Lecture 6: Emittance Compensation

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Overview

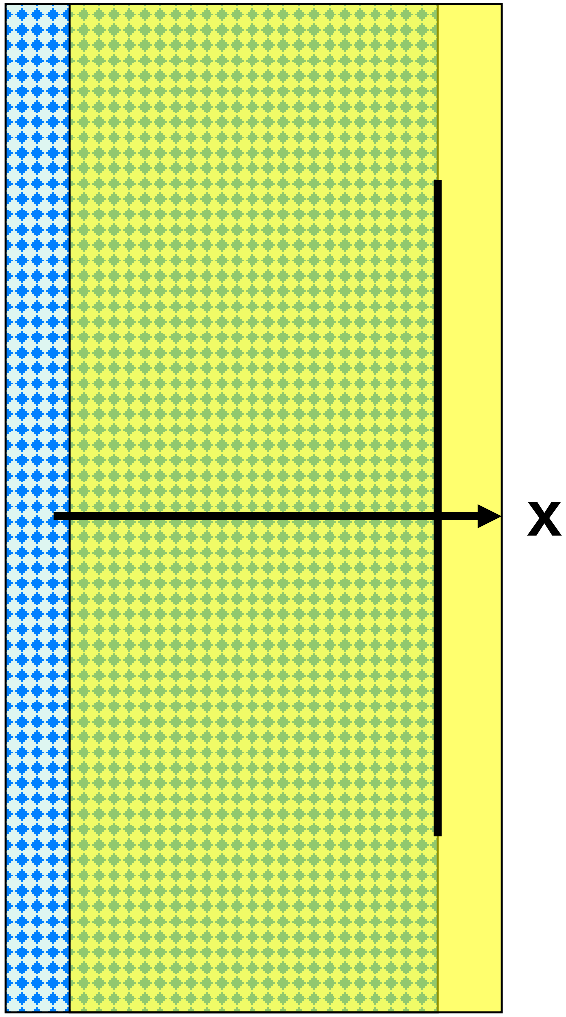
- Plasma oscillations
- Slice versus projected emittance
- Envelope equation
  - Slice properties
  - Coasting beam
  - Accelerated beam
- Emittance compensation

Plasma oscillations

- 1D theory
  - Standard 2-component plasma: Background ions give restoring force
  - Single component plasma: Restoring force provided by background fields
    - also counterstreaming ions, ionized residual gas, etc.
    - ‘lumped’ focusing elements result in modulation of restoring force

- Phase space picture

Plasma oscillations, II



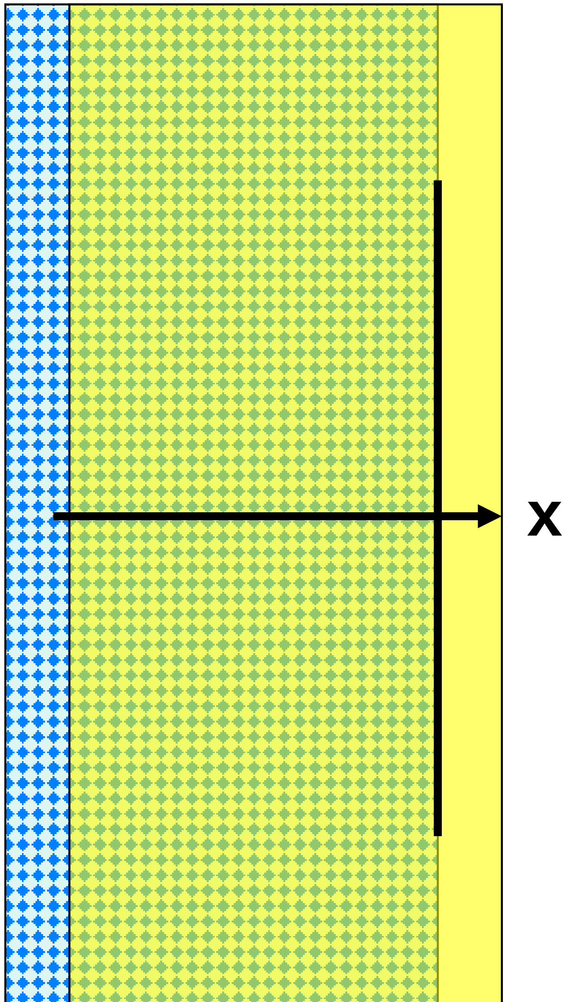
Ions (blue) and electrons (yellow) fill x<0.

The charge density is each species is equal ($N_e=N_i=N$), so the region is charge neutral.

A small displacement (Δx) of the electron cloud produces a surface charge density $\sigma = -N\, e\, \Delta x$

This surface charge generates an electric field $E_x = N\, e\, \Delta x / \varepsilon_0$

Plasma oscillations, III



The electric field acts back upon the individual electrons via the Lorentz force: dp/dt = qE (no magnetic field in this picture)

$$d(mv_x)/dt = -eE_x = -e^2 N x / \varepsilon_0$$

Simplifying a bit,

$$d^2x/dt^2 = - e^2 N x / m \varepsilon_0 = - \omega_p^2 x$$

The (nonrelativistic) plasma frequency is

$$\omega_p^2 = e^2 N / m \varepsilon_0$$

Relativistically: $\omega_p^2 = e^2 N / \gamma^3 m \varepsilon_0$

Plasma oscillations, IV

That's all very nice, but what does this have to do with electron *beams* ?!?

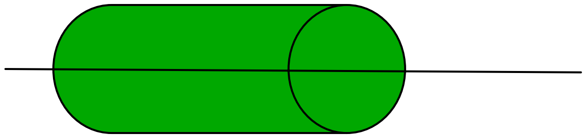
An electron beam is a non-neutral, single component plasma (with a nonzero average velocity)

Instead of a uniform background field of ions, we find ourselves with (largely linear) restoring forces, in the form of focusing and accelerating elements.

The self-fields of the electrons in the beam fight containment. These forces are proportional to charge density. A plasma picture is very useful.

Beam Charge Model

- Following Reiser (Ch. 4.2.1) we consider an axisymmetric model of our beam,
  - Carries charge Q and current I
  - Uniformly distributed over a radius 'a'
  - Bunch length L >> R
  - Travels with constant velocity v along the longitudinal direction.



