Lecture 8

Resonators

A. Nassiri - ANL

June 19, 2003

Field Expressions For TE Modes – Rec. WG

m,n = 0,1,2,... but not both zero

$$E_z = 0$$

$$H_z = A\,cos\left(\frac{m\pi}{a}x\right)cos\left(\frac{m\pi}{b}y\right)e^{\mp j\beta_z z}$$

$$E_x = j\frac{\lambda_c^2}{4\pi^2}\omega\mu\frac{m\pi}{a}A\,cos\left(\frac{m\pi}{a}x\right)sin\left(\frac{m\pi}{b}y\right)e^{\mp j\beta_z z}$$

$$E_y = -j\frac{\lambda_c^2}{4\pi^2}\omega\mu\frac{m\pi}{b}A\,sin\left(\frac{m\pi}{a}x\right)cos\left(\frac{m\pi}{b}y\right)e^{\mp j\beta_z z}$$

$$H_x = \mp\frac{E_y}{\eta_g} \qquad H_y = \pm\frac{E_x}{\eta_g}$$

Field Expressions For TE Modes – Rec. WG

$$f_c = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$$\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

$$\lambda_g = \frac{\lambda}{\sqrt{1-(\lambda/\lambda_c)^2}} = \frac{\lambda}{\sqrt{1-(f_c/f)^2}}$$

$$v_{pz} = \frac{1}{\sqrt{\mu\varepsilon}\sqrt{1-(f_c/f)^2}} = \frac{1}{\sqrt{\mu\varepsilon}\sqrt{1-(\lambda/\lambda_c)^2}}$$

$$\eta_g = \frac{\sqrt{\mu/\varepsilon}}{\sqrt{1-(f_c/f)^2}} = \frac{\sqrt{\mu/\varepsilon}}{\sqrt{1-(\lambda/\lambda_c)^2}}$$

Field Expressions For TM Modes – Rec. WG

$$m,n = 1,2,3,\ldots$$

$$H_z = 0$$

$$E_z = A\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{\mp j\beta_z z}$$

$$E_x = \mp j\frac{\lambda_c^2}{2\pi\lambda_g}\frac{m\pi}{a}A\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{\mp j\beta_z z}$$

$$E_y = \mp j\frac{\lambda_c^2}{2\pi\lambda_g}\frac{n\pi}{b}A\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{\mp j\beta_z z}$$

$$H_x = \mp\frac{E_y}{\eta_g} \qquad H_y = \pm\frac{E_x}{\eta_g}$$

Field Expressions For TM Modes – Rec. WG

$$f_c = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \qquad \lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

$$\lambda_g = \frac{\lambda}{\sqrt{1-(\lambda/\lambda_c)^2}} = \frac{\lambda}{\sqrt{1-(f_c/f)^2}}$$

$$\nu_{pz} = \frac{1}{\sqrt{\mu\varepsilon}\sqrt{1-(f_c/f)^2}} = \frac{1}{\sqrt{\mu\varepsilon}\sqrt{1-(\lambda/\lambda_c)^2}}$$

$$\eta_g = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{1-(f_c/f)^2} = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{1-(\lambda/\lambda_c)^2}$$

Determine the lowest four cutoff frequencies of the dominant mode for three cases of rectangular wave guide dimensions *b/a=1*, *b/a=1/2*, and *b/a =1/3*. Given *a=3 cm*, determine the propagating mode(s) for *f=9 GHz* for each of the three cases.

The expression for the cutoff wavelength for the $TE_{m\,n}$ mode where m=0,1,2,3,.. and n=0,1,2,3,.. But not both m and n equal to zero and for TM_{mn} mode where m=1,2,3,.. And n=1,2,3,.. is given by

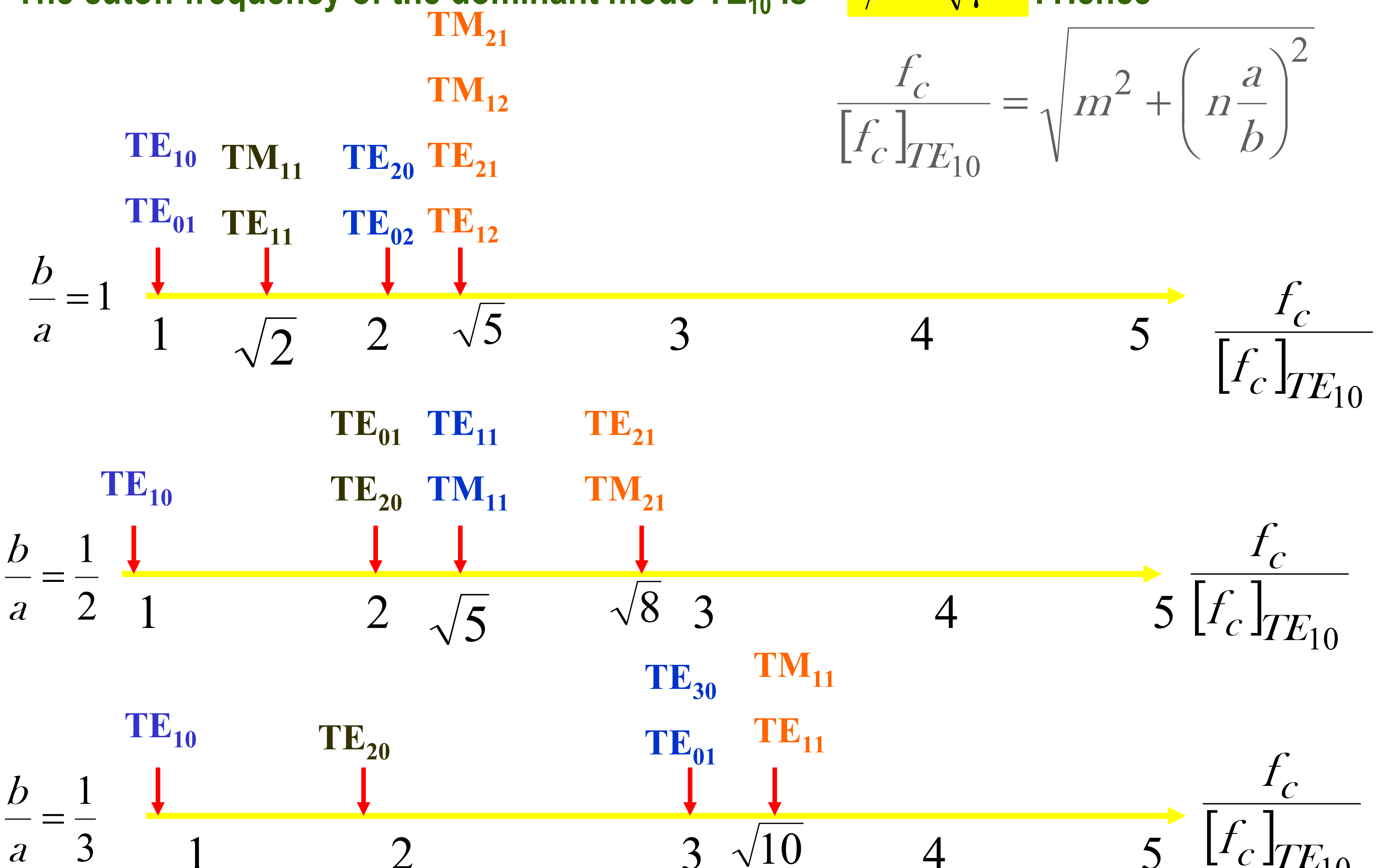
$$\lambda_c = \frac{1}{\sqrt{\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2}}$$

