Spectral Properties of Synchrotron Radiation

US Particle Accelerator School
January 14-18, 2008

- Motivation
- Conceptual View of SR Emission
- Angular Spectral Power Density
- Practical Applications
- Visible Light
- Laboratory Time Scales

"The Physics of Synchrotron Radiation" A. Hofmann
"Synchrotron Radiation" H. Wiedemann

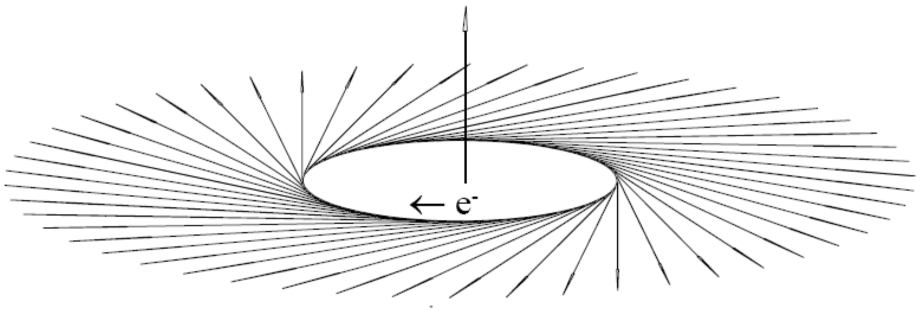
Motivation for Understanding Field Pattern

For engineering, SR science applications and diagnostic purposes we need to know...

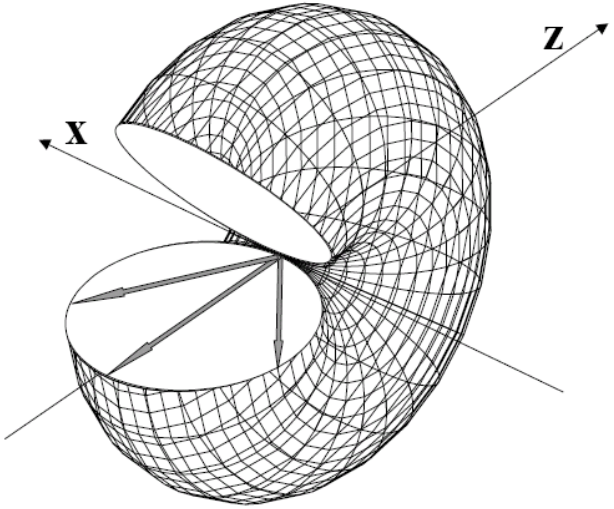
- photon beam frequency spectrum
- photon beam angular distribution (vertical)
- total photon beam power
- power in a given bandwidth
- photon flux in a given bandwidth
- photon brightness in a given bandwidth
- photon beam coherence, polarization, etc

Synchrotron Radiation Basics

radiation emission from a storage ring

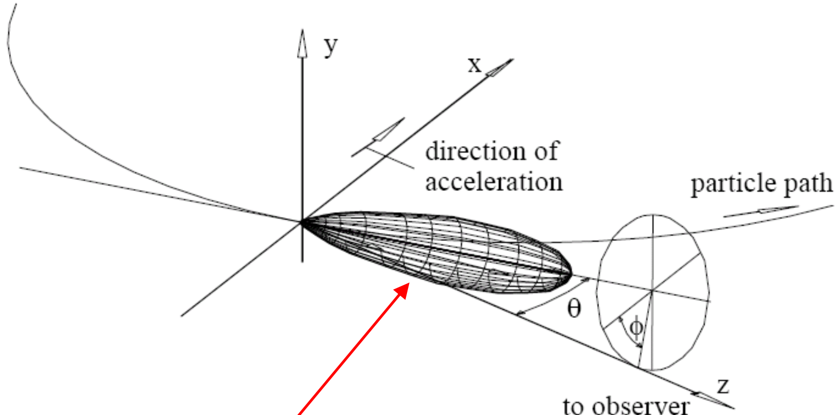


radiation emission in particle system



SR Basics (cont'd)

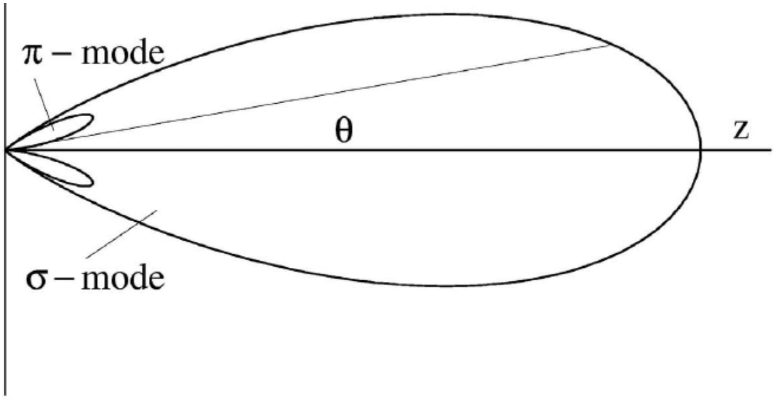
radiation emission in laboratory system



infrared to x-ray spectrum!

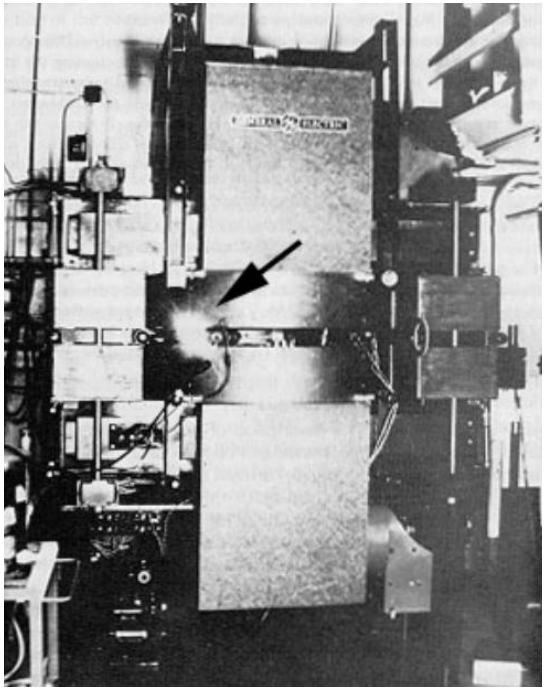
(ordinary heat to passage-through-matter)

Intensity of two modes: σ and π



SR Basics (cont'd)