- The continuity equation (conservation of charge) relates the beam charge densities to the current and velocity

$$\partial\rho/\partial t + \nabla \bullet \vec{J} = 0 \Longrightarrow \mathbf{J}=\rho\mathbf{v}$$

$$J = I / \pi a^2$$

$$\rho = \rho_0 = I / \pi a^2 v$$

Beam Electric and Magnetic Fields

Applying Gauss' and Ampere's Laws to the model we obtain the transverse field components:

$$\int \varepsilon_0 \vec{E} \cdot d\vec{S} = \int \rho dV \quad \begin{cases} E_r = \dfrac{\rho_0 r}{2\varepsilon_0} = \dfrac{Ir}{2\pi\varepsilon_0 a^2 v}, & r \le a \\ E_r = \dfrac{I}{2\pi\varepsilon_0 r v}, & r > a \end{cases}$$

$$\int \vec{B} \cdot d\vec{l} = \int \mu_0 \vec{J} \cdot d\vec{S} \quad \begin{cases} B_\theta = \mu_0 \dfrac{Ir}{2\pi a^2}, & r \le a \\ B_\theta = \mu_0 \dfrac{I}{2\pi r}, & r > a \end{cases}$$

Beam self-field forces

- We can calculate the forces on a beam particle from the Lorentz force components.

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \Rightarrow \begin{cases} F_r = q\left(E_r - c\beta_z B_\theta\right) \\ F_\theta = F_z = 0 \end{cases}$$

With $E_r = \dfrac{Ir}{2\pi\varepsilon_0 a^2 v}$ $\quad B_\theta = \mu_0 \dfrac{Ir}{2\pi a^2}$

$$\boxed{F_r = e\left(E_r - c\beta_z B_\theta\right) = \frac{eIr}{2\pi\varepsilon_0 \gamma^2 \beta c a^2}}$$

Single Particle Equations of Motion

Single particle dynamics follows from the Newton-Lorentz force law:

$$\frac{d\vec{P}}{dt} = \vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

The mechanical momentum is related to the particle mass and velocity via

$$\vec{P} = \gamma m\vec{v}$$

The change in energy of a particle follows from the usual expression

$$\frac{dE_T}{dt} = \frac{d\vec{P}}{dt}\cdot\vec{v} = q\vec{E}\cdot\vec{v}$$

Where the total, kinetic, and rest energy are related by

$$E_T = mc^2 + T = \gamma mc^2$$

Cylindrical Coordinates

- In systems with azimuthal symmetry, we often use cylindrical coordinates where the position and velocity vectors have definitions

$$\vec{x} = (r,\theta,z)$$

$$\vec{v} = (\dot{r}, r\dot{\theta}, \dot{z})$$

- This leads to the Newton-Lorentz equations of motion:

$$\frac{d}{dt}(\gamma m\dot{r}) - \gamma m r\dot{\theta}^2 = q\left(E_r + r\dot{\theta}B_z - \dot{z}B_\theta\right)$$

$$\frac{1}{r}\frac{d}{dt}\left(\gamma m r^2\dot{\theta}\right) = q\left(E_\theta + \dot{z}B_r - \dot{r}B_z\right)$$

$$\frac{d}{dt}(\gamma m\dot{z}) = q\left(E_z + \dot{r}B_\theta - r\dot{\theta}B_r\right)$$

Simplifications

- The front end of injectors are designed with axisymmetry

$$\partial/\partial\theta \to 0$$

- The external fields from rf structures and solenoid magnets are (to lowest order) described by a small number of components

$$E_z(z, r=0) = E(z) \qquad B_z(z, r=0) = B(z)$$

$$E_r \cong -\frac{r}{2}\frac{\partial E}{\partial z} \qquad B_r \cong -\frac{1}{2} r \frac{dB}{dz}$$

- We are more interested in derivatives with respect to beamline position (z) rather than time

$$\dot{\gamma} = \gamma' \dot{z} = \gamma' \beta c$$

$$\dot{r} = r' \dot{z} = r' \beta c$$

$$\dot{\theta} = \theta' \beta c = -\frac{qB}{2\gamma m \beta c} + \frac{p_\theta}{m c \gamma \beta r^2}$$

$$\ddot{r} = \beta c \frac{d}{dz}\left(r' \beta c\right) = r'' \beta^2 c^2 + r' \beta' \beta c^2 = r'' \beta^2 c^2 + r' \frac{\gamma'}{\gamma^3} c^2$$

Particle motion from self-fields

In our model the beam self-field forces are entirely radial.

$$F_r = e\left(E_r - c\beta_z B_\theta\right) = \frac{eIr}{2\pi\varepsilon_0\gamma^2\beta c a^2} = \frac{d}{dt}\left(\gamma m\dot{r}\right) \cong \gamma m\ddot{r}$$

From $\ddot{r} = \beta^2 c^2 r''$ we have

$$r'' = \ddot{r}/\beta^2 c^2 = \frac{eIr}{2\pi\varepsilon_0\gamma^3\beta^3 m c^3 a^2}$$

Define a characteristic electron current $I_0 = \frac{4\pi\varepsilon_0 m c^3}{e} \approx 17kA$

$$r'' = \frac{eIr}{2\pi\varepsilon_0\gamma^3\beta^3 m c^3 a^2} = \frac{2\,I/I_0}{\gamma^3\beta^3}\frac{r}{a^2}$$

Equivalent self-field descriptions

This is still not as simple as we might like.

Recall the *plasma frequency* $\omega_p^2 = e^2 N / \gamma^3 m \varepsilon_0$, which can be expressed (in terms of beam parameters) as

$$\omega_p^{\ 2} = \frac{eI}{\pi \varepsilon_0 m c \beta \gamma^3 a^2} \qquad \text{(which gives } r'' = \frac{\omega_p^{\ 2}}{2} r \text{)}$$

Another parameter, the *generalized perveance (K)*, is also a useful descriptor

$$K = \frac{I}{I_0} \frac{2}{\gamma^3 \beta^3} = \frac{\omega_p^{\ 2} a^2}{2 \beta^2 c^2}$$

The particle motion then follows $r'' = \dfrac{K}{a^2} r$

Paraxial Ray Approximation

The paraxial approximation describes the 1st-order beam dynamics by assuming that particle orbits are largely parallel to the optical axis, and that transverse excursions have small (forward) angles.

$$(Radial\ \ velocity)\ \ r' << 1\ \ or\ \ \dot{r} << \dot{z}$$

$$(Azimuthal\ \ velocity)\ \ r\dot{\theta} << \dot{z}$$

With this approximation, we may neglect terms of 2nd-order and higher (r^2, rr', r'^2, etc.) in the description of the transverse field expansions and particle motion. Additional linear forces are included by additional force terms in the Newton-Lorentz equations.

This approximation can faithfully reproduce the vast majority or observable beam dynamics in injectors. Regimes where the paraxial approximation breakdown are rare.