The corresponding expression for the cutoff frequency is

$$f_c = \frac{\nu_p}{\lambda_c} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2} = \frac{1}{2a\sqrt{\mu\varepsilon}}\sqrt{m^2 + \left(n\frac{a}{b}\right)^2}$$

The cutoff frequency of the dominant mode TE_{10} is $1/2a\sqrt{\mu\varepsilon}$. Hence

$$\frac{f_c}{[f_c]_{TE_{10}}} = \sqrt{m^2 + \left(n\frac{a}{b}\right)^2}$$



Hence for a signal of frequency f=9GHz, all the modes for which $\dfrac{f_c}{[f_c]_{TE_{10}}}$ is less that 1.8 propagate. These modes are:

TE_{10},TE_{01},TM_{11},TE_{11} for b/a=1

TE_{10} for b/a=1/2

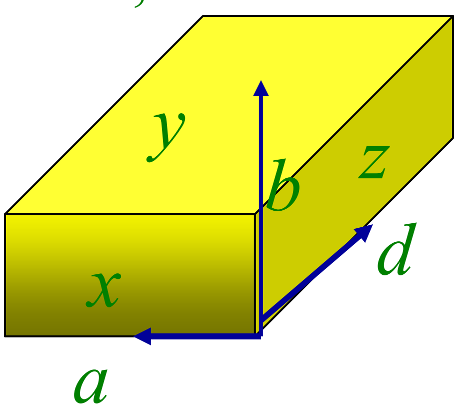
TE_{10} for b/a=1/3

So for b/a≤1/2, the second lowest cutoff frequency which corresponds to that of the TE_{20} mode is twice of the cutoff frequency of the dominant TE_{10} . For this reason, the dimension b of the a rectangular wave guide is generally chosen to be less than or equal to a/2 in order to achieve single-mode transmission over a complete octave(factor of two) range of frequencies.

Rectangular Cavity Resonator

- Add two perfectly conducting walls in z-plane separated by a distance d.
- For B.C's to be satisfied, d must be equal to an integer multiple of $\lambda_g/2$ from the wall.
- Such structure is known as a cavity resonator and is the counterpart of the low-frequency lumped parameter resonant circuit at microwave frequencies, since it supports oscillations at frequencies foe which the foregoing condition, that is

$d = l\,\lambda_g/2, \quad l = 1, 2, 3, \ldots$ is satisfied.



Rectangular Cavity Resonator

Substituting for λ_g and rearranging, we obtain

$$\frac{2d}{p} = \frac{\lambda}{\sqrt{1-(\lambda/\lambda_c)^2}}$$

$$\frac{1}{\lambda^2} - \frac{1}{\lambda_c^2} = \left(\frac{p}{2d}\right)^2$$

Substituting for λ_c gives

$$\frac{1}{\lambda^2} = \left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2 + \left(\frac{p}{2d}\right)^2 \qquad \lambda = \frac{1}{\sqrt{\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2 + \left(\frac{p}{2d}\right)^2}}$$

$$f_{osc} = \frac{\nu_p}{\lambda} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2 + \left(\frac{p}{2d}\right)^2}$$

Quality Factor Q

The quality factor is in general a measure of the ability of a resonator to store energy in relation to time-average power dissipation. Specifically, the Q of a resonator is defined as

$$Q = 2\pi \frac{\text{Maximum energy stored}}{\text{Energy dissipated per cycle}} = \omega_\circ \frac{\overline{W}_{str}}{P_{wall}}$$

$$\overline{W}_{str} = \overline{W}_e + \overline{W}_m$$

Consider the TE_{101} mode:

$$\overline{W}_e = \frac{\varepsilon}{4}\int_v \left|E_y\right|^2 dv = \frac{\varepsilon}{4}\left(\frac{\omega\mu a}{\pi}\right)^2 H_\circ^2 \int_0^d \int_0^b \int_0^a \sin^2\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{\pi z}{d}\right) dx\,dy\,dz$$

$$\overline{W}_e = \frac{abd\mu H_\circ^2}{16}\left[\frac{a^2}{d^2}+1\right]$$

$$\omega^2 = \omega_{101}^2 = \frac{\pi^2}{\mu\varepsilon}\left[\frac{1}{a^2}+\frac{1}{d^2}\right] \quad \text{and} \int_0^a \sin^2\left(\frac{\pi x}{a}\right) dx = \frac{a}{2}$$

The time average stored magnetic energy can be found as

$$\overline{W}_m = \frac{\mu}{4}\int_V |H_x|^2 + |H_z|^2 dv$$

$$= \frac{\mu}{4} H_\circ^2 \int_0^d \int_0^b \int_0^a \frac{a^2}{d^2} sin^2\left(\frac{\pi x}{a}\right) cos^2\left(\frac{\pi z}{d}\right) + cos^2\left(\frac{\pi x}{a}\right) sin^2\left(\frac{\pi z}{d}\right) dxdydz$$

$$\overline{W}_m = \frac{abd\,\mu H_\circ^2}{16}\left[\frac{a^2}{d^2} + 1\right]$$

Note that the the time-average electric and magnetic energies are precisely equal. This should be true in general simply follows from the complex Poyting's theorem. Physically, the fact that energy cycles between being purely electric, partly electric and partly magnetic, and purely magnetic storage, such that on the average over a period, it is shared equally between the electric and magnetic forms. The total time-average stored energy is

$$\overline{W}_{str} = \overline{W}_e + \overline{W}_m = \frac{abd\mu H_\circ^2}{8}\left[\frac{a^2}{d^2} + 1\right]$$

We now need to evaluate the power dissipated in the cavity walls. This dissipation will be due to the surface currents on each of the six walls as induced by the tangential magnetic fields, that is $\bar{J}_s = \hat{n} \times \overline{H}$. Note that the power dissipation is given by $\frac{1}{2}\left|\bar{J}_s\right|^2 R_s$ and that $\left|\bar{J}_s\right| = \left|\overline{H}_{tan}\right|$

$R_s = \sqrt{\pi f \mu_m / \sigma}$ is the surface resistance.