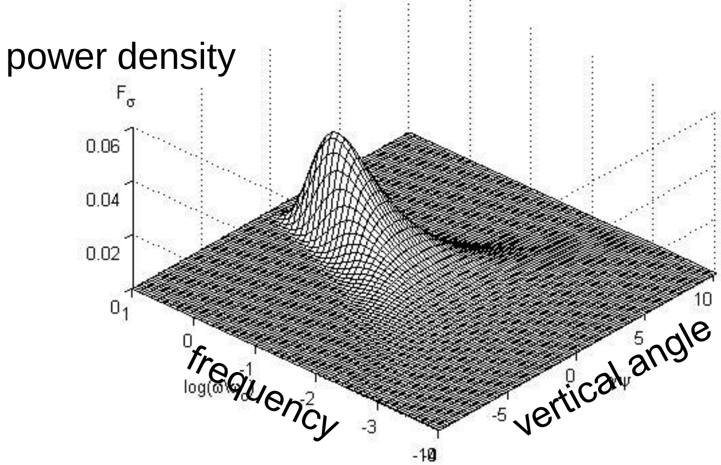
First light - GE synchrotron



A Billion dollar user machine

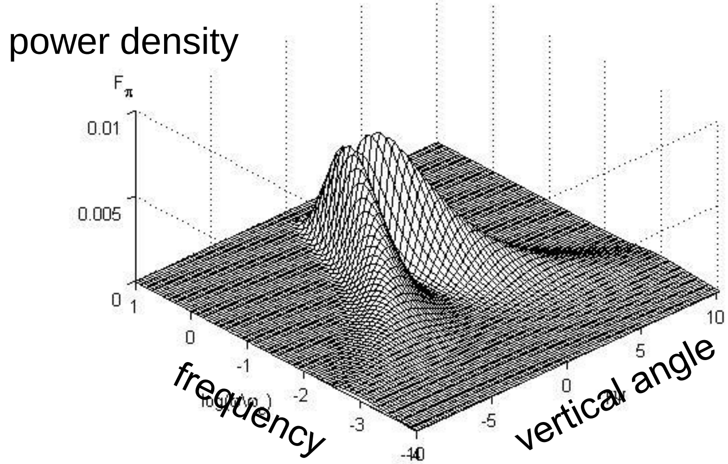
Angular Spectral Power Density Functions

σ-mode polarization

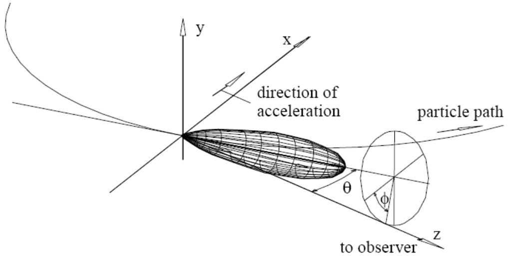


$$F_{s\sigma}(\omega,\psi)=\left(\frac{3}{2\pi}\right)^3\left(\frac{\omega}{2\omega_c}\right)^2\left(1+\gamma^2\psi^2\right)^2 K_{2/3}^2\left(\frac{\omega}{2\omega_c}\left(1+\gamma^2\psi^2\right)^{3/2}\right)$$

π-mode polarization



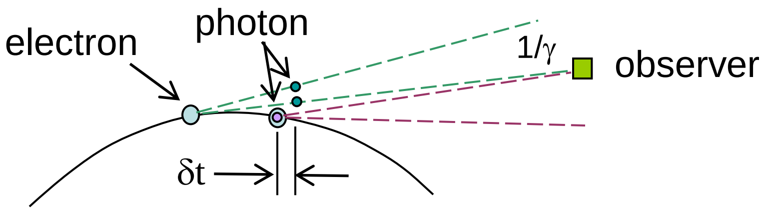
$$F_{s\pi}(\omega,\psi)=\left(\frac{3}{2\pi}\right)^3\left(\frac{\omega}{2\omega_c}\right)^2\gamma^2\psi^2\left(1+\gamma^2\psi^2\right)K_{1/3}^2\left(\frac{\omega}{2\omega_c}\left(1+\gamma^2\psi^2\right)^{3/2}\right)$$



$$\frac{d^2P}{d\Omega d\omega}=\frac{P_s\gamma}{\omega_c}\left[F_{s\sigma}(\omega,\psi)+F_{s\pi}(\omega,\psi)\right]$$

Can we derive these equations, Prof. Schwinger?
Surely, You're Joking...

1/γ and the Critical Frequency



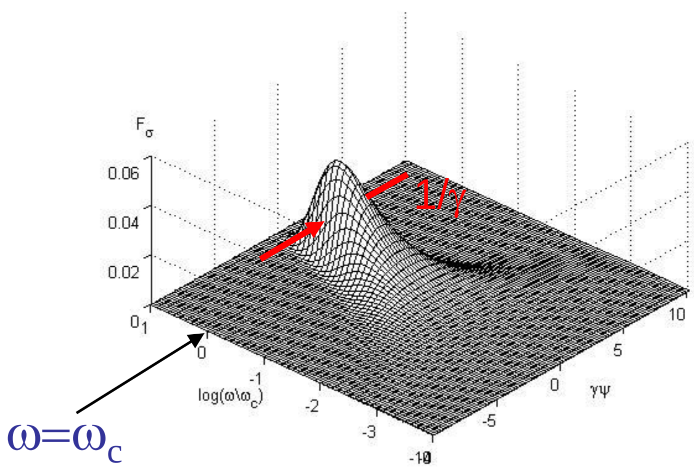
$$\sin(x) \approx x - \frac{x^3}{6}$$

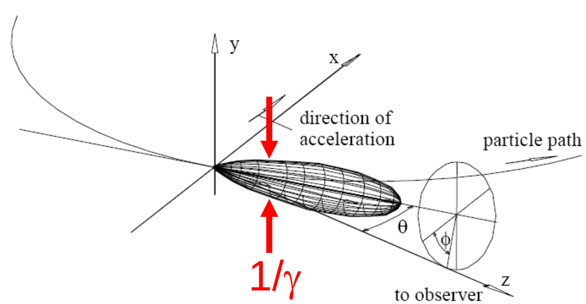
$x=1/\gamma$

$$\delta t = \frac{4\rho}{3c\gamma^3} \quad \delta t = 10^{-19}\ sec!$$

$$\omega_c = \frac{3c\gamma^3}{2\rho} \quad f_c = 10^{19}\ Hz!$$

$$E_c = 7keV\ @\ 3GeV$$





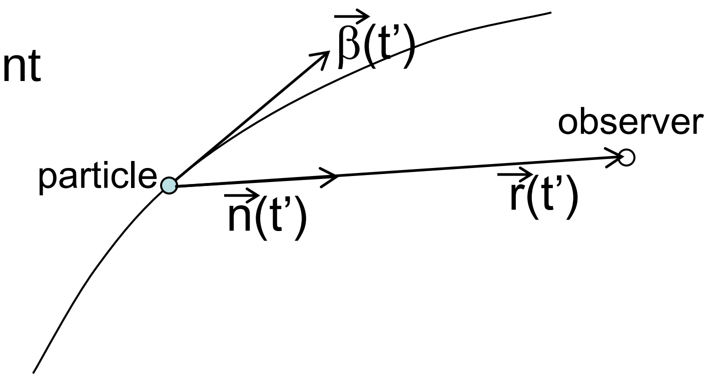
Derivation of Angular Spectral Power Density