Paraxial Ray Equation

We are generally interested in obtaining a dynamical equation for the particle's *radial* motion. Using the previous approximations and symmetry we can derive the *Paraxial Ray Equation*:

$$r'' + \frac{\gamma'}{\gamma\beta^2} r' + \frac{\gamma''}{2\gamma\beta^2} r + \left(\frac{qB}{2\gamma\beta mc}\right)^2 r - \left(\frac{p_\theta}{\gamma\beta mc}\right)^2 \frac{1}{r^3} - \frac{K}{a^2} r = 0$$

Change in particle trajectory slope

Focusing from axial electric field component

Focusing from radial electric field component

Magnetic focusing

Defocusing from centrifugal forces

Defocusing from self-fields

Modification from beam emittance

The paraxial ray equation is a *single particle* equation of motion. BUT, it requires knowledge of the beam envelope (‘*a*’) behavior and evolution.

In previous studies of beam dynamics the effect of emittance on the beam envelope was also important and could be described by

$$a'' = \frac{\varepsilon^2}{a^3} = \frac{\varepsilon_n{}^2}{\gamma^2\beta^2 a^3}$$

The envelope also follows an equation of motion, obtained by replacing ‘*r*’ with ‘*a*’ in the paraxial equation and adding a term to reflect the beam emittance:

$$a'' + \frac{\gamma'}{\gamma\beta^2}a' + \frac{\gamma''}{2\gamma\beta^2}a + \left(\frac{qB}{2\gamma\beta mc}\right)^2 a - \left(\frac{p_\theta}{\gamma\beta mc}\right)^2 \frac{1}{a^3} - \frac{\varepsilon_n{}^2}{\gamma^2\beta^2}\frac{1}{a^3} - \frac{K}{a} = 0$$

Envelope description

- Useful for studying beam transport
- 2nd order, nonlinear ODE (or set of ODEs)
- Based on KV distribution and edge values, but extendable to other distributions when using RMS measures
- Assumes constant normalized emittance and current
  - Useful description if slice (thermal) emittance is invariant
  - Evolving phase space distribution requires full Vlasov treatment
  - Evolving local emittance (local scattering) may require Fokker-Planck description
- Basis of many tracking codes (e.g. HOMDYN for photoinjectors)

RMS Moments and Dynamics

Beam distributions are generally NOT uniform in charge density with simple geometric cross-sections. An RMS description is valuable to connect theory to measurements.

The K-V beam distribution (if applicable) produces the envelope equation that describes the beam edge dynamics.

Lapostolle and Sacherer discovered that the 2nd order moments (i.e. RMS values) of *any* reasonable beam distribution evolve according to an envelope equation, when the envelope is interpreted in an RMS sense.

RMS Envelope Equation

We define the RMS values for the beam spot and emittance

$$\sigma_r^{\ 2} \equiv \left\langle r^2 \right\rangle = \left\langle x^2 + y^2 \right\rangle$$

$$\varepsilon_r^{\ 2} = \frac{1}{4}\left[\left\langle r^2 \right\rangle\left\langle r'^2 \right\rangle - \left\langle rr' \right\rangle^2\right] = \ \varepsilon_x^{\ 2} = \left[\left\langle x^2 \right\rangle\left\langle x'^2 \right\rangle - \left\langle xx' \right\rangle^2\right]$$

where the brackets indicate averages over the distribution. The RMS envelope equation can be written in the compact form

$$\boxed{\sigma_r'' + \frac{\gamma'}{\gamma}\sigma_r' + k_{eff}^{\ 2}\sigma_r - \frac{\varepsilon_{eff}^{\ 2}}{\sigma_r^{\ 3}} - \frac{K}{\sigma_r} = 0}$$

$$k_{eff}^{\ 2} = \frac{\gamma''}{2\gamma} + k_{RF}^2 + k_L^2 \qquad k_L^2 = \left(\frac{eB_z}{2\gamma mc}\right)^2 \qquad \varepsilon_{eff}^{\ 2} = 4\varepsilon_r^{\ 2} + \frac{\left\langle p_\theta^{\ 2} \right\rangle}{\gamma^2\beta^2 m^2 c^2}$$

$$k_{RF}^2 = \frac{1}{2}\left(\frac{eE}{\gamma mc^2}\right)^2 \eta(\phi) \approx \frac{1}{2}\left(\frac{\gamma'}{\gamma}\right)^2 \qquad = 4\varepsilon_r^{\ 2} + \frac{e^2\left(B_{z,cathode}\sigma_{r,cathode}\right)^2}{\gamma^2\beta^2 m^2 c^2}$$

Slice Properties

- Divergence of longitudinal and transverse dynamics
  - Transverse forces proportional to $1/\gamma$
  - Longitudinal forces proportional to $1/\gamma^3$
  - Once electrons are moderately relativistic $\gamma > \sim 2$
- Longitudinal beam variations
  - Charge, spot size, energy, transverse offset and momentum
  - Transverse distribution, angular momentum
  - Emittance, Twiss parameters
  - . . .

We'll now look at equilibrium and quasi-equilibrium states of the beam.

Envelope equation parameterized by slice coordinate

Accept variation in slice parameters (current, spot size, emittance, etc.)

$$\sigma_{r,s}'' + \frac{\gamma_s'}{\gamma_s}\sigma_{r,s}' + k_{eff,s}^2\sigma_{r,s} - \frac{\varepsilon_{eff,s}^2}{\sigma_{r,s}^3} - \frac{K_s}{\sigma_{r,s}} = 0$$

Each longitudinal slice follows an individual orbit, governed by a separate instance of the envelope equation (labeled by ‘s’).

We will assume (here) that particles do not leave their respective slices -- no mixing -- and that particle orbits do not cross the axis (laminar flow). This entails a ***space charge dominated*** rather than an ***emittance dominated*** beam. This follows if

$$\sigma_{r,s}^2 K_s >> \varepsilon_{eff,s}^2$$

Individual slice equilibria

Equilibria are sometimes easy to find. We look for steady-state solutions to the slice envelope equation. These solutions represent the physical state where the external focusing completely balances the internal forces.