$$P_{wall} = \frac{R_S}{2} \int_{wall} \left|H_{tan}\right|^2 ds =$$

$$\frac{R_S}{2}\left[\underbrace{2\int_0^b \int_0^a \left|H_X\right|^2_{Z=0} dxdy}_{front,back} + \underbrace{2\int_0^d \int_0^b \left|H_Z\right|^2_{X=0} dydz}_{right,left} + \underbrace{2\int_0^d \int_0^a \left[\left|H_Z\right|^2 + \left|H_X\right|^2\right] dxdz}_{top,bottom}\right]$$

After completing the integration steps, we obtain:

$$P_{wall} = \frac{R_s H_\circ^2 d^2}{4}\left[\left(\frac{a}{d}\right)\left(\frac{a^2}{d^2}+1\right)+\left(\frac{2b}{d}\right)\left(\frac{a^3}{d^3}+1\right)\right]$$

Therefore the quality factor Q, is

$$Q = \frac{\omega_\circ W_{str}}{P_{wall}} = \frac{\pi\mu f_{101} ab}{R_s D}\frac{\left(\frac{a^2}{d^2}+1\right)}{\left[\left(\frac{a}{d}\right)\left(\frac{a^2}{d^2}+1\right)+\left(\frac{2b}{d}\right)\left(\frac{a^3}{d^3}+1\right)\right]}$$

Substituting for f_{101}, gives

$$Q = \frac{\pi^2 \eta b}{2 R_s d}\frac{\left(\frac{a^2}{d^2}+1\right)^{3/2}}{\left[\left(\frac{a}{d}\right)\left(\frac{a^2}{d^2}+1\right)+\left(\frac{2b}{d}\right)\left(\frac{a^3}{d^3}+1\right)\right]}$$

For a cubical resonator with *a* = *b* = *d*, we have

$$f_{101} = a^{-1}\sqrt{1/(2\mu\varepsilon)} \qquad \left(f_{101} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\frac{1}{a^2}+\frac{1}{d^2}} \right)$$

$$Q_{cube} = \frac{\pi\mu f_{101} a}{3R_s} = \frac{a\mu}{3\mu_m \delta} \qquad \left[\delta = (\pi f \mu_m \sigma)^{-1/2} \right]$$

Skin depth of the surrounding metallic walls, where μ_m is the permeability of the metallic walls.

Air-filled cubical cavity

We consider an air-filled cubical cavity designed to be resonant in TE_{101} mode at 10 GHz (free space wavelength λ=3cm)with silver-plated surfaces (σ=6.14$\times 10^{7}$S-m^{-1}, $\mu_m = \mu_0$. Find the quality factor.

$$f_{101} = a^{-1}\sqrt{1/(2\mu\varepsilon)} \Rightarrow a = \frac{1}{f_{101}}\sqrt{\frac{1}{2\mu_\circ\varepsilon_\circ}} = \frac{c}{f_{101}\sqrt{2}} = \frac{\lambda}{2} \approx 2.12cm$$

At 10GHz, the skin depth for the silver is given by

$$\delta = \left(\pi \times 10 \times 10^{9} \times 4\pi \times 10^{-7} \times 6.14 \times 10^{7}\right)^{-1/2} \approx 0.642\mu m$$

and the quality factor is

$$Q = \frac{a}{3\delta} \cong \frac{2.12cm}{3 \times 0.642\mu m} \cong 11{,}000$$

Observations

Previous example showed that very large quality factors can be achieved with normal conducting metallic resonant cavities. The Q evaluated for a cubical cavity is in fact representative of cavities of other simple shapes. Slightly higher Q values may be possible in resonators with other simple shapes, such as an elongated cylinder or a sphere, but the Q values are generally on the order of magnitude of the volume-to-surface ratio divided by the skin depth.

$$Q = \omega_\circ \frac{\overline{W}_{str}}{P_{wall}} = \frac{\omega_\circ 2\overline{W}_m}{P_{wall}} = \frac{(2\pi f_\circ)\frac{\mu}{2}\int_V H^2 dv}{\frac{R_s}{2}\oint_s H_t^2 ds} \cong \frac{2}{\delta}\frac{V_{cavity}}{S_{cavity}}$$

Where S_{cavity} is the cavity surface enclosing the cavity volume V_{cavity}.

Although very large Q values are possible in cavity resonators, disturbances caused by the coupling system (loop or aperture coupling), surface irregularities, and other perturbations (e.g. dents on the walls) in practice act to increase losses and reduce Q.

Observations

Dielectric losses and radiation losses from small holes may be especially important in reducing Q. The resonant frequency of a cavity may also vary due to the presence of a coupling connection. It may also vary with changing temperature due to dimensional variations (as determined by the thermal expansion coefficient). In addition, for an air-filled cavity, if the cavity is not sealed, there are changes in the resonant frequency because of the varying dielectric constant of air with changing temperature and humidity.

Additional losses in a cavity occur due to the fact that at microwave frequencies for which resonant cavities are used most dielectrics have a complex dielectric constant $\varepsilon = \varepsilon' - j\varepsilon''$. A dielectric material with complex permittivity draws an effective current $J_{eff} = \omega_\circ \varepsilon'' E$, leading to losses that occur effectively due to $E \cdot J_{eff}^*$

The power dissipated in the dielectric filling is

$$P_{dielectric} = \frac{1}{2}\int_V E \cdot J_{eff}^* dv = \frac{1}{2}\int_V E \cdot \omega\varepsilon'' E^* dv$$

$$= \frac{\omega_\circ \varepsilon''}{2}\int_0^a \int_0^b \int_0^d \left|E_y\right|^2 dydxdz$$

Dielectric Losses

Using the expression for E_y for the TE_{101} mode, we have

$$P_{dielectric} = \frac{\varepsilon''}{\varepsilon'}\omega_\circ \frac{\mu H_\circ^2 abd}{8}\left[\frac{a^2}{d^2}+1\right]$$

$$Q_d = \omega_\circ \frac{\overline{W}_{str}}{P_d} = \frac{\varepsilon'}{\varepsilon''}$$

$$\overline{W}_{str} = 2\overline{W}_m = \frac{\varepsilon'}{2}\int_V \left|E_y\right|^2 dv$$

$$\text{and} \quad P_{dielectric} = \frac{\omega_\circ \varepsilon''}{2}\int_V \left|E_y\right|^2 dv$$

The total quality factor due to dielectric losses is

$$\frac{1}{Q} = \frac{1}{Q_d} + \frac{1}{Q_c}$$

Teflon-filled cavity

We found that an air-filled cubical shape cavity resonating at 10 GHz has a Q_c of 11,000, for silver-plated walls. Now consider a Teflon-filled cavity, with $\varepsilon = \varepsilon_0(2.05 - j0.0006)$. Find the total quality factor Q of this cavity.