Start with Lienard-Wiechert Potentials

$$V(t) = \int \frac{\rho(t')}{r(t)} dV \Big|_{ret}$$ scalar potential from charge

p 14

$$\mathbf{A}(t) = \int \frac{\mathbf{J}(t')}{r(t)} dV \Big|_{ret}$$ vector potential from current



$$\mathbf{E} = -\nabla V - \frac{d\mathbf{A}}{dt}$$ electric field

p 11

$$\mathbf{B} = \nabla \times \mathbf{A}$$ magnetic field

$$\mathbf{S} = \frac{\mathbf{E} \times \mathbf{B}}{\mu_o}$$ Poynting Vector (power flux)

Electro-magnetic field (cont'd)

$$\mathbf{E} = -\nabla V - \frac{d\mathbf{A}}{dt}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

After a Jacksonian derivation

$$\mathbf{E}(t) = \frac{1}{4\pi\varepsilon_o c}\left\{\frac{\left[\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\right]\times\dot{\boldsymbol{\beta}}\right]}{r\left(1-\mathbf{n}\cdot\boldsymbol{\beta}\right)^3}\right\}_{\text{ret}}$$

p 17,20

$$\mathbf{B}(t) = \left.\frac{\mathbf{n}\times\mathbf{E}}{c}\right|_{\text{ret}}$$

Prof. Hofmann

'difficulty evaluating the above equations'
'advantageous to calculate Fourier transforms'

Fourier Transform Field Equations

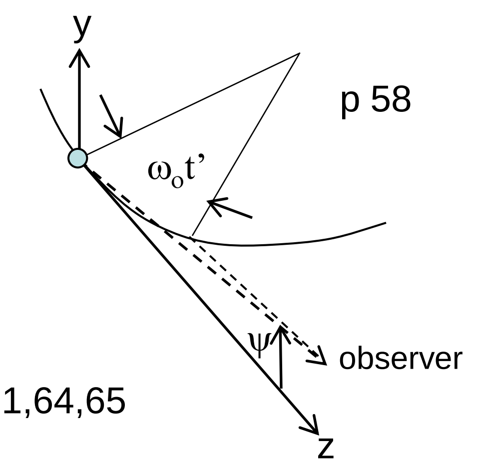
$$\tilde{\mathbf{E}}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathbf{E}(t)e^{-i\omega t}dt$$

$$\tilde{\mathbf{E}}(\omega)=\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{4\pi\varepsilon_o c}\int_{-\infty}^{\infty}\left\{\frac{[\mathbf{n}\times[\mathbf{n}-\boldsymbol{\beta}]\times\boldsymbol{\beta}]}{r(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right\}e^{-i\omega(t'-r(t')/c)}dt$$ p 35

Still looks bad but $\vec{n}$ and $\vec{r}$ are approximately constant, except |r| in phase term

Integrate by parts

$$\tilde{\mathbf{E}}(\omega)=\frac{i\omega}{\sqrt{2\pi}}\cdot\frac{1}{4\pi\varepsilon_o c}\int_{-\infty}^{\infty}[\mathbf{n}\times[\mathbf{n}\times\boldsymbol{\beta}]]e^{-i\omega(t'-r(t')/c)}dt$$ p 36



Decompose vector into components

$$\tilde{\mathbf{E}}(\omega)=\frac{i\omega e}{\sqrt{2\pi}}\cdot\frac{1}{4\pi\varepsilon_o c r_c}\int_{-\infty}^{\infty}[\underbrace{-\omega_o t',\psi,0}_{\text{relativistic approximation}}]e^{-i\omega(t'-r(t')/c)}dt'$$ p 61,64,65

Small Angle and Relativistic Approximations

$$\tilde{E}(\omega) = \frac{i\omega e}{\sqrt{2\pi}} \cdot \frac{1}{4\pi\varepsilon_o c r_c} \int_{-\infty}^{\infty} [-\omega_o t', \psi, 0] e^{-i\omega(t' - r(t')/c)} dt'$$

$$t' - \frac{r(t')c}{c} \approx t' - \frac{\cos\psi \rho \sin(\omega_o t')}{c}$$

$$\cos\psi \approx 1 - \frac{\psi^2}{2} \qquad \sin(\omega_o t') \approx \omega_o t' - \frac{(\omega_o t)^3}{6}$$

p 60-65*

$$1 - \beta \approx \frac{1}{2\gamma^2} \qquad \omega_o = \frac{\beta c}{\rho}$$

$$t' - \frac{r(t')c}{c} \approx t' \left(\frac{1 + \gamma^2 \psi^2}{2\gamma^2} \right) + 6 \frac{c^2 t'^3}{2\rho^2}$$

$$\tilde{E}(\omega) = \frac{i\omega e}{\sqrt{2\pi}} \cdot \frac{1}{4\pi\varepsilon_o c r_c} \int_{-\infty}^{\infty} [-\omega_o t, \psi, 0] \exp(-i\omega(\frac{t'(1 + \gamma^2 \psi^2)}{2\gamma^2} + \frac{c^2 t'^3}{6\rho^2}) dt'$$

p 65

Change Variables and Integrate

$$\tilde{E}(\omega) = \frac{i\omega e}{\sqrt{2\pi}} \cdot \frac{1}{4\pi\varepsilon_o c r_c} \int_{-\infty}^{\infty} [-\omega_o t, \psi, 0] \exp(-i\omega(\frac{t'(1+\gamma^2\psi^2)}{2\gamma^2} + \frac{c^2 t'^3}{6\rho^2}) dt'$$