Setting $\sigma_{r,s}{}' = 0 = \sigma_{r,s}{}'' \Rightarrow$

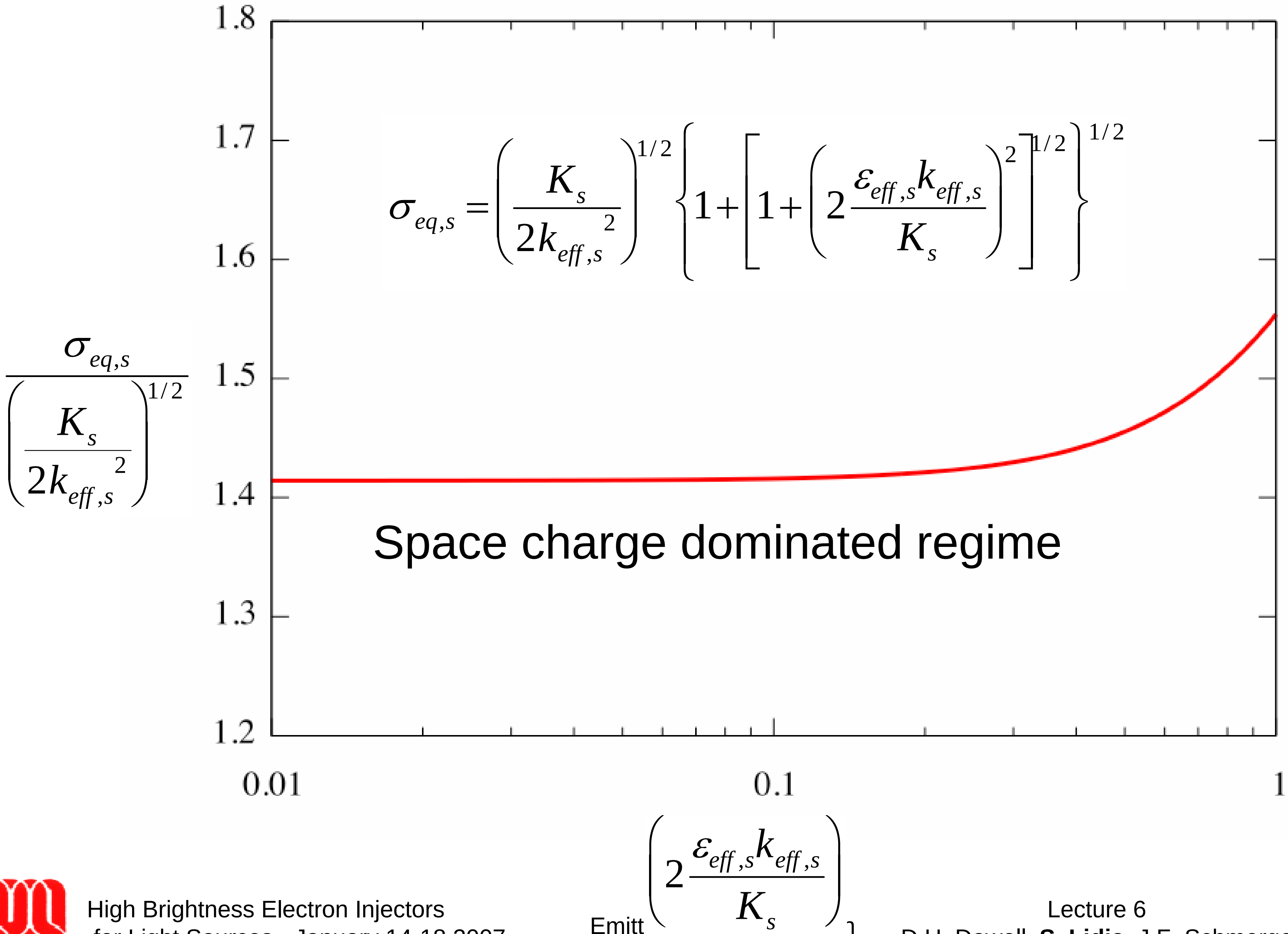
$$\Rightarrow \; k_{eff,s}{}^2 \sigma_{r,s} - \frac{\varepsilon_{eff,s}{}^2}{\sigma_{r,s}{}^3} - \frac{K_s}{\sigma_{r,s}} = 0$$

This algebraic equation has (real) solution

$$\sigma_{eq,s} = \left(\frac{K_s}{2k_{eff,s}{}^2} \right)^{1/2} \left\{ 1 + \left[1 + \left(2\frac{\varepsilon_{eff,s} k_{eff,s}}{K_s} \right)^2 \right]^{1/2} \right\}^{1/2}$$

$$\left(2\frac{\varepsilon_{eff,s} k_{eff,s}}{K_s} \right) \leq 1$$

Space-charge dominated beam



$$\sigma_{eq,s} = \left(\frac{K_s}{2k_{eff,s}^2}\right)^{1/2}\left\{1+\left[1+\left(2\frac{\varepsilon_{eff,s}k_{eff,s}}{K_s}\right)^2\right]^{1/2}\right\}^{1/2}$$
$\dfrac{\sigma_{eq,s}}{\left(\dfrac{K_s}{2k_{eff,s}^2}\right)^{1/2}}$
1.8
1.7
1.6
1.5
1.4
1.3
1.2
0.01
0.1
1
$\left(2\dfrac{\varepsilon_{eff,s}k_{eff,s}}{K_s}\right)$
Space charge dominated regime

Slice equilibria scaling

In the space charge dominated regime the slice equilibrium solutions

$$\sigma_{eq,s} = \left(\frac{K_s}{2k_{eff,s}^{\ 2}}\right)^{1/2} \left\{1 + \left[1 + \left(2\frac{\varepsilon_{eff,s} k_{eff,s}}{K_s}\right)^2\right]^{1/2}\right\}^{1/2} \approx \left(\frac{K_s}{2k_{eff,s}^{\ 2}}\right)^{1/2} O(1)$$

$$k_{eff,s}^2 = \frac{\gamma_s''}{2\gamma_s} + \frac{1}{2}\left(\frac{\gamma_s'}{\gamma_s}\right)^2 + \left(\frac{eB_z}{2\gamma_s mc}\right)^2 \propto \frac{1}{\gamma_s^{\ 2}} \qquad K_s = \frac{I_s}{I_0}\frac{2}{\gamma_s^{\ 3}\beta_s^{\ 3}} \propto \frac{I_s}{\gamma_s^{\ 3}}$$

. . . have the scaling with parameters

$$\boxed{\sigma_{eq,s} \propto \left(\frac{I_s}{\gamma_s^{\ 3}}\gamma_s^{\ 2}\right)^{1/2} \approx \frac{I_s^{\ 1/2}}{\gamma_s^{\ 1/2}}}$$

Linear oscillations about equilibria

We are very interested in what happens to our beam when the slices are slightly 'out of equilibrium'. We expect the equilibrium spot size to have some variation with parameters