$$f_\circ = [f_{101}]_{a=d} \cong \frac{1}{2\sqrt{\mu\varepsilon'}}\sqrt{\frac{2}{a^2}} = \frac{c}{a\sqrt{2\mu_r\varepsilon_r}} \Rightarrow a = \frac{c}{\sqrt{2}f_\circ\sqrt{\varepsilon'_r}}$$

μ_r=1 for Teflon. This shows that the the cavity is $\sqrt{\varepsilon'_r}$ smaller, or a=b=d=1.48 cm. Thus we have

$$Q_c = \frac{a}{3\delta} \cong 7684$$

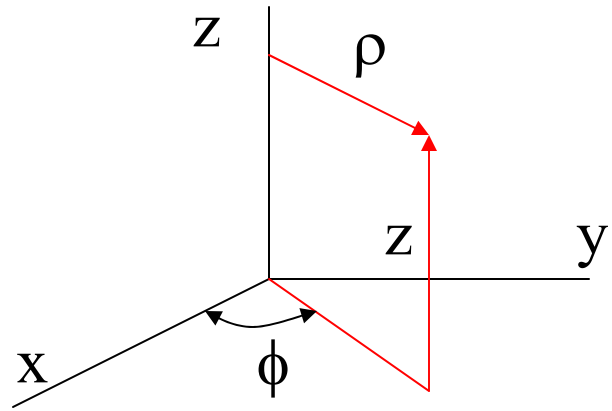
Or $\sqrt{\varepsilon'_r}$ times lower than that of the air-filled cavity. The quality factor Q_d due to the dielectric losses is given by

$$Q = \frac{Q_d Q_c}{Q_d + Q_c} \cong 2365$$

Thus, the presence of the Teflon dielectric substantially reduces the quality factor of the resonator.

Cylindrical Wave Functions

The Helmholtz equation in cylindrical coordinates is



$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2}+\frac{\partial^2\psi}{\partial z^2}+k^2\psi=0$$

The method of separation of variables gives the solution of the form

$$\frac{1}{\rho R}\frac{d}{d\rho}\frac{\rho\, dR}{d\rho}+\frac{1}{\rho^2\Phi}\frac{d^2\Phi}{\partial\phi^2}+\frac{1}{Z}\frac{d^2 Z}{\partial z^2}+k^2=0$$

Cylindrical Wave Functions

$$\frac{1}{Z}\frac{d^2 Z}{\partial z^2} = -k_z^2$$

$$\frac{\rho}{R}\frac{d}{d\rho}\frac{\rho\, dR}{d\rho} + \frac{1}{\Phi}\frac{d^2\Phi}{\partial\phi^2} + \left(k^2 - k_z^2\right)\rho^2 = 0$$

$$\frac{1}{\Phi}\frac{d^2\Phi}{\partial\phi^2} = -n^2$$

$$\frac{\rho}{R}\frac{d}{d\rho}\frac{\rho\, dR}{d\rho} - n^2 + \left(k^2 - k_z^2\right)\rho^2 = 0$$

Cylindrical Wave Functions

Define k_p *to satisfy*

$$k_\rho^2 + k_z^2 = k^2$$

$$\frac{\rho}{R}\frac{d}{d\rho}\frac{\rho\, dR}{d\rho} + \left[(k_\rho \rho)^2 - k_z^2\right]R = 0$$

$$\frac{d^2\psi}{\partial\phi^2} + n^2\Phi = 0$$

$$\frac{d^2 Z}{\partial z^2} + \boldsymbol{K}_z^2 Z = 0$$

Cylindrical Wave Functions

These are harmonic equations. Any solution to the harmonic equation we call harmonic functions and here is denoted by $h(n\phi)$ and $h(k_z z)$. Commonly used cylindrical harmonic functions are:

$$B_n(k_\rho \rho) \sim J_n(k_\rho \rho), N_n(k_\rho \rho), H_n^1(k_\rho \rho), H_n^2(k_\rho \rho)$$

Where $J_n(k_\rho \rho)$ is the Bessel function of the first kind, $N_n(k_\rho \rho)$ Is the Bessel function of the second kind, $H_n^1(k_\rho \rho)$ is the Hankel function of the first kind, and $H_n^2(k_\rho \rho)$ is the Hankel function of the second kind.