$$\tilde{E}_x(\omega) = \frac{-e\gamma}{(2\pi)^{3/2}\varepsilon_o c r_c} \cdot \left(\frac{3\omega}{4\omega_c}\right)^{1/3} \int_{-\infty}^{\infty} u \sin\left(\left(\frac{3\omega}{4\omega_c}\right)^{2/3} \left(1+\gamma^2\psi^2\right) u + \frac{u^3}{3}\right) du$$

$$\tilde{E}_y(\omega) = \frac{ie\gamma^2}{(2\pi)^{3/2}\varepsilon_o c r_c} \cdot \left(\frac{3\omega}{4\omega_c}\right)^{2/3} \int_{-\infty}^{\infty} \cos\left(\left(\frac{3\omega}{4\omega_c}\right)^{2/3} \left(1+\gamma^2\psi^2\right) u + \frac{u^3}{3}\right) du$$

p 66

$$\tilde{E}_x(\omega) = \frac{-\sqrt{3}e\gamma}{(2\pi)^{3/2}\varepsilon_o c r_c} \cdot \frac{\omega}{2\omega_c} \cdot \left(1+\gamma^2\psi^2\right) K_{2/3}\left(\frac{\omega}{2\omega_c}\left(1+\gamma^2\psi^2\right)^{3/2}\right)$$

$$\tilde{E}_y(\omega) = \frac{i\sqrt{3}e\gamma}{(2\pi)^{3/2}\varepsilon_o c r_c} \cdot \frac{\omega}{2\omega_c} \cdot \gamma\psi\left(1+\gamma^2\psi^2\right) K_{1/3}\left(\frac{\omega}{2\omega_c}\left(1+\gamma^2\psi^2\right)^{3/2}\right)$$

p 67

Flux, Energy and Power Density

Return to Poynting's vector for power flux

$$\mathbf{S} = \frac{\mathbf{E}\times\mathbf{B}}{\mu_o} = \frac{\mathbf{E^2}}{\mu_o c} = \frac{1}{r^2}\frac{d^2U}{d\Omega dt}$$ (where U is the radiated energy) p 41,51

1. Solve for radiated energy U into a unit solid angle

$$\frac{dU}{d\Omega} = r^2\int S dt = \frac{r^2}{\mu_0 c}\int E(t)^2 dt = \frac{r^2}{\mu_0 c}\int E(\omega)^2 d\omega$$ (by Parseval's theorem) p 51

2. Differentiate with repect to ω to find angular spectral <u>energy</u> density

$$\frac{dU}{d\Omega d\omega} = \frac{2r^2}{\mu_0 c}E(\omega)^2$$ (factor of 2 from positive frequencies only) p 52

3. Note that power=energy/time: $P = \frac{\omega_o}{2\pi}U$ (time interval is one turn)

$$\frac{d^2P}{d\Omega d\omega} = \frac{\omega_0}{2\pi}\frac{d^2U}{d\Omega d\omega} = \frac{2r^2\omega_0\left|\tilde{\mathbf{E}}(\omega)\right|^2}{2\pi\mu_0 c}.$$

Angular Spectral Power Density p 52

Angular Spectral Power Density

$$\frac{d^2P}{d\Omega d\omega} = \frac{\omega_0}{2\pi}\frac{d^2U}{d\Omega d\omega} = \frac{2r^2\omega_0\left|\tilde{\mathbf{E}}(\omega)\right|^2}{2\pi\mu_0 c}.$$

Angular Spectral Power Density

there are two polarizations

$$\frac{d^2P}{d\Omega d\omega} = \frac{d^2P_\sigma}{d\Omega d\omega} + \frac{d^2P_\pi}{d\Omega d\omega} = \frac{2r^2}{2\pi\mu_0\rho}\left(\left|\tilde{E}_x(\omega)\right|^2 + \left|\tilde{E}_y(\omega)\right|^2\right).$$

p 68

where from before

$$\tilde{E}_x(\omega) = \frac{-\sqrt{3}e\gamma}{(2\pi)^{3/2}\varepsilon_o c r_c}\cdot\frac{\omega}{2\omega_c}\cdot\left(1+\gamma^2\psi^2\right)K_{2/3}\left(\frac{\omega}{2\omega_c}\left(1+\gamma^2\psi^2\right)^{3/2}\right)$$

$$\tilde{E}_y(\omega) = \frac{i\sqrt{3}e\gamma}{(2\pi)^{3/2}\varepsilon_o c r_c}\cdot\frac{\omega}{2\omega_c}\cdot\gamma\psi\left(1+\gamma^2\psi^2\right)K_{1/3}\left(\frac{\omega}{2\omega_c}\left(1+\gamma^2\psi^2\right)^{3/2}\right)$$

we still have to square these expressions

The F_σ and F_π Functions

$$\frac{d^2P}{d\Omega d\omega} = \frac{d^2P_\sigma}{d\Omega d\omega} + \frac{d^2P_\pi}{d\Omega d\omega} = \frac{2r^2}{2\pi\mu_0\rho}\left(\left|\tilde{E}_x(\omega)\right|^2 + \left|\tilde{E}_y(\omega)\right|^2\right).$$

$$\frac{d^2P}{d\Omega d\omega} = \frac{P_s\gamma}{\omega_c}\left[F_{s\sigma}(\omega,\psi) + F_{s\pi}(\omega,\psi)\right] = \frac{P_s\gamma}{\omega_c}F_s(\omega,\psi).$$

p 83

where $P_s = \frac{2r_0 c m_0 c^2 \gamma^4}{3\rho^2}.$ is the total radiated power for one particle

and we define

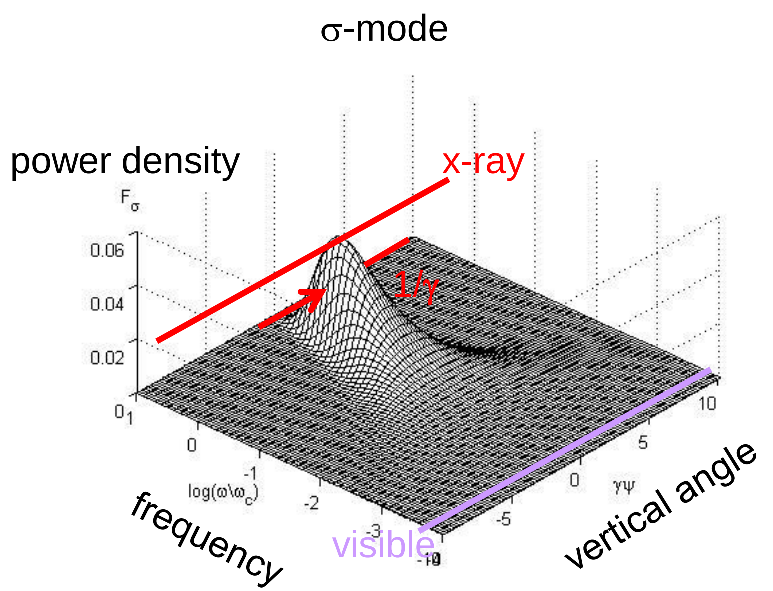
$$F_{s\sigma}(\omega,\psi) = \left(\frac{3}{2\pi}\right)^3 \left(\frac{\omega}{2\omega_c}\right)^2 \left(1+\gamma^2\psi^2\right)^2 K_{2/3}^2\left(\frac{\omega}{2\omega_c}\left(1+\gamma^2\psi^2\right)^{3/2}\right).$$