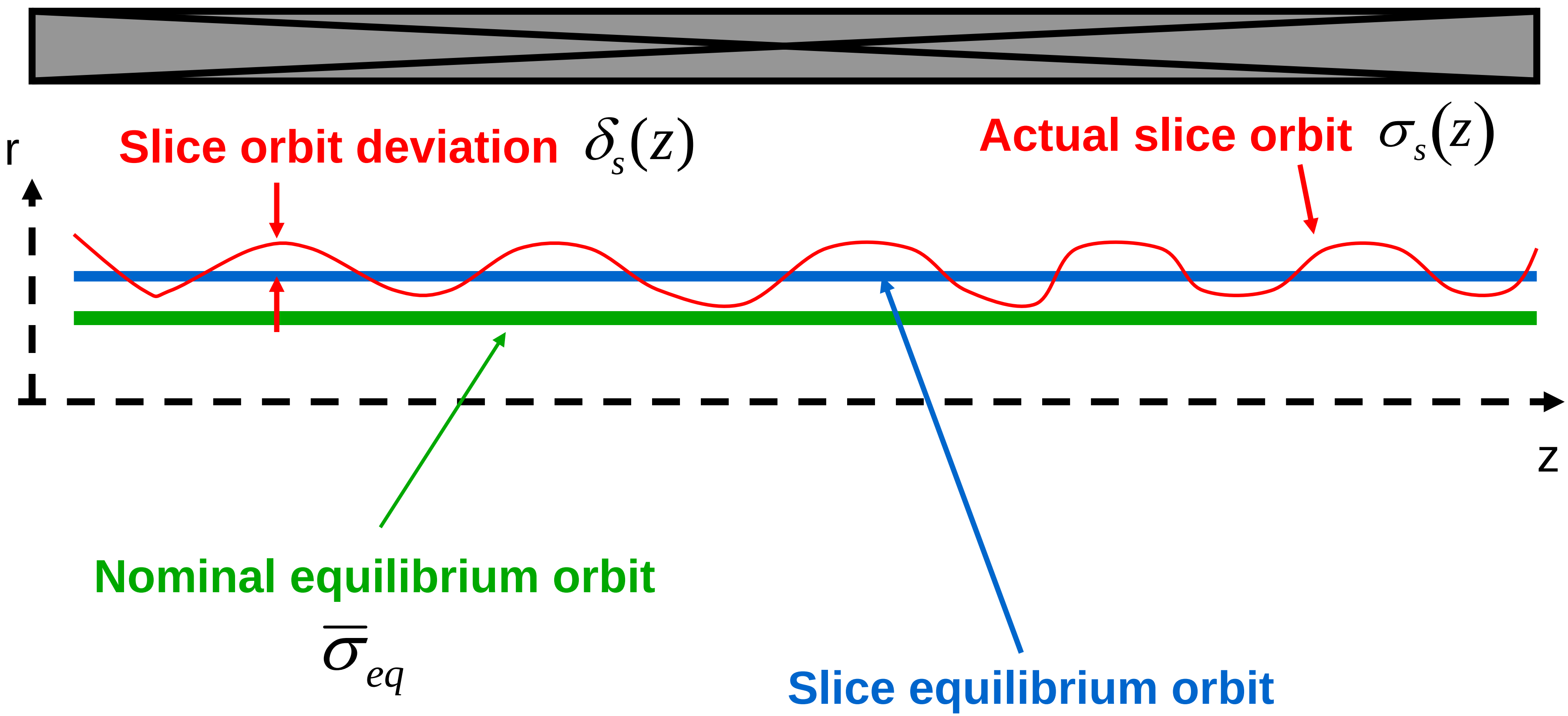
$$\sigma_{eq,s} \propto \frac{I_s^{1/2}}{\gamma_s^{1/2}} \;\Rightarrow\; d\sigma_{eq,s} = \partial\sigma_{eq,s} + \sigma_{eq,s}\left(\frac{1}{2}\frac{\partial I}{I_s} - \frac{1}{2}\frac{\partial\gamma}{\gamma_s}\right)$$

We look for a description of the beam evolution in the presence of very small deviations from equilibrium spot size, current, and energy.

$$\sigma_s(z) = \bar{\sigma}_{eq} + \bar{\sigma}_{eq}\left(\frac{1}{2}\frac{\delta I(s)}{I} - \frac{1}{2}\frac{\delta\gamma(s)}{\gamma}\right) + \delta_s(z) \;=\; \sigma_{eq}(s) + \delta_s(z)$$

where $\sigma_{eq}(s) = \bar{\sigma}_{eq} + \bar{\sigma}_{eq}\left(\frac{1}{2}\frac{\delta I(s)}{I} - \frac{1}{2}\frac{\delta\gamma(s)}{\gamma}\right)$ is the slice variation in the equilibrium envelope, measured from the nominal slice current and energy and $\delta_s(z)$ measures the mismatch from the slice equilibrium.

Continuous solenoid channel



$$\sigma_{eq}(s) = \overline{\sigma}_{eq} + \overline{\sigma}_{eq}\left(\frac{1}{2}\frac{\delta I(s)}{I} - \frac{1}{2}\frac{\delta\gamma(s)}{\gamma}\right)$$

Linear oscillations about equilibrium, II

We substitute the expression describing the slice equilibrium envelope and small deviations back into the envelope equation and, retaining only linear terms, we obtain an equation of motion for the slice variations

$$\delta_s'' + \frac{\gamma'}{\gamma}\delta_s' + \bar{k}^2\delta_s = 0$$

where $\bar{k}^2 = k_{eff}^2 + 3\frac{\varepsilon_{eff}^2}{\sigma_{eq}^4} + \frac{K}{\sigma_{eq}^2} = 4\frac{\varepsilon_{eff}^2}{\sigma_{eq}^4} + 2\frac{K}{\sigma_{eq}^2} \approx 2k_{eff}^2$

This linear, 2nd order ordinary differential equation describes the orbit deviations of the individual slices (parameterized by '*s*') along the beamline coordinate ('*z*').

We will make a few case studies.

Coasting beam

We first consider the case of a beam nearly matched into a continuous focusing channel, without any acceleration present, i.e. $\gamma'=0$ and k_{eff} is constant.

The equation of motion $\delta_s'' + \bar{k}^2 \delta_s = 0$

admits solutions

$$\delta_s(z) = \delta_{s0} \cos \bar{k}z + \frac{\delta_{s0}'}{\bar{k}} \sin \bar{k}z$$

$$\delta_s'(z) = \delta_{s0}' \cos \bar{k}z - \bar{k}\delta_{s0} \sin \bar{k}z$$

We should pause a moment and observe that the oscillation wavenumber $\bar{k}$ has no dependence on the slice parameter variations, in the linear approximation.

We expect **all** slices, regardless of current, spot size, emittance to oscillate with the same frequency regardless of amplitude.

This is rather important.