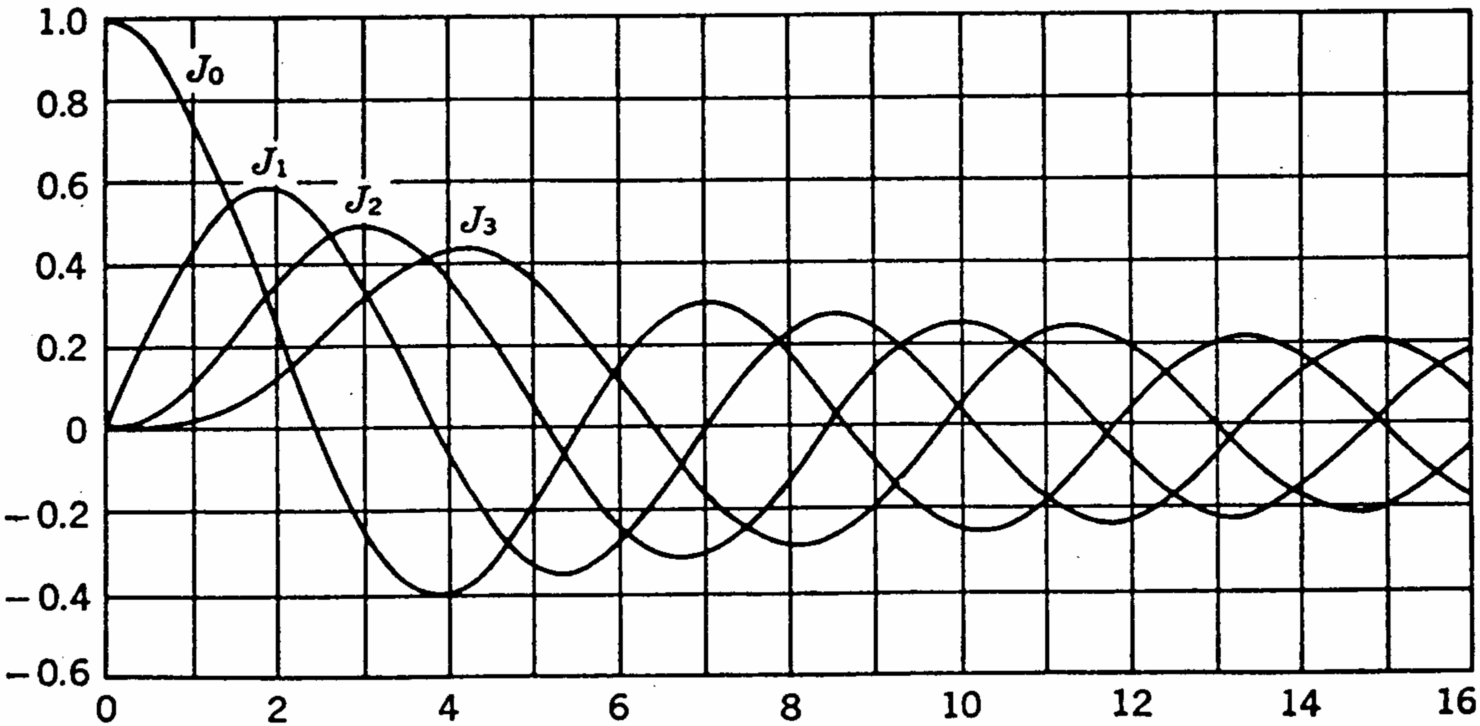
Cylindrical Wave Functions

- Any two of these are linearly independent.
- A constant times a harmonic function is still a harmonic function
- Sum of harmonic functions is still a harmonic function

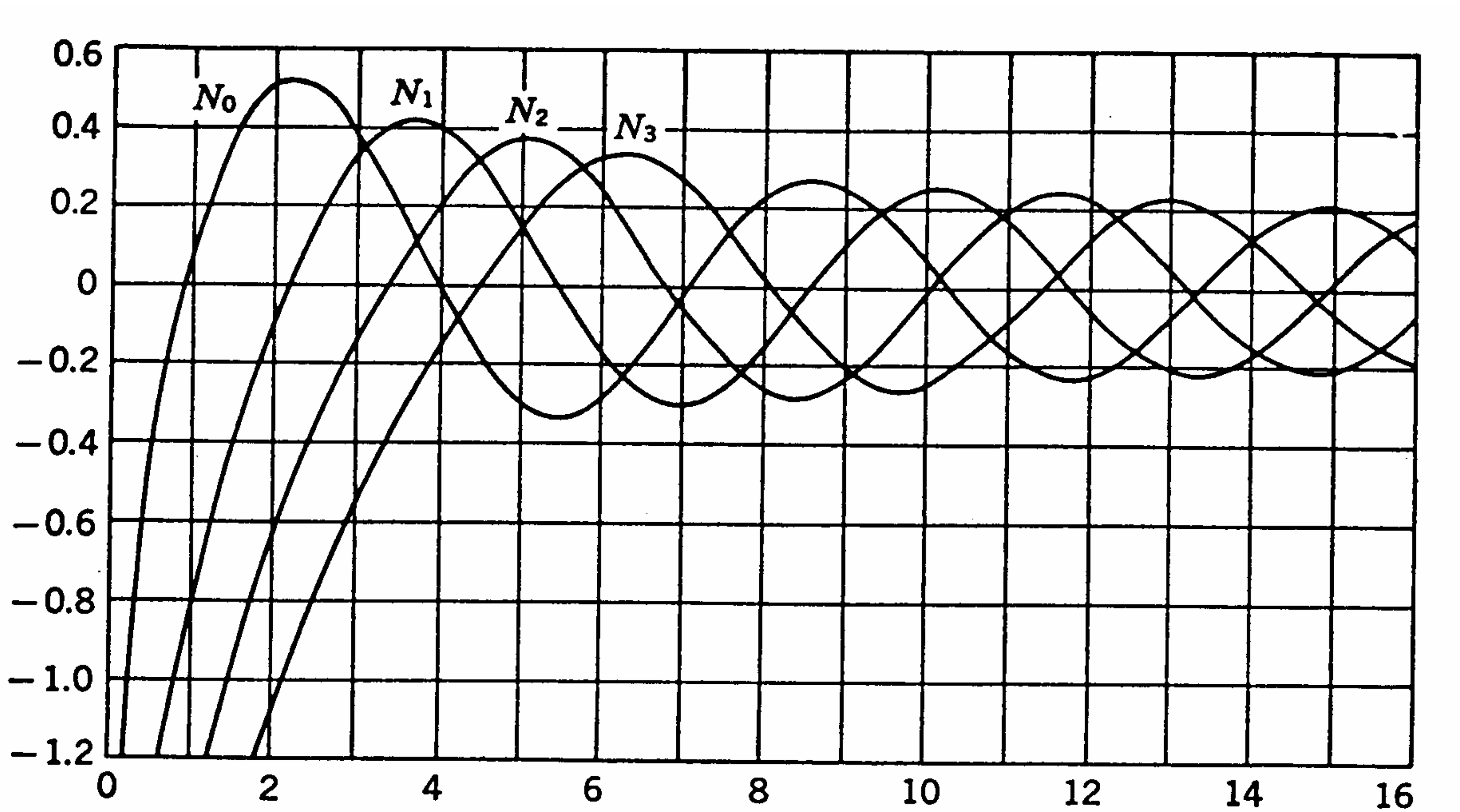
We can write the solution as :

$$\psi_{k_\rho,n,k_z} = B_n(k_\rho \rho)h(n\phi)h(k_z z)$$

Bessel functions of 1st kind



Bessel functions of 2nd kind



Bessel functions

The $J_n(k_\rho \rho)$ are nonsingular at ρ=0. Therefore, if a field is finite at ρ=0, $B_n(k_\rho \rho)$ must be $J_n(k_\rho \rho)$ and the wave functions are

$$\psi_{k_\rho, n, k_z} = J_n(k_\rho \rho) e^{jn\phi} e^{jk_z z}$$

The $H_n^{(2)}(k_\rho \rho)$ are the only solutions which vanish for large ρ. They represent outward-traveling waves if k_ρ is real. Thus $B_n(k_\rho \rho)$ must be $H_n^{(2)}(k_\rho \rho)$ if there are no sources at ρ→∞. The wave functions are

$$\psi_{k_\rho, n, k_z} = H_n^{(2)}(k_\rho \rho) e^{jn\phi} e^{jk_z z}$$

Bessel functions

$$J_n(k_\rho \rho) \quad \textit{analogous to} \quad \cos k\rho$$

$$N_n(k_\rho \rho) \quad \textit{analogous to} \quad \sin k\rho$$

$$H_n^{(1)}(k_\rho \rho) \quad \textit{analogous to} \quad e^{jk_\rho}$$

$$H_n^{(2)}(k_\rho \rho) \quad \textit{analogous to} \quad e^{-jk_\rho}$$

Bessel functions

The $J_n(k_\rho \rho)$ and $N_n(k_\rho \rho)$ functions represent cylindrical standing waves for real k as do the sinusoidal functions. The $H_n^{(1)}(k_\rho \rho)$ and $H_n^{(2)}(k_\rho \rho)$ functions represent traveling waves for real k as do the exponential functions. When k is imaginary (k = -jα)it is conventional to use the modified Bessel functions:

