}\,\mathcal{M}\,\mathcal{U}^{-1}e^{-:f_3(\vec{u}):}e^{-:f_4(\vec{u}):}=I_4$$

$$e^{:h_4(u\bar{u}):}\Lambda^{-1}e^{:f_4(\vec{u}):}\Lambda\Lambda^{-1}e^{:f_3(\vec{u}):}\mathcal{U}\,\mathcal{M}\,\mathcal{U}^{-1}e^{-:f_3(\vec{u}):}e^{-:f_4(\vec{u}):}=I_4$$

$$e^{:h_4(u\bar{u}):}\Lambda^{-1}e^{:f_4(\vec{u}):}\Lambda I_3 e^{-:f_4(\vec{u}):}=I_4$$

$$e^{:h_4(u\bar{u}):}e^{:f_4(\Lambda^{-1}\vec{u}):}I_3 e^{-:f_4(\vec{u}):}=I_4$$

It is easy to see the solution is

$$f_4(\vec{u})-f_4(\Lambda^{-1}\vec{u})=\bar{f}_4(\vec{u})$$

$$h_4(u\bar{u})=-\bar{h}_4(u\bar{u})$$

Nonlinear Normal Form

This procedure can be continued until the right hand side becomes identity due to the truncation (n-th order) of the Taylor Map. The result is the normal form presentation of map

$$\mathcal{M} = \mathcal{U}^{-1} e^{-:f_3:} \ldots e^{-:f_{n+1}:} \Lambda e^{:h_3+\ldots+h_{n+1}:} e^{:f_{n+1}:} \ldots e^{:f3:} \mathcal{U}$$

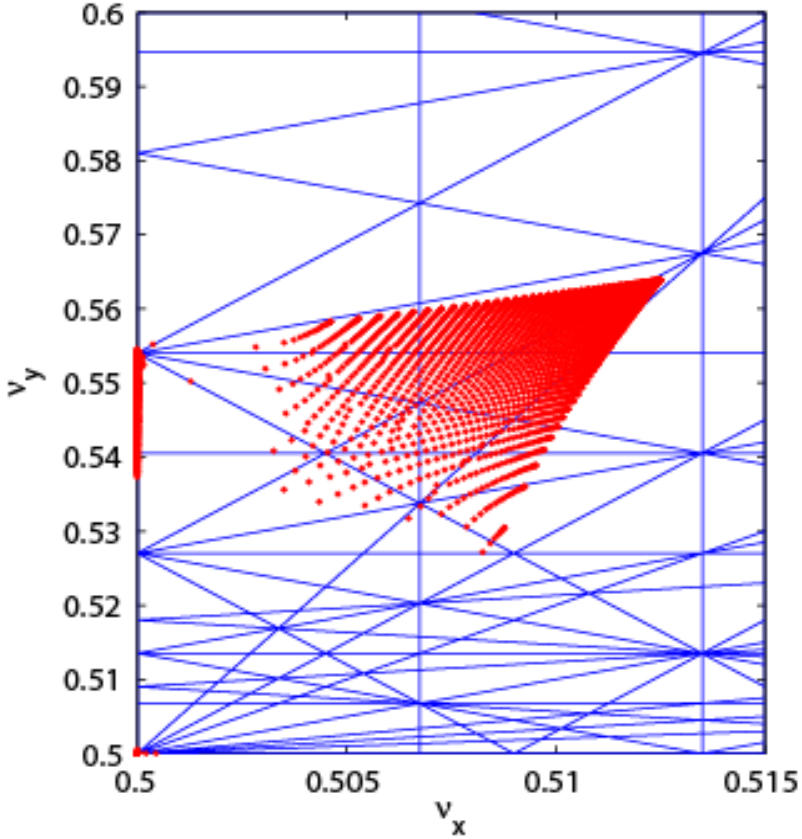
It is clear from the expression that we should perform linear transformation and then order-by-order nonlinear transformation to the nonlinear normal form. It is also easier to see the resonances in the complex coordinates. To go back to real space, we substitute $\mathcal{U}=\mathcal{K}^{-1}\mathcal{A}$