Projected emittance of a coasting beam

Let's calculate the projected emittance of our beam. To simplify the calculation, we will make an assumption on the initial conditions: all slices start at a waist (i.e. zero slice radial divergence). The slice radii then evolve as

$$\sigma_s(z) = \sigma_{eq}(s) + \delta_{s0}\cos\bar{k}z \ , \ \sigma_s'(z) = -\bar{k}\delta_{s0}\sin\bar{k}z$$

The projected emittance is then calculated from $\varepsilon_{proj}^2 = \left\langle \sigma_r^{\ 2} \right\rangle_s \left\langle \sigma_r'^{\,2} \right\rangle_s - \left\langle \sigma_r \sigma_r' \right\rangle_s^2$

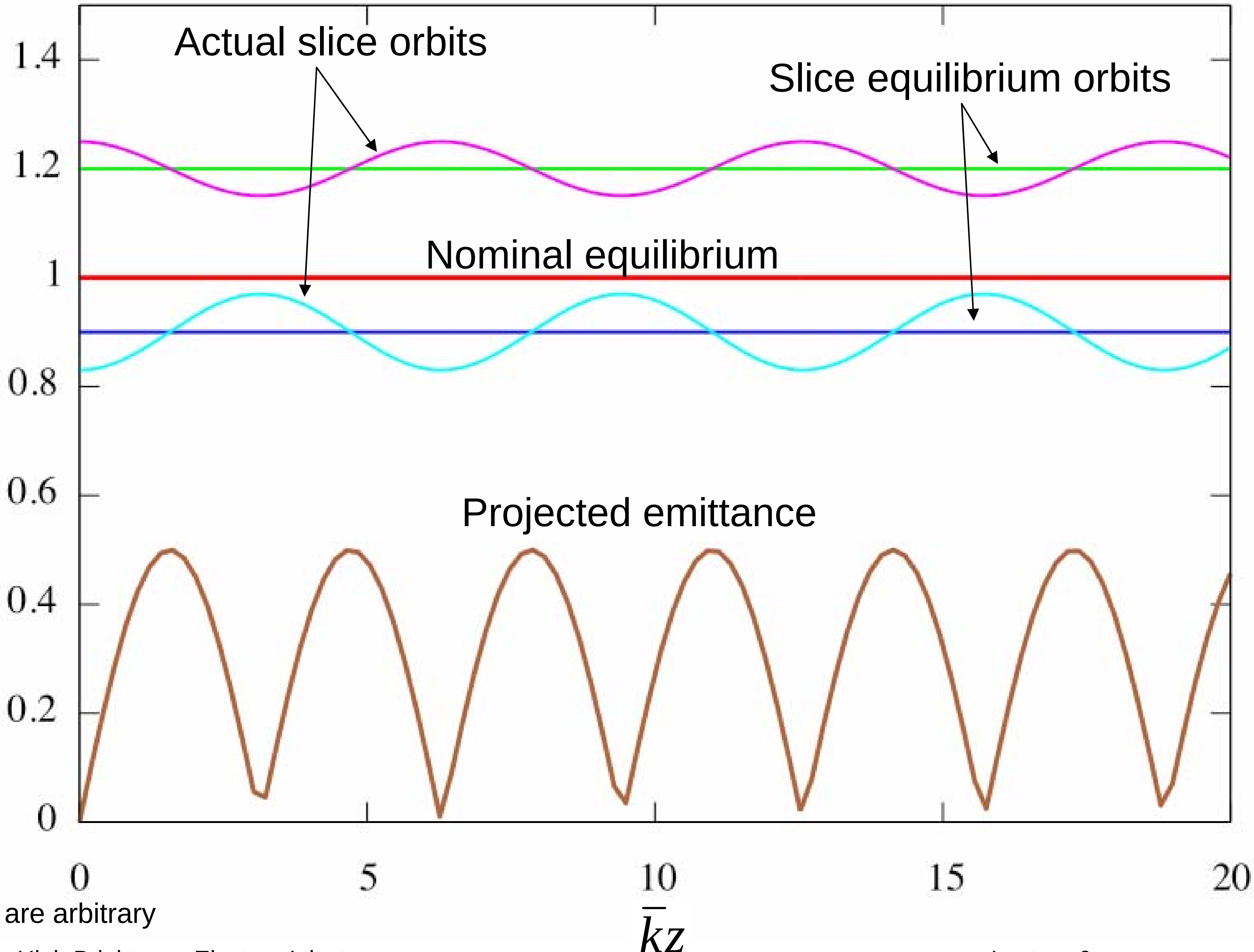
Carrying out the computation yields

$$\varepsilon_{proj} = \sigma_{eq}\bar{k} \left|\sin\bar{k}z\right| \left(\left\langle \delta_{s0}^{\ 2} \right\rangle - \left\langle \delta_{s0} \right\rangle^2 \right)^{1/2}$$

Recalling the definition of the radial spot size variation yields

$$\sigma_s(z) - \bar{\sigma}_{eq} = \delta_s(z) + \bar{\sigma}_{eq}\left(\frac{1}{2}\frac{\delta I(s)}{I} - \frac{1}{2}\frac{\delta\gamma(s)}{\gamma} \right) \ \Rightarrow$$

$$\varepsilon_{proj} = \bar{\sigma}_{eq}\bar{k}\left\langle (\delta_0)^2 \right\rangle^{1/2} \left\{ 1 + \frac{1}{4}\left[\left\langle \left(\frac{\partial I}{I} \right)^2 \right\rangle + \left\langle \left(\frac{\partial \gamma}{\gamma} \right)^2 \right\rangle \right] \right\}^{1/2} \left|\sin\bar{k}z\right|$$



Units are arbitrary

Adding acceleration

The envelope equation in it's original form includes acceleration terms. We would like to re-write it in a form that permits easier analysis.

$$\sigma_{r,s}'' + \frac{\gamma_s'}{\gamma_s}\sigma_{r,s}' + k_{eff,s}^2\sigma_{r,s} - \frac{\varepsilon_{eff,s}^2}{\sigma_{r,s}^3} - \frac{K_s}{\sigma_{r,s}} = 0$$

If we define a *scaled rms envelope radius* $\hat{\sigma} = \sigma_r\sqrt{\gamma}$ the linear portion of the envelope equation transforms as

$$\sigma_r'' + \frac{\gamma'}{\gamma}\sigma_r' + k_{eff}^2\sigma_r = \frac{\hat{\sigma}'' + \hat{k}^2\hat{\sigma}}{\gamma^{1/2}}$$

where

$$\hat{k}^2 = k_{eff}^2 - \frac{1}{4}\left(\frac{\gamma'}{\gamma}\right)^2 - \frac{1}{2}\left(\frac{\gamma'}{\gamma}\right)' \Rightarrow \hat{k}^2 = k_{rf}^2 + k_L^2 + \frac{(\gamma^2+2)(\gamma')^2}{4\gamma^4} \cong k_L^2 + \frac{3}{4}\left(\frac{\gamma'}{\gamma}\right)^2$$

$$k_{eff}^2 = k_{es}^2 + k_{rf}^2 + k_L^2 \qquad k_{es}^2 = \frac{\gamma''}{2\gamma}\ ,\ k_{rf}^2 = \frac{1}{2}\left(\frac{\gamma'}{\gamma}\right)^2\ ,\ k_L^2 = \left(\frac{eB_z}{2\gamma mc}\right)^2$$