$$I_n(\alpha\rho) = j^n J_n(-j\alpha\rho)$$

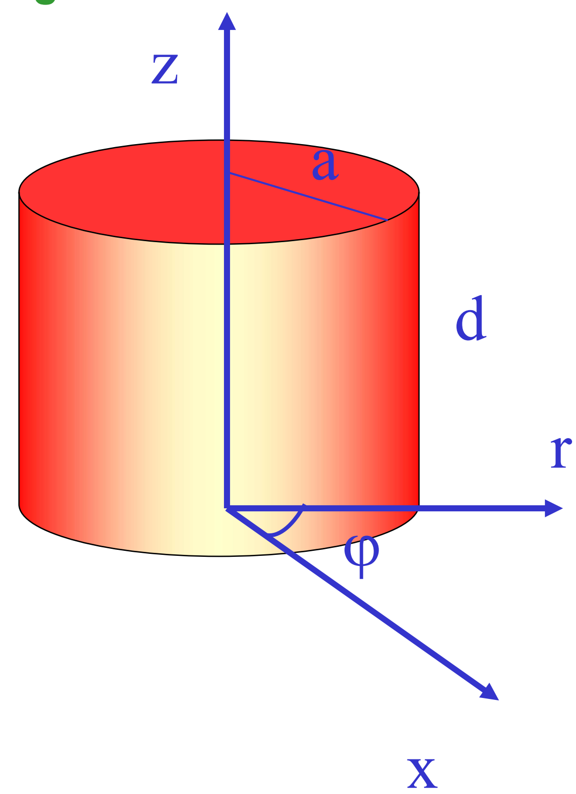
$$K_n(\alpha\rho) = \frac{\pi}{2}(-j)^{n+1} H_n^{(2)}(-\alpha\rho)$$

$I_n(\alpha\rho)$ *analogous to* $e^{\alpha\rho}$

$K_n(\alpha\rho)$ *analogous to* $e^{-\alpha\rho}$

Circular Cavity Resonators

As in the case of rectangular cavities, a circular cavity resonator can be constructed by closing a section of a circular wave guide at both ends with conducting walls.



The resonator mode in an actual case depends on the way the cavity is excited and the application for which it is used. Here we consider TE_{011}mode, which has particularly high Q.

Circular Cavity Resonators

$$\psi_{mnq}^{TM} = J_n\left(\frac{x_{mn}\rho}{a}\right)\begin{Bmatrix} \sin m\phi \\ \cos m\phi \end{Bmatrix}\cos\left(\frac{q\pi z}{d}\right)$$

$$where\ m = 0,1,2,3,\ldots; n = 1,2,3,\ldots; q = 0,1,2,3,\ldots$$

$$\psi_{mnq}^{TE} = J_n\left(\frac{x'_{mn}\rho}{a}\right)\begin{Bmatrix} \sin m\phi \\ \cos m\phi \end{Bmatrix}\sin\left(\frac{q\pi z}{d}\right)$$

$$where\ m = 0,1,2,3,\ldots; n = 1,2,3,\ldots; q = 1,2,3,\ldots$$

Circular Cavity Resonators

The separation constant equation becomes

$$\left(\frac{x_{mn}}{a}\right)^2 + \left(\frac{q\pi}{d}\right)^2 = k^2$$

$$\left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{q\pi}{d}\right)^2 = k^2$$

For the TM and TE modes, respectively. Setting $k = 2\pi f\sqrt{\mu\varepsilon}$, we can solve for the resonant frequencies

Circular Cavity Resonators

$$\frac{f_{r_{mnq}}}{f_{r_{dominant}}}$$ for the circular cavity of radius a and length d

d/a	TM_{010}	TE_{111}	TM_{110}	TM_{011}	TE_{211}	TM_{111} TE_{011}	TE_{112}	TM_{210}	TM_{020}
0.00	1.00	∞	1.59	∞	∞	∞	∞	2.13	2.29
.50	1.00	2.72	1.59	2.80	2.90	3.06	5.27	2.13	2.29
1.00	1.00	1.50	1.59	1.63	1.80	2.05	2.72	2.13	2.29
2.00	1.00	1.00	1.59	1.19	1.42	1.72	1.50	2.13	2.29
3.00	1.13	1.00	1.80	1.24	1.52	1.87	1.32	2.41	2.60
4.00	1.20	1.00	1.91	1.27	1.57	1.96	1.20	2.56	3.00
∞	1.31	1.00	2.08	1.31	1.66	2.08	1.00	2.78	3.00

Circular Cavity Resonators

Ordered zeros X_{mn} of $J_n(X)$

n \ m	0	1	2	3	4	5
1	2.405	3.832	5.136	6.380	7.588	8.771
2	5.520	7.016	8.417	9.761	11.065	12.339
3	8.654	10.173	11.620	13.015	14.372	
4	11.792	13.324	14.796			

Ordered zeros $X`_{mn}$ $J`_n(X)$

n \ m	0	1	2	3	4	5
1	3.832	1.841	3.054	4.201	5.317	6.416
2	7.016	5.331	6.706	8.015	9.282	10.520
3	10.173	8.536	9.969	11.346	12.682	13.987
4	13.324	11.706	13.170			

Circular Cavity Resonators

Cylindrical cavities are often used for microwave frequency meters. The cavity is constructed with movable top wall to allow mechanical tuning of the resonant frequency, and the cavity is loosely coupled to a wave guide with a small aperture.

The transverse electric fields (E_ρ, E_ϕ) of the TE_{mn} or TM_{mn} circular wave guide mode can be written as

$$\bar{E}_t(\rho,\phi,z) = \bar{\mathcal{E}}(\rho,\phi)\left[A^+ e^{-j\beta_{mn}z} + A^- e^{j\beta_{mn}z}\right]$$

The propagation constant of the TE_{nm} mode is

$$\beta_{mn} = \sqrt{\kappa^2 - \left(\frac{x'_{mn}}{a}\right)^2}$$

While the propagation constant of the TM_{nm} mode is

$$\beta_{mn} = \sqrt{\kappa^2 - \left(\frac{x_{mn}}{a}\right)^2}$$

Circular Cavity Resonators

Now in order to have E_t =0 at z=0, d, we must have *A^+= -A^-*, and *$A^+ \sin \beta_{nm} d$=0 or*

$\beta_{mn} d$ =$l\pi$, *for l=0,1,2,3,…,* **which implies that the wave guide must be an integer number of half-guide wavelengths long. Thus, the resonant frequency of the TE_{mnl} mode is**

$$f_{mnq} = \frac{c}{2\pi\sqrt{\mu_r \varepsilon_r}} \sqrt{\left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{q\pi}{d}\right)^2}$$