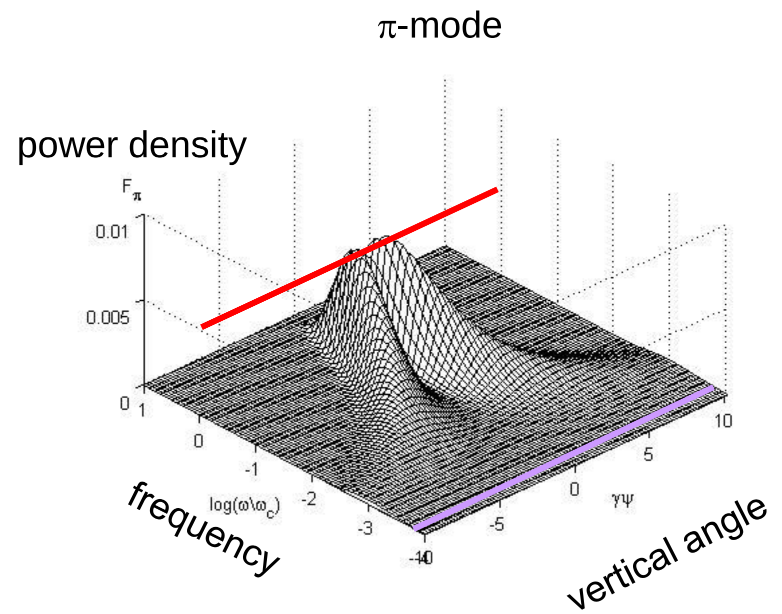
$$F_{s\pi}(\omega,\psi) = \left(\frac{3}{2\pi}\right)^3 \left(\frac{\omega}{2\omega_c}\right)^2 \gamma^2\psi^2\left(1+\gamma^2\psi^2\right) K_{1/3}^2\left(\frac{\omega}{2\omega_c}\left(1+\gamma^2\psi^2\right)^{3/2}\right).$$

p 84

plot as a function of normalize frequency and normalized angle...

The Angular Spectral Power Density Functions



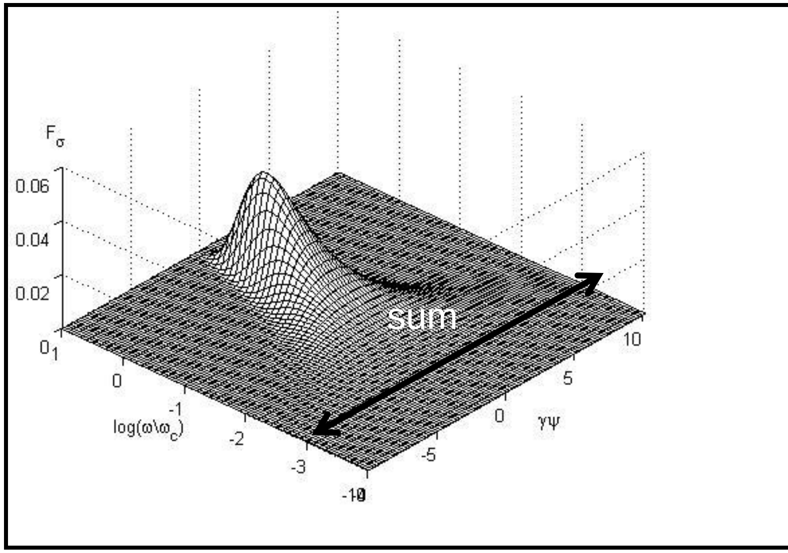


$$\frac{d^2P}{d\Omega d\omega} = \frac{P_s\gamma}{\omega_c}\left[F_{s\sigma}(\omega,\psi) + F_{s\pi}(\omega,\psi)\right]$$

$$F_{s\sigma}(\omega,\psi) = \left(\frac{3}{2\pi}\right)^3 \left(\frac{\omega}{2\omega_c}\right)^2 \left(1+\gamma^2\psi^2\right)^2 K_{2/3}^2\left(\frac{\omega}{2\omega_c}\left(1+\gamma^2\psi^2\right)^{3/2}\right)$$

$$F_{s\pi}(\omega,\psi) = \left(\frac{3}{2\pi}\right)^3 \left(\frac{\omega}{2\omega_c}\right)^2 \gamma^2\psi^2\left(1+\gamma^2\psi^2\right) K_{1/3}^2\left(\frac{\omega}{2\omega_c}\left(1+\gamma^2\psi^2\right)^{3/2}\right)$$

Integral Over Vertical Angle



universal dipole flux curve

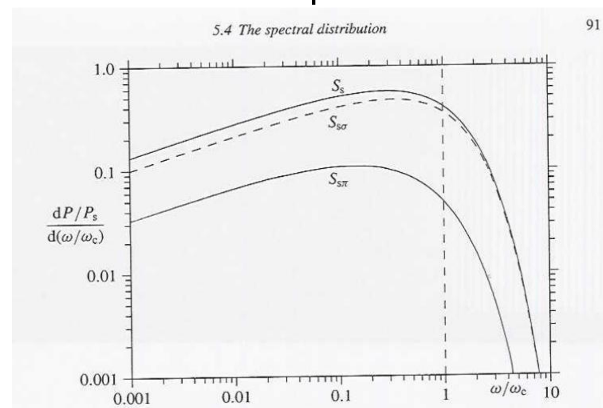


Fig. 5.5. The normalized power spectrum in double-logarithmic representation.

