Use of Null Vectors (Corrector Ironing)

When there are more correctors than BPMS a multiplicity of solutions is possible. The response matrix $R$ now has dimensionality $m < n$ (more columns than rows). For simplicity, assume $R$ has rank $m$. Since $(\text{rank of the column space}) = (\text{rank of the row space})$, the additional $n - m$ column vectors do not extend the dimensionality. The singular value decomposition looks like:

\[
R = U \cdot W \cdot V^T = \begin{bmatrix}
| & | & | & | & | & | & | & | \\
| & u_p & | & | & | & | & w & | & 0 \\
| & | & | & | & u_n & | & - & + & - \\
| & | & | & | & | & 0 & 0 & | & - v_p & - \\
| & | & | & | & | & | & | & - v_n & - \\
\end{bmatrix}
\]

The corresponding sets of orbit- and corrector-eigenvectors have been partitioned into particular eigenvectors and null eigenvectors. Application of a linear combination of particular corrector-eigenvectors is observable at the BPMS. Application of a null corrector-eigenvector is not observable at the BPMS (hence the zero singular value). In terms of linear algebra, the particular corrector-eigenvectors span the row space of $R$ while the remaining vectors span the complementary null space of $R$.

Typically, the total corrector pattern in a storage ring may contain both particular eigenvector and null eigenvector components:

\[
\theta_i = \sum_i a_i \nu_i^p + \sum_i b_j \nu_j^n
\]

The null vector components are sometimes called 'fighting' correctors and can be reduced or eliminated. The orbit distortion caused by the correctors is

\[
x = R \cdot \theta = (U \cdot W \cdot V^T) (\sum_i a_i \nu_i^p + \sum_i b_j \nu_j^n)
\]

Since the null vectors do not contribute to observable orbit distortion

\[
x = (U \cdot W \cdot V^T) \sum_i a_i \nu_i^p
\]

or, by applying the orthonormal properties of vectors $v_i$,

\[
x = \sum_i (w_i a_i) u_i .
\]

In practice, we can replace the complete corrector set $\theta_i = \sum_i a_i \nu_i^p + \sum_i b_j \nu_j^n$ with a reduced set
\[ \theta_p = \sum_i a_i v_i^p \]

This procedure is often referred to as 'corrector ironing' [Ziemann].

To isolate to scalar coefficients \( a_i \), use SVD matrix mechanics as before:

\[ x = R \cdot \theta_p = (U \cdot W \cdot V^T) \left( \sum_i a_i v_i^p \right) \]

\[ (V \cdot W^{-1} \cdot U^T) \cdot x = \sum_i a_i v_i^p \]

(matrix inverse)

\[ V \cdot W^{-1} \cdot U \]

\[ x = \sum_i a_i v_i^p \]

(expand U)

\[ V \cdot \begin{bmatrix} w_1^{-1} & 0 & & \\ & \ddots & & \\ & & w_n^{-1} & \end{bmatrix} \cdot \begin{bmatrix} u_1 \cdot x \\ \vdots \\ u_n \cdot x \end{bmatrix} = \sum_i a_i v_i^p \]

(multiply x, expand W)

\[ a_i = \frac{u_i \cdot x}{w_i} \]

(expand V)

\[ \text{(equate coefficients of V)} \]

In physical terms, the measured orbit is projected into the non-null eigenvectors (row-space) and scaled by the corresponding singular value to find the coefficients \( a_i \). These coefficients are used to generate the corrector vector \( \theta_p = \sum_i a_i v_i^p \). The new corrector set contains no null-vector components.

An example of where this technique was used on SPEAR is the case where the position of 10 photon beams positions were held constant while the strength of approximately 30 corrector magnets were reduced (10 constraints, 30 variables).

Note - to see that the solution \( \theta_p \) is the minimum-square corrector strength set, compute the modulus of the corrector vector:

\[ (\theta_p + \theta_n) \cdot (\theta_p + \theta_n) = \theta_p^2 + 2\theta_p \cdot \theta_n + \theta_n^2 \]

Since the \( \theta_n \) component is rejected, the expression is minimized.