And for TM_{nml} mode is

$$f_{mnq} = \frac{c}{2\pi\sqrt{\mu_r \varepsilon_r}} \sqrt{\left(\frac{x_{nm}}{a}\right)^2 + \left(\frac{q\pi}{d}\right)^2}$$

Circular Cavity Resonators

Then the dominant TE mode is the TE_{111} mode, while the dominant TM mode is the TM_{110} mode. The fields of the TM_{nml} mode can be written as

$$H_z = H_\circ J_n\left(\frac{x'_{mn}\rho}{a}\right)\cos m\phi \sin\frac{q\pi z}{d}$$

$$H_\rho = \frac{\beta a H_\circ}{x'_{mn}} J'_n\left(\frac{x'_{mn}\rho}{a}\right)\cos m\varphi \cos\frac{q\pi z}{d}$$

$$H_\phi = \frac{-\beta a^2 m H_\circ}{(x'_{mn})^2\rho} J_n\left(\frac{x'_{mn}\rho}{a}\right)\sin m\varphi \cos\frac{q\pi z}{d}$$

$$E_\rho = \frac{jk\eta a^2 m H_\circ}{(x'_{mn})^2\rho} J_n\left(\frac{x'_{mn}\rho}{a}\right)\sin m\varphi \sin\frac{q\pi z}{d}$$

$$E_\phi = \frac{jk\eta a H_\circ}{x'_{mn}} J'_n\left(\frac{x'_{mn}\rho}{a}\right)\cos m\varphi \sin\frac{q\pi z}{d}$$

$$E_z = 0$$

$$\boxed{\eta = \sqrt{\mu/\varepsilon} \ \text{ and } H_\circ = -2jA^+}$$

Circular Cavity Resonators

Since the time-average stored electric and magnetic energies are equal, the total stored energy is

$$W = 2W_e = \frac{\varepsilon}{2}\int_0^d \int_0^{2\pi} \int_0^a \left(|E_\rho|^2 + |E_\phi|^2 \right) \rho\, d\rho\, d\phi\, dz$$

$$= \frac{\varepsilon \kappa^2 \eta^2 a^2 \pi d H_\circ^2}{4 (x'_{mn})^2} \int_{\rho=0}^a \left[J_n'^2\left(\frac{x'_{mn}\rho}{a}\right) + \left(\frac{ma}{x'_{mn}}\right)^2 J_n^2\left(\frac{x'_{mn}\rho}{a}\right) \right] \rho\, d\rho$$

$$= \frac{\varepsilon \kappa^2 \eta^2 a^4 \pi d H_\circ^2}{8 (x'_{mn})^2} \left[1 - \left(\frac{m}{x'_{mn}}\right)^2 \right] J_n^2(x'_{mn})$$

Circular Cavity Resonators

The power loss in the conducting walls is

$$P_c = \frac{R_s}{2}\int_S |H_t|^2 ds = \frac{R_s}{2}\left\{\int_{z=0}^{d}\int_{\phi=0}^{2\pi}\left[|H_\phi(\rho=a)|^2 + |H_z(\rho=a)|^2\right] a\,d\phi\,dz \right.$$

$$\left. + 2\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\left[|H_\rho(z=0)|^2 + |H_\phi(z=0)|^2\right]\rho\,d\rho\,d\phi\right\}$$

$$= \frac{R_s}{2}\pi H_\circ^2 J_n^2(x'_{mn})\left\{\frac{da}{2}\left[1+\left(\frac{\beta a m}{(x'_{mn})^2}\right)^2\right] + \left(\frac{\beta a^2}{x'_{nm}}\right)^2\left(1-\frac{m^2}{(x'_{mn})^2}\right)\right\}$$

$$Q_c = \frac{\omega W}{P_c} = \frac{(\kappa a)^3 \eta a d}{4(x'_{mn})^2 R_s}\frac{1-\left(\dfrac{m}{x'_{mn}}\right)^2}{\left\{\dfrac{ad}{2}\left[1+\left(\dfrac{\beta a m}{(x'_{mn})^2}\right)^2\right] + \left(\dfrac{\beta a^2}{x'_{mn}}\right)^2\left(1-\dfrac{m^2}{(x'_{mn})^2}\right)\right\}}$$

Circular Cavity Resonators

To compute the Q due to dielectric loss, we must compute the power dissipated in the dielectric. Thus,

$$P_d = \frac{1}{2}\int_V J \cdot E^* dv = \frac{\omega\varepsilon''}{2}\int_V \left[\left|E_\rho\right|^2 + \left|E_\phi\right|^2 \right] dv$$

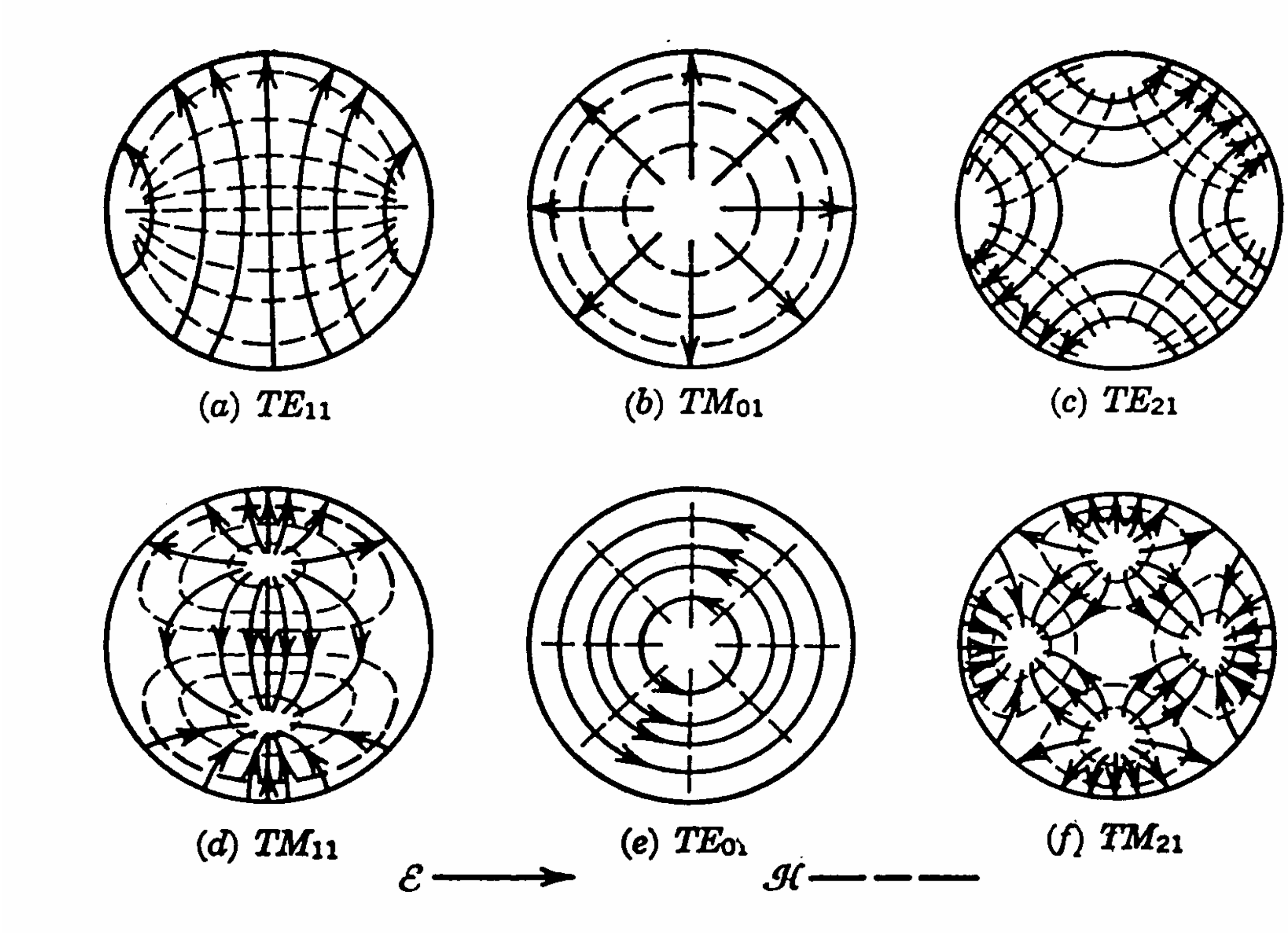
$$= \frac{\omega\varepsilon''\kappa^2\eta^2 a^2 H_\circ^2 \pi d}{4\left(x'_{mn}\right)^2}\int_{\rho=0}^{a}\left[\left(\frac{ma}{x'_{mn}\rho}\right)^2 J_n^2\left(\frac{x'_{mn}\rho}{a}\right) + J_n'^2\left(\frac{x'_{mn}\rho}{a}\right)\right]\rho\, d\rho$$