$$\frac{dP}{d\omega} = \int_0^\infty \frac{d^2P}{d\Omega d\omega} d\Omega = \frac{P_s}{\omega_c}\left[S_\sigma\left(\frac{\omega}{\omega_c}\right) + S_\pi\left(\frac{\omega}{\omega_c}\right)\right] = \frac{P_s}{\omega_c} S\left(\frac{\omega}{\omega_c}\right)$$

where

$$S_\sigma\left(\frac{\omega}{\omega_c}\right) = \frac{9\sqrt{3}}{16\pi}\frac{\omega}{\omega_c}\left(\int_{\frac{\omega}{\omega_c}}^\infty K_{5/3}(z')dz' + K_{2/3}\left(\frac{\omega}{\omega_c}\right)\right)$$

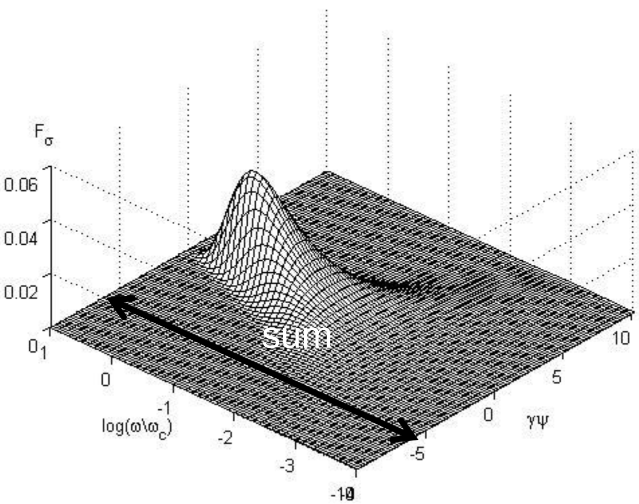
p 89,90

$$S_\pi\left(\frac{\omega}{\omega_c}\right) = \frac{9\sqrt{3}}{16\pi}\frac{\omega}{\omega_c}\left(\int_{\frac{\omega}{\omega_c}}^\infty K_{5/3}(z')dz' - K_{2/3}\left(\frac{\omega}{\omega_c}\right)\right)$$

$$S\left(\frac{\omega}{\omega_c}\right) = \frac{9\sqrt{3}}{8\pi}\frac{\omega}{\omega_c}\int_{\frac{\omega}{\omega_c}}^\infty K_{5/3}(z')dz'$$

Sands

Integral Over Frequency



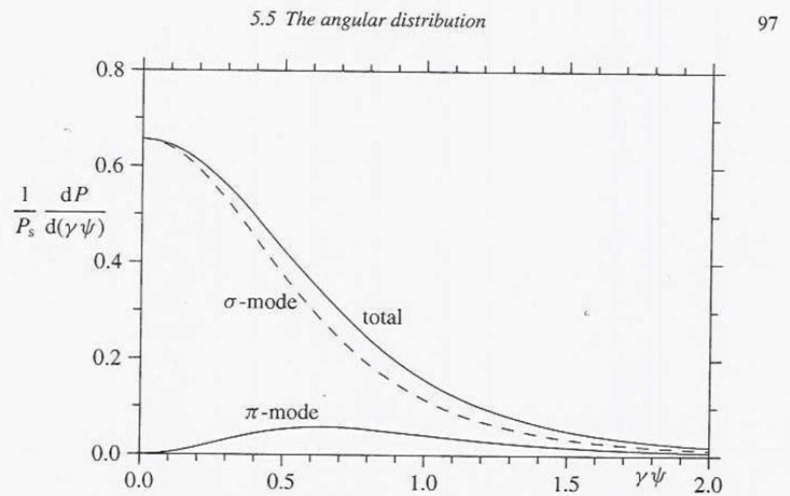


Fig. 5.9. The angular distribution after integrating over frequencies.

$$\frac{dP}{d\Omega} = \int_0^\infty \frac{d^2P}{d\Omega d\omega} d\omega = \frac{P_s\gamma}{\omega_c}\int_0^\infty [F_{s\sigma}(\omega,\psi) + F_{s\pi}(\omega,\psi)] d\omega.$$

$$\frac{dP_\sigma}{d\Omega} = \frac{P_s\gamma}{2\pi}\frac{21}{32}\frac{1}{(1+\gamma^2\psi^2)^{5/2}}.$$

$$\frac{dP_\pi}{d\Omega} = \frac{P_s\gamma}{2\pi}\frac{15}{32}\frac{\gamma^2\psi^2}{(1+\gamma^2\psi^2)^{7/2}}.$$

p 96

RMS Opening Angle as a Function of Frequency

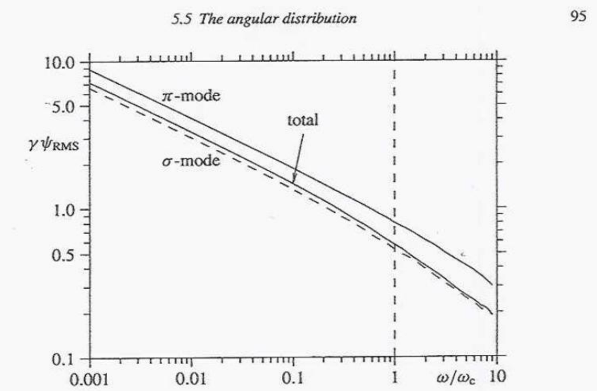


Fig. 5.8. The vertical RMS opening angle versus the frequency.

$$\left\langle \gamma^2\psi^2 \right\rangle_\sigma = \frac{\int \gamma^2\psi^2 \frac{d^2 p_\sigma}{d\Omega d\omega} d\Omega}{\int \frac{d^2 p_\sigma}{d\Omega d\omega} d\Omega} = \frac{2\pi}{S_{s\sigma}} \int_{-\infty}^{\infty} \gamma^2\psi^2 F_{s\sigma}(\omega,\psi) d(\gamma\psi)$$

$$(\gamma\psi)_{RMS} = \sqrt{\left\langle \gamma^2\psi^2 \right\rangle}$$ p 94

$$\left\langle \gamma^2\psi^2 \right\rangle_\pi = \frac{\int \gamma^2\psi^2 \frac{d^2 p_\pi}{d\Omega d\omega} d\Omega}{\int \frac{d^2 p_\sigma}{d\Omega d\omega} d\Omega} = \frac{2\pi}{S_{s\pi}} \int_{-\infty}^{\infty} \gamma^2\psi^2 F_{s\pi}(\omega,\psi) d(\gamma\psi)$$

at long wavelengths $$\sqrt{\left\langle \psi^2 \right\rangle} = 0.449 \left(\frac{\lambda}{\rho}\right)^{1/3}$$ p 96

$$\sqrt{\left\langle \psi^2 \right\rangle} = 0.449 \left(\frac{550nm}{8m}\right)^{1/3} = 2mr$$ (opening angle of green light)