Transforming the equation

We transform the rest of the beam parameters in our accelerated frame

Original	*Transformed*	
σ_r	$\hat{\sigma} = \sigma_r \gamma^{1/2}$	
k_{eff}^2	$\hat{k}^2 = k_L^2 + \frac{3}{4}\left(\frac{\gamma'}{\gamma}\right)^2$	
ε_{eff}^2	$\varepsilon_n^2 = \varepsilon_{eff}^2 \gamma^2$	
K	$\kappa = K\gamma^3$	$\kappa = 2I/I_0$

And find the *invariant envelope equation*

$$\hat{\sigma}_s'' + \hat{k}_s^2 \hat{\sigma}_s - \frac{\varepsilon_n^2}{\hat{\sigma}_s^3} - \frac{\kappa_s}{\gamma^2 \hat{\sigma}_s} = 0$$

Coasting beam correspondence

The invariant envelope accounts for the expected and observed decrease of the beam radius with increasing energy, as well as the finite (non-zero) values of the divergence. It redefines what it means to have a 'matched beam' solution in the presence of acceleration.

$$\hat{\sigma}'' = 0 \;\Rightarrow\; \sigma'' + \left(\frac{\gamma'}{\gamma}\right)\sigma' - \left(\frac{\gamma'}{2\gamma}\right)^2 \sigma = 0$$

$$\hat{\sigma}' = 0 \;\Rightarrow\; \sigma' = -\left(\frac{\gamma'}{2\gamma}\right)\sigma$$

Matched beam conditions

Nevertheless, there is a formal equivalence between the coasting beam solution we already found, and the solutions for our invariant envelope equation.

Invariant envelope 'equilibrium'

Equilibrium or 'matched' beam envelope solutions will exhibit

$$\hat{\sigma}_{eq} = \left(\frac{\kappa}{2\gamma^2\hat{k}^2}\right)^{1/2} \left\{1 + \left[1 + \left(\frac{2\gamma^2\hat{k}\varepsilon_n}{\kappa}\right)^2\right]^{1/2}\right\}^{1/2}$$

The space-charge dominated regime occurs at lower energies where

$$\gamma << \sqrt{\frac{\kappa}{2\hat{k}\varepsilon_n}} \;\Rightarrow\; \hat{\sigma}_{s-c} = \left(\frac{4}{3}\frac{\kappa}{\gamma'^2}\right)^{1/2} \Rightarrow \sigma_{s-c} = \frac{1}{\gamma'}\left(\frac{4}{3}\frac{\kappa}{\gamma}\right)^{1/2}$$

At high energies the beam envelope is emittance dominated

$$\gamma >> \sqrt{\frac{\kappa}{2\hat{k}\varepsilon_n}} \;\Rightarrow \hat{\sigma}_{emit} = \left(\frac{\varepsilon_n}{\hat{k}}\right)^{1/2} = \left(\frac{2\gamma\varepsilon_n}{\sqrt{3}\gamma'}\right)^{1/2} \Rightarrow \sigma_{emit} = \left(\frac{2\varepsilon_n}{\sqrt{3}\gamma'}\right)^{1/2}$$

Scaling

In the space charge dominated regime the matched or equilibrium beam envelope has the limiting behavior

$$\hat{\sigma}_{eq,s-c} = \left(\frac{4}{3}\frac{\kappa}{\gamma'^2}\right)^{1/2} \Rightarrow \sigma_{eq,s-c} = \frac{1}{\gamma'}\left(\frac{4}{3}\frac{\kappa}{\gamma}\right)^{1/2}$$

The invariant envelope only scales as $I^{1/2}$, since the dependence on γ is factored out. Small variations to the invariant envelope size have two sources - initial invariant spot size mismatches, and any modulation in the current

$$\hat{\sigma}_{eq,s-c} \propto I^{1/2} \;\; \Rightarrow \;\; d\hat{\sigma}_{eq,s-c} = \partial\hat{\sigma}_{eq,s-c} + \left(\frac{1}{2}\frac{\partial I}{I}\right)\hat{\sigma}_{eq,s-c}$$

The invariant envelope for individual slices

$$\hat{\sigma}_s(z) = \hat{\bar{\sigma}}_{eq} + \left(\frac{1}{2}\frac{\delta I(s)}{I}\right)\hat{\bar{\sigma}}_{eq} + \hat{\delta}_s(z) \;=\; \hat{\sigma}_{eq}(s) + \hat{\delta}_s(z)$$

where $\hat{\sigma}_{eq}(s) \;=\; \hat{\bar{\sigma}}_{eq} + \left(\frac{1}{2}\frac{\delta I(s)}{I}\right)\hat{\bar{\sigma}}_{eq}$ is the slice variation in equilibrium envelope

and $\hat{\delta}_s(z)$ measures the mismatch from the slice equilibrium.

Projected emittance under acceleration

Recall our definition of the slice-averaged projected emittance for a coasting beam (no acceleration)

$$\varepsilon_{proj}^2 = \left\langle \sigma_r^{\,2} \right\rangle \left\langle \sigma_r'^{\,2} \right\rangle - \left\langle \sigma_r \sigma_r' \right\rangle^2$$

We will adopt the same definition of projected emittance in the case of a beam undergoing acceleration, using scaled envelope measures:

$$\hat{\varepsilon}_{proj}^2 = \left\langle \hat{\sigma}_r^{\,2} \right\rangle \left\langle \hat{\sigma}_r'^{\,2} \right\rangle - \left\langle \hat{\sigma}_r \hat{\sigma}_r' \right\rangle^2$$

For envelope orbits near the matched invariant envelope solution we can show that

$$\hat{\varepsilon}_{proj}^2 = \gamma^2 \varepsilon_{proj}^2 = \varepsilon_{n,proj}^2$$

and we recover the *normalized* projected emittance.

Emittance oscillations with acceleration

Carrying out a similar calculation to that for the coasting beam, we can compute the normalized, projected emittance for a nearly matched, accelerated, space-charge dominated beam. By ‘nearly matched’ we mean matched in the average sense, but allowing rms variations.