$$= \frac{\omega\varepsilon''\kappa^2\eta^2 a^4 H_\circ^2}{8\left(x'_{mn}\right)^2}\left[1 - \left(\frac{m}{x'_{mn}}\right)^2\right] J_n^2\left(x'_{mn}\right)$$

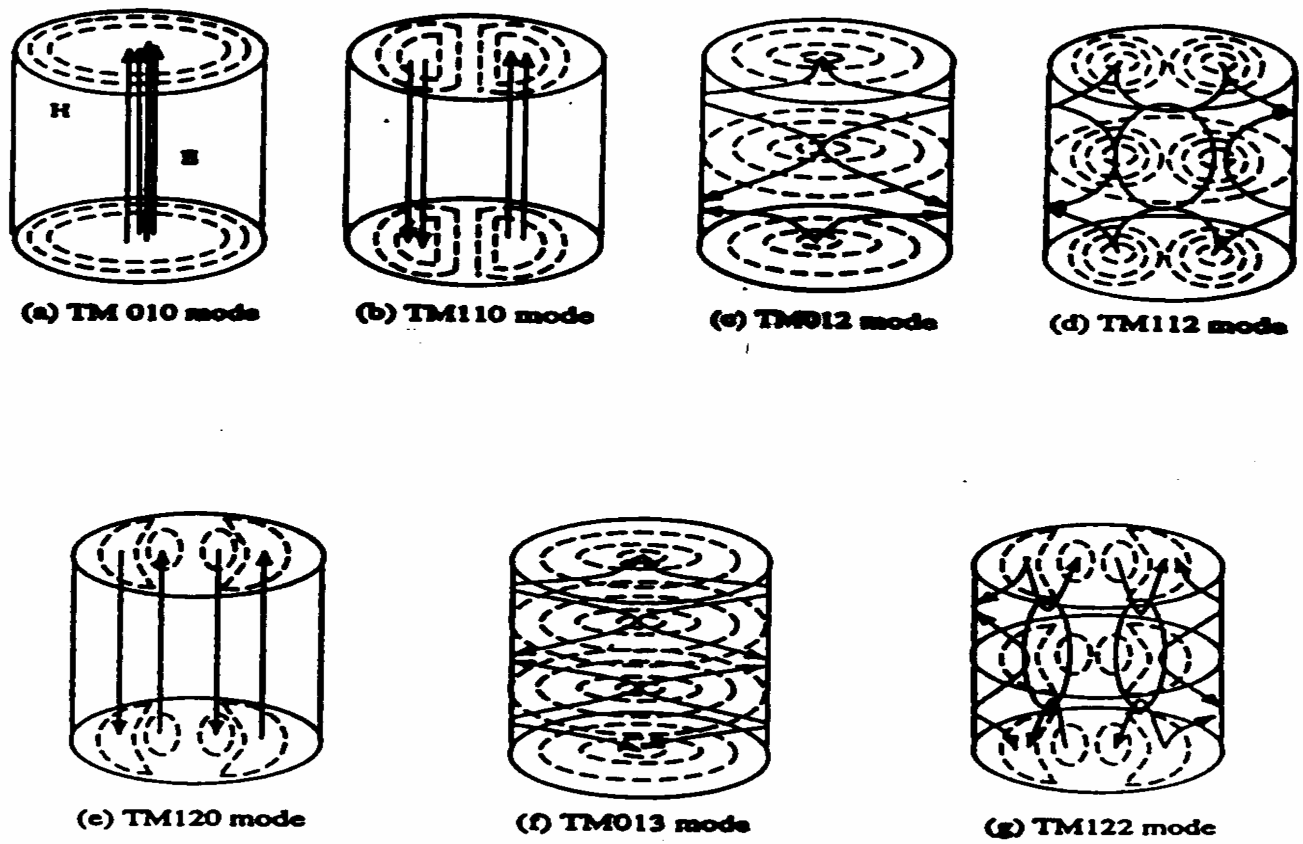
$$Q_d = \frac{\omega W}{P_d} = \frac{\varepsilon}{\varepsilon''} = \frac{1}{tan\,\delta}$$

Where $tan\,\delta$ is the loss tangent of the dielectric. This is the same as the result of Q_d for the rectangular cavity.

Cavity wave guide mode patterns



Cylindrical Cavity mode patterns



Loop or Probe Coupling

For a probe coupler the electric flux arriving on the probe tip furnishes the current induced by a cavity mode:

$$I = \omega\varepsilon SE$$

where E is the electric field from a mode averaged over probe tip and S is the antenna area. The external Q of this simple coupler terminated on a resistive load R for a mode with stored energy W is

$$Q_{ext} = \frac{2W}{R\omega\varepsilon^2 S^2 E^2}$$

In the same way for a loop coupler the magnetic flux going through the loop furnishes the voltage induced in the loop by a cavity mode:

$$V = \omega\mu SH$$