Total Integrals – A Reality Check

total power $$\frac{d^2P}{d\Omega d\omega} = \frac{P_s\gamma}{\omega_c}\left[F_{s\sigma}(\omega,\psi) + F_{s\pi}(\omega,\psi)\right]$$

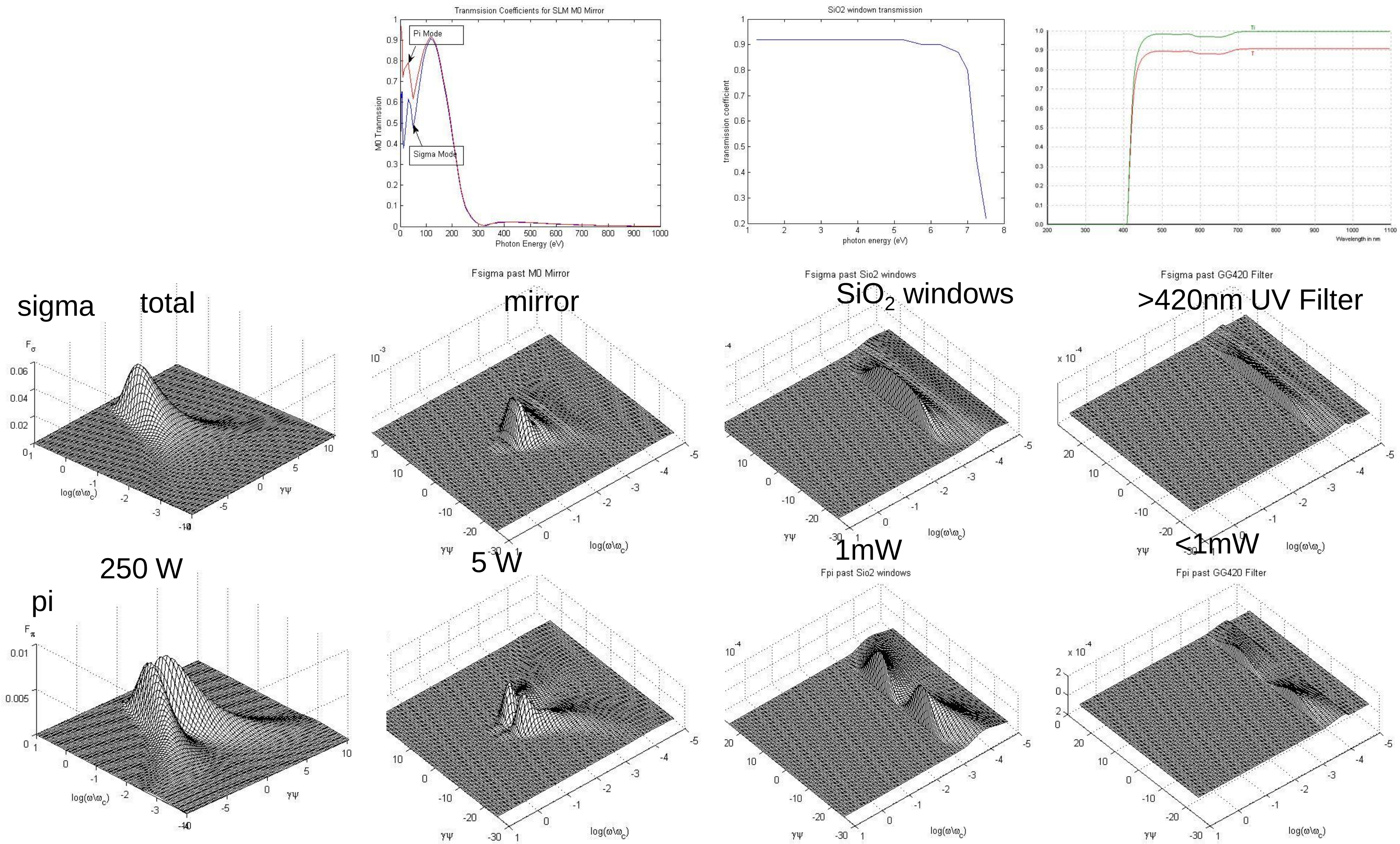
$$\int_0^{2\pi} d\phi \int_{-\infty}^{\infty} d(\gamma\psi) \int_0^{\infty} F_s(\omega,\psi)\, d(\omega/\omega_c) = 1.$$

$$\iint \frac{d^2P}{d\Omega d\omega} d\Omega d\omega = P_s \qquad P_s = \frac{2r_0 c m_0 c^2 \gamma^4}{3\rho^2}.$$

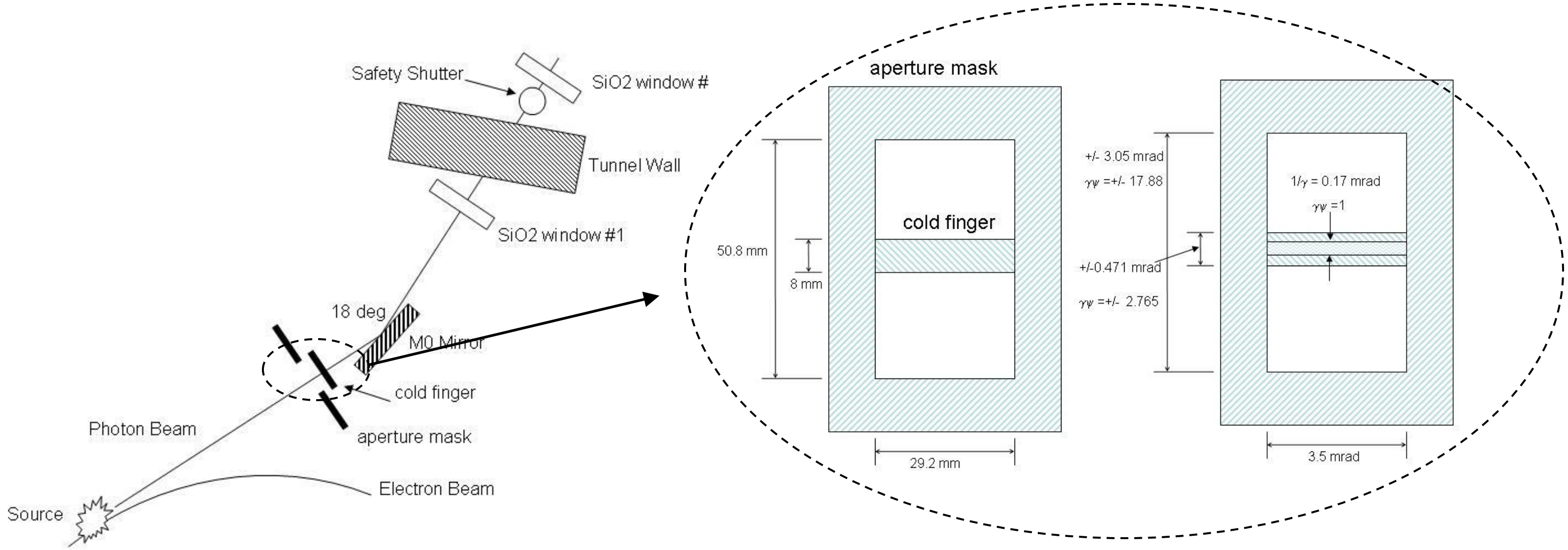
$$U_o = \int P_s dt = \int P_s ds/c = 88.5 \frac{E^4(GeV)}{\rho(m)}$$