$$\begin{aligned}
\mathcal{M} &= \mathcal{A}^{-1} \mathcal{K} e^{-:f_3:} \ldots e^{-:f_{n+1}:} \Lambda e^{:h_3+\ldots+h_{n+1}:} e^{:f_{n+1}:} \ldots e^{:f3:} \mathcal{K}^{-1} \mathcal{A} \\
&= \mathcal{A}^{-1} \mathcal{K} e^{-:f_3:} \mathcal{K}^{-1} \mathcal{K} \ldots e^{-:f_{n+1}:} \mathcal{K}^{-1} \mathcal{K} \Lambda \mathcal{K}^{-1} \mathcal{K} e^{:h_3+\ldots+h_{n+1}:} e^{:f_{n+1}:} \ldots \mathcal{K}^{-1} \mathcal{K} e^{:f3:} \mathcal{K}^{-1} \mathcal{A} \\
&= \mathcal{A}^{-1} e^{-:\widehat{f}_3:} \ldots e^{-:\widehat{f}_{n+1}:} \mathcal{R} \, \mathrm{e}^{:\widehat{h}_3+\ldots+\widehat{h}_{n+1}:} e^{:\widehat{f}_{n+1}:} \ldots e^{:\widehat{f}3:} \mathcal{A}
\end{aligned}$$

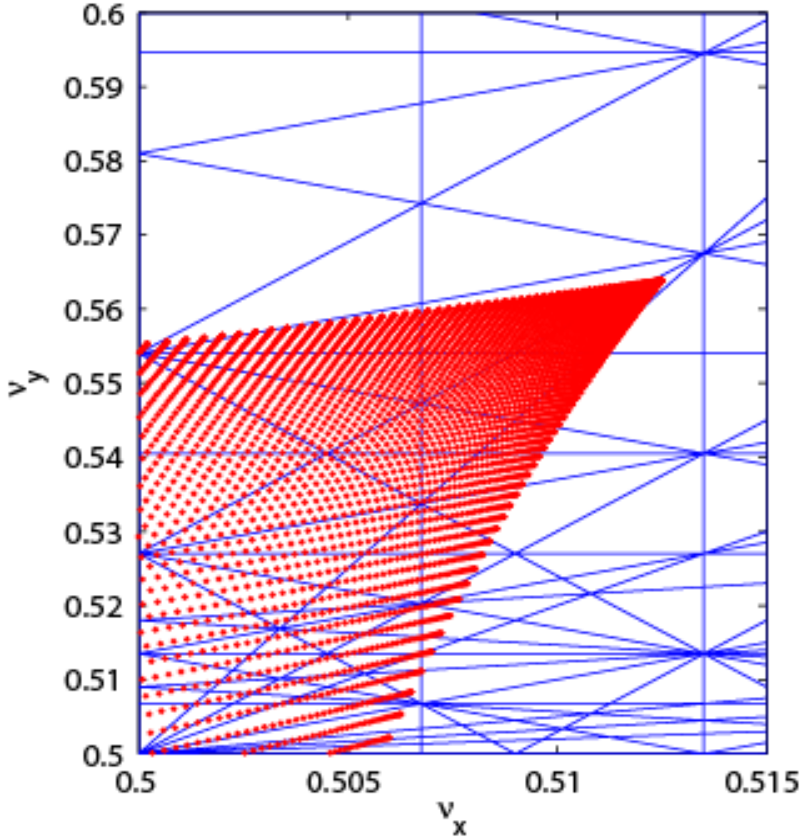
Be careful, here we used map notation, so the left acts first. In real coordinates, we have

$$\widehat{f}_n(\vec{x}) = f_n(\mathcal{K}\vec{u}), \widehat{h}_n(\vec{x}) = h_n(\mathcal{K}\vec{u})$$

Footprint in Tune Space



Frequency analysis
Tracking & FFT



Normal form analysis
Taylor map & Lie form

Summary

- Concept of truncated Taylor map is important. The map analysis should not go beyond the order of the map when it is extracted from an accelerator.
- Dragt-Finn factorization is fundamental in map analysis. The Lie factors can be compared to the analytic calculation.
- Normal form gives us the beam footprint in the tune space. It is an essential metric in design of the storage rings.

References

1) Alex J. Dragt and John M. Finn, "Normal form for mirror machine Hamiltonian," J. Math. Phys. 20(12), (1979).
2) Alexander W. Chao, "Evaluation of beam distribution parameters in an electron storage ring," J. Appl. Phys. **50**, 595 (1979).
3) M. Berz, "Differential algebraic description of beam dynamics to very high orders," Particle Accelerator **24**, pp. 109-124 (1989).
4) E. Forest, M. Berz, and J. Irwin, "Normal form methods for complicated periodic systems: a complete solution using differential algebra," Particle Accelerator **24**, pp. 91-107 (1989).

Acknowledgements

Many thanks to Alex Chao, John Irwin, Etienne Forest, Yiton Yan form who I have learned over the years.