Defining the perturbation wavenumber

$$\bar{\hat{k}}^2 = \hat{k}^2 + 3\frac{\varepsilon_n^2}{\hat{\sigma}_{eq}^4} + \frac{\kappa}{\gamma^2 \hat{\sigma}_{eq}^2} = 4\frac{\varepsilon_n^2}{\hat{\sigma}_{eq}^4} + 2\frac{\kappa}{\gamma^2 \hat{\sigma}_{eq}^2} \approx 2\hat{k}^2 = 2\left[k_L^2 + \frac{3}{4}\left(\frac{\gamma'}{\gamma}\right)^2\right] \propto \gamma^{-2}$$

Making the simplifying assumption that all slices are perfectly matched in their divergences, the normalized projected emittance follows

$$\varepsilon_{n,proj} = \bar{\hat{\sigma}}_{eq} \bar{\hat{k}} \left\langle \left(\hat{\delta}_0\right)^2 \right\rangle^{1/2} \left\{1 + \frac{1}{4}\left[\left\langle \left(\frac{\partial I}{I}\right)^2 \right\rangle\right]\right\}^{1/2} \left|\sin \bar{\hat{k}} z\right|$$

Normalized emittance evolution

In terms of the initial slice variations and energy at the entrance to the accelerating channel, the normalized emittance

$$\varepsilon_{n,proj} = \frac{(2\overline{\kappa})}{\gamma_0 + \gamma' z} \left\langle (\delta_0)^2 \right\rangle^{1/2} \gamma_0^{1/2} \left\{ 1 + \frac{1}{4} \left[\left\langle \left(\frac{\partial I}{I} \right)^2 \right\rangle + \left\langle \left(\frac{\partial \gamma}{\gamma} \right)^2 \right\rangle \right] \right\}^{1/2} \left| \sin\left(\sqrt{\frac{3}{2}} \frac{\gamma' z}{\gamma_0 + \gamma' z} \right) \right|$$

$$\varepsilon_{n,proj} \propto \frac{\left| \sin\left(\sqrt{\frac{3}{2}} \frac{\gamma' z}{\gamma_0 + \gamma' z} \right) \right|}{\gamma_0 + \gamma' z} \Rightarrow \frac{\sin\left(\sqrt{\frac{3}{2}} \right)}{\gamma} \qquad z \Rightarrow \infty$$

At increasing energies, the rotation of the beam slices in phase space slows to a negligible rate, and the projected emittance is ‘frozen in’.

Any residual projected emittance will remain with the beam

Emittance compensation

Here's our challenge as injector designers -

Given the requirements for beam charge (current), energy, etc., how do we obtain beams with the lowest possible projected emittance?

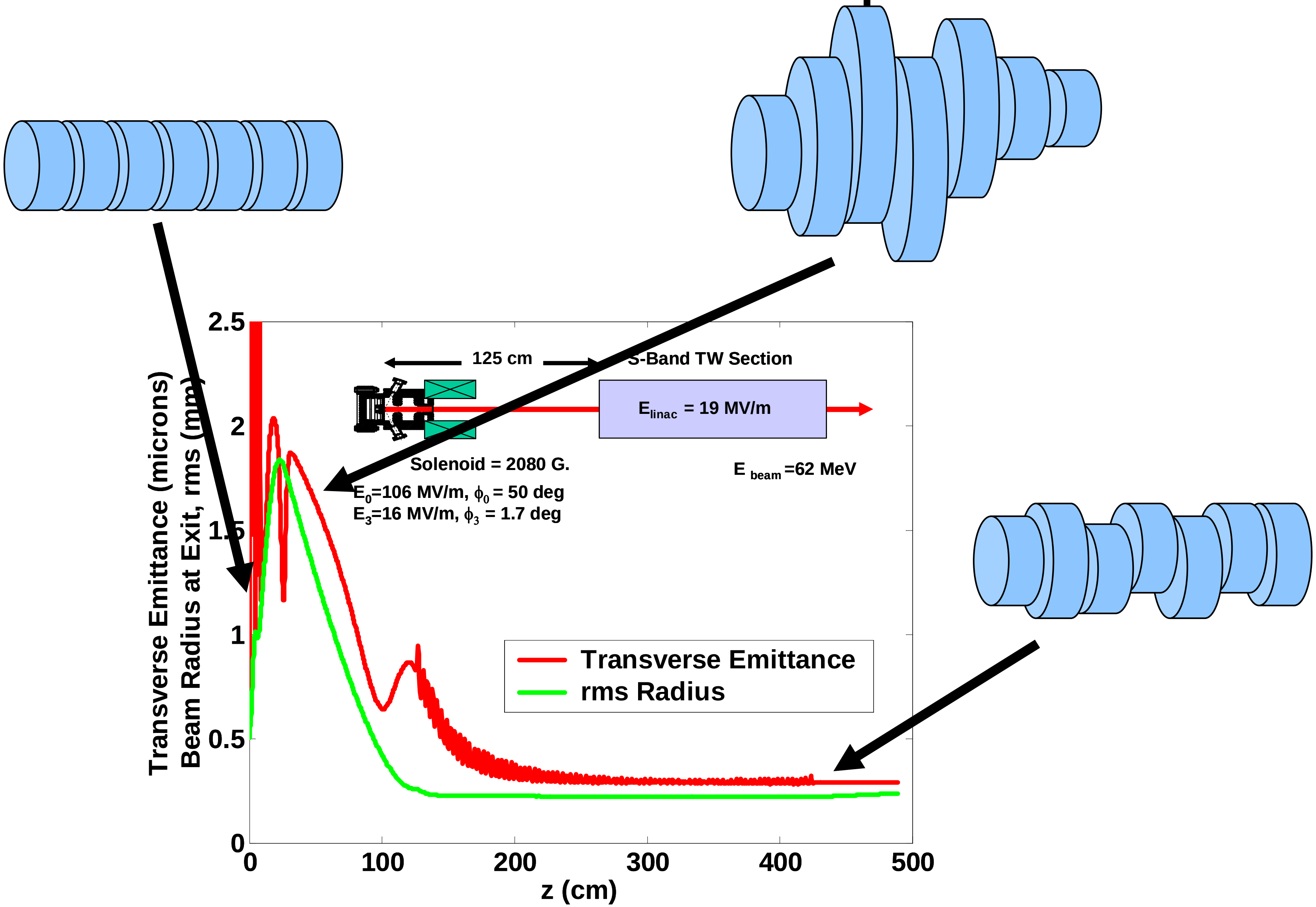
Answer: **emittance compensation!**

$$\varepsilon_{n,proj} = \bar{\hat{\sigma}}_{eq} \bar{\hat{k}} \left\langle \left(\hat{\delta}_0\right)^2 \right\rangle^{1/2} \left\{ 1 + \frac{1}{4} \left[\left\langle \left(\frac{\partial I}{I} \right)^2 \right\rangle \right] \right\}^{1/2} \left| \sin \bar{\hat{k}} z \right|$$

By controlling the focusing lattice parameters and the schedule of acceleration down the beamline, we have the ability to tune the phase of the emittance oscillation.

We must, then arrange the emittance oscillation phase to approach $n\pi$ (where *n* is an integer) as the energy increases to the point where the oscillation stops.

Picture of emittance compensation



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