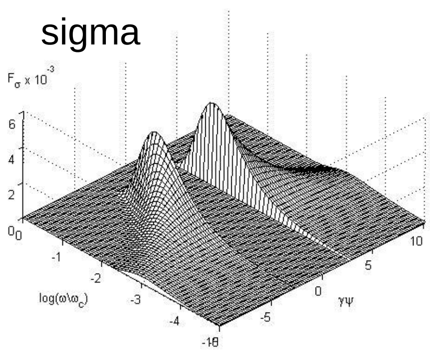
for I=200ma, E=3GeV, ρ=7.5m, P=200kW, U=1MeV/turn

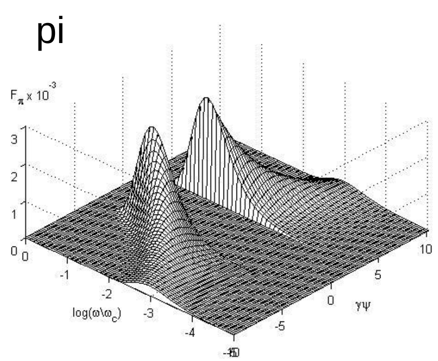
Power Through a Diagnostic Beam Line



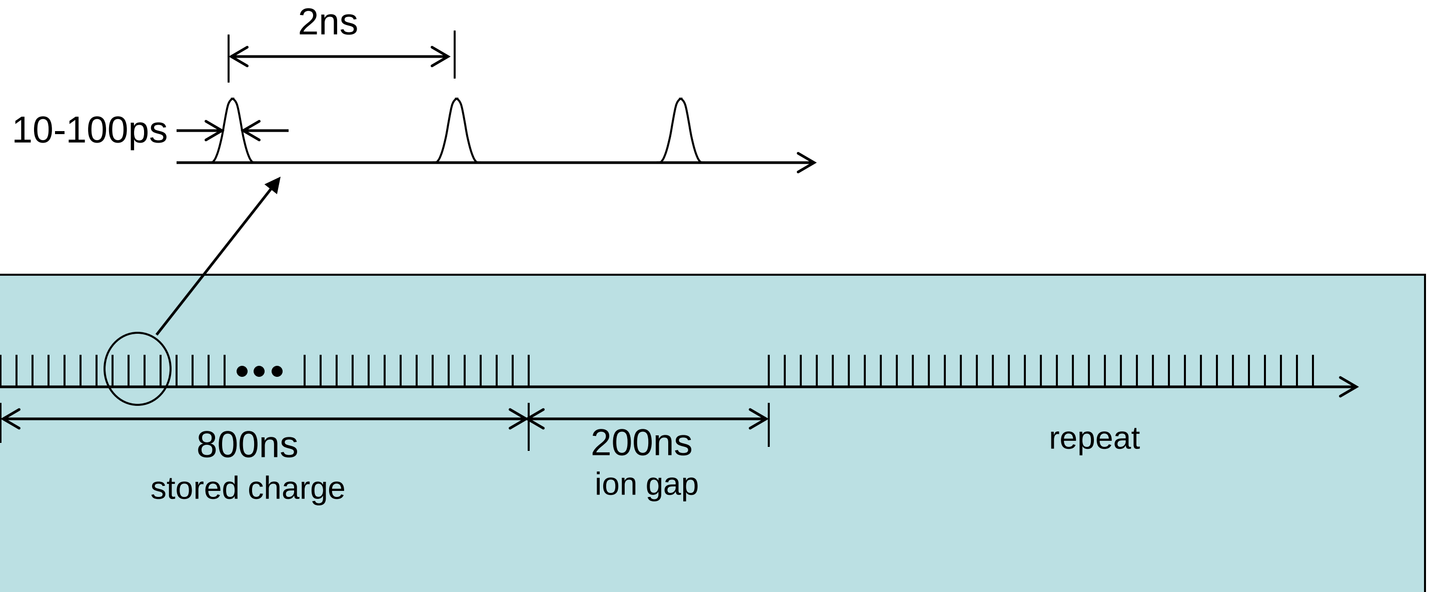
Cold Finger for Mirror Protection







Timing – Revisited (Alan and Walter will say more)

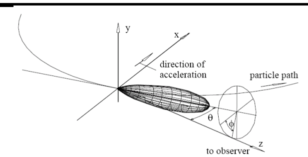


Important timescales

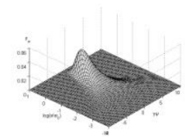
10^{-3} 1s, Hz	booster, vibrations, slow camera read-out
10^{-3} ms, kHz	video, vibrations, power lines, fast camera read-out
10^{-6} μs, MHz	ring clock, TTL pulses, instabilities, spectrum analyzers
10^{-9} ns, GHz	RF, fast-gate cameras, syncroscan, scopes
10^{-12} ps, THz	bunch lengths, IR radiation, jitter, single-bunch instabilities
10^{-15} fs, pHz	visible radiation, FEL pulses, jitter
...	UV radiation, electron energy levels
	critical frequency, xrays, electron dynamics

Summary – Synchrotron Radiation Properties and Timing

- SR Emission Cone



- Spectral Angular Power Density



$$\frac{d^2P}{d\Omega d\omega}.$$

- Practical Applications

- Visible Light $\sqrt{\langle\psi^2\rangle} = 0.449\left(\frac{550nm}{8m}\right)^{1/3} = 2mr$ <1mW

- Laboratory Time Scales